Homework 1 answer key

#1 − #3 ; see Maple file

#4 : Done in class

#5 :

$$P(t+\Delta t, X) = a\,P(t, x-\Delta x) + b\,P(t, x+\Delta x)$$

$$\cancel{a}\,P(t+\Delta t, x) = P(t,x) + \Delta t\cdot\frac{\partial P}{\partial t}(t,x) + \tfrac{1}{2}(\Delta t)^2\frac{\partial^2 P}{\partial t^2}(t+\xi\cdot\Delta t, x)$$

$$aP(t, x-\Delta x) = aP(t,x) - a\cdot\Delta x\frac{\partial P}{\partial x}(t,x) + \frac{a}{2}(\Delta x)^2\frac{\partial^2 P}{\partial x^2}(t,x) + (\Delta x)^3(\ \ )$$

$$bP(t, x+\Delta x) = bP(t,x) + b\cdot\Delta x\frac{\partial P}{\partial x}(t,x) + \frac{b}{2}(\Delta x)^2\frac{\partial^2 P}{\partial x^2}(t,x) + (\Delta x)^3(\ \ )$$

$$(a+b=1) \Rightarrow \quad \Delta t\cdot\frac{\partial P}{\partial t}(t,x) + (\Delta t)^2\cdot(\ \ ) = (b-a)(\Delta x)\frac{\partial P}{\partial x}(t,x) + \tfrac{1}{2}(\Delta x)^2\frac{\partial^2 P}{\partial x^2}(t,x) + (\Delta x)^3(\ \ )$$

$$\frac{\partial P}{\partial t}(t,x) + (\Delta t)\cdot(\ \ ) = \frac{(b-a)\Delta x}{\Delta t}\frac{\partial P}{\partial x}(t,x) + \frac{(\Delta x)^2}{2\Delta t}\frac{\partial^2 P}{\partial x^2}(t,x) + \frac{(\Delta x)^3}{\Delta t}(\ \ )$$

We assume that $\Delta t \to 0$, $\Delta x \to 0$, $\frac{(\Delta x)^2}{2\Delta t} \to D$

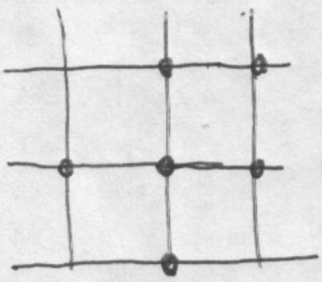
$\frac{(b-a)\Delta x}{\Delta t} \to -V$ (since $b<0.5$, $a>0.5$) $V>0$.

This can be achieved if we choose $(\Delta x)_n = 10^{-n}$, $(\Delta t)_n = \frac{D}{2}\cdot 10^{-2n}$

~~Then~~ and $(b-a)_n = \frac{D}{2}\cdot(-V)\cdot 10^{-n}$

Then in the limit : $\displaystyle\frac{\partial P}{\partial t} = -V\cdot\frac{\partial P}{\partial x} + D\frac{\partial^2 P}{\partial x^2}$

#6 :



$$P(t+\Delta t, x, y) = \tfrac{1}{4}P(t, x-\Delta x, y) + \tfrac{1}{4}P(t, x+\Delta x, y) + \tfrac{1}{4}P(t, x, y-\Delta y) + \tfrac{1}{4}P(t, x, y+\Delta y)$$

$$P(t+\Delta t, x, y) = P(t,x,y) + (\Delta t)\frac{\partial P}{\partial t}(t,x,y) + (\Delta t)^2(\ \ \ )$$

$$\tfrac{1}{4}P(t, x-\Delta x, y) = \tfrac{1}{4}P + \tfrac{1}{4}(-\Delta x)\frac{\partial P}{\partial x} + \tfrac{1}{8}(\Delta x)^2\frac{\partial^2 P}{\partial x^2} + (\ \ )$$

$$\tfrac{1}{4}P(t, x+\Delta x, y) = \tfrac{1}{4}P + \tfrac{1}{4}(\Delta x)\frac{\partial P}{\partial x} + \tfrac{1}{8}(\Delta x)^2\frac{\partial^2 P}{\partial x^2} + (\ \ )$$

$$\tfrac{1}{4}P(t, x, y-\Delta y) = \tfrac{1}{4}P + \tfrac{1}{4}(-\Delta y)\frac{\partial P}{\partial y} + \tfrac{1}{8}(\Delta y)^2\frac{\partial^2 P}{\partial y^2} + (\ \ )$$

$$\tfrac{1}{4}P(t, x, y+\Delta y) = \tfrac{1}{4}P + \tfrac{1}{4}(\Delta y)\frac{\partial P}{\partial y} + \tfrac{1}{8}(\Delta y)^2\frac{\partial^2 P}{\partial y^2} + (\ \ )$$

$$(\Delta t)\frac{\partial P}{\partial t} + (\Delta t)^2(\quad) = \frac{1}{4}(\Delta x)^2 \frac{\partial^2 P}{\partial x^2} + \frac{1}{4}(\Delta y)^2 \frac{\partial^2 P}{\partial y^2} + (\Delta x)^3(\quad) + (\Delta y)^3(\quad)$$

$$\frac{\partial P}{\partial t} + (\Delta t)(\quad) = \frac{(\Delta x)^2}{4\Delta t}\frac{\partial^2 P}{\partial x^2} + \frac{(\Delta y)^2}{4\Delta t}\frac{\partial^2 P}{\partial y^2} + \frac{(\Delta x)^3}{\Delta t}(\quad) + \frac{(\Delta y)^3}{\Delta t}(\quad)$$

We assume $\Delta t \to 0$, $\Delta x \to 0$, $\Delta y \to 0$

$$\frac{(\Delta x)^2}{4\Delta t} \longrightarrow D_1 \qquad \frac{(\Delta y)^2}{4\Delta t} \longrightarrow D_2$$

Then we have $\dfrac{\partial P}{\partial t} = D_1 \dfrac{\partial^2 P}{\partial x^2} + D_2 \dfrac{\partial^2 P}{\partial y^2}$

If $D_1 = D_2$ $(\Delta x = \Delta y)$, then $\dfrac{\partial^2 P}{\partial t} = D\left(\dfrac{\partial^2 P}{\partial x^2} + \dfrac{\partial^2 P}{\partial y^2}\right)$

#8 $\cancel{\dfrac{\partial A}{\partial t} = D_A \Delta A}$

$$\frac{\partial X}{\partial t} = D_X \Delta X + k_1 A + k_3 X^2 Y - k_4 X$$

$$\frac{\partial Y}{\partial t} = D_Y \Delta Y - k_3 X^2 Y$$

#9 (a) $C(0,x)=0$, $C(t,0)=C_1$, $C(t,L)=C_2$

(b) $C(0,x) = \dfrac{C_3}{L}(L-x)$, $\dfrac{\partial C(t,0)}{\partial x} = 0$, $C(t,L)=0$