Test 2

Math 490

A. Consider a reaction diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \lambda \left[u(1-u) - \dfrac{au}{1+u} \right], & t > 0,\ x \in (0,1), \\ u(t,0) = 0,\ u(t,1) = 0, \\ u(0,x) = u_0(x), \quad x \in (0,1), \end{cases} \tag{1}$$

where $\lambda > 0$ and $a > 0$.

1. $u(t,x) = 0$ is an equilibrium solution. Consider

$$\begin{cases} \phi''(x) + \lambda f'(0)\phi(x) = \mu\phi(x), \quad x \in (0,1), \\ \phi(0) = \phi(1) = 0, \end{cases} \tag{2}$$

   where $f(u) = u(1-u) - \dfrac{au}{1+u}$. Determine for which (λ, a), $u = 0$ is asymptotically stable, and for which (λ, a), $u = 0$ is unstable.

2. For fixed $a > 0$, we define a bifurcation point as the value of λ where $u = 0$ changes from stable to unstable. From part 1, determine the bifurcation point when $a = a_0$ (a positive number). (Note that for some a, there is no bifurcation point.)

3. When $a = 0.5$, the bifurcation point is $\lambda = 2\pi^2$. Near the bifurcation point, if $\varepsilon = \lambda - 2\pi^2$, then $u(\varepsilon, x) = \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \cdots$. Find $u_1(x)$. (Hint: in expansion of $au/(1+u)$, you should use the Taylor expansion: $\dfrac{1}{1+u} = 1 - u + u^2 - u^3 + \cdots$.)

B. Suppose that $u(x)$ is a positive solution of

$$\begin{cases} u'' + u^2 = 0, & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{3}$$

Prove that $u(x)$ is unstable. (Hint: use the idea in proof of Proposition 3.9.)

C. Consider a reaction diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \lambda \left[u(3-u) - \dfrac{au}{1+u} \right], & t > 0,\ x \in (0,1), \\ u(t,0) = 0,\ u(t,1) = 0, \\ u(0,x) = u_0(x), \quad x \in (0,1), \end{cases} \tag{4}$$

where $\lambda > 0$ and $a > 0$.

1. When $a = 3.5$, use `Maple` to draw the global bifurcation diagram of the positive equilibrium solutions. What is the smallest value of $u_{max} = u(1/2)$? And what is the smallest λ such that there is a positive equilibrium solution?

2. There is an $a_0 > 0$ such that when $a > a_0$, the equation has no positive equilibrium solution. Find the smallest a_0.