Review of Multi-variable calculus:

The functions in all models depend on two variables: time t and spatial variable x, (x, y) or (x, y, z).

The spatial variable represents the environment where the species is living (bacteria:tank in lab, rabbits and foxes:woods, birds: the space).

The time variable is one dimension, we call it **time interval**. Usually it is $(-\infty, \infty)$, $[0, \infty)$ or $[0, T]$.

In mathematics we call the environment **spatial domain** (or simply **domain**) , or **region**.

Domains

The choice of domain in a model depends on the nature of the problem.

Most of time, domain is bounded. (lab tank, woods, island, earth, universe?). And it has a boundary.

Mathematically we assume that a bounded domain is an interval (a, b) in 1-d, the region enclosed by a circular curve in 2-d, or the region enclosed by a spherical surface in 3-d.

Sometime for simplicity, or to observe certain phenomenon clearer, we also consider the whole space $\mathbf{R} = (-\infty, \infty)$, $\mathbf{R}^2$ or $\mathbf{R}^3$.

We will call a domain Ω.

Functions

Functions in the models are defined for (time interval $\times$ domain).

Let X be x, (x, y) or (x, y, z). Then the function is in a form of $f(t, X)$.

Example: Let D be a 2-d domain. (a woods)
$R(t, x, y)$ =the density of rabbit population at location (x, y) and time t
$F(t, x, y)$ =the density of fox population at location (x, y) and time t

$$\text{Population density} = \frac{\text{total population in an area}}{\text{area}}$$

Example: population density is $50,000$ per square kilometer in NYC, and it is $5,000$ in Williamsburg

Graph of the function: (hard to draw in 2-d or 3-d)

graph: $(x, y, f(x, y))$ (Maple), $(x, y, z, f(x, y, z))$.

level curve (contour): the graph of $f(x, y) = c$. (Maple)

level surface: the graph of $f(x, y, z) = c$. (Maple)

Derivatives: partial derivatives $\frac{\partial f(t, x, y)}{\partial t} = f_t$, $\frac{\partial f(t, x, y)}{\partial x} = f_x$

Gradient: $\nabla f(x, y) = (\partial f / \partial x, \partial f / \partial y)$

Gradient at one point is a vector; gradient function is a vector field; gradient vector is perpendicular to the level curve

Vector field: (a vector) $F(x,y) = (f(x,y), g(x,y))$

Jacobian: (a matrix) $J = \begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix}$

Divergent of a vector field: (a scalar)
for $F(x,y) = (f(x,y), g(x,y))$, $div(F) = f_x + g_y$

Laplacian of a function: (a scalar)
for a function $f(x,y)$, $\Delta f = div(\nabla f) = div(f_x, f_y) = f_{xx} + f_{yy}$

Hessian of a function: (a matrix)
for a function f(x,y), Jacobian of ∇f, $H = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{pmatrix}$

Example: $f(x,y) = x^2 + 2y^2 - 2xy$.
(1) Find ∇f; (2) Find Hessian of f; (3) Find Δf.

Different kinds of functions:

$P(t)$: function (one variable, one function)

$P(x, y)$: multi-variable function (two variables, one function)

$(P(t), Q(t))$: vector valued function (one variable, two functions)

$(P(x, y), Q(x, y))$: vector field (two variables, two functions)

Integral of functions: Ω: two-dimensional domain, boundary $\partial\Omega$ a closed curve, $X = (x, y)$

$$\int_\Omega f(x,y)dX = \int\int_\Omega f(x,y)dxdy \qquad \int_\Omega 1dX = \text{area of } \Omega$$

Divergence Theorem:

Let $\vec{F}(x, y)$ be a vector field, and let $\vec{n}(x, y)$ be the unit <u>outer</u> normal vector at (x, y), a boundary point on $\partial\Omega$. Then $\int_{\partial\Omega} \vec{F}(x,y) \cdot \vec{n}(x,y)ds$ is the total flux of $\vec{F}$ over the curve $\partial\Omega$.

$$\int_{\partial\Omega} \vec{F}(x,y) \cdot \vec{n}(x,y)ds = \int_\Omega div(\vec{F}(x,y))dX.$$

1-d: $F(b) - F(a) = \int_a^b F'(x)dx$

Green's Identities:

$$\int_\Omega u\Delta v dX = \int_{\partial\Omega} u\nabla v \cdot \vec{n} ds - \int_\Omega \nabla u \cdot \nabla v dX$$

$$\int_\Omega u\Delta v dX - \int_\Omega v\Delta u dX = \int_{\partial\Omega} u\nabla v \cdot \vec{n} ds - \int_{\partial\Omega} v\nabla u \cdot \vec{n} ds$$

Example: Let $F(x, y) = (x + y, e^{x-y})$, and let Ω be a square $(0, 1) \times (0, 1)$.

(1) Calculate $\int_\Omega div(\vec{F}(x, y))dX$
(2) calculate $\int_{\partial\Omega} \vec{F}(x, y) \cdot \vec{n}(x, y)ds$