

# **Population Biology of Infectious Diseases**

(Mathematical Epidemiology)

**Microparasitic** diseases: caused by viruses and bacteria  
(viruses: influenza, measles, rubella(German measles), chicken pox)

(bacteria: tuberculosis, meningitis, gonorrhoea)

Transmission: directly from human to human

**Macroparasitic** diseases: caused by worms or insects  
(malaria(mosquitos), schistosomiasis, black plague(rats))

Transmission: human to agents (worms, insects) then to human

Epidemic: sudden outbreak of a disease

Endemic: a disease always present

First Attempt: predator(viruses)-prey(human) model  
(unsuccessful)

Widely used model: **Compartmental model**  
(Kermack-Mckendrick, 1927)

**Susceptible** population  $S(t)$ : who are not yet infected

**Infective** population  $I(t)$ : who are infected at time  $t$  and are able to spread the disease by contact with susceptible

**Removed** population  $R(t)$ : who have been infected and then removed from the possibility of being infected again or spreading  
(Methods of removal: isolation or immunization or recovery or death)

The total population  $N$

(assumed to be a constant, so birth and death are ignored)

$$N = S(t) + I(t) + R(t)$$

## Assumptions

1. Total population is a constant  $N$   
(except death from the disease)
2. A average infective makes contact sufficient to transmit infection with  $rN$  others per unit time
3. A fraction  $a$  of infectives leave the infective class per unit time

## SIR model:

$$S' = -rSI$$

$$I' = rSI - aI$$

$$R' = aI$$

You can drop the third equation since  $N = S + I + R$ .

## Qualitative analysis:

$$S' = -rSI$$

$$I' = rSI - aI$$

(1) If  $S(0) < a/r$ , then  $I(t)$  is a decreasing function which tends to 0, and  $S(t)$  is also decreasing and tends to a constant level greater than 0.

(2) If  $S(0) > a/r$ ,  $S(t)$  is also decreasing and tends to a constant level greater than 0, but  $I(t)$  will first increase in a time period  $(0, T_0)$ , then decrease and tends to 0 after  $T_0$ .

Define a dimensionless quantity  $R_0 = \frac{rS(0)}{a}$ . This is a threshold quantity called reproduction rate. If we introduce a small number of infectives  $I(0)$  into the a susceptible population, then an epidemic will occur if  $R_0 > 1$ .  $1/a$  is the average infectious period.

## Analytic results

$$\frac{I'}{S'} = \frac{dI}{dS} = \frac{(rS - a)I}{-rSI} = -1 + \frac{a}{rS}, \quad S(0) = S_0, \quad I(0) = I_0, \quad R(0) = 0$$

$$I(t) = -S(t) + \frac{a}{r} \ln S(t) + I(0) + S(0) - \frac{a}{r} \ln S(0)$$

Usually:  $S(0) \approx N$  and  $I(0) \approx 0$ ,

then  $\lim_{t \rightarrow \infty} I(t) = 0$  and  $\lim_{t \rightarrow \infty} S(t) = S_\infty$ ,

and  $N - \frac{a}{r} \ln S(0) = S_\infty - \frac{a}{r} \ln S_\infty$ .

$S_\infty$  is the eventual population who were never infective.

Maximum infective population (climax of epidemic):

$$I_{max} = -\frac{a}{r} + \frac{a}{r} \ln \frac{a}{r} + N - \frac{a}{r} \ln S(0)$$

**Example:** A flu occurs in a boys boarding school with a total of 763 boys. Of these 512 were confined to bed during the epidemic, which lasted from 22nd January to 4th February 1978. It seems that one infected boy initiated the epidemic.

This example fits the requirements of the model well: (i) the disease is of short duration, thus birth and death rate does not need to be considered; (ii) the population is confined to one location, so spatial effect is not considered.

### **Prevention of an Epidemic:**

Reduce  $R_0 = \frac{rS(0)}{a}$

Immunization: reduce  $S(0)$ , transfer  $S$  to  $R$

Isolation: isolate the known infective, reduce  $r$

Medical improvement: increase  $r$

## Geographic spread of epidemics

$$\frac{\partial S}{\partial t} = D \frac{\partial^2 S}{\partial x^2} - rSI$$

$$\frac{\partial I}{\partial t} = D \frac{\partial^2 I}{\partial x^2} + rSI - aI$$

$S(t, x)$ : density of susceptible population

$I(t, x)$ : density of infectious population

Nondimensionalized equation: ( $S = S/S_0$ )

$$\frac{\partial S}{\partial t} = \frac{\partial^2 S}{\partial x^2} - SI$$

$$\frac{\partial I}{\partial t} = \frac{\partial^2 I}{\partial x^2} + SI - \lambda I$$

$1/\lambda = rS_0/a$ : basic reproduction rate. That is the number of secondary infections produced by one primary infective in a susceptible population.

**Look for a traveling wave solution:** introducing a number of infectives into a uniform population with initial homogeneous susceptible density, and determining the geotemporal spread of the disease

$$I(t, x) = I(x - ct), \quad S(t, x) = S(x - ct):$$

$$S'' + cS' - IS = 0, \quad I'' + cI' + SI - \lambda I = 0,$$
$$S(-\infty) \geq 0, \quad S(\infty) = 1, \quad I(-\infty) = I(\infty) = 0$$

Four-dimensional Dynamical system:

$$S' = U, \quad U' = SI - cU, \quad I' = V, \quad V' = \lambda I - SI - cV$$
$$(S, U, I, V)(-\infty) = (?, 0, 0, 0), \quad (S, U, I, V)(\infty) = (1, 0, 0, 0)$$

### **Existence of epidemic wave:**

It exists if  $\lambda = a/(rS_0) < 1$ , and it does not exist if  $\lambda > 1$ .

Wave speed:  $c = 2\sqrt{1 - \lambda}$ . So when  $\lambda < 1$ , the epidemic occurs, and from the origin of the infective, two wave fronts of infective propagate to the left and right with speed  $V = 2\sqrt{rS_0D(1 - \lambda)}$ .

**Prevention of spread of epidemics:** reduce  $\lambda = a/(rS_0)$

Isolation: isolate the known infective, reduce  $r$

Medical improvement: increase  $r$

Immunization: reduce  $S_0$  near the origin of the infective

1. Epidemic is more likely to occur in a densely populous area than a less populous area.
2. A sudden influx of susceptible can initiate an epidemic.
3. An epidemic will usually only spread in one metro area.
4. Diffusion constant has no effect on whether the epidemic occurs but it has effect on spread speed if the epidemic occurs.

**Application:** the spread of Black Death (plague pandemic)

Time: mid-14th century (1347-1350) Place: Europe

Disease: bubonic plague, caused by *Bacillus pestis* and transmitted by fleas, mainly from black rats, to man.

Spread: introduced to Italy in about Dec. 1347, brought there by ship from the East where it had been raging for years. In the next few years, it spread up through Europe at approximately 200 – 400 miles a year. About 25% of the population died and 80% died within 2-3 days.

Great Plague of London (1665)

worldwide plague (1850-1959)

## Mathematical study of Black Death (Noble 1974)

$S_0 = 50/\text{mile}^2$  (population density in 1347 in Europe)

$r = 0.4\text{mile}^2/\text{year}$

$a = 15/\text{years}$  (Infectious period two weeks)

$D = 10^4\text{miles}/\text{year}^2$  (spread speed of news at that time)

$$\lambda = a/(rS_0) = 0.75$$

$$V = (rS_0D)^{1/2}c = 2 \left( rS_0D \left( 1 - \frac{a}{rS_0} \right) \right)^{1/2} = 140\text{miles}/\text{year}$$

Real speed: 200-400 miles