Chapter 4

Nonlinear reaction diffusion equations

In predicting most population, Malthus model can only be used in a short time. Due to the limited resource, the growth rate of population will slow down, and the population may saturate to a maximum level. Thus it is natural to use a density dependent growth rate per capita, and the simplest and mostly used is the logistic growth rate. Together with the diffusion of the species, we consider the following diffusive logistic equation:

$$\frac{\partial P}{\partial t} = D\Delta P + aP\left(1 - \frac{P}{N}\right), \tag{4.1}$$

where $a > 0$ is the maximum growth rate per capita and $N > 0$ is the carrying capacity. Diffusive logistic equation is the simplest *nonlinear* reaction-diffusion equation, but the methods of analyzing it can be used in many other problems. For the simplicity, we will mainly consider the spatial dimension $n = 1$.

4.1 Nondimensionalization

Let's start with diffusive logistic equation with Dirichlet boundary condition on $(0, 1)$:

$$\begin{cases} u_t = Du_{xx} + au\left(1 - \dfrac{u}{N}\right), & t > 0, \quad x \in (0, L), \\ u(t, 0) = u(t, L) = 0, & \\ u(0, x) = f(x), & x \in (0, L). \end{cases} \tag{4.2}$$

The method of separation of variables fails here to find an explicit expression of the solution to the equation, and that is not surprised due to the nonlinear nature of the equation. It is possible to write a formula for the equilibrium solution of the equation in an elliptic integral, but that is mainly because of the special form of the quadratic growth rate. Interested readers should find that calculation in [Ske51] or [Ban94] (page 363). We will mainly use qualitative and numerical methods to analyzing the equation, although sometimes this will still require analytic tools to achieve.

In both qualitative and numerical methods, the dependence of solutions on the parameters plays an important role, and there are always more difficulties when there are more parameters. Hence usually we need to reduce the number of parameters by converting the equation into nondimensionized version. Here we reduce the number of parameters in the equation by a dimension analysis and a nondimensionalization substitution. In the following table, we list the dimension of all variables and parameters

Variables	Dimension	Parameters	Dimension
t	τ	D	$\tau^{-1}\theta^2$
x	θ	a	τ^{-1}
u	μ	N	μ
		L	θ

To reduce the number of parameters, we introduce dimensionless variables:

$$s = \frac{Dt}{L^2}, \quad y = \frac{x}{L}, \quad v = \frac{u}{N}. \tag{4.3}$$

Then from the chain rule, we have

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial t} = \frac{DN}{L^2}\frac{\partial v}{\partial s}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial v} \cdot \frac{\partial^2 v}{\partial y^2} \cdot \left(\frac{\partial y}{\partial x}\right)^2 = \frac{N}{L^2}\frac{\partial^2 v}{\partial y^2},$$

and the equation becomes

$$\begin{cases} \dfrac{\partial v}{\partial s} = \dfrac{\partial^2 v}{\partial y^2} + \lambda v(1-v), \quad s > 0, \ y \in (0,1), \\ v(t,0) = v(t,1) = 0, \\ v(0,y) = f_1(y), \end{cases} \tag{4.4}$$

where

$$\lambda = \frac{aL^2}{D}, \quad f_1(y) = \frac{f(Ly)}{N}. \tag{4.5}$$

In the following sections, we will use (4.4) in the analysis, but since we normally use variables (t, x, u) instead of (s, y, v). Thus we rewrite (4.4) in the old variables:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \lambda u(1-u), \qquad t > 0, \ x \in (0,1), \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = f(x), \ x \in (0,1). \end{cases} \tag{4.6}$$

The way of reducing parameters is not unique. The new equation (4.6) is sometimes called a *nonlinear eigenvalue problem* with λ as a bifurcation parameter. Another nondimensionalization will result in an equation with the length of interval as the parameter (see the exercises.)

4.2 Numerical method for 1-d problem

Investigation of many hard problems starts from a numerical computation, and a successful numerical study will lead to the discovery of the mathematical theory behind the phenomena displayed by the numerical simulations. Numerical computation of reaction diffusion equations itself can well be the subject of a whole book, but here we will just introduce the most natural one, which may not be the most efficient or accurate, but an easy one to learn.

The simplest numerical algorithm for an ordinary differential equation is Euler's method. For an equation

$$\frac{du}{dt} = f(t,u), \quad u(t_0) = u_0, \tag{4.7}$$

the linear approximation gives the simple iteration formula:

$$t_{n+1} = t_n + \Delta t, \quad u_{n+1} = u_n + f(t_n, u_n)\Delta t, \tag{4.8}$$

where Δt is the step size. For reaction diffusion equation, we can design a similar algorithm, but now we have to take the spatial variable into the consideration. Again we use a set of finite points to represent the continuous variable x. Let's take the example of diffusive logistic equation for our numerical computation by considering equation (4.6).

We divide the interval $[0,1]$ to n equal subintervals $[x^i, x^{i+1}]$, $i = 1, 2, \cdots, n, n+1$, with $x^i = x^{i-1} + \Delta x$, $x^1 = 0$ and $\Delta x = 1/n$ (which we call *grid size*.) We also denote the *step size* (in the time direction) by Δt. The initial data now is

$$u_0^1 = f(x^1),\ u_0^2 = f(x^2), \cdots, u_0^{n-1} = f(x^{n-1}),\ u_0^n = f(x^n), \tag{4.9}$$

where

$$u_j^i = u(t_j, x^i), \quad t_j = t_{j-1} + \Delta t, x^i = x^{i-1} + \Delta x, \quad j \geq 1,\ 1 \leq i \leq n+1. \tag{4.10}$$

Because of the Dirichlet boundary condition, u_1^j and u_{n+1}^j should always be zero, thus we only need to calculate $u_{i,j}$ with $2 \leq i \leq n$. Let

$$v_j = [u_j^1, u_j^2, \cdots, u_j^{n+1}]^T, \tag{4.11}$$

where T is the transpose of the matrix (since we would like to have column vector instead of row vector.) Then the idea is very similar to Euler method, as we try to get vector v_{j+1} from v_j.

The differential equation can be approximated by difference equations from the following relations:

$$\frac{\partial u(t_j, x^i)}{\partial t} \approx \frac{u_{j+1}^i - u_j^i}{\Delta t}, \quad \frac{\partial u(t_j, x^i)}{\partial x} \approx \frac{u_j^{i+1} - u_j^i}{\Delta x}, \tag{4.12}$$

$$\frac{\partial^2 u(t_j, x^i)}{\partial x^2} \approx \frac{u_x(t_j, x^{i+1}) - u_x(t_j, x^i)}{\Delta x} \approx \frac{u_j^{i+1} - 2u_j^i + u_j^{i-1}}{(\Delta x)^2}. \tag{4.13}$$

Therefore the differential equation $u_t = Du_{xx} + \lambda g(u)$ is now

$$\frac{u_{j+1}^i - u_j^i}{\Delta t} = D\frac{u_j^{i+1} - 2u_j^i + u_j^{i-1}}{(\Delta x)^2} + \lambda g(u_j^i). \tag{4.14}$$

Define

$$r = \frac{D\Delta t}{(\Delta x)^2}, \quad c = 1 - 2r. \tag{4.15}$$

Then (4.14) can be rewritten to

$$u_{j+1}^i = ru_j^{i+1} + (1-2r)u_j^i + ru_j^{i-1} + \lambda g(u_j^i)\Delta t. \tag{4.16}$$

Notice that for $i = 2$ or $i = n$, (4.16) still hold since $u_j^1 = u_j^{n+1} = 0$ from the Dirichlet boundary condition. If we use matrix notation, we obtain (assuming $n = 10$)

$$\begin{bmatrix} u_{j+1}^2 \\ u_{j+1}^3 \\ u_{j+1}^4 \\ u_{j+1}^5 \\ u_{j+1}^6 \\ u_{j+1}^7 \\ u_{j+1}^8 \\ u_{j+1}^9 \\ u_{j+1}^{10} \end{bmatrix} = \begin{pmatrix} c & r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ r & c & r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r & c & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r & c & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r & c & r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r & c & r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r & c & r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r & c & r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r & c \end{pmatrix} \cdot \begin{bmatrix} u_j^2 \\ u_j^3 \\ u_j^4 \\ u_j^5 \\ u_j^6 \\ u_j^7 \\ u_j^8 \\ u_j^9 \\ u_j^{10} \end{bmatrix} + \lambda\Delta t \begin{bmatrix} g(u_j^2) \\ g(u_j^3) \\ g(u_j^4) \\ g(u_j^5) \\ g(u_j^6) \\ g(u_j^7) \\ g(u_j^8) \\ g(u_j^9) \\ g(u_j^{10}) \end{bmatrix}. \tag{4.17}$$

If we denote the matrix in (4.17) by A, then (4.17) can be written as

$$V_{j+1} = A \cdot V_j + \lambda G(V_j)\Delta t, \tag{4.18}$$

where G is the vector valued function given in (4.17). In general, V_j is an $(n-2)$-vector, and A is an $(n-1) \times (n-2)$ matrix. The algorithm can easily be implemented in `Maple` either based on matrix computation like (4.18) or the iteration computation in (4.16).

Now we adapt the algorithm to Neumann boundary value problem with $u_x(t,x) = 0$ at the endpoints. In this case, the boundary values are not fixed, thus we should consider an $(n+1)$-vector V_i to include the boundary points and an $(n+1) \times (n+1)$ matrix B. For each j, u_{j+1}^i can be determined by (4.16) except at $i = 1$ and $i = n$. We have to extend the solution $u(t,x)$ in a reasonable way beyond $i = 1$ and $i = n$. Since homogeneous Neumann boundary condition is "reflecting", then we can define $u_j^0 = u_j^2$ and $u_j^{n+2} = u_j^n$. Thus

$$u_{j+1}^1 = ru_j^2 + (1-2r)u_j^1 + ru_j^0 + \lambda g(u_j^1)\Delta t = 2ru_j^2 + (1-2r)u_j^1 + \lambda g(u_j^1)\Delta t. \tag{4.19}$$

And same for u_{j+1}^{n+1}. In matrix notation, the matrix B is of form (in the case $n = 10$)

$$\begin{pmatrix} c & 2r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ r & c & r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r & c & r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r & c & r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r & c & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r & c & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r & c & r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r & c & r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r & c & r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r & c & r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2r & c \end{pmatrix}. \tag{4.20}$$

In numerical analysis, the method we describe in this section is called finite difference method. Here we use a forward difference for u_t in (4.12), and a centered difference for u_{xx} in (4.13). A backward difference is

$$\frac{\partial u(t_j, x^i)}{\partial t} \approx \frac{u_j^i - u_{j-1}^i}{\Delta t}. \tag{4.21}$$

We should notice that if we choose Δt and Δx improperly, the result of the numerical approximation may be ridiculous, similar to Euler's method in ordinary differential equations. For example, when $D = 1$, $\Delta t = \Delta x = 1$, then (4.12) becomes

$$u_{j+1}^i = u_j^{i+1} - 2u_j^i + u_j^{i-1}. \tag{4.22}$$

Suppose that we start with an initial value $u_1 = [0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0]$, and consider the Dirichlet boundary value problem. Then the iteration sequence is

$$\begin{aligned} u_2 &= [0, 0, 0, 0, 1, -2, 1, 0, 0, 0, 0] \\ u_3 &= [0, 0, 0, 1, -4, 6, -4, 1, 0, 0, 0] \\ u_4 &= [0, 0, 1, -6, 15, -20, 15, -6, 1, 0, 0] \\ &\cdots\cdots \end{aligned} \tag{4.23}$$

However as you remember, the solution of diffusion equation with a positive initial value should keep positive! Such an iteration will not provide an approximation to the true solution, thus it is called *unstable*. It can be show that the above forward finite difference method is stable if

$$r = \frac{D\Delta t}{(\Delta x)^2} \leq \frac{1}{2}. \tag{4.24}$$

The stability condition (4.24) means that the time step size Δt has to be small relative to $(\Delta x)^2$. For example when $\Delta = 0.1$ and $D = 1$, then Δt must be less than 0.005. To reach $T = 1$, 200 iterations are needed. On the other hand, more advanced numerical methods are available for the approximations of these types of partial differential equations, and some do not require such stability criterion. But the forward finite difference method is the simplest and most straightforward.

4.3 Diffusive logistic equation on an interval

In this section, we use `Maple` program to do some numerical experiments to motivate some of our qualitative analysis later. We first consider the Dirichlet problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \lambda u(1-u), & t > 0,\ x \in (0,1), \\ u(t,0) = u(t,1) = 0, & \\ u(0,x) = f(x), & x \in (0,1). \end{cases} \tag{4.25}$$

We use the algorithm described in the last section, and grid size $\Delta x = 0.1$. In the program, we assume $\lambda = 1$ and $f(x) = \sin(\pi x)$. The simulation shows a sequence of decreasing arches, and the height of the arch decreases from $u = 1$ when $t = 0$ to $u \approx 0.05$ when $t = 0.3$ (see Figure 4.1.)

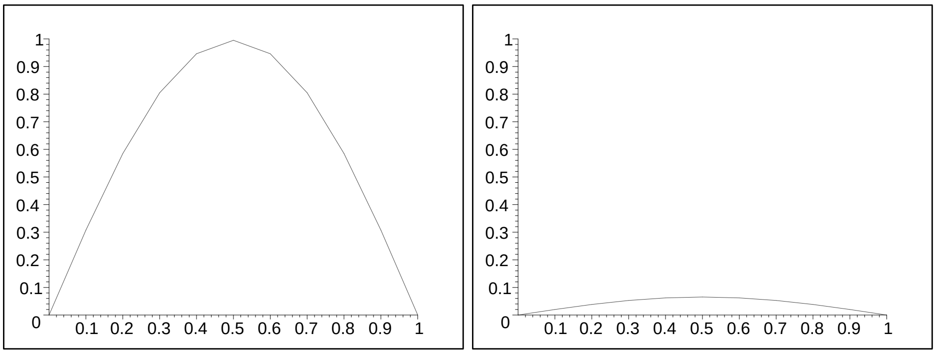


Figure 4.1: **(a)** $u(0, x)$; **(b)** $u(0.3, x)$ when $\lambda = 1$.

If we increase the number of time iterations, we will see that decreasing trend will continue so that the solution appears to approach $u = 0$ as $t \to \infty$. We could connect this with our early analysis of diffusive Malthus equation (2.76). In the Malthus model, the solution will also tend to $u = 0$ as $t \to \infty$ since the critical number of survival or extinction is $a_0 = \pi^2$ when $L = 1$, and $\lambda = 1 < \pi^2$. On the other hand the logistic growth is slower than the linear growth in Malthus model, thus the population in diffusive logistic equation will also become extinct for all $\lambda < \pi^2$. This intuitive idea can be showed in a more mathematically rigorous way in later sections, and this argument is also very valuable in our later analysis.

So our natural question is, when $\lambda > \pi^2$, what will happen? Will the population have an exponential growth or still die out? Let's turn to our program again, but change the parameter to $\lambda = 20$, which is larger than π^2, and try two different initial data: $u(0, x) = sin(\pi x)$ and $u(0, x) = 0.1sin(\pi x)$.

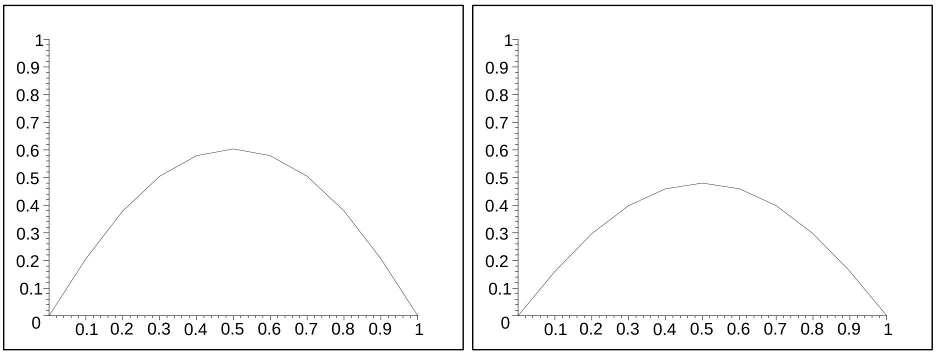


Figure 4.2: **(a)** $u(0.3, x)$ when $u(0, x) = sin(\pi x)$; **(b)** $u(0.3, x)$ when $u(0, x) = 0.1sin(\pi x)$. Both are when $\lambda = 20$.

From the simulation, we can see that in the first case, the solution decreases to the profile in 4.2 (a), and in the second case, the solution increases to the profile in 4.2 (b). So zero or infinity? We are more confused now. Let's push the computational ability of our Pentium IV computer to limit—try the step number `m=1000`. Now let's see what we will have after a long 0.5 time unit:

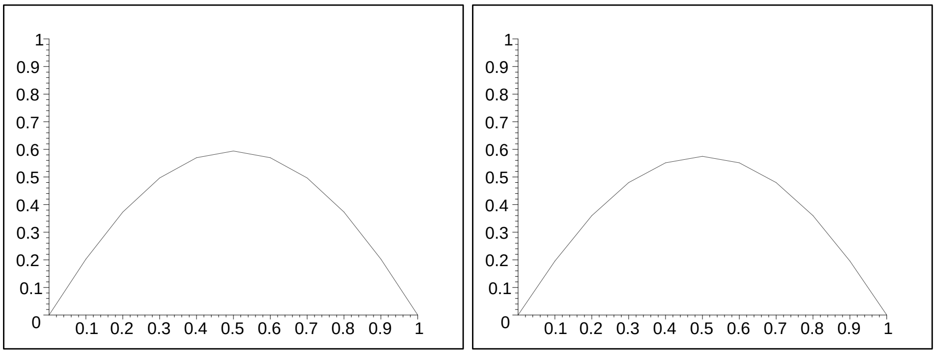


Figure 4.3: **(a)** $u(0.5,x)$ when $u(0,x) = sin(\pi x)$; **(b)** $u(0.5,x)$ when $u(0,x) = 0.1sin(\pi x)$. Both are when $\lambda = 20$.

Two almost identical functions! To convince yourself, you may try change the initial function to another function, and you will that the solution will again tend to the function in Figure 4.3 (a) or (b). Thus our numerical experiment for $\lambda = 20$ can reach to a conclusion: the solution eventually will approach a limit function with maximum value at about $u = 0.55$. This limit function does not change when t changes, thus it has to be an equilibrium solution of the equation, and it satisfies

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \lambda u(1-u) = 0, & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{4.26}$$

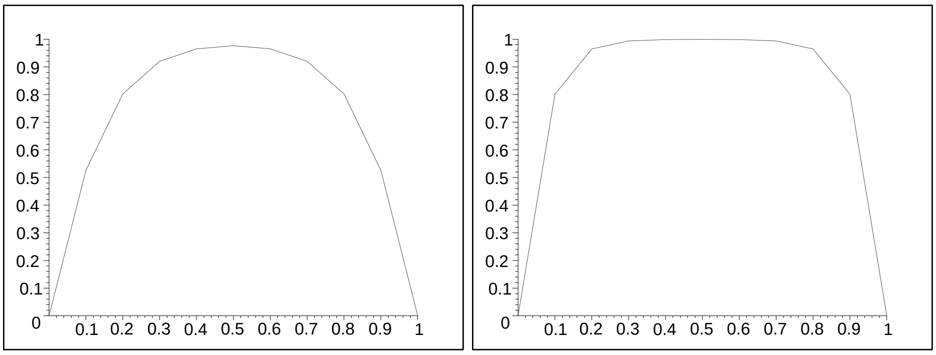


Figure 4.4: **(a)** $u(0.5,x)$ when $u(0,x) = sin(\pi x)$ and $\lambda = 100$; **(b)** $u(0.5,x)$ when $u(0,x) = sin(\pi x)$ and $\lambda = 400$.

If we change the value of λ again (but to values greater than π^2), then we will find similar phenomena only the equilibrium solutions have different profiles. In fact, we can observe that the equilibrium solution for larger λ is larger, and when λ is really large, the equilibrium solution almost equals to 1 for all x except two narrow intervals near $x = 0$ and $x = 1$. Such a solution is usually called a *boundary layer solution*. We can now summarize our numerical exploitation:

1. When $\lambda < \pi^2$, all solutions of (4.25) tends to the equilibrium solution $u = 0$;

2. When $\lambda > \pi^2$, all solutions of (4.25) tends to an equilibrium solution $u_\lambda(x)$.

We recall that in the original model, $\lambda = aL^2/D$, where a is the maximum growth rate per capita, D is the diffusion constant, and L is the size of the habitat. Then we can see that the concept of critical patch size is still relevant here, and $L_0 = \sqrt{D/a}\pi$ is the critical patch size when a and D are fixed. But in the case of $L > L_0$, the population will not grow exponentially as in the diffusive Malthus model. Instead the population will reach an upper limit, just similar to the case of ordinary differential equation $P' = P(1-P)$. Moreover, when L is very large (corresponding to λ very large), the equilibrium solution almost equals to 1 in most of interior of the habitat, and not surprisingly $P = 1$ is also the carrying capacity of $P' = P(1-P)$. Thus even though some population is lost due to the outer flux at $x = 0$ and $x = 1$, when the habitat is large enough to reproduce new lives, the population will still saturate to a level close to the carrying capacity.

As a comparison, we now turn to the Neumann boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \lambda u(1-u), & t > 0,\ x \in (0,1), \\ u_x(t,0) = u_x(t,1) = 0, & \\ u(0,x) = f(x), & x \in (0,1). \end{cases} \tag{4.27}$$

We leave the numerical experiments to the reader. We will find that no matter what value λ is, all solutions will tend to the equilibrium solution $u = 1$, that is exactly the carrying capacity. A curious question is that in the cases of Dirichlet problem with $\lambda > \pi^2$ and Neumann problem, there are at least two equilibrium solutions for the equation: $u = 0$, and $u = u_\lambda$ for Dirichlet, and $u = 0, 1$ for Neumann. Why do all solutions tend to only one of these two solutions, not the other one? Or maybe there are some solutions which will approach to the other equilibrium solution, and we are just cheated by the computer? We shall answer this question in the next section.

4.4 Stability of equilibrium solutions

In diffusive Malthus equation, there is a critical patch size $L_0 = \sqrt{D/a}\pi$ such that all solutions are decaying to zero when $L < L_0$, and all solutions are exponentially growing when $L > L_0$. This phenomenon can also be explained in term of the stability of the equilibrium solution $u = 0$. We can say that $u = 0$ is *stable* when $L < L_0$ and it is *unstable* when $L > L_0$.

Now we define a stable equilibrium solution. Suppose that $v(x)$ is an equilibrium solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} + g(u), & t > 0,\ x \in (0,L), \\ u(t,0) = u(t,L) = 0, & \\ u(0,x) = f(x),\ x \in (0,L), & \end{cases} \tag{4.28}$$

which means $v(x)$ is a solution of

$$\begin{cases} D\dfrac{\partial^2 v}{\partial x^2} + g(v) = 0, & x \in (0,L), \\ v(0) = v(L) = 0. & \end{cases} \tag{4.29}$$

$v(x)$ is said to be *stable* if for any $\delta > 0$, there exists $\varepsilon > 0$ such that whenever

$$|f(x) - v(x)| \leq \delta, \quad \text{for all } x \in [0, L], \tag{4.30}$$

we have

$$|u(t, x) - v(x)| \leq \varepsilon, \quad \text{for all } x \in [0, L], \tag{4.31}$$

where $u(t, x)$ is the solution of (4.28) with $u(0, x) = f(x)$. If an equilibrium solution $v(x)$ is stable, and moreover

$$\lim_{t\to\infty} |u(t, x) - v(x)| = 0, \quad \text{uniformly for } x \in [0, L], \tag{4.32}$$

then we say $v(x)$ is *asymptotically stable*. An equilibrium solution $v(x)$ is *unstable* if it is not stable. Roughly speaking, if all solutions of (4.28) with initial values near $v(x)$ stay close to $v(x)$, then $v(x)$ is stable; if all solutions of (4.28) with initial values near $v(x)$ will approach $v(x)$ as $t \to \infty$, then $v(x)$ is asymptotically stable. If $v(x)$ is asymptotically stable, then it is also stable. The stability for other boundary conditions are defined similarly.

To illustrate the notion of stability, we consider several examples of the linear diffusion equations where solution can be explicitly found:

1. (Dirichlet problem of diffusion equation) $u_t = Du_{xx}$, $u(t, 0) = u(t, L) = 0$, $u(0, x) = f(x)$; and the solution is $\sum_{n=1}^{\infty} c_n exp(-Dn^2\pi^2 t/L^2) \sin(n\pi x/L)$. The only equilibrium solution is $u = 0$, and it is asymptotically stable.

2. (Neumann problem of diffusion equation) $u_t = Du_{xx}$, $u_x(t, 0) = u_x(t, L) = 0$, $u(0, x) = f(x)$; and the solution is $c + \sum_{n=1}^{\infty} c_n exp(-Dn^2\pi^2 t/L^2) \cos(n\pi x/L)$, where $c = L^{-1} \int_0^L f(x)dx$. Any constant function $u = c$ is an equilibrium solution, and all of them are stable from the formula; but none of these equilibrium solution is asymptotically stable, since for any $u = c$, there is always other constant solutions nearby.

3. (Dirichlet problem of diffusive Malthus equation) $u_t = Du_{xx} + au$, $u(t, 0) = u(t, L) = 0$, $u(0, x) = f(x)$; and the solution is $\sum_{n=1}^{\infty} c_n exp(at - Dn^2\pi^2 t/L^2) \sin(n\pi x/L)$. The equilibrium solution $u = 0$ is asymptotically stable when $L < L_0$ and it is unstable when $L > L_0$.

The stability of equilibrium solutions for linear diffusion equation can easily be determined, so the question is: how about nonlinear equation? Let's recall the situation of ordinary differential equations.

1. (scalar equation) Suppose that $u = u_0$ is an equilibrium solution of $u' = g(u)$, then it is stable if $g'(u_0) < 0$, and it is unstable if $g'(u_0) > 0$.

2. (planar systems) Suppose that $(u, v) = (u_0, v_0)$ is an equilibrium solution of $u' = g(u, v)$, $v' = h(u, v)$, then it is stable if both eigenvalues of Jacobian $J(u_0, v_0)$ have negative real part, and it is unstable if both eigenvalues have positive real part.

In both cases, the stability is determined by the *linearization* of the right hand side of the equation, and that would also be true for the case of reaction diffusion equation. The stability of an equilibrium solution can be determined by the following criteria:

Theorem 4.1. *Suppose that $v(x)$ is an equilibrium solution of (4.28). Then*

1. *The equation*
$$\begin{cases} D\phi''(x) + g'(v(x))\phi(x) = \mu\phi, & x \in (0, L), \\ \phi(0) = \phi(L) = 0, \end{cases} \tag{4.33}$$
has an sequence of eigenvalues μ_1, μ_2, $\cdots$ such that μ_i is a real number, $\mu_{i+1} < \mu_i$, and $\lim_{i\to\infty} \mu_i = -\infty$;
2. *The eigenfunction $\phi_i(x)$ changes its sign exactly $(i-1)$ times in $(0, L)$;*
3. *If all eigenvalues μ_i are negative, then $v(x)$ is a stable equilibrium solution; if there is a positive eigenvalue, then $v(x)$ is an unstable equilibrium solution.*

You can certainly find that the theorem resembles its easier counterparts for scalar equations and planar systems. The fact that (4.33) has only real eigenvalues is because (4.33) is also a scalar equation, and later we will learn that the linearization of solutions of reaction diffusion systems can also have complex eigenvalues as planar systems. The proof of the theorem is beyond the scope of this notes, thus it will not be given here. Interested readers can find the proof of the first two parts of the theorem in Strauss [Str92] Chapter 11, and the proof of the last part in ?. In the following, we will apply Theorem 4.1, and one of its consequence: the eigenfunction ϕ_1 corresponding to the first eigenvalue μ_1 of (4.33) does not change sign, thus it can be assumed as a positive function.

Now let's apply Theorem 4.1 to diffusive logistic equation with Dirichlet boundary condition. $u = 0$ is an equilibrium solution of (4.25). The linearized eigenvalue problem at $u = 1$ is

$$\begin{cases} \phi''(x) + \lambda g'(0)\phi(x) = \mu\phi, & x \in (0, 1), \\ \phi(0) = \phi(1) = 0, \end{cases} \tag{4.34}$$

where $g(u) = u(1-u)$. Since $g'(0) = 1$, then ϕ satisfies

$$\begin{cases} \phi''(x) = (\mu - \lambda)\phi, & x \in (0, 1), \\ \phi(0) = \phi(1) = 0. \end{cases} \tag{4.35}$$

From Section 2.1, $\mu - \lambda$ must be $-(n\pi)^2$, $n \geq 1$. So $\mu_n = \lambda - n^2\pi^2$. In particular, $\mu_1 = \lambda - \pi^2$. Thus $u = 0$ is asymptotically stable if $\lambda < \pi^2$, and it is unstable if $\lambda > \pi^2$.

The stability of a constant equilibrium solution is easy to determine by Theorem 4.1, but for the non-constant ones, it is usually not so easy. However we are able to show that the positive equilibrium solution which we observe in the last section must be asymptotically stable via the following calculation. (Note that we have *not* shown that that equilibrium solution actually exists, which we will in the following sections, so here we prove that it is asymptotically stable under the assumption of its existence.)

Proposition 4.2. *Suppose that $u(x)$ is a positive equilibrium solution of (4.25) for $\lambda > \pi^2$. Then $u(x)$ is asymptotically stable.*

Proof. We prove it using a trick of calculus: integral by parts. Let (μ_1, ϕ_1) be the first eigenvalue-eigenfunction pair (μ_1 is the largest eigenvalue) of

$$\begin{cases} \phi''(x) + \lambda g'(u(x))\phi(x) = \mu\phi, & x \in (0,1), \\ \phi(0) = \phi(1) = 0, \end{cases} \tag{4.36}$$

where $g(u) = u(1-u)$. From Theorem 4.1, we could assume that $\phi_1(x) > 0$ for all $x \in (0,1)$, and we only need to show that μ_1 is negative (hence all other eigenvalues are also negative.) We multiply (4.25) ($u'' + \lambda g(u) = 0$) by $\phi_1(x)$, and multiply (4.36) by $u(x)$, subtract and integrate on $[0,1]$, then we obtain

$$\begin{aligned} &\int_0^1 [u''(x)\phi_1(x) - \phi_1''(x)u(x)]dx \\ +\lambda &\int_0^1 [g(u(x)) - u(x)g'(u(x))]\phi_1(x)dx = -\mu_1 \int_0^1 u(x)\phi_1(x)dx. \end{aligned} \tag{4.37}$$

We handle the first integral by integrating by parts as follows:

$$\begin{aligned} &\int_0^1 u''(x)\phi_1(x)dx = \int_0^1 \phi_1(x)du'(x) \\ =&\phi_1(x)u'(x)|_0^1 - \int_0^1 u'(x)\phi'(x)dx \\ =&-\int_0^1 u'(x)\phi'(x)dx, \end{aligned} \tag{4.38}$$

here the terms on end points vanishes because $\phi_1(0) = \phi_1(1) = 0$; and the second part can be manipulated similarly:

$$\begin{aligned} &-\int_0^1 \phi_1''(x)u(x)dx = -\int_0^1 u(x)d\phi_1'(x) \\ =&-u(x)\phi_1'(x)|_0^1 + \int_0^1 u'(x)\phi'(x)dx \\ =&\int_0^1 u'(x)\phi'(x)dx. \end{aligned} \tag{4.39}$$

Thus

$$\int_0^1 [u''(x)\phi_1(x) - \phi_1''(x)u(x)]dx = 0, \tag{4.40}$$

which makes the identity (4.37) becoming

$$\lambda \int_0^1 [g(u(x)) - u(x)g'(u(x))]\phi_1(x)dx = -\mu_1 \int_0^1 u(x)\phi_1(x)dx. \tag{4.41}$$

Since $g(u) = u(1-u)$, then $g(u) - ug'(u) = u^2$, and the integral on the left hand side of (4.40) now is $\int_0^1 [u(x)]^2 \phi_1(x)dx$, which is positive since we choose $\phi_1(x)$ to be positive. On the other hand, if $\phi_1(x)$ is positive, then $\int_0^1 u(x)\phi_1(x)dx$ is also positive since $u(x)$ is a positive solution. Therefore, μ_1 must be negative from the expression in (4.41). □

4.5 Motivation from algebraic equation

In many aspects, the reaction diffusion equation $u_t = u_{xx} + g(u)$ is similar to the ordinary differential equation $u' = g(u)$, and the behavior of the equilibrium solutions of the two equations is one of them. In this section we consider an equation of equilibrium solutions of $u' = g(u)$, that is an algebraic equation $g(u) = 0$. But the methods we use here can be later used in more complicated reaction diffusion equation as well.

Consider

$$x(x^5 + x + 1 - \varepsilon) = 0, \tag{4.42}$$

where ε is a parameter. Clearly $x = 0$ is always a solution to the equation no matter what ε is. Other solutions of the equation satisfy $x^5 + x + 1 - \varepsilon = 0$, which is not solvable algebraically. Here we apply two advanced mathematical methods: *perturbation* and *bifurcation* to this problem.

First we discuss the solutions of $x(x^5 + x + 1 - \varepsilon) = 0$ when ε is near $\varepsilon_0 = 1$. When $\varepsilon = 1$, $x = 0$ is the only solution of $f(x) = x(x^5 + x + 1 - \varepsilon) = 0$, and it is a double zero of $f(x)$ in the sense that $f'(0) = 0$. If you draw the graph of $f(x)$ for $\varepsilon = -0.1$, 0 and 0.1, you will find that $f(x) = 0$ has another zero near $x = 0$ when $\varepsilon \neq 0$ but close to 0. Thus a bifurcation occurs at $\varepsilon = 0$: when $\varepsilon < 1$, $f(x) = 0$ has two zeros, $x = 0$ and a negative one; when $\varepsilon = 1$, $x = 0$ is the only one; and when $\varepsilon > 1$, $f(x) = 0$ also has two zeros, $x = 0$ and a positive one. This is called a *transcritical bifurcation*, and it can be depicted as in Figure 4.5 (a). The question is, what is the value of the other zero of $f(x)$ when ε is near 1? We know it is close to $x = 0$, but in applications, we may want to know the value more precisely.

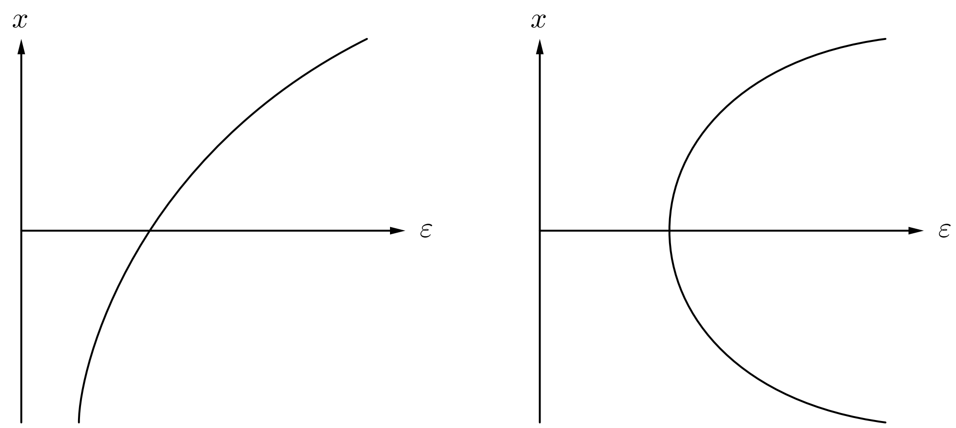


Figure 4.5: **(a)** Transcritical bifurcation; **(b)** pitchfork bifurcation.

Now we use the perturbation method: suppose that $r = \varepsilon - 1$ is small, we assume that the zero point x_ε has a form:

$$x_\varepsilon = 0 + rx_1 + r^2 x_2 + r^3 x_3 + r^4 x_4 + r^5 x_5 + \textit{higher order terms}, \tag{4.43}$$

and substitute this form into $x^5 + x + 1 - \varepsilon = 0$, then we obtain

$$(rx_1 + r^2 x_2 + \cdots)^5 + rx_1 + r^2 x_2 + r^3 x_3 + r^4 x_4 + r^5 x_5 + \cdots + 1 - (1 + r) = 0, \tag{4.44}$$

thus

$$x_1 = 1, \;\; x_2 = x_3 = x_4 = 0, \;\; x_5 = -1, \cdots \tag{4.45}$$

and we get an approximate value of x_ε:

$$x_\varepsilon = 1 - \varepsilon - (1 - \varepsilon)^5 + R(\varepsilon), \tag{4.46}$$

with the remainder $R(\varepsilon)$ is in an order smaller than $(1 - \varepsilon)^5$. The equation (4.46) gives a very accurate estimate of x_ε, but from a rigorous mathematical point of view, the reason that $R(\varepsilon)$ is indeed a small quantity and hence whether the expansion of x_ε is convergent is questionable. But nevertheless, we gain useful information from this simple calculation. In fact, this argument can be made rigorous in analytic mathematics via one of most important theorem in mathematical analysis, the *implicit function theorem.*

To introduce implicit function theorem, we think the function $x^5 + x + 1 - \varepsilon$ is a function depending on two variables ε and x: $F(\varepsilon, x) = x^5 + x + 1 - \varepsilon$. Then $(\varepsilon, x) = (1, 0)$ is a zero point of F. The level set $F(\varepsilon, x) = 0$ near $(1, 0)$ is a curve if the gradient $\nabla F = (\partial F/\partial\varepsilon, \partial F/\partial x)$ is not zero at $(1, 0)$, and the points on the curve are solutions of $F(\varepsilon, x) = 0$. If we use the linearization of the level curve, we obtain

$$0 = F(\varepsilon, x) \approx F(1, 0) + \frac{\partial F(1, 0)}{\partial\varepsilon}(\varepsilon - 1) + \frac{\partial F(1, 0)}{\partial x}x, \tag{4.47}$$

Thus a solution (ε, x) of $F(\varepsilon, x)$ near $(1, 0)$ has a rough form

$$x = -\frac{(\partial F/\partial x)(1, 0)}{(\partial F/\partial\varepsilon)(1, 0)}(\varepsilon - 1). \tag{4.48}$$

You can verify that (4.48) and (4.46) are consistent. A simple version of implicit function theorem is as follows:

Theorem 4.3. *Suppose that $F(\varepsilon, x)$ is a differentiable function near $(\varepsilon_0, x_0) \in \mathbf{R}^2$, and $f(\varepsilon, x_0) = 0$. If $(\partial F/\partial\varepsilon)(\varepsilon_0, x_0) \neq 0$, then for each ε near ε_0, there exists a unique $x(\varepsilon)$ such that $f(\varepsilon, x(\varepsilon)) = 0$, and*

$$x = x_0 - \frac{(\partial F/\partial x)(\varepsilon_0, x_0)}{(\partial F/\partial\varepsilon)(\varepsilon_0, x_0)}(\varepsilon - \varepsilon_0) + R(\varepsilon), \tag{4.49}$$

where $\lim_{\varepsilon\to\varepsilon_0} \frac{R(\varepsilon)}{\varepsilon - \varepsilon_0} = 0$. Moreover $(\varepsilon, x(\varepsilon))$ is a smooth curve in (ε, x) space.

The implicit function theorem can also be used in calculating the solution of (4.42) when ε is near 3. When $\varepsilon = 3$, $x = 1$ is a solution. Using the perturbation method or implicit function theorem, we can calculate the solution x_ε of the equation for $\varepsilon \approx 3$:

$$x_\varepsilon = 1 + \frac{1}{6}(\varepsilon - 3) - \frac{25}{432}(\varepsilon - 3)^2 + \cdots . \tag{4.50}$$

The calculation of (4.50) is left as exercise.

The perturbation and bifurcation method can be used to solve the equation near a known solution, and both methods only work in a local neighborhood of the known solution. Neither gives information of global bifurcation diagram of $F(\varepsilon, x) = 0$. In fact, there is an easy way to do that: instead of express the solution x in term of parameter ε, we can solve ε in term of x easily: the solutions of (4.124) are

$$\text{either } x = 0, \quad \text{or } \varepsilon = x^5 + x + 1. \tag{4.51}$$

(4.51) is especially useful when you draw the bifurcation diagram of the problem, although it does not provide the information of x in term of ε. So from our observation of an algebraic equation, we learn the following lessons:

1. Bifurcation points are the points where $f_x(\varepsilon, x) = 0$.
2. Perturbation method can be used to find approximate solutions near a known solution.
3. When you cannot solve the solution x in term of parameter ε, you may solve ε in term of x.

All three ideas will be useful when we return to the discussion of equilibrium solutions of reaction diffusion equation.

4.6 Perturbation and local bifurcation

In this section we use the ideas in the last section to study the equilibrium solutions of diffusive logistic equation, so we consider the equation:

$$\begin{cases} u''(x) + \lambda u(1-u) = 0, & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{4.52}$$

$u = 0$ is always an equilibrium solution for any $\lambda > 0$, so we would like to know equilibrium solutions other than $u = 0$. First, on the bifurcation points of the equation, we have

Proposition 4.4. *If* $(\lambda, 0) = (\lambda_0, 0)$ *is a bifurcation point of the equation* (4.52)*, then* $\mu = 0$ *must be an eigenvalue of*

$$\begin{cases} \phi''(x) + \lambda g'(0)\phi(x) = \mu\phi, & x \in (0,1), \\ \phi(0) = \phi(1) = 0, \end{cases} \tag{4.53}$$

where $g(u) = u(1-u)$.

This is similar to $F(\varepsilon, x) = 0$, where the bifurcation points must satisfy $F_x(\varepsilon, x) = 0$. The rigorous proof of Proposition 4.4 requires some advanced knowledge, which we will not try here. The basic idea is the following: when 0 is not an eigenvalue of (4.53) when $\lambda = \lambda_0$, then the solutions of (4.52) near $(\lambda_0, 0)$ must on a curve $(\lambda, u(\lambda))$, which must be the zero solutions $(\lambda, 0)$.

By using Proposition 4.4, we can now determine the possible bifurcation points: since $g'(0) = 1$, $\mu - \lambda = -n^2\pi^2$, thus to make $\mu = 0$, λ must be $n^2\pi^2$. Recall that when $\lambda < \pi^2$, $u = 0$ is a stable equilibrium solution, and when $\lambda > \pi^2$, $u = 0$ is unstable. And from our numerical experiments, we also find that when $\lambda > \pi^2$, (4.52) has a stable positive equilibrium solution. Now we can use perturbation method to determine the analytic property of this stable solution. Since $(\lambda, u) = (\pi^2, 0)$ is a bifurcation point, then we look for a solution (λ, u) which is close to $(\pi^2, 0)$. We introduce a small parameter ε, and we look for a solution with form:

$$\lambda = \pi^2 + a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + \cdots, \quad u = 0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots, \tag{4.54}$$

where a_i is a real number and u_i is a function. We substitute the form (4.54) into equation (4.52), then the equation becomes

$$\begin{aligned} 0 =& u'' + \lambda u(1-u) = (\varepsilon u_1'' + \varepsilon^2 u_2'' + \cdots) \\ &+ (\pi^2 + a_1\varepsilon + a_2\varepsilon^2 + \cdots)(\varepsilon u_1 + \varepsilon^2 u_2 + \cdots)(1 - \varepsilon u_1 - \varepsilon^2 u_2 + \cdots) \\ =& \varepsilon[u_1'' + \pi^2 u_1] + \varepsilon^2[u_2'' + \pi^2 u_2 + a_1 u_1 - \pi^2 u_1^2] + \cdots, \end{aligned} \tag{4.55}$$

and the boundary condition becomes

$$u(0) = \varepsilon u_1(0) + \varepsilon^2 u_2(0) + \cdots = 0 = \varepsilon u_1(1) + \varepsilon^2 u_2(1) + \cdots = u(1). \tag{4.56}$$

Thus u_1 and u_2 satisfy

$$u_1'' + \pi^2 u_1 = 0, \quad x \in (0,1), \quad u_1(0) = u_1(1) = 0, \tag{4.57}$$

$$u_2'' + \pi^2 u_2 + a_1 u_1 - \pi^2 u_1^2 = 0, \quad x \in (0,1), \quad u_2(0) = u_2(1) = 0. \tag{4.58}$$

The solution of (4.57) must be $u_1(x) = k\sin(\pi x)$, but the constant k can not be determined from (4.57). To determine k, we have to look further for u_2. The solvability of u_2 is not a necessity. Here let's play the multiply-multiply-subtract-integrate trick in the proof of Proposition 4.2: we multiply (4.58) by u_1, multiply (4.57) by u_2, subtract the two equations and integrate over $[0,1]$, then we get

$$\int_0^1 [a_1 u_1^2(x) - \pi^2 u_1^3(x)]dx = 0. \tag{4.59}$$

Notice that the equality (4.59) contains only function u_1, so that is an equation which you can use to solve the constant k! By substituting $u_1 = k\sin(\pi x)$ to (4.59), we obtain

$$\pi^2 k^3 \int_0^1 \sin^3(\pi x)dx = a_1 k^2 \int_0^1 \sin^2(\pi x)dx, \tag{4.60}$$

thus $k = 3a_1/(8\pi)$. Notice that we can select a_1 arbitrarily because of scaling, so we can assume that $a_1 = 1$. Then u_2 can also be solved (but with another undetermined constant):

$$u_2(x) = -\frac{3}{32}\cos(\pi x) + k_1\sin(\pi x) + \frac{3}{128\pi^2}\cos(2\pi x) + \frac{3}{16\pi^2}x\cos(\pi x) - \frac{9}{128\pi^2}.$$

and k will be determined in the equation of u_3. The formula of u_2 above is calculated by `Maple`. Put together all results in this section, we have the following local bifurcation picture of (4.52):

Theorem 4.5. *Near $(\lambda, u) = (\pi^2, 0)$, the solution set of (4.52) consists of two parts: a line of constant solutions $(\lambda, 0)$ and a curve of non-constant solutions $(\lambda(\varepsilon), u(\varepsilon))$, which crosses the line of constant solutions at $(\pi^2, 0)$; $u(\varepsilon)$ is positive when $\lambda > \pi^2$, and it is negative when $\lambda < \pi^2$; Moreover,*

$$\lambda(\varepsilon) = \pi^2 + \varepsilon + a_2\varepsilon^2 + \textit{higher order term}$$
$$u(\varepsilon)(x) = \frac{3}{8\pi}\varepsilon \sin(\pi x) + \varepsilon^2\left(-\frac{3}{32}\cos(\pi x) + \frac{3}{128\pi^2}\cos(2\pi x) + \frac{3}{16\pi^2}x\cos(\pi x) - \frac{9}{128\pi^2} + k_1\sin(\pi x)\right) + \cdots \tag{4.61}$$

4.7 Global Bifurcation

We have found that it is hard to express the equilibrium solutions in term of parameter λ. In this section, we try the opposite: express the parameter in term of the solution u. To do that, we use a dynamical system approach. The differential equation $u'' + \lambda g(u) = 0$ can be viewed as a system of first order ordinary differential equations:

$$u' = v, \quad v' = -\lambda g(u), \tag{4.62}$$

and there is a rich geometric theory about the phase plane of the system (4.62). A solution of the boundary value problem (4.52) is the right-half of an orbit of (4.62) such that $u(0) = u(1) = 0$. Now we show that the parameter λ can be solved in term of $u(0.5)$, the maximum point of $u(x)$.

To derive a formula of λ in term of u_{max}, we consider a more general equation:

$$\begin{cases} Du''(x) + g(u) = 0, & x \in (0, L), \\ u(0) = u(L) = 0, \end{cases} \tag{4.63}$$

and we can always convert (4.63) to nondimensionalized version

$$\begin{cases} v''(x) + \lambda g(v) = 0, & x \in (0, 1), \\ v(0) = v(1) = 0, \end{cases} \tag{4.64}$$

where $\lambda = L^2/D$. The equation in (4.63) can be rewritten as

$$u' = v, \quad v' = -D^{-1}g(u). \tag{4.65}$$

A positive solution $u(x)$ of (4.63) is equivalent to a solution orbit $(u(x), v(x))$ of (4.65) such that

1. $u(0) = u(L) = 0$;
2. $u(L/2) = \max u(x)$, $v(L/2) = 0$;
3. $u(x) > 0$, for $x \in (0, L)$, $v(x) > 0$ for $x \in [0, L/2)$, and $v(x) < 0$ for $x \in (L/2, L]$.

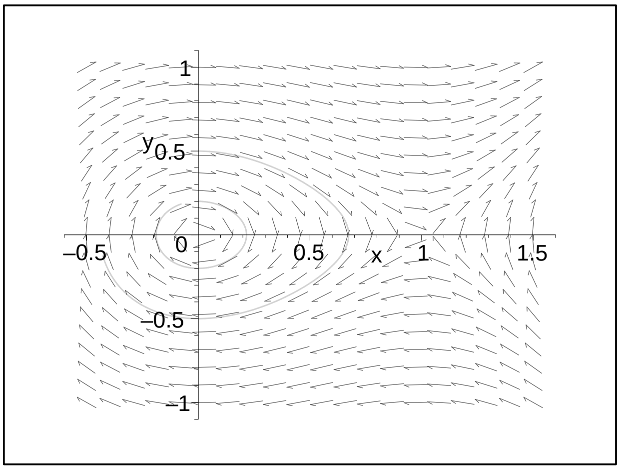


Figure 4.6: Phase portrait of diffusive logistic equation

From the phase portrait of (4.65) (where $g(u) = u(1-u)$), the possible values of $u(L/2)$ are from $u = 0$ to $u = 1$, so we consider a solution $(u(x), v(x))$ of (4.65) with $0 < u(L/2) < 1$. We multiply (4.63) by $u'(x)$, and integrate it over $(x, L/2)$ (for some $x \in [0, L/2)$), then we obtain

$$\begin{aligned} 0 =& \int_x^{L/2} [Du''(x)u'(x) + g(u(x))u'(x)]dx \\ =& D\int_x^{L/2} \frac{d}{dx}\left(\frac{1}{2}[u'(x)]^2\right)dx + \int_x^{L/2} g(u(x))du(x) \\ =& \frac{D}{2}[u'(L/2)]^2 - \frac{D}{2}[u'(x)]^2 + \int_{u(x)}^{u(L/2)} g(u)du \\ =& -\frac{D}{2}[u'(x)]^2 + G(u(L/2)) - G(u(x)), \end{aligned} \tag{4.66}$$

where $G(u) = \int_0^u g(t)dt$. Since $u(L/2)$ is a constant, we denote it by s. The equality (4.66) can be written as

$$[u'(x)]^2 = \frac{2}{D}[G(s) - G(u(x))]. \tag{4.67}$$

So after an integration, we obtain a new equation (4.67), which is a first order ordinary differential equation:

$$\frac{du}{dx} = \sqrt{\frac{2}{D}}\sqrt{G(s) - G(u)}. \tag{4.68}$$

Here we take du/dx to be positive since $x \in [0, L/2)$. We integrate (4.68) from $u = 0$ to $u = s$, then we obtain

$$\frac{L}{2} = \int_0^{L/2} 1dx = \sqrt{\frac{D}{2}}\int_0^s \frac{1}{\sqrt{G(s) - G(u)}}du, \tag{4.69}$$

or

$$L = \sqrt{2D}\int_0^s \frac{1}{\sqrt{G(s) - G(u)}}du. \tag{4.70}$$

Using the relation $\lambda = L^2/D$, we obtain

$$\lambda = 2\left(\int_0^s \frac{1}{\sqrt{G(s)-G(u)}}du\right)^2 \tag{4.71}$$

for (4.64). Therefore, if u is a solution of (4.64) with $u(L/2) = s$, then the parameter λ can be expressed in term of s as in (4.71). In another word, the global bifurcation diagram can be drawn as a graph $(\lambda(s), s)$, where $\lambda(s)$ is defined in (4.71), and $s \in (0,1)$. The integral in (4.70) is often called *time-mapping* as it maps a solution to L, which can be thought as the half of the time traveled from $u = 0$ to the next time $u = 0$.

The integral in (4.71) is not easy to evaluate. For example, when $g(u) = u(1-u)$, the expression of $\lambda(s)$ is

$$\lambda(s) = 2\left(\int_0^s \frac{\sqrt{6}}{\sqrt{3s^3 - 2s^2 - 3u^2 + 2u^3}}du\right)^2. \tag{4.72}$$

This integral is improper since when $u = s$, the integrand has zero denominator, but this improper integral is convergent, since the integral near $u = s$ is approximately (for $\delta > 0$ small)

$$\begin{aligned}
&\int_{s-\delta}^s \frac{du}{\sqrt{3s^3 - 2s^2 - 3u^2 + 2u^3}}du \\
=&\int_{s-\delta}^s \frac{du}{\sqrt{(s-u)(3s+3u-2s^2-2su-2u^2)}}du \\
\approx&\frac{1}{\sqrt{6s-6s^2}}\int_{s-\delta}^s \frac{du}{\sqrt{s-u}} = \frac{2}{\sqrt{6s-6s^2}}\sqrt{\delta},
\end{aligned} \tag{4.73}$$

thus the whole integral is convergent. On the other hand, from the phase plane, the integral is proportional to the time that the solution travels from $(0, u'(0))$ to $(s, 0)$, which is a finite time. `Maple` is able to calculate this integral for any $s \in (0,1)$, thus we can obtain numerically a graph of the function $\lambda(s)$.

Even with the numerical help of `Maple`, a little more calculation of the function $\lambda(s)$ would help us to better understand this bifurcation diagram and generate a better numerical bifurcation diagram. Here we use some integral calculus to show two properties of the bifurcation diagram. The key here is the calculation of the time-mapping

$$T(s) = \int_0^s \frac{1}{\sqrt{G(s)-G(u)}}du, \tag{4.74}$$

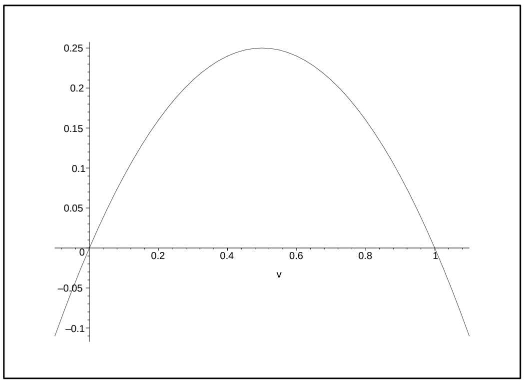
since $\lambda(s) = 2[T(s)]^2$, and $\lambda'(s) = 4T(s)T'(s)$. Thus critical points of $T(s)$ and $\lambda(s)$ are equivalent, and if $T(s)$ is monotone, so is $\lambda(s)$. To calculate $T(s)$, we use a change of variables $u = sw$ to convert the integral in (4.74) to

$$T(s) = \int_0^s \frac{1}{\sqrt{G(s)-G(u)}}du = \int_0^1 \frac{s}{\sqrt{G(s)-G(sw)}}dw. \tag{4.75}$$

The advantage of the new integral is that the bounds of the integral are now independent of s variable, that would be helpful when we consider the derivative of $T(s)$. We differentiate $T(s)$, and simplifying the expression, we obtain

$$T'(s) = \int_0^1 \frac{H_g(s) - H_g(sw)}{2[G(s) - G(sw)]^{3/2}} dw, \tag{4.76}$$

where $H_g(v) = G(v) - vg(v)$. Now if we let $g(v) = v - v^2$, then $H_g(v) = (1/6)v^2(4v - 3)$ and $H_g'(v) = 2v^2 - v > 0$ if $v \in (0, 1)$. In particular, $H_g(v)$ is a strictly increasing function when $v \in (0, 1)$, and $H_g(s) - H_g(sw) > 0$ for all $w \in (0, 1)$ and $s \in (0, 1)$. Therefore the integral in (4.76) is positive since the integrand is positive, and $T'(s) > 0$ for all $s \in (0, 1)$.



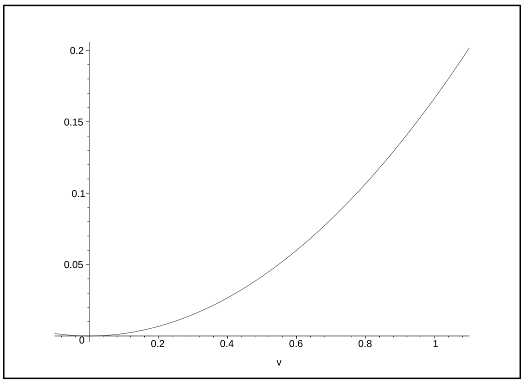


Figure 4.7: (a) Graph of $g(v) = v - v^2$; (b) Graph of $H_g(v) = \frac{1}{6}v^2(4v - 3)$

The domain of the function $T(s)$ and $\lambda(s)$ is $s \in (0, 1)$, from our observation of the phase portrait in Figure 4.7. So if we can determine the behavior of the time-mapping when $s \to 0^+$ and $s \to 1^-$, we will have a complete qualitative picture for $T(s)$ and $\lambda(s)$. We have found previously that $\lambda = \pi^2$ is the bifurcation point, but this conclusion can also be obtained through the calculation of time-mapping. In fact, when $s \to 0^+$, $G(s) \approx G(0) + G'(0)s + 0.5G''(0)s^2 = 0.5s^2$, thus

$$\lim_{s \to 0^+} T(s) = \int_0^1 \sqrt{\frac{2}{1 - w^2}} dw = \frac{\pi}{\sqrt{2}} \tag{4.77}$$

and thus $\lim\limits_{s \to 0^+} \lambda(s) = \pi^2$. So once again we find that $\lambda = \pi^2$ is the bifurcation point. The integral $T(s)$ approaches to ∞ as $s \to 1^-$ from the calculation in (4.73). Summarizing the above, we find that

$$\lim_{s \to 0^+} \lambda(s) = \pi^2, \lim_{s \to 1^-} \lambda(s) = \infty, \text{ and } \lambda'(s) > 0, \quad s \in (0, 1). \tag{4.78}$$

Now we use `Maple` to visualize the bifurcation diagram $(\lambda(s), s)$. Here is the `Maple` code:

`> with(plots):`
`> b:=1; f:=x*(b-x);` (define the nonlinearity $f(u)$)
`> F1:=int(f,x); F:=unapply(F1,x);` (find the antiderivative $F(u)$ of $f(u)$)
`> fsolve(f,x);` (solve the zeros of $f(u)$, which determines the domain of the time-mapping)
`> Initial:=0; End:=1;` (domain of the time-mapping)
`> numplots:=100;` (the number of plotting points)

```
> lambda:=array(1..numplots); u:=array(1..numplots);
> lambda[1]:=evalf(Pi^2); u[1]:=0; (the bifurcation point)
```

```
> for n from 2 to numplots-1 do
> s:=Initial+(End-Initial)*(n-1)/numplots:
> evalf(Int(1/(sqrt(F(s)-F(x))), x=0..s));
> lambda[n]:=2*(%^2); u[n]:=s;
> end  do:
```

```
>plot([seq([lambda[n],u[n]],n=1..numplots-1)],0..100,tickmarks=[10,11]);
```

Now we obtain the bifurcation diagram in (λ, s) coordinate system:

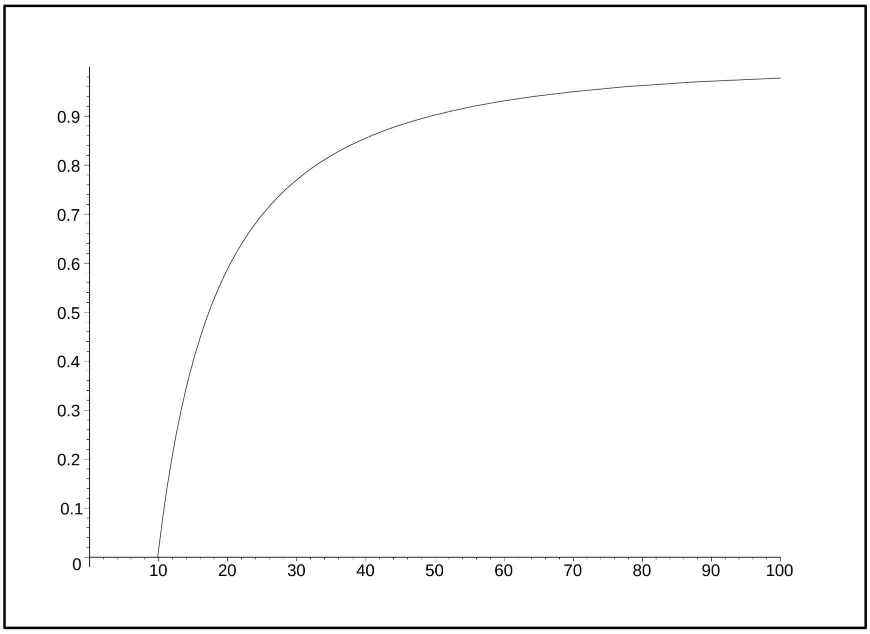


Figure 4.8: Bifurcation diagram of $u'' + \lambda u(1-u) = 0$, $u(0) = u(1) = 0$.

From Figure 4.8 and previous discussions, we shall conclude our investigation of the diffusive logistic equation with Dirichlet boundary condition:

Theorem 4.6. *Consider*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \lambda u(1-u), & t > 0,\ x \in (0,1), \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = f(x) > 0,\ x \in (0,1). \end{cases} \tag{4.79}$$

1. *When $0 < \lambda < \pi^2$, (4.72) has only one non-negative equilibrium solution $u = 0$, which is asymptotically stable. For any $f(x) > 0$, $\lim_{t\to\infty} u(t,x) = 0$;*
2. *When $\lambda > \pi^2$, (4.72) has exactly two non-negative equilibrium solution $u = 0$, which is*

unstable, and $u = u_\lambda(x)$, which is positive and asymptotically stable. For any $f(x) > 0$, $\lim_{t\to\infty} u(t,x) = u_\lambda(x)$.

Biologically, diffusive logistic equation provides a more reasonable alternative to diffusive Malthus equation, where population has an exponential growth when the size of the habitat is larger than the critical patch size. In logistic model, the population will approach a positive equilibrium solution if the size of the habitat is larger than the critical patch size, which play a similar role as the carrying capacity as the ODE case.

Mathematically, we have sampled a few powerful tools in studying nonlinear evolution equation: perturbation, bifurcation, time-mapping and numerical methods. An important omission here is the *maximum principle* due to the time constrain, and it is the key to complete the proof of Theorem 4.6 in proving the convergence of all solutions to the equilibrium solutions.

4.8 Allee effect

In this section, we introduce some variants of the diffusive logistic equation. In logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) = P \cdot g(P), \tag{4.80}$$

the growth rate per capita $g(P) = k(1 - P/N)$ is a strictly decreasing function with respect to the population density P, which considers the crowding effect. For some species, a small or sparse population may not be favorable, since for example, mating may be difficult. This is called *Allee effect*. Mathematically $g(P)$ will not have maximum value at $P = 0$. If the growth rate per capita $g(P)$ is negative when P is small, we call such a growth pattern has a *strong Allee effect*. A typical example is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)\left(\frac{P}{M} - 1\right) = P \cdot g(P), \tag{4.81}$$

where $0 < M < N$, M is the sparsity constant and N is the carrying capacity. The graph of an example of growth rate $f(P)$ and growth rate per capita $g(P)$ is in Figure 4.10. If the growth rate per capita $g(P)$ is smaller than the maximum but still positive for small P, we call such a growth pattern has a *weak Allee effect*. A typical example is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)\left(\frac{P}{M} - 1\right) = P \cdot g(P), \tag{4.82}$$

where $M < 0 < N$. The graph of a example of growth rate $f(P)$ and growth rate per capita $g(P)$ is in Figure 4.9. The dynamics of strong and weak Allee effects are different: the weak Allee effect has the qualitatively same dynamics as the logistic equation, and all positive solutions tend to $P = N$ as $t \to \infty$; but in the strong Allee effect case, there is a positive unstable equilibrium point $P = M$ which is a threshold value, and there are two non-negative stable equilibrium points $P = 0$ and $P = N$. In the latter case, if $0 < P(0) < M$, then $\lim_{t\to\infty} P(t) = 0$, and if $M < P(0) < N$, then $\lim_{t\to\infty} P(t) = N$.

In this section, we consider the corresponding spatial model. Suppose that we make the same nondimensionalization as in Section 3.1, then the nondimensionalized equation is

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \lambda u(1-u)(u-a), \quad t > 0,\ x \in (0,1), \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = u_0(x),\ x \in (0,1), \end{cases} \tag{4.83}$$

where $-\infty < a < 1$. In particular, we have strong Allee effect if $1 > a > 0$, and we have weak Allee effect if $a < 0$.

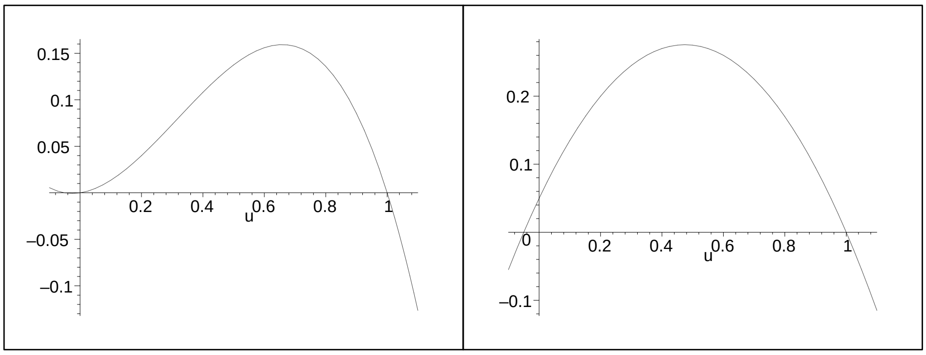


Figure 4.9: Weak Allee effect: growth rate (left), growth rate per capita (right).

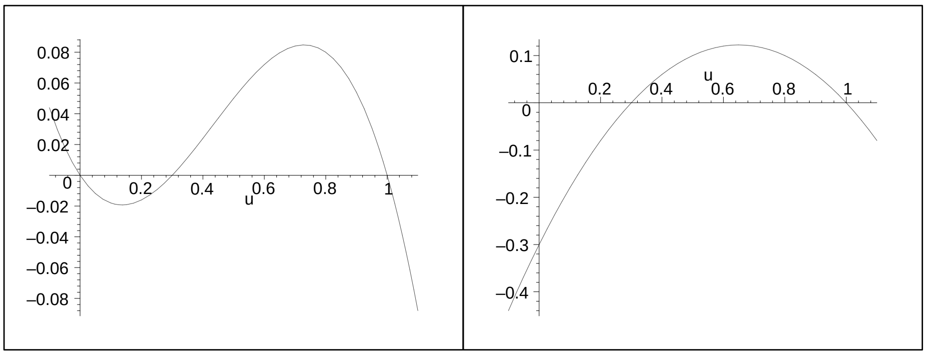


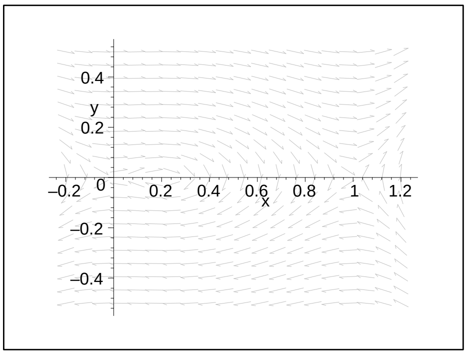
Figure 4.10: Srong Allee effect: growth rate (left), growth rate per capita (right).

The mathematical tools developed in the last few sections can be used here for a complete study of the equation, which will be done over a series of exercises. But we want to point out two new phenomena which does not happen in the case of diffusive logistic equation.

Existence of equilibrium solutions. First we consider the case when a is close to 1. This is the case of strong Allee effect with the sparsity constant close to the carrying capacity. We can use the phase portrait of the first order dynamical system to study the existence of equilibrium solutions. The equilibrium solutions are equivalent to an orbit starting from and ending on $v = 0$. The system in this case is

$$u' = v, \quad v' = -\lambda u(1-u)(u-a). \tag{4.84}$$

But from the phase portrait (Figure 4.11 (a)), there is no orbit starting from $v = 0$ will hit $v = 0$ again. So there is no positive equilibrium solutions.



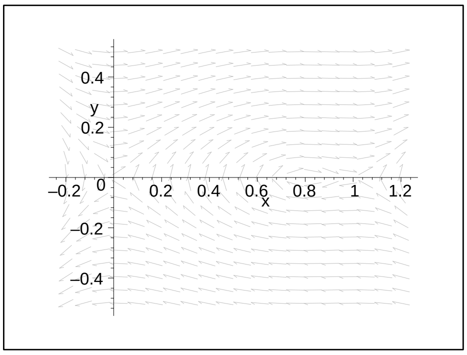


Figure 4.11: Phase portraits: $u'' + u(1-u)(u-0.2) = 0$ (left), $u'' + u(1-u)(u-0.8) = 0$ (right)

So the question is when the equation has an positive equilibrium solution. We recall the time-mapping formula:

$$\lambda = 2\left(\int_0^s \frac{1}{\sqrt{G(s)-G(u)}}du\right)^2, \tag{4.85}$$

where $s = u(1/2)$. To make this formula meaningful, we must have $G(s)-G(u) \geq 0$ for all $u \in [0, s]$. In this case, it means

$$G(s) - G(u) = \int_u^s g(t)dt \geq 0, \quad \text{where } g(t) = u(1-u)(u-a). \tag{4.86}$$

Let's check the graph of $G(u)$. By integration, we obtain

$$G(u) = -\frac{1}{4}u^4 + \frac{a+1}{3}u^3 - \frac{a}{2}u^2 = u^2\left(-\frac{1}{4}u^2 + \frac{a+1}{3}u - \frac{a}{2}\right). \tag{4.87}$$

From the graph of $G(u)$ when $a = 0.8$ (Figure 4.12 (a)), one can see that there is no any s satisfying (4.86), thus there is no any valid s for $u(1/2)$.

On the other hand, when a is closer to 0 than 1, for example, $a = 0.2$, (see Figure 4.12 (b),) then there are s satisfying (4.86), and from the phase portrait (Figure 4.11 (a)), there are orbits starting from and also ending on $v = 0$. The valid values for $s = u(1/2)$ is from where $G(s) = 0$ to $s = 1$. For $s > 1$, $g(s) < 0$, there is no such solution with $u(1/2) = s$. Therefore, summarizing the observation above, we can conclude

Proposition 4.7. *Suppose that $u(x)$ is a positive solution of*

$$\begin{cases} u''(x) + \lambda g(u) = 0, & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{4.88}$$

and $u(1/2) = s$. Then s satisfies

$$g(s) > 0, \quad \int_u^s g(t)dt \geq 0, \quad \textit{for } u \in [0, s]. \tag{4.89}$$

When $0 < a < 1$, this is only possible when the positive hump of the function has larger area than the negative hump, *i.e.* $0 < a < 0.5$. Hence, when $0.5 < a < 1$, there is no positive equilibrium solution no matter what λ is. When $0 < a < 0.5$, there exists positive equilibrium solutions for some λ, which we discuss below.

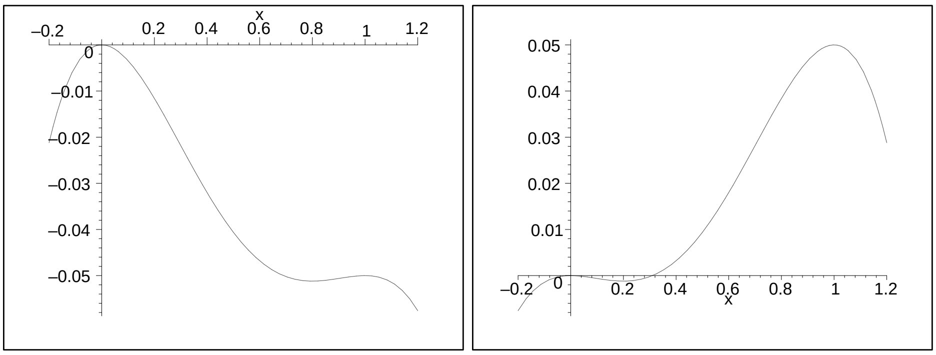


Figure 4.12: **(a)** graph of $G(u)$ when $a = 0.8$; **(b)** graph of $G(u)$ when $a = 0.2$

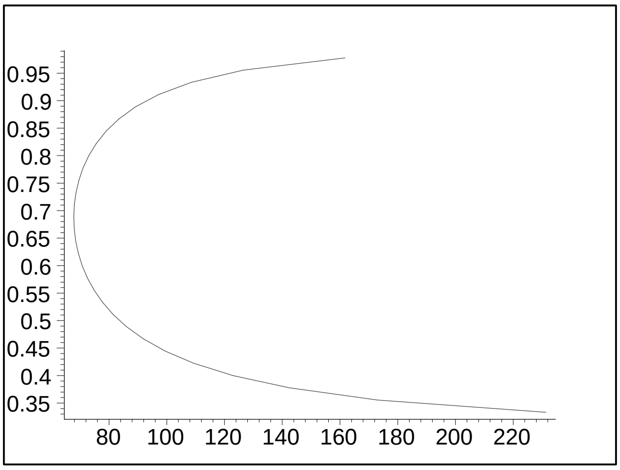


Figure 4.13: bifurcation diagram of $u'' + \lambda u(1-u)(u-0.2) = 0$, $u(0) = u(1) = 0$.

Bistable equilibrium solutions. When $0 < a < 0.5$, the global bifurcation diagram (Figure 4.13) shows that there exists $\lambda_* > 0$ such that there is no positive equilibrium solutions when $\lambda < \lambda_*$, there is one when $\lambda = \lambda_*$ and there are two positive equilibrium solutions when $\lambda > \lambda_*$. Remember that $u(x) = 0$ is always an equilibrium solution. Then we have three non-negative equilibrium solutions when $\lambda > \lambda_*$.

This bifurcation diagram reveals several biologically interesting phenomena. First, the existence of λ_* implies the existence of a critical patch size $L_* = \sqrt{\lambda_*}$, but this L_* is not obtained from linearization like diffusive logistic equation. From the bifurcation diagram, we estimate $\lambda_* = 70$ which is much larger than the one in diffusive logistic equation $\lambda_* = \pi^2 \approx 10$. That shows the negative impact of allee effect to the population. The critical patch size is larger so the species needs a larger living environment to survive.

Secondly, even when $\lambda > \lambda_*$, the species still may not survive. In this case, there are three equilibrium solutions 0, $u_1(x)$ (with larger $u(1/2)$) and $u_2(x)$ (with smaller $u(1/2)$). You can check that $u = 0$ is always an asymptotically stable equilibrium solution, so when the initial population is too small in some sense, then the population will still drop to zero. But if the initial population is large, then it will tend to an positive equilibrium solution, which is in fact the larger one among the two positive equilibrium solutions. Therefore there are two stable equilibrium solutions, and either one can be the asymptotic limit of the population evolution. This is similar to the ODE case, but more complicated. To understand the threshold phenomenon here, we need to introduce the concept of the *basin of attraction*.

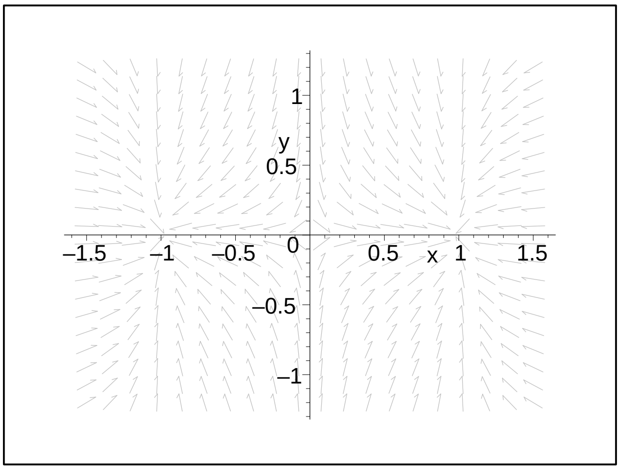


Figure 4.14: phase portrait of $x' = x(1-x)(1+x)$, $y' = -y$

Suppose that $u(x)$ is an asymptotically stable equilibrium solution. Then the *basin of attraction* of $u(x)$ is the set of initial functions $u_0(x)$ such that $\lim_{t\to\infty} u(t,x) = u(x)$, where $u(t,x)$ is the solution of reaction diffusion equation with $u(0,x) = u_0(x)$. From numerical experiment, we can easily find initial values belonging to the basins of attraction of 0 or $u_1(x)$. From a deep result in the dynamical system, if $u_0(x) < u_2(x)$, then $u_0(x)$ belongs to the basin of attraction of $u = 0$, and if $u_0(x) > u_2(x)$, then it belongs to the basin of attraction of $u_1(x)$. But there are initial values which may not larger or smaller than $u_2(x)$, and there is no complete results regarding these solutions. But it is known that the two basins of attraction are separated by a "surface" in the space of all continuous functions. If the initial value is one of the functions on this "surface" then the limit of the solution is neither 0 nor $u_1(x)$, but $u_2(x)$! This "surface" is called *stable manifold* of the equilibrium solution $u_2(x)$. Although in the experiment, it is vary hard to catch a solution on the stable manifold, but the stable manifold serves as the role of the threshold here. This situation is similar to the following ODE system: (see Figure 4.14.)

$$x' = x(1-x)(1+x), \quad y' = -y. \tag{4.90}$$

On this phase plane, the line $x = 0$ is the stable manifold of the saddle point $(0,0)$, and it separates the basins of attraction of stable equilibrium points $(-1,0)$ and $(1,0)$.

Biologically, the population for strong Allee effect has a *conditional persistence* for $\lambda > \lambda_*$, in which the survival of the population depends on initial population distributions. In the logistic case, the population has a *unconditional persistence* for $\lambda > \pi^2$. The "surface" mentioned above is called *threshold manifold*. This threshold manifold separates the set of all non-negative initial distributions into two disconnected subsets, which we can call "above threshold" (A) and "below thresold" (B) sets. For any initial distribution in A, the asymptotic state is the stable steady state u_1 (On the upper branch of the bifurcation diagram); for any initial distribution in B, the asymptotic state is the stable steady state 0.

To conclude this section, we mention that the bifurcation diagram for the weak Allee effect case ($f(u) = u(1-u)(u-a)$ where $a \in (-1, 0)$) is a combination of the logistic and strong Allee effect cases. There are two critical values $\lambda_1 = \pi^2/f'(0)$ and $\lambda_* < \lambda_1$ which divide the parameter space into three parts: *extinction regime* $\lambda \in (0, \lambda_*)$, *conditional persistence regime* $\lambda \in (\lambda_*, \lambda_1)$, and *unconditional persistence regime* $\lambda \in (\lambda_1, \infty)$. In the following numerical bifurcation diagram, $\lambda_1 = \pi^2/f'(0) = 10\pi^2 \approx 98.70$, and $\lambda_* \approx 33.53$.

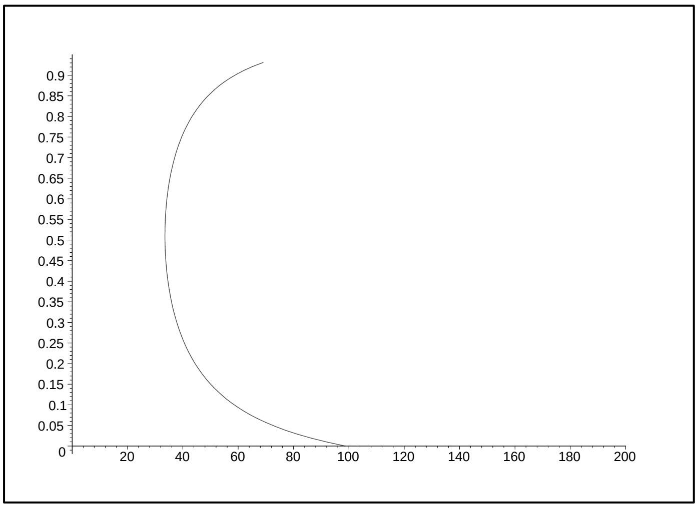


Figure 4.15: Bifurcation diagram of $u'' + \lambda u(1-u)(u+0.1) = 0$, $u(0) = u(1) = 0$.

The bifurcation diagram in Figure 4.15 allows for the possibility of *hysteresis* as the diffusion coefficient λ or the habitat size is varies. Suppose we start with a large size habitat, and then slowly decrease the size. Then initially the population will stabilize at the unique steady state solution $u_1(\lambda)$. However when the habitat is too small (when $\lambda < \lambda_*$), the population collapses quickly to zero. To salvage the population, we may attempt to restore the habitat by slightly increasing λ so that $\lambda > \lambda_*$. But if the population has dropped below the threshold $u_2(\lambda)$ at that moment, then the population cannot be saved since it is now still in the basin of attraction of the stable steady state $u = 0$.

4.9 Pattern formation

Reaction diffusion equation is used to model natural phenomena of individuals moving and reproducing in the habitat. Models are formulated and computed, and then we compare the simulated data with experimental data. If a particular population distribution appears in the experimental data often, then it may be of important value, since this distribution could occur in future again. The model based on the phenomenon is more reliable if the same density distribution can be generated by the model. Such particular density distribution which persists and can be observed in nature or experiment is called a *pattern*.

Reaction diffusion equation is one of the simplest model which considers the spatial distribution of a substance or species. Thus it is also thought as a basic mechanism of generating patterns by physicists, chemists, biologist and engineers.

Suppose there is a reaction diffusion model. What is a pattern? In natural science, any observable phenomena could be called a pattern, which could be temporal, spatial or spatial-temporal. Mathematically, the definition of pattern is not clear. In this section, we use a naive way to define it:

Definition 4.8. A pattern is a non-constant stable equilibrium solution of a reaction diffusion equation.

For a reaction diffusion equation on a bounded domain, this definition seems quite reasonable, because from our studies so far, the solution of the equation always has a limit as $t \to \infty$ unless it goes to infinity. In the case of Dirichlet problems, the limit is either a constant $u = 0$, which we do not consider as a pattern, or an equilibrium solution, which is a pattern. In the case of Neumann problems, the limit is always a constant for all numerical experiments we did so far. In this section, we confirm these observations mathematically by establishing the following results:

1. The limit of a bounded solution of a reaction diffusion equation is always an equilibrium solution, which is either a constant or a pattern.

2. There is no pattern for the Neumann boundary value problem of reaction diffusion equation.

The first result is similar to that of ordinary differential equation $\frac{du}{dt} = f(u)$. For the ODE, any solution is monotonic, and if it is also bounded, then it has to approach an equilibrium point. For the reaction diffusion equation, the solutions also have a certain monotonicity, which can be best explained using the energy method. The equation $\frac{du}{dt} = f(u)$ can be viewed as describing a physical process with a potential energy $-F(u) = \int f(t)dt$. Suppose that $u(t)$ is a solution of the equation, then the rate of change of energy along the solution orbit is

$$\frac{d}{dt}[-F(u(t))] = -F'(u(t))u'(t) = -f(u(t))u'(t) = -[u'(t)]^2 \leq 0. \tag{4.91}$$

Thus the potential energy decreases along an orbit, and each solution will eventually fall to a local minimum of the energy function $-F(u)$, which is equivalent to a zero point of $-f(u) = -F'(u)$ with $-f'(u) = -F''(u) > 0$.

We can find a similar formulation for the reaction diffusion equation. Suppose that $u(t, x)$ is a solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + f(u), & t > 0,\ x \in (0,1), \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = u_0(x) > 0,\ x \in (0,1). \end{cases} \tag{4.92}$$

The energy function is

$$E(u(t,x)) = \int_0^1 \left(\frac{1}{2} u_x^2(t,x) - F(u(t,x)) \right) dx, \tag{4.93}$$

where $F(u) = \int f(t)dt$. Then the rate of change of energy along the solution orbit is

$$\begin{aligned} \frac{d}{dt} E(u(t,x)) &= \int_0^1 \left(\frac{1}{2} \cdot 2u_x(t,x)u_{tx}(t,x) - F'(u(t,x))u_t(t,x) \right) dx \\ &= \int_0^1 (u_x(t,x)u_{tx}(t,x) - f(u(t,x))u_t(t,x))dx. \end{aligned} \tag{4.94}$$

Integral by parts again provides a big help:

$$\begin{aligned} &\int_0^1 u_x(t,x)u_{tx}(t,x)dx = \int_0^1 u_x(t,x)du_t(t,x) \\ =& u_x(t,x)u_t(t,x)|_0^1 - \int_0^1 u_t(t,x)du_x(t,x) \\ =& 0 - \int_0^1 u_t(t,x)u_{xx}(t,x)dx \\ =& -\int_0^1 u_t(t,x)u_{xx}(t,x)dx. \end{aligned} \tag{4.95}$$

Here the boundary terms are zero since $u_t(t,0) = u_t(t,1) = 0$ for the Dirichlet boundary condition. From (4.94) and (4.95), we obtain

$$\begin{aligned} \frac{d}{dt} E(u(t,x)) &= -\int_0^1 u_t(t,x)[u_{xx}(t,x) + f(u(t,x))]dx \\ &= -\int_0^1 u_t^2(t,x)dx \le 0. \end{aligned} \tag{4.96}$$

Here in the second step we use the fact that u is a solution of $u_t = u_{xx} + f(u)$! Therefore the energy $E(u)$ is decreasing along an orbit $u(t,x)$. (For this reason, sometimes $E(u)$ is called a Lyapunov function.) If $u(t,x)$ is also bounded, then the energy $E(u(t,x))$ is also bounded, and it must have a limit, which in fact must be a critical point of the energy function. The critical point here means a function $u(t,x)$ such that $dE(u(t,x))/dt = 0$. From (4.96), this critical point must satisfy $\int_0^1 u_t^2(t,x)dx = 0$, thus it must be an equilibrium solution. Thus we have proved (though not completely rigorously)

Proposition 4.9. *Suppose that $u(t,x)$ is a bounded solution of* (4.92). *Then there exists an equilibrium solution $v(x)$ such that $\lim_{t\to\infty} u(t,x) = v(x)$.*

This property of reaction diffusion equation tells us why the equilibrium solutions are so important. In fact, the limit equilibrium solution is usually a local minimum point of the energy function, which is equivalent to stable equilibrium solutions. So stable equilibrium solutions (patterns or stable equilibrium solutions) are even more important. However in the next result, we show that there is no pattern in Neumann boundary value problems.

Theorem 4.10. *Suppose that $u(x)$ is a non-constant solution of*

$$\begin{cases} Du''(x) + g(u) = 0, & x \in (0, L), \\ u'(0) = u'(L) = 0. \end{cases} \tag{4.97}$$

Then u is unstable.

Proof. Since $u(x)$ is a non-constant solution on $[0, L]$, then $u(x)$ must be one of the following cases: (a) $u(x)$ is increasing on $[0, L]$; (b) $u(x)$ is decreasing on $[0, L]$; or (c) $u(x)$ is neither increasing nor decreasing.

If $u(x)$ is increasing on $[0, l]$, then $u'(x) \geq 0$ for $x \in (0, L)$, and

$$u'(0) = 0, \text{ and } u'(L) = 0, \tag{4.98}$$

because of Neumann boundary condition. Also $u'(x) \geq u'(0)$ for any $x > 0$, so $u''(0) \geq 0$ ($u'(x)$ is increasing at $x = 0$), and similarly $u''(L) \leq 0$. Suppose that $\phi_1(x)$ is the eigenfunction of the largest eigenvalue μ_1, then

$$\begin{cases} D\phi_1''(x) + g'(u(x))\phi_1 = \mu_1\phi_1, & x \in (0, L), \\ \phi_1'(0) = \phi_1'(L) = 0. \end{cases} \tag{4.99}$$

From Theorem 4.1, $\phi_1(x) > 0$ for $x \in [0, L]$. On the other hand, if we differentiate (4.97) with respect to x, we obtain

$$\begin{cases} D(u')''(x) + g'(u(x))u' = 0, & x \in (0, L), \\ u'(0) = u'(L) = 0. \end{cases} \tag{4.100}$$

We apply the multiply-multiply-subtract-integrate trick to ϕ_1 and u' on $[0, L]$, then we obtain

$$\int_0^L (\phi_1''u' - (u')''\phi_1)dx = \mu_1 \int_0^L \phi_1(x)u'(x)dx. \tag{4.101}$$

The left hand side does not completely disappear. In fact, by using the boundary conditions, we have

$$\int_0^L (\phi_1''u' - (u')''\phi_1)dx = -u''(L)\phi_1(L) + u''(0)\phi_1(0) \geq 0, \tag{4.102}$$

since $u''(L) \leq 0$, $\phi_1(L) > 0$, $u''(0) \geq 0$ and $\phi_1(0) > 0$. On the other side,

$$\int_0^L \phi_1(x)u'(x)dx > 0 \tag{4.103}$$

because both $\phi_1(x)$ and $u'(x)$ are positive. Thus by comparison, $\mu_1 > 0$ and $u(x)$ is unstable.

If $u(x)$ is decreasing, the proof above can also be carried out in an exact same way. If $u(x)$ is neither increasing nor decreasing, then there exists a subinterval $[x_1, x_2]$ of $[0, L]$ such that $u'(x_1) = u'(x_2) = 0$, $u''(x_1) \geq 0$, $u''(x_2) \leq 0$ and $u'(x) \geq 0$ in (x_1, x_2). Then we can apply the proof above to the interval $[x_1, x_2]$. Hence $u(x)$ is unstable in any case. □

4.10 Traveling waves

Let's consider the Neumann boundary problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \lambda u(1-u), & t > 0,\ x \in (0,1), \\ u_x(t,0) = u_x(t,1) = 0, \\ u(0,x) = u_0(x),\ x \in (0,1). \end{cases} \tag{4.104}$$

From the results in last section, the asymptotic behavior of the solution is pretty clear: (a) the limit of any solution is an equilibrium solution; (b) since any non-constant solution and $u = 0$ are all unstable equilibrium solutions, then most likely, the limit is $u = 1$, which is stable. However, we can observe some interesting dynamics if we choose the initial function $u_0(x)$ to be a function which is almost 0 on the whole interval $[0, 1]$, but is positive near $x = 1$. The particular function we choose here is $u_0(x) = 2e^{-40x^2}$. We could assume that this is a population concentrating near $x = 0$, and completely absent on $[0.5, 1]$. (see Figure 4.16 (a).)

Once the evolution according to the diffusive logistic equation begins, the peak of the population quickly drop to near $u = 1$, but the remaining part of the profile hardly changes except the tail extends from $x = 0.4$ to $x = 0.45$ (this is only approximation since we only takes 10 grid points). Since then, the shape of the solution does not change much–the maximum is at $x = 0$, near $x = 0$ the solution is nearly flat at $u = 1$, the solution transits from $u = 1$ to $u = 0$ in an interval of length 0.3, then it is almost 0 on the subinterval near $x = 1$. But the transition part (which we call a transition layer) moves as time elapse, initially ($t = 0.001$) the center of the transition layer is at $x = 0.24$ (Figure 4.16 (b)), then it moves to $x = 0.43$ at $t = 0.005$ (Figure 4.16 (c)), and to $x = 0.68$ at $t = 0.01$ (Figure 4.16 (d)). Here we define the center of the transition layer to be the value of x such that $u(x) = 0.5$. But after that, the transition layer breaks off (Figure 4.17 (e)), and the solution converges to $u(x) = 1$ quickly (Figure 4.17 (f)). Biologically, the evolution of the solution from $t = 0$ to about $t = 0.02$ simulates an *invasion* of a species from its original habitat to the whole region. The profile of the solution during the the invasion is almost same, and it looks like a wave propagating from the left to the right.

Now we derive more analytical information about this wave propagating phenomenon. We notice that during the invasion, the transition layer moves at an almost constant velocity. Suppose this velocity is c, then at the point (t_1, x_1) and (t_2, x_2), if $x_2/t_2 = x_1/t_1 = c$, then $u(t_1, x_1) = u(t_2, x_2)$. So $u(t, x)$ should be in a form of $w(x - ct)$. We notice that during the invasion, the impact of boundary condition is not important, so we consider just the equation, and on the whole space

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda u(1-u),\ t > 0,\ x \in (-\infty, \infty), \tag{4.105}$$

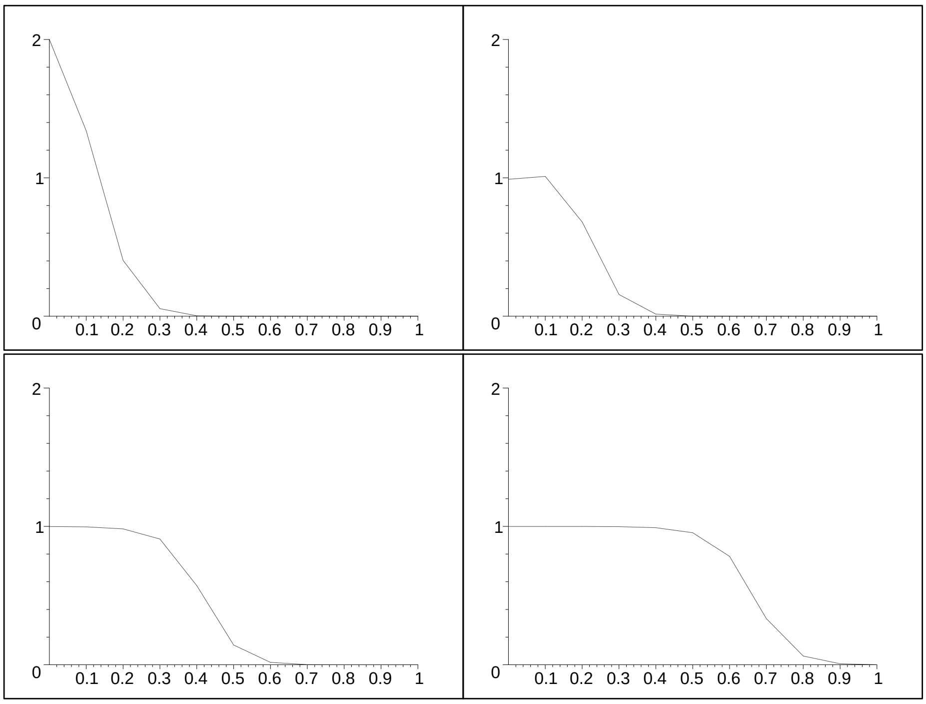


Figure 4.16: Solution profile **(a)** $t = 0$; **(b)** $t = 0.001$; **(c)** $t = 0.005$; **(d)** $t = 0.010$.

and we looks for a special solution with form $u(t, x) = w(x - ct)$. We call such a solution a *traveling wave solution*. Traveling wave solutions are similar to equilibrium solutions, in fact, equilibrium solutions can be regarded as traveling wave solution with velocity 0. If your observation point moves with the traveling solution at the same speed, then in your eyes, the traveling wave does not move, just like an equilibrium solution.

Let $y = x - ct$. Then

$$\frac{\partial u}{\partial t} = w'(y) \cdot (-c), \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = w''(y). \tag{4.106}$$

Thus equation (4.105) becomes an ordinary differential equation

$$-cw' = w'' + \lambda w(1 - w), \tag{4.107}$$

but with constrains

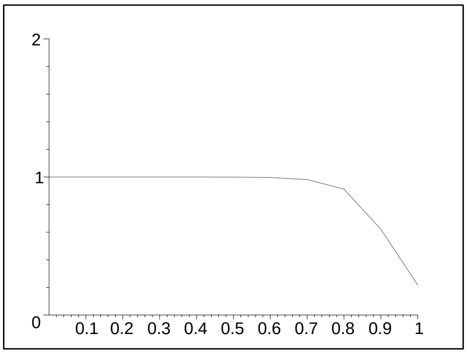
$$w'(y) < 0, \quad \lim_{y \to -\infty} w(y) = 1, \quad \lim_{y \to \infty} w(y) = 0. \tag{4.108}$$

The equation can be converted to a planar system:

$$w' = v, \quad v' = -\lambda w(1 - w) - cv, \tag{4.109}$$

and the constrains now are

$$v < 0, \quad \lim_{y \to -\infty} (w(y), v(y)) = (1, 0), \quad \lim_{y \to \infty} (w(y), v(y)) = (0, 0). \tag{4.110}$$



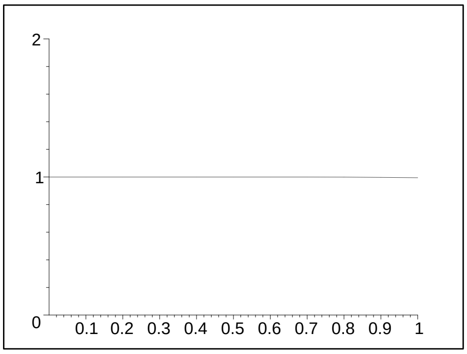


Figure 4.17: Solution profile **(e)** $t = 0.015$; **(f)** $t = 0.020$

By phase plane analysis, we find that $(1, 0)$ and $(0, 0)$ are the only equilibrium points of system (4.109), thus the traveling wave solution is a connecting orbit between two equilibrium points. The question is: is there such an orbit, and what is the velocity c? We can achieve that by a little more phase plane analysis. We linearize the system at both equilibrium points, then the Jacobians are

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -\lambda & -c \end{pmatrix}, \quad J(1,0) = \begin{pmatrix} 0 & 1 \\ \lambda & -c \end{pmatrix}. \tag{4.111}$$

Thus $(1, 0)$ is always a saddle point, and the type of $(0, 0)$ depends on the values of λ and c. The eigenvalues of $J(0, 0)$ are

$$\mu = \frac{-c \pm \sqrt{c^2 - 4\lambda}}{2}. \tag{4.112}$$

If $c^2 - 4\lambda < 0$, then $(0, 0)$ is a spiral sink. And we do not have an orbit satisfying (4.110) since all solution approaching $(0, 0)$ are in an oscillatory fashion, then $v(y)$ cannot be always negative. If $c^2 - 4\lambda \geq 0$, then $(0, 0)$ is a sink, and the unstable solution coming out of the saddle point $(1, 0)$ hits $(0, 0)$ directly. (see Fig. 4.18.) This solution $(u(z), v(z))$ satisfies (4.110), so $u(x - ct)$ is a desired traveling wave solution.

For the existence of the traveling wave solutions, we have the following result from phase plane analysis above:

Theorem 4.11. (4.105) *has a family of traveling wave solutions* $u_c(t, x)$, $c \geq \sqrt{4\lambda}$, *and for each* $t \in (-\infty, \infty)$,

$$\lim_{x\to-\infty} u(t, x) = 1, \quad \lim_{x\to-\infty} u(t, x) = 0. \tag{4.113}$$

In biological applications, an important question is the speed of the population wave. From Theorem 4.11, we find that the minimum speed is $c = 2\sqrt{\lambda}$. But in the application, which speed does the population "prefer"? Let's go back to the example in the beginning. From the coordinate of the center of the wave, we can estimate that the wave speed is about $c \approx 52$, and the theoretical minimum wave speed from Theorem 4.11 is $c = \sqrt{4000} \approx 63$ (since $\lambda = 1000$). Considering the calculation errors and the impact of the boundary conditions, we can guess that this population is invading at the minimum speed $c = \sqrt{4\lambda}$. Indeed this fact was proved in a pioneer paper by Kolmogolov, Petrovki and Pisconov in 1936.

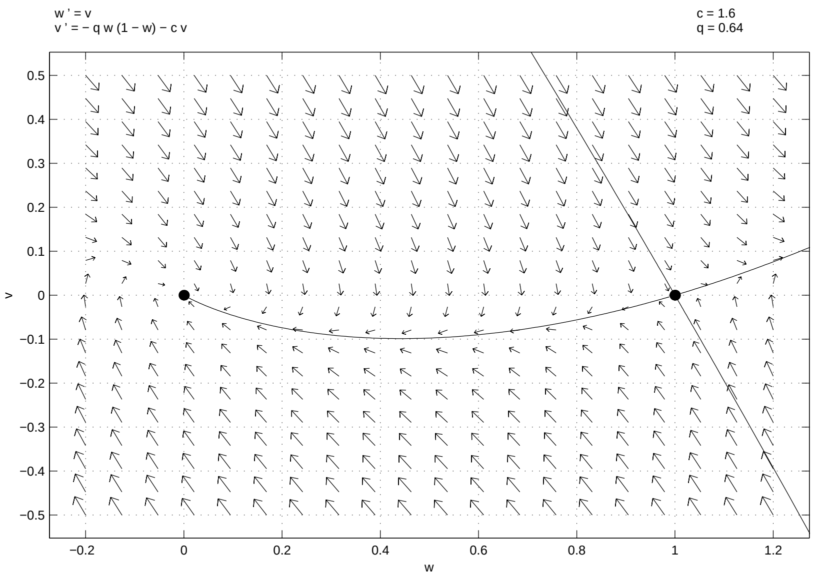


Figure 4.18: Phase portrait of traveling wave

We could compare this to the diffusive Malthus equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda u, \ t > 0, \ x \in (-\infty, \infty). \tag{4.114}$$

In Section 3.4, we have obtain that the front of the population wave approximately advances at the speed $R'(t) = \sqrt{4\lambda}$, which is same as the minimum wave speed of the logistic case. Intuitively, the wave speed of logistic equation should always be smaller than the Malthus case (the growth rate is smaller), here we show that asymptotically the two speeds are same. But in the Malthus case, population has an exponential growth in the "metro" area, and the population tends to the carrying capacity in the center part for the logistic equation. The traveling wave solution here can be used to model the biological invasion of a foreign species into a new territory.

4.11 Standing Wave

4.12 Summary

In this chapter we studied the behavior of the solution of reaction diffusion equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + f(u), \quad t > 0, \ x \in (0,1), \\ u(t,0) = u(t,1), (\text{or } u_x(t,0) = u_x(t,1) = 0,) \\ u(0,x) = f(x), \ x \in (0,1). \end{cases} \tag{4.115}$$

Mathematically, our investigation can be summarized as follows:

1. For a solution $u(t,x)$, if it remains bounded, then it approaches an equilibrium solution of the equation, and most likely, this equilibrium solution is a stable one.

2. For Dirichlet problem, the stable equilibrium solution is either a positive one, or zero; for Neumann problem, the stable equilibrium solution is a positive constant.

Biologically, we can observe several phenomena for different growth rates, different diffusion constants, different size of the domains, or even different initial values:

1. *Extinction*: the population approaches an equilibrium zero.

2. *Persistence*: the population approaches a positive equilibrium solution.

3. *Invasion*: the population invades the new territory at a more or less constant velocity.

Chapter 4 Exercises

1. Use the substitution
$$s = at, \quad y = \sqrt{\frac{a}{D}}x, \quad v = \frac{u}{N} \tag{4.116}$$
to deduce a new equation from
$$\begin{cases} u_t = Du_{xx} + au\left(1 - \dfrac{u}{N}\right), & t > 0, \quad x \in (0, L), \\ u(t,0) = u(t,L) = 0, & \\ u(0,x) = f(x), & x \in (0, L). \end{cases} \tag{4.117}$$
Express the new parameters and initial values in terms of old ones.

2. Find a substitution different from (4.3) and (4.116) to deduce a nondimensionalized equation from (4.117).

3. Consider diffusive population model with Allee effect
$$\begin{cases} u_t = Du_{xx} + au\left(1 - \dfrac{u}{N}\right)\left(\dfrac{u}{M} - 1\right), & t > 0, \quad x \in (0, L), \\ u(t,0) = u(t,L) = 0, & \\ u(0,x) = f(x), & x \in (0, L). \end{cases} \tag{4.118}$$
Use a table to list the dimensions of all quantities and parameters, and convert the equation to a dimensionless equation.

4. Modify the numerical algorithm in Section 4.2 for non-homogeneous boundary condition
$$u_x(t,0) = c_1, \; u_x(t,1) = c_2. \tag{4.119}$$

5. Modify the numerical algorithm in Section 4.2 for Robin boundary condition

$$u_x(t,0) = bu(t,0),\ u_x(t,1) = -bu(t,1),\ \ b > 0. \tag{4.120}$$

Write a `Maple` program to implement your algorithm.

6. Write a `Maple` program to numerically compute the solutions of

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \lambda u(1-u), & t > 0,\ x \in (0,1), \\ u_x(t,0) = u_x(t,1) = 0, & \\ u(0,x) = f(x), & x \in (0,1). \end{cases} \tag{4.121}$$

Use positive initial data $f(x) > 0$. Confirm that for all positive initial data and all $\lambda > 0$ which you try, the solution will always tend to $u = 1$.

7. Write a `Maple` program to numerically compute the solutions of

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \lambda u(1-u)(u-0.05), & t > 0,\ x \in (0,1), \\ u(t,0) = u(t,1) = 0, & \\ u(0,x) = f(x), & x \in (0,1). \end{cases} \tag{4.122}$$

Use the following parameters: (a) $\lambda = 20$, any $f(x) > 0$; (b) $\lambda = 80$, $f(x) = 0.3\sin(\pi x)$, $f(x) = 0.6\sin(\pi x)$ and $f(x) = \sin(pix)$. Describe the qualitative behavior of the solutions.

8. For the Dirichlet problem of diffusive Malthus equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \lambda u, & t > 0,\ x \in (0,1), \\ u(t,0) = u(t,1) = 0, & \\ u(0,x) = f(x), & x \in (0,1). \end{cases} \tag{4.123}$$

What is the stability of $u = 0$ when $\lambda = \pi^2$? Also show that $u(x) = \sin(\pi x)$ is also an equilibrium solution when $\lambda = \pi^2$. What is the stability of this equilibrium solution?

9. Use the technique in the proof of Proposition 4.2 to show that when $\lambda \le \pi^2$, (4.121) has no positive equilibrium solution. (Hint: suppose there is one, say $u(x)$, and use the multiply-multiply-subtract-integrate trick to $u(x)$ and $\sin(\pi x)$, remember that $v = \sin(\pi x)$ satisfies $v'' + \pi^2 v = 0$.)

10. Consider the equation:

$$x^6 + x^2 + x - \varepsilon x = 0. \tag{4.124}$$

For ε near $\varepsilon = 3$, the equation has a solution with a form $x_\varepsilon = 1 + r x_1 + r^2 x_2 + \textit{higher order terms}$, where $r = \varepsilon - 3$.

   (a) Use perturbation method to calculate x_1 and x_2.
   (b) Use the formula in implicit function theorem to calculate x_1.

11. Consider
$$u'' + \lambda u(1-u) = 0, \;\; x \in (0,1), \;\; u(0) = u(1) = 0. \tag{4.125}$$
$(\lambda, u) = (9\pi^2, 0)$ is also a bifurcation point. Use the perturbation method to find the approximate form of solutions near the bifurcation point:
$$\lambda = 9\pi^2 + a_1\varepsilon + a_2\varepsilon^2 + \cdots, \;\; u(x) = \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \textit{higher order terms.} \tag{4.126}$$
Show that $u_1(x) = k\sin(3\pi x)$, for some constant k, and determine k by solving the equation of u_2 and assuming $a_1 = 1$.

12. Consider again (4.125), but at $\lambda = 4\pi^2$. The perturbation method which works at $\lambda = \pi^2$ and $\lambda = 9\pi^2$ does not work exactly same way at $\lambda = 4\pi^2$. By using the same perturbation method in (4.126) (with $9\pi^2$ replaced by $4\pi^2$), show that $a_1 = 0$ and $u_1 = 0$. Use `Maple` to further expand the series, show that $a_2 > 0$, and assuming that $a_2 = 1$, calculate the first nonzero u_i.

13. Use perturbation method to calculate the approximate solutions of
$$u'' + \lambda u(1-u) = 0, \;\; x \in (0,1), \;\; u'(0) = u'(1) = 0, \tag{4.127}$$
at a bifurcation point $(\lambda, u) = (\pi^2, 0)$.

14. Prove the following Fredholm alternative theorem by using the "multiply-multiply-subtract-integrate" technique: suppose that
$$y'' + \pi^2 y = f(x), \;\; y(0) = 0, \;\; y(1) = 0, \tag{4.128}$$
has a solution $y(x)$, then $\displaystyle\int_0^1 f(x)\sin(\pi x)dx$ must equal to zero.

15. Consider
$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \lambda\left[u(1-u) - \dfrac{au}{1+bu}\right], & t > 0, \; x \in (0,1), \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = f(x), \; x \in (0,1), \end{cases} \tag{4.129}$$
where $a, b > 0$. This equation models a species which is harvested or predated, and the rate of harvesting has a saturation limit as $u \to \infty$ (which is also called Holling's type II functional response.) Write a `Maple` program to numerically compute the solutions of (4.129). Use positive initial data $f(x) > 0$.

16. Show that for any $\lambda > 0$, $u = 1$ is an asymptotically stable equilibrium solution of (4.27), and $u = 0$ is an unstable equilibrium solution of (4.27).

17. Show that if $u = c$ is an unstable (stable) equilibrium of $\dfrac{du}{dt} = f(u)$, then $u(x) = c$ is an unstable (asymptotically stable) equilibrium of
$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + f(u), & t > 0, \; x \in (0,1), \\ u_x(t,0) = u_x(t,1) = 0, \\ u(0,x) = f(x), \; x \in (0,1). \end{cases} \tag{4.130}$$

18. (a) Use `Maple` program to draw the bifurcation diagrams of the equilibrium solutions of (4.83) when the parameter is the following values:

$$\begin{aligned} &(1)\ a = 0.4,\ (2)\ a = 0.1,\ (3)\ a = 0, \\ &(4)\ a = -0.2,\ (5)\ a = -1,\ (6)\ a = -10. \end{aligned} \tag{4.131}$$

We have known that $a = 0.5$ is a bifurcation point where the bifurcation diagram has a qualitative change (from no positive solutions to exist positive solutions). From the various bifurcation diagrams, guess other bifurcation points.

(b) When $a < 0$ (weak Allee effect), the positive solutions may bifurcate from the zero solutions for some a. Determine for which a, this is possible, and where is the bifurcation point λ_a for each possible a. For which a the bifurcation is supercritical, and subcritical?

(c) Explain the biological significance for each different bifurcation diagram.

19. When a type-II functional response is added to the diffusive logistic equation, the new equation is

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \lambda \left[u(1-u) - \dfrac{au}{1+bu} \right], \quad t > 0,\ x \in (0,1), \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = f(x) > 0,\ x \in (0,1). \end{cases} \tag{4.132}$$

(a) Determine for which values of (a, b) that the growth function is of strong Allee effect, or weak Allee effect.

(b) Choose two sample parameter sets (a, b) which are of strong Allee effect, and weak Allee effect respectively, and use Maple to draw the corresponding bifurcation diagrams of equilibrium solutions.

20. Consider

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u),\ t > 0,\ x \in (-\infty, \infty), \tag{4.133}$$

where $f(u) = u$ when $0 < u < 1/2$ and $f(u) = 1 - u$ when $1/2 < u < 1$. Solve the exact traveling wave solution $u(t, x) = v(x - ct)$.

21. Consider

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^3,\ t > 0,\ x \in (-\infty, \infty). \tag{4.134}$$

This equation has a traveling wave solution with speed $c = 0$. Find this solution by solving the corresponding ordinary differential equation.

22. Consider

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda u^2(1-u),\ t > 0,\ x \in (-\infty, \infty). \tag{4.135}$$

(a) Write the ordinary differential equation satisfied by the traveling wave equation of (4.114).

(b) Find the range of speed c for which (4.114) has a traveling wave with that speed.

(c) Let $\lambda = 1$. Use the perturbation method to find the leading term in the asymptotic expansion of the traveling wave solution, *i.e.* $u(t,x) = v(x-ct)$, $v(z) = V(y)$, $z = x-ct$, $y = z/c = \sqrt{\varepsilon}z$, and $V(y) = V_0(y)+\varepsilon V_1(y)+\cdots$. Find $V_0(y)$. (Hint: `f:=1/(x^2*(1-x));` and `convert(%,parfrac,x);` can do partial fraction in `Maple`, and the answer for V_0 may be an implicit function.)