Test 1

Math 490

Due: March 5th, 5pm

1. Consider a diffusion equation with convection

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial u}{\partial x} + 8u, \quad t > 0, \ x \in (0, L), \\ u(t, 0) = 0, \ u(t, L) = 0, \\ u(0, x) = u_0(x), \quad x \in (0, L). \end{cases} \tag{1}$$

   Find the solution of the equation with the following steps:

   (a) Use separation of variables method to show that if $u(t, x) = U(t)V(x)$ is a solution, then for some constant k, U and V satisfy

   $$U'(t) = kU(t), \quad V''(x) + 4V'(x) + 8V(x) = kV(x), \quad V(0) = V(L) = 0.$$

   (b) Find the eigenvalues and eigenfunctions of

   $$V''(x) + 4V'(x) + 8V(x) = kV(x), \quad V(0) = V(L) = 0.$$

   (Hint: treat the cases of $k > 4$, $k = 4$ and $k < 4$ separately.)

   (c) Find the solution of the equation in a series form. (Hint: $c_n = \dfrac{2}{L}\displaystyle\int_0^L e^{2x} u_0(x) \sin\left(\dfrac{n\pi x}{L}\right) dx$.)

   (d) Determine the critical patch L_0 of the problem, and describe the population distribution qualitatively when $L > L_0$.

   (e) Suppose that the species lives in a river with length L, and the convection is due to the drifting of the river. Explain your mathematical results in this context.

   (f) Use `Maple` simulation to estimate when $L = 2L_0$, what is the time that the maximum of the population density reach 100 if $u_0(x) = \sin(\pi x/(2L_0))$. (You could either use the simulation of the series solution or the simulation based on numerical solution of the PDE. see `http://www.resnet.wm.edu/~jxshix/math490/index.html`)

2. (Spread of gypsy moths) Gypsy moths (*Lymantria dispar*) were brought to Massachusetts from Europe around 1870 in connection with silkworm development research. Needless to say, some of the moths escaped from breeding cages but somehow large growths and widespread dispersal were kept under control for a number of years. However, around 1900 there was a drastic increase of gypsy moth population in the Boston area which quickly spread to adjacent regions. By 1925 or so, when dispersal was finally halted, gypsy moths covered all of New England and parts of New York state and Canada. There was severe damage to forests throughout the region. The following are the cumulative areas corresponding to the dispersion fronts. According to the studies by Elton, there was no significant expansion of the front after 1925.

Year	1900	1905	1910	1915	1920	1925
Area (km^2)	1290	9080	26960	58840	79770	113320

   Use the data above to estimate the value of aD and the year when the area was zero in this example. (Hint: modify the `Maple` program for muskrat population, use function $A(x - B)^2$ instead of Ax^2, and notice that Boston is a coastal city, so the spread areas are semicircular instead of circular.)