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EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS TO NONLINEAR SCHRÖDINGER EQUATIONS ON A BRIDGE TYPE UNBOUNDED GRAPH

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Abstract. The existence of positive standing wave solutions to a nonlinear Schrödinger equation on a bridge type unbounded metric graph (a domain of multiple half-lines with two junctions connected by a line segment with arbitrary length) is showed, and under certain conditions, the existence of multiple positive solutions is proved. Similar results also hold for the equation with bistable nonlinearity.

1. Introduction and main results

The nonlinear Schrödinger equation (NLS)

$$i\frac{\partial \phi}{\partial t} + r\Delta\phi + \chi h(|\phi|^2)\phi = 0,\ t > 0,\ x \in \mathbb{R}^N \tag{1.1}$$

arises as a canonical model of physics from the studies of continuum mechanics, condensed matter, nonlinear optics, plasma physics [10, 35]. A standing wave solution of (1.1) is in a form of $\phi(x,t) = \Phi(x)e^{-\lambda i t}$ and Φ satisfies a nonlinear elliptic equation

$$r\Delta\Phi + \lambda\Phi + \chi h(|\Phi|^2)\Phi = 0,\ x \in \mathbb{R}^N \tag{1.2}$$

which has been extensively considered in the last few decades [6, 7, 34]. Here r is interpreted as the normalized Planck constant, χ describes the strength of the attractive interactions, λ is the wavelength and h is a real-valued function. Standing wave solutions of more general Schrödinger type equation have also been studied in [9, 11, 14, 16, 28, 31, 36].

While the standard spatial setting for the nonlinear Schrödinger equation is the Euclidean space $\mathbb{R}^N$ for $N = 1, 2, 3$, there have been recent interests on wave propagation on thin graph like domains which can be approximated by metric graphs (or quantum graphs) [2–5, 12, 13, 15, 17, 18, 20–23, 25–27, 32, 33, 37], see also surveys [1, 24].

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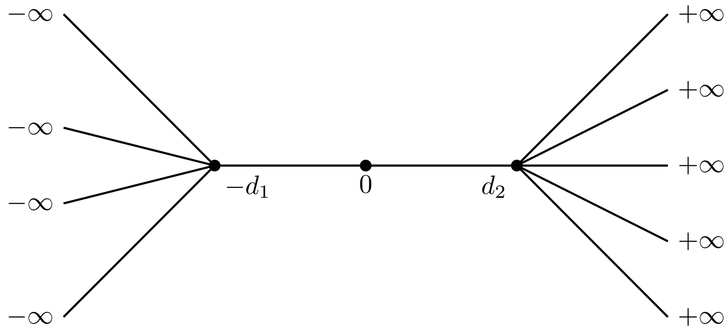

FIGURE 1. Metric graph with a single bridge

In this paper we consider the standing waves of the nonlinear Schrödinger equation on an unbounded metric graph with two vertices, namely, a domain of multiple half-lines with two junctions connected by a line segment. We state the precise definition of the domain below. The domain Ω consists of a single line segment $I_0 = (-d_1, d_2)$ with endpoints which have coordinate $x = -d_1$ and $x = d_2$ $(d_1, d_2 > 0)$, and two families of half-lines $\{I_{-i} = (-\infty, -d_1) : i = 1, \cdots, l\}$ and $\{I_j = (d_2, \infty) : j = 1, \cdots, k\}$ $(l, k \geq 2)$ such that

$$\Omega = \overline{I_0} \bigcup (\bigcup_{i=1}^{l} I_{-i}) \bigcup (\bigcup_{j=1}^{k} I_j), \ I_{-i} \bigcap I_j = \emptyset, \text{ for } i \in \{1, \cdots, l\} \text{ and } j \in \{1, \cdots, k\},$$

$$\overline{I_0} \backslash \{I_0\} = \{-d_1, d_2\}, \ \bigcap_{i=1}^{l} \overline{I_{-i}} = \{-d_1\}, \ \bigcap_{j=1}^{k} \overline{I_j} = \{d_2\}.$$

See Figure 1 for an illustration of the domain Ω.

Here assuming $d_1, d_2 > 0$ and $l, k \geq 2$, we set the local coordinates in the domain Ω as

$$\begin{cases} I_0 = \{-d_1 < x_0 < d_2\}, \\ I_{-i} = \{-\infty < x_i < -d_1\}, & i \in \{1, \cdots, l\}, \\ I_j = \{d_2 < x_j < +\infty\}, & j \in \{1, \cdots, k\}. \end{cases} \tag{1.3}$$

Since the coordinate domain for I_0, I_{-i} and I_j do not overlap, we will also use $-\infty < x < \infty$ in the following as a global coordinate for the solution u which is the same for all the left half-lines and for all the right half-lines. By denoting $u_i(x_i) = u|_{\overline{I_{-i}}}(x_i)$, $i = 0, 1, \cdots, l$, and $v_j(x_j) = u|_{\overline{I_j}}(x_j)$, $j = 1, \cdots, k$, $u = (u_0, u_1, \cdots, u_l, v_1, \cdots, v_k)$ is a function defined in Ω. We investigate the existence of standing wave solutions to a nonlinear Schrödinger equation on the unbounded graph Ω with two vertices:

$$\begin{cases} -u_0'' = g(u_0), & -d_1 < x_0 < d_2, \\ -u_i'' = g(u_i), & -\infty < x_i < -d_1, \ i \in \{1, \cdots, l\}, \\ -v_j'' = g(v_j), & d_2 < x_j < +\infty, \ j \in \{1, \cdots, k\}, \end{cases} \tag{1.4}$$

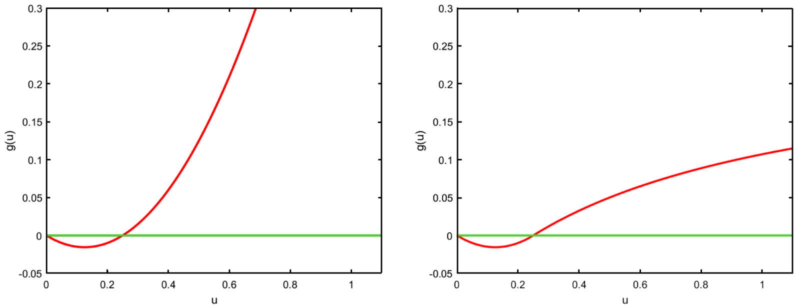

FIGURE 2. Graphs of functions in Example 1.1. Left: $g_1(u)$ with $p = 2$ and $a = 1/4$; Right: $g_2(u)$ with $a = 1/4$

with the compatibility conditions:

$$\begin{cases} u_0(-d_1) = u_i(-d_1), & i \in \{1, \cdots, l\}, \\ u_0(d_2) = v_j(d_2), & j \in \{1, \cdots, k\}, \\ u_0'(-d_1 + 0) = \sum_{i=1}^{l} u_i'(-d_1 - 0), \\ u_0'(d_2 - 0) = \sum_{j=1}^{k} v_j'(d_2 + 0). \end{cases} \tag{1.5}$$

Here $w'(d + 0)$ and $w'(d - 0)$ are defined as the left and right hand side limits of the function $w'(x)$ at $x = d$. The first two conditions in (1.5) imply the continuity of the function u at $x = -d_1$ and $x = d_2$, while the last two conditions indicate the conservation of the flux at $x = -d_1$ and $x = d_2$ (Kirchhoff condition).

In (1.4) $g : [0, \infty) \to \mathbb{R}$ is a continuous function satisfying the following condition $(\mathcal{G})$ or $(\mathcal{G}')$ listed as bellow.

$(\mathcal{G})$ For fixed $a > 0$,
- (g1) $g(0) = g(a) = 0$;
- (g2) $g(u) < 0$ in $(0, a)$ and $g(u) > 0$ in (a, ∞);
- (g3) there exists $\xi > a$ such that $G(\xi) = 0$, where $G(t) = \int_0^t g(s)ds$.

$(\mathcal{G}')$ For fix $0 < a < 1$,
- (g1') $g(0) = g(a) = g(1) = 0$;
- (g2') $g(u) < 0$ in $(0, a) \cup (1, \infty)$ and $g(u) > 0$ in $(a, 1)$;
- (g3') there exists $\xi \in (a, 1)$ such that $G(\xi) = 0$, where $G(t) = \int_0^t g(s)ds$;
- (g4) $G(1) + 3G(a) \geq 0$.

When $(\mathcal{G}')$ is satisfied, from (g4), we define

$$\mathcal{A} := \sqrt{\frac{G(1)}{-G(a)} + 1} \geq 2. \tag{1.6}$$

We note that (g3') implies that $G(1) > G(\xi) = 0$, geometrically it means that the area under the graph of $g(u)$ from a to 1 is larger than the area under the graph of $|g(u)|$ from 0 to a. That is $G(1) - G(a) > |G(a)| = -G(a)$. And (g4) implies that $G(1) - G(a) \geq 4|G(a)| = -4G(a)$, so the area under the graph of $g(u)$ from a to 1 is at least four times of the area under the graph of $|g(u)|$ from 0 to a. Condition (g4) guarantees the existence of a positive solution of (1.4)–(1.5) with $l, k \geq 2$.

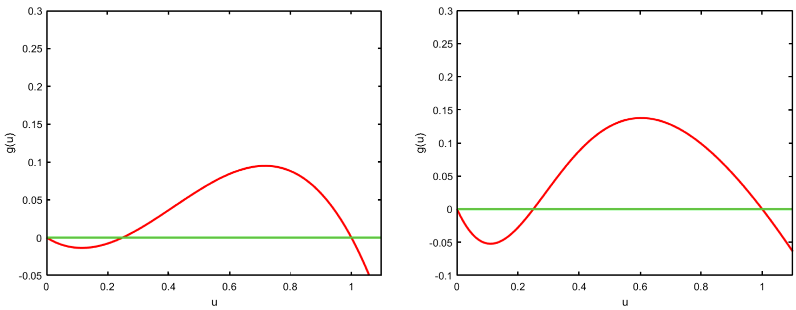

FIGURE 3. Graphs of functions in Example 1.2. Left: $g_3(u)$ with $a = 1/4$; Right: $g_4(u)$ with $b = 1/4$

Example 1.1. We list some examples satisfying $(\mathcal{G})$. Here $0 < a < 1$. (see Figure 2)

(1) $g_1(u) = -a^{p-1}u + u^p$, where $p > 1$;

(2) $g_2(u) = \begin{cases} -au + u^2 & 0 \le u \le a; \\ a(u-a)/(1+u-a) & u > a. \end{cases}$

Example 1.2. We list some examples satisfying $(\mathcal{G}')$. (see Figure 3)

(1) $g_3(u) = -u(u-a)(u-1)$ where $0 < a < a_0$ and $a_0 \approx 0.37318$ is the smallest positive root of $3a^4 - 6a^3 - 2a + 1 = 0$;

(2) $g_4(u) = -u + (1+b)u^2/(b+u^2)$ where $b > 0$ is small.

Note that a function satisfying $(\mathcal{G})$ could be Schrödinger type like $g_1(u)$ which is asymptotically superlinear, and it can be asymptotically sublinear like $g_2(u)$. On the other hand, a function satisfying $(\mathcal{G}')$ is a bistable nonlinearity with $u = 0$ and $u = 1$ both being stable for the corresponding ODE $u' = g(u)$. The bistable nonlinearity arises from the studies of neuron propagation (Fitz-Hugh Nagumo equation) or population ecology (strong Allee effect).

For some $l, k \ge 2$, we look for a symmetric positive solution $u(x)$ of (1.4) and (1.5) in a form of $(u_1, u_2, \cdots, u_l, u_0, v_1, v_2, \cdots, v_k)$ satisfying

$$u_1 = u_2 = \cdots = u_l \text{ and } v_1 = v_2 = \cdots = v_k.$$

Here a symmetric solution u does not mean $u(-x) = u(x)$ as in many other work, and it actually represents a solution that is the same on all the left half-lines and on all the right half-lines. When $l = k = 1$, the problem (1.4) and (1.5) is reduced to

$$-u'' = g(u),\ x \in \mathbb{R},\ u'(0) = 0,\ \lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} u'(x) = 0. \tag{1.7}$$

Since g satisfies the condition $(\mathcal{G})$ or $(\mathcal{G}')$, (1.7) has a unique positive solution $\omega(x)$ which is symmetric with respect to $x = 0$, positive, strictly increasing for $x < 0$, and decaying exponentially at the infinity [6, 8]. Moreover $\omega(0) = \xi$ where $\xi > a$ satisfies $G(\xi) = \int_0^\xi g(s)ds = 0$ as in (g3) or (g3').

In the following we consider (1.4) and (1.5) with $l \geq 2$ and $k \geq 2$. For some $\tau > 0$, we convert the problems (1.4) and (1.5) with a symmetric solution to

$$\begin{cases} -u_0'' = g(u_0), -d_1 < x_0 < 0, \\ u_0'(0) = 0, u_0(0) = \tau, \end{cases} \tag{1.8}$$

$$\begin{cases} -u_1'' = g(u_1), -\infty < x_1 < -d_1, \\ u_1(-d_1) = u_0(-d_1), u_1(-\infty) = 0, \end{cases} \tag{1.9}$$

with

$$u_0'(-d_1 + 0) = l u_1'(-d_1 - 0); \tag{1.10}$$

and

$$\begin{cases} -u_0'' = g(u_0), 0 < x_0 < d_2, \\ u_0'(0) = 0, u_0(0) = \tau, \end{cases} \tag{1.11}$$

$$\begin{cases} -v_1'' = g(v_1), d_2 < x_2 < \infty, \\ v_1(d_2) = u_0(d_2), v_1(\infty) = 0, \end{cases} \tag{1.12}$$

with

$$u_0'(d_2 - 0) = k v_1'(d_2 + 0). \tag{1.13}$$

Namely, we look for an appropriate $\tau \in (\xi, \infty)$ which allows a solution (u_0, u_1) to (1.8) and (1.9) satisfying (1.10) which is increasing in $(-\infty, 0)$, and a solution (u_0, v_1) to (1.11) and (1.12) satisfying (1.13) which is decreasing in $(0, \infty)$. Then a desired solution $u^{l,k}(x)$ stated in the following theorems is given by

$$u^{l,k}(x) = \begin{cases} u_0(x_0), & -d_1 \leq x_0 \leq d_2, \\ u_1(x_i), & -\infty < x_i < -d_1, \ i = 1, \cdots, l \\ v_1(x_j), & d_2 < x_j < \infty, \ j = 1, \cdots, k. \end{cases} \tag{1.14}$$

First we have the following results regarding the existence of a positive solution of (1.4) and (1.5) on the bridge type graph Ω for Schrödinger type $g(u)$.

Theorem 1.3. *Consider the stationary problem* (1.4) *and* (1.5) *in* Ω *with the coordinates given by* (1.3)*, and* $d_1, d_2 > 0$ *and* $l, k \geq 2$*. Assume the hypothesis* $(\mathcal{G})$ *is satisfied and impose the conditions at infinity of the domain as*

$$\lim_{x_i \to -\infty} u_i(x_i) = \lim_{x_j \to \infty} v_j(x_j) = 0, \ i \in \{1, \cdots, l\}, \ j \in \{1, \cdots, k\}. \tag{1.15}$$

(1) *For* $l = k \geq 2$ *and* $d_1 = d_2 = d > 0$ *given,* (1.4) *and* (1.5) *admit at least one positive solution satisfying* $u_0(-x) = u_0(x)$ *and* $u_i(-x) = v_j(x)$*;*
(2) *For* $l \geq 2$ *and* $d_1 > 0$ *given,* (1.8) *and* (1.9) *admit at least one positive solution* $u_L^l(x)$ *in* $(-d_1, 0)$ *and* $(-\infty, -d_1)$ *respectively satisfying* (1.10)*, and* $u_L^l(0) = \tau > \xi$ *where* ξ *is defined in (g3); there exists* $2 \leq k_l \leq l$ *such that for each* $k \geq k_l$*, there exists* $d_2^k > 0$ *so that* (1.11) *and* (1.12) *admit at least one positive solution* $u_R^k(x)$ *in* $(0, d_2^k)$ *and* (d_2^k, ∞) *respectively satisfying* (1.13) *and* $u_R^k(0) = \tau = u_L^l(0)$*;* $u^{l,k}(x) = (u_L^l(x), u_R^k(x))$ *is a positive solution of* (1.4) *and* (1.5) *with* $d_1 > 0$ *and* $d_2 = d_2^k > 0$*.*

Note that for the solution $u^{l,k}(x) = (u^l_L(x), u^k_R(x))$ in Theorem 1.3, $u^l_L(x)$ is defined for $x \le 0$ and $u^k_R(x)$ is defined for $x \ge 0$, while the solution $u^{l,k}(x)$ in (1.14) is defined in three distinct sub-intervals $(-\infty, -d_1)$, $[-d_1, d_2]$ and (d_2, ∞). Each of the two definitions can be easily converted to the other one. Similarly for the bistable nonlinearity, we have

Theorem 1.4. *Consider the stationary problem* (1.4) *and* (1.5) *in* Ω *with the coordinates given by* (1.3)*, and* $d_1, d_2 > 0$ *and* $l, k \in \mathbb{N}$ *satisfying* $2 \le l, k \le [\mathcal{A}]$ *where* $\mathcal{A}$ *is defined in* (1.6)*, and* $[\mathcal{A}]$ *is the greatest integer less than or equal to* $\mathcal{A}$*. Assume the hypothesis* $(\mathcal{G}')$ *is satisfied and impose the conditions at infinity of the domain as* (1.15)*.*

(1) *For* $l = k$ *and* $d_1 = d_2 = d > 0$ *given,* (1.4) *and* (1.5) *admit at least one positive solution satisfying* $u_0(-x) = u_0(x)$ *and* $u_i(-x) = v_j(x)$*;*
(2) *For* $l \ge 2$ *and* $d_1 > 0$ *given,* (1.8) *and* (1.9) *admit at least one positive solution* $u^l_L(x)$ *in* $(-d_1, 0)$ *and* $(-\infty, -d_1)$ *respectively satisfying* (1.10)*, and* $u^l_L(0) = \tau > \xi$ *where* ξ *is defined in (g3'); there exists* $2 \le k_l \le l$ *such that for each* k *satisfying* $k_l \le k \le [\mathcal{A}]$*, there exists* $d^k_2 > 0$ *so that* (1.11) *and* (1.12) *admit at least one positive solution* $u^k_R(x)$ *in* $(0, d^k_2)$ *and* (d^k_2, ∞) *respectively satisfying* (1.13) *and* $u^k_R(0) = \tau = u^l_L(0)$*;* $u^{l,k}(x) = (u^l_L(x), u^k_R(x))$ *is a positive solution of* (1.4) *and* (1.5) *with* $d_1 > 0$ *and* $d_2 = d^k_2 > 0$*.*

Indeed a more careful analysis of the length d_1 and d_2 for a solution u in the above results implies the following existence of a positive solution of (1.4) and (1.5) for Schrödinger type g on the bridge type graph Ω with any number of "legs" on the left and right, and any length $d = d_1 + d_2$ of the bridge.

Theorem 1.5. *For* (1.4) *and* (1.5) *in* Ω *with the coordinates given by* (1.3)*, assume the hypothesis* $(\mathcal{G})$ *is satisfied and the conditions at infinity is imposed as* (1.15)*. Then for any* $l \ge 2$*,* $k \ge 2$*, and* $d > 0$*, there exists at least one positive solution* $u^{l,k}_1(x)$ *of* (1.4) *and* (1.5) *with* $|I_0| = d_1 + d_2 = d$*; and there exists* $d_* > 0$ *such that when* $d > d_*$ *there exists at least two positive solutions* $u^{l,k}_j(x)$ $(j = 2, 3)$ *of* (1.4) *and* (1.5) *with* $|I_0| = d_1 + d_2 = d$ *and these two solutions are different from* $u^{l,k}_1(x)$*; and when* $d = d_*$ *there exists at least one positive solution* $u^{l,k}_2(x)$ *of* (1.4) *and* (1.5) *with* $|I_0| = d_1 + d_2 = d$ *that is different from* $u^{l,k}_1(x)$*. Moreover for* $j = 1, 2, 3$*,* $u^{l,k}_j(x)$ *is strictly increasing for* $x \in (-\infty, 0)$ *and* $u^{l,k}_j(x)$ *is strictly decreasing for* $x \in (0, \infty)$*.*

Similar results also hold for bistable type g but the number of "legs" on the left and right is restricted by the quantity $\mathcal{A}$ defined in (1.6).

Theorem 1.6. *For* (1.4) *and* (1.5) *in* Ω *with the coordinates given by* (1.3)*, assume the hypothesis* $(\mathcal{G}')$ *is satisfied and the conditions at infinity is imposed as* (1.15)*. Then for any* $d > 0$ *and* $2 \le l, k \le [\mathcal{A}]$*, where* $\mathcal{A}$ *is defined in* (1.6)*, and* $[\mathcal{A}]$ *is the greatest integer less than or equal to* $\mathcal{A}$*, there exists at least one positive solution* $u^{l,k}_1(x)$ *of* (1.4) *and* (1.5) *with* $|I_0| = d_1 + d_2 = d$*; and there exists* $d_* > 0$ *such that when* $d > d_*$ *there exist at least two positive solutions* $u^{l,k}_j(x)$ $(j = 2, 3)$ *of* (1.4) *and* (1.5) *with* $|I_0| = d_1 + d_2 = d$ *and these two solutions are different from* $u^{l,k}_1(x)$*; and when* $d = d_*$ *there exists at least one positive solution* $u^{l,k}_2(x)$ *of* (1.4) *and*

(1.5) *with* $|I_0| = d_1 + d_2 = d$ *that is different from* $u_1^{l,k}(x)$. *Moreover for* $j = 1, 2, 3$, $u_j^{l,k}(x)$ *is strictly increasing for* $x \in (-\infty, 0)$ *and* $u_j^{l,k}(x)$ *is strictly decreasing for* $x \in (0, \infty)$.

We remark that the results in Theorem 1.6 when $l = k$ have been proved in [19, Theorem 1.5]. Theorems 1.5 and 1.6 show the effect of different nonlinearities on the existence of positive solutions on the bridge type metric graphs, which is different from the case of $l = k = 1$ (equivalent to $\Omega = \mathbb{R}$ and (1.7)). It also shows that (1.4) and (1.5) have at least three distinctive positive solutions when the length d of the bridge is sufficiently long.

We also have the following results regarding the multiplicity of positive solutions of (1.4) and (1.5) for a fixed height (maximum value of the solution) $\tau = u(0)$. For Schrödinger type $g(u)$ we have

Theorem 1.7. *Consider the stationary problem* (1.4) *and* (1.5) *in* Ω *with the coordinates given by* (1.3). *Assume the hypothesis* $(\mathcal{G})$ *is satisfied and the conditions at infinity is imposed as* (1.15). *Then there exists a sequence* $\{w_m : m = 1, 2, \cdots\}$ *such that*

$$\xi = w_1 < w_2 < \cdots < w_m < w_{m+1} < \cdots,$$

and for any positive integer $m \geq 2$,

(1) *when* $\tau \in (w_1, w_m)$ *and any integers* $l, k \geq m$, (1.4) *and* (1.5) *admit at least four positive solutions* $u^{l,k}(x)$ *with* $u^{l,k}(0) = \tau$;

(2) *when* $\tau = w_m$, *and either* (i) $l \geq m + 1$ *and* $k = m$, *or* (ii) $l = m$ *and* $k \geq m + 1$, (1.4) *and* (1.5) *admit at least two positive solutions* $u^{l,k}(x)$ *with* $u^{l,k}(0) = \tau$;

(3) *when* $\tau = w_m$ *and* $l = k = m$, (1.4) *and* (1.5) *admit at least one positive solution with* $u^{l,k}(x)$ *with* $u^{l,k}(0) = \tau$.

Similarly for bistable type $g(u)$ we have

Theorem 1.8. *Consider the stationary problem* (1.4) *and* (1.5) *in* Ω *with the coordinates given by* (1.3). *Assume the hypothesis* $(\mathcal{G}')$ *is satisfied and the conditions at infinity is imposed as* (1.15). *Then there exists a finite sequence* $\{w_m : m = 1, 2, \cdots, [\mathcal{A}]\}$ *such that*

$$\xi = w_1 < w_2 < \cdots < w_m < w_{m+1} < \cdots < w_{[\mathcal{A}-1]} < w_{[\mathcal{A}]} = 1,$$

and for any positive integer $m \geq 2$,

(1) *when* $\tau \in (w_1, w_m)$ *and any integers* $m \leq l, k \leq [\mathcal{A}]$, (1.4) *and* (1.5) *admit at least four positive solutions* $u^{l,k}(x)$ *with* $u^{l,k}(0) = \tau$;

(2) *when* $\tau = w_m$, *and either* (i) $m + 1 \leq l \leq [\mathcal{A}]$ *and* $k = m$, *or* (ii) $l = m$ *and* $m + 1 \leq k \leq [\mathcal{A}]$, (1.4) *and* (1.5) *admit at least two positive solutions* $u^{l,k}(x)$ *with* $u^{l,k}(0) = \tau$;

(3) *when* $\tau = w_m$ *and* $l = k = m$, (1.4) *and* (1.5) *admit at least one positive solution with* $u^{l,k}(x)$ *with* $u^{l,k}(0) = \tau$.

In Theorems 1.7 and 1.8 part (1), if $l = k \geq m$ (or $m \leq l = k \leq [\mathcal{A}]$), then two of four positive solutions are symmetric in the sense that $d_1 = d_2$, $u_0(-x) = u_0(x)$ and $u_i(-x) = v_j(x)$, and the other two are not symmetric with $d_1 \neq d_2$. The one positive solution in Theorems 1.7 and 1.8 part (3) is also a symmetric one. For the Schrödinger type g, the "height" $u(0) = \tau$ for a positive solution is unbounded, while the "height" of a positive for bistable type g is bounded by $u = 1$ from the Maximum Principle. Theorems 1.7 and 1.8 also show an relation between the "height" $\tau = u(0)$ and the number l, k of "legs" on the two ends of the bridge.

The solutions defined in the theorems above are related to the unique positive solution ω of (1.7). The existence and uniqueness of the solution ω of (1.7) with g satisfying $(\mathcal{G})$ or $(\mathcal{G}')$ follows from phase portrait analysis in [8] or variational approach in [6]. Here we show that for a bridge type graph, such a positive solution still exists but it may not be unique. The exact multiplicity of positive solutions of a semilinear elliptic equation with bistable nonlinearity on a ball domain was studied in [29, 30].

We prove the main results stated in the Introduction in Section 2.

2. Ground states for a bridge graph

In this section, we first prove the existence of a solution to (1.8)–(1.10). Suppose that u is the solution of initial value problem

$$\begin{cases} -u'' = g(u), \\ u(0) = \tilde{u}_0, u'(0) = \tilde{\omega}_0, \end{cases} \tag{2.1}$$

where $\tilde{u}_0 > 0$ and $\tilde{\omega}_0 \in \mathbb{R}$. Let $\theta(x) = u'(x)$. Then $(u(x), \theta(x))$ is the solution of the initial value problem

$$\begin{cases} u' = \theta, \\ \theta' = -g(u), \\ u(0) = \tilde{u}_0, \theta(0) = \tilde{\omega}_0. \end{cases} \tag{2.2}$$

The solution (u, θ) can be extended to $x \in (-T, T)$ which is the maximum interval of existence of the solution and $T \in (0, \infty]$. Note that (2.2) is a first order Hamiltonian ODE system with the Hamiltonian

$$H(u, \theta) = G(u) + \frac{1}{2}\theta^2. \tag{2.3}$$

Hence for a solution (u, θ) of (2.2),

$$\frac{d}{dx}H(u(x), \theta(x)) = \frac{\partial H}{\partial u}u' + \frac{\partial H}{\partial \theta}\theta' = 0.$$

In particular, $H(u(x), \theta(x)) = H(u(0), \theta(0))$ for all $x \in (-T, T)$.

Now, we consider a solution of (2.1) with $\tilde{\omega}_0 = 0$. Multiplying (2.1) by u' and integrating on $[x, 0]$, we obtain that

$$0 = \int_x^0 [u''u' + g(u)u']dy = -\frac{1}{2}[u'(x)]^2 + G(u(0)) - G(u(x)). \tag{2.4}$$

We consider a solution u of (2.1) satisfying $u'(0) = 0$, $u'(x) > 0$ for $x < 0$. Then (2.4) implies that

$$u'(x) = \sqrt{2}\sqrt{G(u(0)) - G(u(x))},\ x < 0$$

or

$$dx = \frac{du}{\sqrt{2}\sqrt{G(\tilde{u}_0) - G(u(x))}},\ x < 0. \tag{2.5}$$

Suppose that $\alpha < 0$, integrating (2.5) for $x \in [\alpha, 0]$, we have

$$\alpha = -\frac{1}{\sqrt{2}} \int_{u(\alpha)}^{\tilde{u}_0} \frac{du}{\sqrt{G(\tilde{u}_0) - G(u)}}. \tag{2.6}$$

Let $\alpha < 0$ and recall that ω is the unique positive solution of (1.7). Let $(P, Q) = (\omega(\alpha), \omega'(\alpha))$. Consider the following system which is equivalent to (1.8) and (1.10):

$$\begin{cases} u' = \theta, \ \rho < x < 0, \\ \theta' = -g(u), \ \rho < x < 0, \\ u(0) = \tilde{u}, \ \theta(0) = 0, \\ u(\rho) = P, \ \theta(\rho) = lQ \end{cases} \tag{2.7}$$

for some $\tilde{u} > 0$, $l \geq 2$ and $\rho < 0$. The following lemmas are important for obtaining a solution to (1.8)–(1.10).

Lemma 2.1.

(i) *If $(\mathcal{G})$ holds, for any $\alpha < 0$ and integer $l \geq 2$, there exists a unique $\tilde{u} = \tilde{u}_l(\alpha) > \xi$ and $\rho = \rho_L(\alpha) < 0$ such that (2.7) has a solution (u, θ) with $u(x) > 0$ and $\theta(x) > 0$ for $x \in (\rho_L(\alpha), 0)$.*

(ii) *If $(\mathcal{G}')$ holds, for any $\alpha < 0$ and integer l satisfying $2 \leq l \leq [\mathcal{A}]$, there exists a unique $\tilde{u} = \tilde{u}_l(\alpha) \in (\xi, 1)$ and $\rho = \rho_L(\alpha) < 0$ such that (2.7) has a solution (u, θ) with $u(x) > 0$ and $\theta(x) > 0$ for $x \in (\rho_L(\alpha), 0)$.*

Proof. Fix $\alpha < 0$ and let $(P, Q) = (\omega(\alpha), \omega'(\alpha))$. Let (u, θ) be the solution of $u' = \theta$ and $\theta' = -g(u)$ with $(u(\rho_L), \theta(\rho_L)) = (P, lQ)$ where $\rho_L < 0$ is to be determined. Then the solution orbit of the solution satisfying $(u(\rho_L), \theta(\rho_L)) = (P, lQ)$ is on the curve

$$H(u, \theta) = H(P, lQ) = G(P) + \frac{l^2}{2} Q^2. \tag{2.8}$$

We claim that the curve $H(u, \theta) = H(P, lQ)$ intersects with $\theta = 0$.

(i) If $(\mathcal{G})$ holds, since $G(P) < H(P, lQ)$ and $\lim\limits_{u \to \infty} G(u) = +\infty > H(P, lQ)$, then there exists $\tilde{u} \in (P, +\infty)$ such that $G(\tilde{u}) = H(P, lQ)$, which implies that $H(\tilde{u}, 0) = H(P, lQ)$. We claim that $\tilde{u} > \xi$. Indeed

$$G(\tilde{u}) = H(P, lQ) > H(P, Q) = 0$$

since $l \geq 2$ and $H(P, Q) = H(0, 0)$ as $\lim\limits_{\alpha \to \infty} \omega(\alpha) = \lim_{\alpha \to \infty} \omega'(\alpha) = 0$. Hence $\tilde{u} > \xi$ as $G(u) < 0$ for $0 < u < \xi$ and $G(u) > 0$ for $u > \xi$. The monotonicity of G implies that such $\tilde{u} = \tilde{u}_l \in (\xi, \infty)$ is unique. We may assume $u(0) = \tilde{u}$ and $\theta(0) = 0$ then $\rho_L < 0$ is uniquely determined by $\alpha < 0$. The solution satisfies $u' = \theta > 0$.

(ii) If $(\mathcal{G}')$ holds, then by using $G(P) + Q^2/2 = 0$ and (g4), for integer l satisfying $2 \leq l \leq [\mathcal{A}]$, we have

$$H(P, lQ) = G(P) + \frac{l^2}{2} Q^2 = G(P) - l^2 G(P) \leq (l^2 - 1)(-G(a)) \leq G(1), \tag{2.9}$$

as $G(a) = \min_{u \in [0,1]} G(u)$. Then again from $G(P) < H(P, lQ)$, there exists a unique $\tilde{u} \in (P, 1)$ such that $G(\tilde{u}) = H(P, lQ)$, which implies that $H(\tilde{u}, 0) = H(P, lQ)$. The proof of that $\tilde{u} > \xi$ is as before and the uniqueness follows with the monotonicity of G in $(\xi, 1)$. ∎

Lemma 2.2. *For any $\alpha < 0$ and integer $l \geq 2$ if ($\mathcal{G}$) holds (or $2 \leq l \leq [\mathcal{A}]$ if ($\mathcal{G}'$) holds), let $\rho_L = \rho_L(\alpha)$ be defined as in Lemma 2.1, we have*

$$\lim_{\alpha\to 0^-} \rho_L(\alpha) = 0 \text{ and } \lim_{\alpha\to -\infty} \rho_L(\alpha) = -\infty.$$

Proof. From (2.6), we have

$$\rho_L(\alpha) = -\frac{1}{\sqrt{2}}\int_{\omega(\alpha)}^{\tilde{u}(\alpha)} \frac{du}{\sqrt{G(\tilde{u}(\alpha)) - G(u)}}. \tag{2.10}$$

For $u \in [\omega(\alpha), \tilde{u}(\alpha))$, from the mean-value theorem, there exists $\eta \in (u, \tilde{u}(\alpha))$ such that

$$G(\tilde{u}(\alpha)) - G(u) = g(\eta)(\tilde{u}(\alpha) - u).$$

Note that $\omega(0) = \xi > a$. Since $P = \omega(\alpha) \to \xi$ and $Q = \omega'(\alpha) \to 0$ as $\alpha \to 0^-$, then $\tilde{u}(\alpha) \to \xi$ and $\eta \to \xi$ as $\alpha \to 0^-$. Hence $g(\eta) > 0$ is bounded as $\alpha \to 0^-$ and

$$\begin{aligned} 0 \geq \rho_L(\alpha) &= -\frac{1}{\sqrt{2}}\int_{\omega(\alpha)}^{\tilde{u}(\alpha)} \frac{du}{\sqrt{g(\eta)}\sqrt{\tilde{u}(\alpha) - u}} \\ &\geq -C\int_{\omega(\alpha)}^{\tilde{u}(\alpha)} \frac{du}{\sqrt{\tilde{u}(\alpha) - u}} \\ &= -2C\sqrt{\tilde{u}(\alpha) - \omega(\alpha)} \to 0, \text{ as } \alpha \to 0^-. \end{aligned}$$

On the other hand, $P = \omega(\alpha) \to 0$ and $Q = \omega'(\alpha) \to 0$ as $\alpha \to -\infty$. then $\tilde{u}(\alpha) \to \xi$ as $\alpha \to -\infty$. Thus (2.10) implies that

$$\lim_{\alpha\to -\infty} \rho_L(\alpha) = -\frac{1}{\sqrt{2}}\int_0^{\xi} \frac{du}{\sqrt{G(\xi) - G(u)}} = -\infty,$$

since the solution of (2.2) with $u(0) = \omega(0)$ and $\theta(0) = 0$ is a homoclinic orbit. □

Now we can prove the existence of a positive solution to (1.8)–(1.10).

Proposition 2.3. *For any $d > 0$ and integer $l \geq 2$ if ($\mathcal{G}$) holds (or $2 \leq l \leq [\mathcal{A}]$ if ($\mathcal{G}'$) holds), the equations* (1.8)–(1.10) *admit a positive solution* (u_0, u_1) *and $u_0'(x) > 0$ in $(-d, 0)$, $u_1'(x) > 0$ in $(-\infty, -d)$.*

Proof. For any $d > 0$, by Lemma 2.2 and the continuity of $\rho_L(\alpha)$, there exists $\alpha \in (-\infty, 0)$ such that $d = -\rho_L(\alpha)$. By Lemma 2.1, there exists a unique $\tilde{u} = \tilde{u}_l(\alpha) > \xi$ such that (2.7) has a solution (u, θ) with $u(0) = \tilde{u}_l(\alpha)$, $u(d) = \omega(\alpha)$ and $\theta(-d) = l\omega'(\alpha)$ where $\omega(x)$ is the unique positive solution of (1.7). Let $\tau = \tilde{u}_l(\alpha)$, then (1.8) admits a solution with $u_0(0) = \tau$, $u_0(-d) = \omega(\alpha)$ and $u_0'(-d) = l\omega'(\alpha)$. Finally we take $u_1(x) = \omega(x + \alpha + d)$ for $x \in (-\infty, -d)$. Hence (u_0, u_1) is a solution to (1.8) and (1.9) satisfying (1.10) and $u_0'(x) > 0$ in $(-d, 0)$, $u_1'(x) > 0$ in $(-\infty, -d)$. □

We remark that when ($\mathcal{G}'$) holds, the equations (1.8)–(1.10) have no positive solution (u_0, u_1) satisfying $u_0'(x) > 0$ in $(-d, 0)$, $u_1'(x) > 0$ in $(-\infty, -d)$ for integer $l > [\mathcal{A}]$. Indeed (2.9) would become

$$G(P) + \frac{l^2}{2}Q^2 = (l^2 - 1)(-G(P)) > G(P'),$$

for any $P' \in (P, 1]$ and l large enough as $G(P) < 0$. Thus, there is not $\tilde{u}$ such that $G(\tilde{u}) = H(P, lQ)$ for any P, Q and $l > [\mathcal{A}]$ and (2.7) has no solution for such l.

For the function $\tilde{u}_l(\alpha)$ defined in Lemma 2.1, we have the following properties.

Proposition 2.4. *Suppose that $l \geq 2$ if $(\mathcal{G})$ holds (or $2 \leq l \leq [\mathcal{A}]$ if $(\mathcal{G}')$ holds), the function $\tilde{u}_l(\alpha)$ defined in Lemma 2.1 satisfies $\tilde{u}_l(\alpha) > \xi$ for $\alpha \in (-\infty, 0)$, admits the maximum value at $\alpha_* \in (-\infty, 0)$ which satisfies $\omega(\alpha_*) = a$, $\tilde{u}_l'(\alpha)(\alpha - \alpha_*) < 0$ for $\alpha \neq \alpha_*$, and $\tilde{u}_l(\alpha_*)$ satisfies*

$$G(\tilde{u}_l(\alpha_*)) = (1 - l^2)G(a). \tag{2.11}$$

Moreover for $l > j \geq 2$, we have $\tilde{u}_l(\alpha) > \tilde{u}_j(\alpha)$ for $\alpha \in (-\infty, 0)$.

Proof. Indeed, from the fact

$$G(\tilde{u}_l(\alpha)) = H(P, lQ),$$

we have

$$G(\tilde{u}_l(\alpha)) = G(\omega(\alpha)) + \frac{l^2}{2}[\omega'(\alpha)]^2. \tag{2.12}$$

Differentiating (2.12) with respect to α, and by using the equation in (1.7), we have

$$g(\tilde{u}_l(\alpha))\tilde{u}_l'(\alpha) = (1 - l^2)\omega'(\alpha)g(\omega(\alpha)).$$

Since $g(\tilde{u}_l(\alpha)) > 0$, $\omega'(\alpha) > 0$ and $1 - l^2 < 0$ for $l \geq 2$, hence

$$\tilde{u}_l'(\alpha) > 0, \quad \text{if } g(\omega(\alpha)) < 0$$

and

$$\tilde{u}_l'(\alpha) < 0, \quad \text{if } g(\omega(\alpha)) > 0.$$

Moreover, $\tilde{u}_l'(\alpha) = 0$ if and only if $g(\omega(\alpha)) = 0$. That is only satisfied when $\alpha = \alpha_*$ such that $\omega(\alpha_*) = a$ for some $\alpha_* \in (-\infty, 0)$. Hence, for $\alpha < \alpha_*$, $\tilde{u}_l$ is increasing and for $\alpha > \alpha_*$, $\tilde{u}_l$ is decreasing. By the fact $\tilde{u}_l(\alpha) \to \xi$ as $\alpha \to 0^-$ and $\alpha \to -\infty$, have $\tilde{u}_l(\alpha) > \xi$ for $\alpha \in (-\infty, 0)$. By (2.12),

$$G(\tilde{u}_l(\alpha_*)) = G(a) + \frac{l^2}{2}[\omega'(\alpha_*)]^2. \tag{2.13}$$

Multiplying (1.7) by u' and integrating from α_* to 0, we have

$$\frac{1}{2}[\omega'(\alpha_*)]^2 = -G(a). \tag{2.14}$$

Combining (2.13) and (2.14), we obtain (2.11). Finally from (2.12), we have

$$G(\tilde{u}_l(\alpha)) - G(\tilde{u}_j(\alpha)) = \frac{l^2 - j^2}{2}[\omega'(\alpha)]^2 > 0.$$

This implies that $\tilde{u}_l(\alpha) > \tilde{u}_j(\alpha)$ if $l > j \geq 2$ for $\alpha \in (-\infty, 0)$, as $G(u)$ is strictly increasing when $u > \xi$. □

Now we are ready to prove Theorem 1.3, and the proof of Theorem 1.4 is similar so it is omitted.

Proof of Theorem 1.3. First we assume that $l = k \geq 2$. From Propositions 2.3 and 2.4, for any $d > 0$ given, (1.8) admits at least one solution $u_0^1(x_0)$ defined in $[-d, 0]$ with $u_0^1(0) > \xi$, (1.9) admits a solution $u_1(x_1)$ defined in $(-\infty, -d)$, and the junction condition (1.10) holds. Since $l = k$, by the same way, we can obtain a solution $u_0^2(x_0) = u_0^1(-x_0)$ defined in $(0, d)$ to (1.11) with $u_0^2(0) = u_0^1(0) > \xi$, a solution $v_1(x_1) = u_1(-x_1)$ defined in (d, ∞) to (1.12) and (1.13) holds. Then $u^* = (u_0, u_1, v_1)$ is a positive solution of (1.8)–(1.13) with $d_1 = d_2 = d$ where $u_0(x_0) = u_0^1(x_0)$ when $x_0 \in [-d, 0]$ and $u_0(x_0) = u_0^2(x_0)$ when $x_0 \in (0, d]$. This proves the existence of a positive solution to (1.4)–(1.5) when $l = k$.

Next we assume that $l \geq 2$ with k to be specified. From Propositions 2.3 and 2.4 again, for any $d_1 > 0$ given, (1.8) admits at least one solution $u_0^1(x_0)$ defined in $[-d_1, 0]$ with $u_0^1(0) > \xi$, (1.9) admits a solution $u_1(x_1)$ defined in $(-\infty, -d_1)$, and the junction condition (1.10) holds. Taking $\tau = u_0^1(0) > \xi$, by Proposition 2.4, there exists $2 \leq k_l \leq l$ such that for each $k \geq k_l$, there exists $\beta_k \in (-\infty, 0)$ satisfying $\tilde{u}_k(\beta_k) = \tau$. Let $d_2^k = -\rho_L(\beta_k) > 0$, then applying the arguments in Proposition 2.3 for $x > 0$, (1.11) admits a solution $u_0^2(x_0)$ with $u_0^2(0) = \tau$ in $(0, d_2^k)$, (1.12) admits a solution $v_1(x_1)$ in (d_2^k, ∞) and (1.13) holds. Then $u^* = (u_0, u_1, v_1)$ is a positive solution of (1.8)–(1.13) with $d_1 > 0$ arbitrary and d_2 determined by d_1, l and k, where $u_0(x_0) = u_0^1(x_0)$ when $x_0 \in [-d_1, 0]$ and $u_0(x_0) = u_0^2(x_0)$ when $x_0 \in (0, d_2]$. This proves the existence of a positive solution to (1.4)–(1.5) when l and k are not necessarily the same. □

If $l = 1$, then, $H(P, Q) = 0$ where $P = \omega(\alpha)$ and $Q = \omega'(\alpha)$ as defined in Lemma 2.1 and $G(\tilde{u}) = 0$ implies that $\tilde{u} = \xi$. On the other hand,

$$0 = G(\xi) = G(\omega(\alpha)) + \frac{k^2}{2}[\omega'(\alpha)]^2 = (1 - k^2)G(\omega(\alpha)). \tag{2.15}$$

If $k = 1$ also holds, then the graph Ω is a line and (1.4)–(1.5) is reduced to (1.7) which has a unique positive solution. If $k \neq 1$, we have that $G(\omega(\alpha)) = 0$ in (2.15). Thus $\alpha = 0$ and the graph Ω is a star which has a unique vertex and $k+1$ half-lines connected to the vertex. The positive solutions of (1.4)–(1.5) on a star graph have been studied in for example [1–4, 26].

We can analyze β_k and d_2^k in the above proof more carefully to prove Theorem 1.5, and again the proof of Theorem 1.6 is similar thus omitted.

Proof of Theorem 1.5. Let $l \geq 2$ and $k \geq 2$. Without loss of generality, we assume that $2 \leq l \leq k$. As in the proof of Theorem 1.3, from Propositions 2.3 and 2.4, for any $d_1 > 0$ given, (1.8) admits at least one solution $u_0^1(x_0)$ defined in $[-d_1, 0]$ with $u_0^1(0) > \xi$, (1.9) admits a solution $u_1(x_1)$ defined in $(-\infty, -d_1)$, and the junction condition (1.10) holds. Let $\tau = u_0^1(0) > \xi$ then $\tau = \tilde{u}_l(\alpha)$ for some $\alpha \in (-\infty, 0)$. We know that when $\alpha = \alpha_*$, $\tilde{u}_l(\alpha)$ achieves the maximum value $\tilde{u}_l(\alpha_*)$ for $\alpha \in (-\infty, 0)$. Then from $l \leq k$, we have $\tilde{u}_k(\alpha) \geq \tilde{u}_l(\alpha)$, and $\tilde{u}_k'(\alpha)(\alpha - \alpha_*) < 0$ and $\tilde{u}_k(\alpha) \to \xi$ as $\alpha \to -\infty$ or $\alpha \to 0^+$ from Proposition 2.4. This implies that for $\tau \in (\xi, \tilde{u}_l(\alpha_*))$, from Proposition 2.4, there exists two values β_k^1 and β_k^2 satisfying

$$-\infty < \beta_k^1 < \alpha_* < \beta_k^2 < 0$$

such that $\tilde{u}_k(\beta_k^i) = \tau$ for $i = 1, 2$. Let $d_{2,1}^k = -\rho_L(\beta_k^1)$ and $d_{2,2}^k = -\rho_L(\beta_k^2)$, then

$$\begin{aligned} d_{2,1}^k = -\rho_L(\beta_k^1) =& \frac{1}{\sqrt{2}} \int_{\omega(\beta_k^1)}^{\tau} \frac{du}{\sqrt{G(\tau) - G(u)}} \\ >& \frac{1}{\sqrt{2}} \int_{\omega(\beta_k^2)}^{\tau} \frac{du}{\sqrt{G(\tau) - G(u)}} = -\rho_L(\beta_k^2) = d_{2,2}^k, \end{aligned} \tag{2.16}$$

as $\tau > \omega(\beta_k^2) > \omega(\beta_k^1)$. Thus (1.11) admits a solution $u_{0,i}^2$ with $u_{0,i}^2(0) = \tau$ in $(0, d_{2,i}^k)$, (1.12) admits a solution $v_{1,i}$ in $(d_{2,i}^k, \infty)$ $i = 1, 2$ and (1.13) holds. Hence for a fixed $d_1 > 0$, there are two distinct $d_2 = d_{2,i}^k$ such that (1.8)–(1.13) has a positive solution with $d_1 > 0$ arbitrary and $d_2 = d_{2,1}^k > d_{2,2}^k > 0$. Note that $d_2 = d_{2,i}^k$ depends on d_1 continuously when l, k are fixed.

Taking $d_1 > 0$ as a varying variable, we have $\tau \to \xi$ when $d_1 \to 0$ or $d_1 \to \infty$. Consequently we have $\beta_k^1 \to 0$ and $\beta_k^2 \to \infty$, which imply that $d_{2,1}^k \to \infty$ and $d_{2,2}^k \to 0$ hold when either $d_1 \to 0$ or $d_1 \to \infty$. Now we define

$$d_L = d_1 + d_{2,2}^k. \tag{2.17}$$

Then (1.8)–(1.13) has a positive solution with $d_1 > 0$ arbitrary and $d_2 = d_{2,2}^k > 0$. When d_1 varies from 0 to ∞, $d_{2,2}^k > 0$ and $d_{2,2}^k \to 0$ as $d_1 \to 0$ or $d_1 \to \infty$, which implies that d_L varies from 0 to ∞ as well. This proves that for any $d = d_L > 0$, there exists at least one positive solution $u_1^{l,k}(x)$ of (1.4) and (1.5) with $|I_0| = d_1 + d_{2,2}^k = d_L$. From the construction above, $u^{l,k}(x) = (u_1(x_1), u_0^1(x_0))$ is strictly increasing for $x \in (-\infty, 0)$ and $u^{l,k}(x) = (u_0^2(x_0), v_1(x_2))$ is strictly decreasing for $x \in (0, \infty)$.

On the other hand, we define

$$d_U = d_1 + d_{2,1}^k. \tag{2.18}$$

Then (1.8)–(1.13) has a positive solution with $d_1 > 0$ arbitrary and $d_2 = d_{2,1}^k > 0$. When d_1 varies from 0 to ∞, $d_{2,1}^k > d_{2,2}^k > 0$ and $d_{2,1}^k \to \infty$ as $d_1 \to 0$ or $d_1 \to \infty$, which implies that $d_U \to \infty$ as $d_1 \to 0$ or $d_1 \to \infty$. Let $d_* = \min_{d_1>0} d_U$. Then for $d > d_*$, there exists at least two positive solutions $u_j^{l,k}(x)$ $(j = 2, 3)$ of (1.4) and (1.5) with $|I_0| = d_1 + d_{2,1}^k = d_U$ from the continuity of d_U with respect to d_1. For $d = d_*$, there exists at least one positive solution of (1.4) and (1.5) with $|I_0| = d_1 + d_{2,1}^k = d_U$. These solutions with $|I_0| = d_U$ are different from $u_1^{l,k}(x)$ with $|I_0| = d_L$ as $d_U > d_L$ for any $d_1 > 0$. □

The existence of multiple positive solutions of (1.4) and (1.5) can be shown following the approach in the proof of Theorem 1.5. Now we will prove Theorem 1.7, and the proof of Theorem 1.8 is similar thus omitted.

Proof of Theorem 1.7. Let $m \in \mathbb{N}$ and let $w_{m+1} = \tilde{u}_{m+1}(\alpha_*)$ where $\tilde{u}_l$ and α_* are defined in Lemma 2.1 and Proposition 2.4. Then from (2.11), we have

$$G(w_{m+1}) = (1 - (m+1)^2)G(a), \qquad m = 1, 2, 3, \cdots. \tag{2.19}$$

Hence from $\tilde{u}_m$ is increasing in m as in Proposition 2.4 and (2.19), we have $\omega_{m+1} > \omega_m$ for $m \in \mathbb{N}$ as $G(u)$ is an increasing function for $u > \xi$.

From Lemma 2.1, Lemma 2.2 and Proposition 2.4, for each $l \geq m + 1$, the function $\tilde{u}_l(\alpha)$ is defined for $\alpha \in (-\infty, 0)$, $\tilde{u}_l'(\alpha)(\alpha - \alpha_*) < 0$ for $\alpha \neq \alpha_*$, $\tilde{u}_l(\alpha)$ achieves a local and global maximum value at $\alpha = \alpha_*$, and

$$\lim_{\alpha \to -\infty} \tilde{u}_l(\alpha) = \lim_{\alpha \to 0^-} \tilde{u}_l(\alpha) = \xi.$$

Moreover when $l > j \geq 2$, we have $\tilde{u}_l(\alpha) > \tilde{u}_j(\alpha)$ for $\alpha \in (-\infty, 0)$.

Suppose that $\tau \in (w_1, w_{m+1})$, then for each $l \geq m + 1$, there exists α_l^1 and α_l^2 satisfying $-\infty < \alpha_l^1 < \alpha_* < \alpha_l^2 < 0$ such that

$$\tau = \tilde{u}_l(\alpha_l^1) = \tilde{u}_l(\alpha_l^2).$$

Let $d_{1,1}^l = -\rho_L(\alpha_l^1)$ and $d_{1,2}^l = -\rho_L(\alpha_l^2)$, then similar as (2.16), we have $d_{1,1}^l > d_{1,2}^l > 0$. Hence the equations (1.8)–(1.10) have two solutions $u_{L,i}^l(x) = (u_{0,i}(x_0), u_{1,i}(x_1))$ for $i = 1, 2$ such that $d_1 = d_{1,1}^l$ for $u_{L,1}^l(x)$ and $d_1 = d_{1,2}^l$ for

$u^l_{L,2}(x)$, and the construction of $u^l_{L,i}(x)$ is similar to the ones in the proof of Theorems 1.3 and 1.5.

Similarly since $k \geq m+1$, by the same argument, there exists β^1_k and β^2_k satisfying $-\infty < \beta^1_k < \alpha_* < \beta^2_k < 0$ such that

$$\tau = \tilde{u}_k(\beta^1_k) = \tilde{u}_k(\beta^2_k).$$

Let $d^k_{2,1} = -\rho_L(\beta^1_k)$ and $d^k_{2,2} = -\rho_L(\beta^2_k)$, then from (2.16), we have $d^k_{2,1} > d^k_{2,2} > 0$. Hence the equations (1.11)–(1.13) have two solutions $u^k_{R,i}(x) = (u_{0,i}(x_0), v_{1,i}(x_2))$ for $i = 1, 2$ such that $d_2 = d^k_{2,1}$ for $u^k_{R,1}(x)$ and $d_2 = d^k_{2,2}$ for $u^k_{R,2}(x)$, and the construction of $u^k_{R,i}(x)$ is similar to the ones in the proof of Theorems 1.3 and 1.5.

Now for $\tau \in (w_1, w_{m+1})$, given that $l \geq m+1$ and $k \geq m+1$, we have four positive solutions of (1.8)–(1.13) as

$$u^{l,k}_{11}(x) = (u^l_{L,1}(x), u^k_{R,1}(x)), \quad u^{l,k}_{12}(x) = (u^l_{L,1}(x), u^k_{R,2}(x)),$$
$$u^{l,k}_{21}(x) = (u^l_{L,2}(x), u^k_{R,1}(x)), \quad u^{l,k}_{22}(x) = (u^l_{L,2}(x), u^k_{R,2}(x)).$$

These give four positive solutions to (1.4) and (1.5). Note that the length of the bridge $d = |I_0| = d_1 + d_2$ for the four solutions satisfy

$$d_{11} \equiv d^l_{1,1} + d^k_{2,1} > d_{21} \equiv d^l_{1,2} + d^k_{2,1} > d_{22} \equiv d^l_{1,2} + d^k_{2,2},$$
$$d_{11} \equiv d^l_{1,1} + d^k_{2,1} > d_{12} \equiv d^l_{1,1} + d^k_{2,2} > d_{22} \equiv d^l_{1,2} + d^k_{2,2}.$$

This proves part (1).

For part (2), if $\tau = w_{m+1}$, $l \geq m+2$ and $k = m+1$, there exists α^1_l and α^2_l satisfying $-\infty < \alpha^1_l < \alpha_* < \alpha^2_l < 0$ and $\beta_k = \alpha_*$ satisfying $\tilde{u}_k(\alpha_*) = w_{m+1}$. Then similar to the case in part (1), (1.4) and (1.5) admit at least two positive solutions $u^{l,k}_{11}(x)$ and $u^{l,k}_{21}(x)$. Similarly if $\tau = w_{m+1}$, $l = m+1$ and $k \geq m+2$, (1.4) and (1.5) admit at least two positive solutions $u^{l,k}_{11}(x)$ and $u^{l,k}_{12}(x)$. Finally for part (3), if $\tau = w_{m+1}$, $l = k = m+1$, then $\alpha_l = \beta_l = \alpha_*$ such that $\tilde{u}_l(\alpha_*) = w_{m+1}$. Thus (1.4) and (1.5) admit at least one positive solution $u^{l,l}_{11}(x)$. □

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