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Global boundedness of solutions to a class of partial differential equations with time delay ☆

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Abstract

A class of diffusive partial differential equations with strongly coupled time delays and diffusion is considered. The global boundedness of weak solutions of the equation is proved by an entropy method that was initially proposed for studying the global boundedness of reaction-diffusion equations with cross-diffusion. The presence of the time delays in the equation prevents the entropy method to be directly applied, and here we extend the entropy method to this class of diffusive partial differential equations with time delays by proving some key entropy inequalities, which further allows us to obtain the estimates of gradient of the solutions. The results can be used to show the global boundedness of solutions of population models with memory effect, which were recently proposed for describing the movement of highly-developed animal species. In addition, we show that the results are also applicable for the classic partial functional differential equations, where the time delays only appear in the reaction terms.

Keywords: Global boundedness; Reaction diffusion equations; Time delay; Entropy method; Memory effect

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1. Introduction

The reaction-diffusion equations play an important role in mathematical modeling, and they have many applications in modeling spatiotemporal phenomena in physics, chemistry and biology [8,23,27]. In recent decades, time delays are often taken into account in the mathematical modeling by reaction-diffusion equations, such as species gestation and maturation in populations [26], memory effect of materials on heat conduction [28], memory effect of animals on its movement [10] and so on. In this paper, we consider the following diffusive partial differential equations with time delays:

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla(A(u)\nabla u) + \nabla\cdot(B(u)\nabla u_\tau) + f(u,u_\sigma), & x\in\Omega,\ t>0, \\ \frac{\partial u}{\partial \mathbf{n}} = 0, & x\in\partial\Omega,\ t>0. \end{cases} \tag{1.1}$$

Here, Ω is a bounded domain with C^2 boundary $\partial\Omega$ in $\mathbb{R}^d$ with $d\in\mathbb{N}$; $u(x,t)\in\mathbb{R}^n$ is the vector-valued densities of individuals of n species with $n\in\mathbb{N}$ at location x and time t; $u_\tau = u(x,t-\tau)$ where τ is called memory-induced time-delay in the context of population models, and $u_\sigma = u(x,t-\sigma)$ where σ represents a time-delay in the growth process due to maturation; $A(u) = (a_{ij}(u))\in\mathbb{R}^{n\times n}$ and $B(u) = (b_{ij}(u))\in\mathbb{R}^{n\times n}$ are the density-dependent diffusion tensor matrices, and the second term on the right hand side of the equation in (1.1) describes the impact of the density of individuals at a past time on their current movement; $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ describes the birth, death, growth and interaction between species and f may depend on a time-delay term u_σ. Note that for $\tau=\sigma=0$, (1.1) is reduced to a classical reaction-diffusion system with possible self-diffusion and/or cross-diffusion model that has been extensively studied in the literature, see [18,20,22,34,38]; for $B=0$ and $\sigma\neq 0$, (1.1) is a reaction-diffusion system with delay effect in the growth and interaction [2,12,24,36]; and when $\sigma=0$ and $\tau\neq 0$, (1.1) becomes the memory-induced diffusion models which were recently proposed in [32,33].

The global boundedness of solutions of differential equations is an important issue, as it is closely related to much qualitative behavior of solutions, such as the asymptotic stability of steady states, the existence of global attractor, the precompactness of solutions and so on, see [5,11,13,17,30,35,43] and references therein. However, the cross-diffusion or time delays may cause the solution to blow up in finite or infinite time. For example, it is well known that some solutions of many chemotaxes (as a special case of cross-diffusion) models may tend to infinite at some finite time for large diffusion rates or in high dimensional spatial domain [6,7,15,16,37,42]. In [12,24], for diffusive Hutchinson's equation, it was proved that the solution is always bounded for $\Omega=[0,L]$ or small time delay in the case of $\dim(\Omega)\geq 2$, while the equation loses its dissipativity property for small diffusion rate and large time delay, even if the negative feedback is imposed in the models. Moreover, the author in [12] showed that there exists a large set of trajectories with their mass going to infinity along periodic path in space. If the diffusion and time delay are strongly coupled, such as the models in [32,33], little is known for the global boundedness of the solution, except [39] where the uniform boundedness of solutions was proved for a predator-prey model with memory-induced diffusion in some special case, i.e., the self-diffusions are absent.

For (1.1), it is noticed that the time delay is not only involved in the reaction term but also appears in the diffusive terms. Therefore, the comparison theorems for parabolic equations do not hold for (1.1), and hence it is not possible to prove the global boundedness of the solution to (1.1) by using the method of upper and lower solutions. On the other hand, if we rewrite (1.1) in an abstract integral form, it is also not easy to obtain the estimation of $\|\nabla u_\tau\|$ when the self-diffusion is not absent, by applying semigroup theory. If we treat the equation as a non-autonomous parabolic equation without delay on the time intervals $[(i-1)\tau, i\tau]$ step by step for $i \in \mathbb{N}$, then it is difficult to obtain the estimation of the gradient of solution in adjacent time intervals. Accordingly, typical existing methods for proving global boundedness of solutions of parabolic type equations cannot be directly applied to (1.1).

In this paper, we will prove the global boundedness of solutions to (1.1) by employing a modified entropy method which was first introduced in [19]. The entropy method has been already successfully applied to (1.1) for the case of $\tau = \sigma = 0$, see [3,4,20,21,44]. Since this method uses implicit Euler method to discretize model (1.1) with respect to time, it reduces the influence of time delays on the estimation of the gradient of solutions to a certain extent. Firstly, we prove the existence of globally bounded weak solutions of model (1.1) for $n = 1$ and $\sigma = 0$, which remains an open problem so far because the self-diffusion term with time delay τ is involved. As mentioned above, the techniques for the global boundedness of solutions in [39] are not applicable in this case. By proving some key entropy inequalities, we show that the entropy method can be applied to this case under some suitable assumptions on $B(u)$, showing the uniform boundedness of solutions. Secondly, for $n \geq 2$ and $\sigma = 0$, the global boundedness of solution is also established for a wider selection of $B(u)$. Here, all the entries in memory-based diffusion tensor matrix $B(u)$ can be nonzero, meaning that both self-diffusion and cross-diffusion with time delay τ are allowed. However, the proof of entropy inequalities for the case of $n = 1$ cannot be extended directly to $n \geq 2$, and if the method of proving entropy inequalities for $n \geq 2$ is applied to the case of $n = 1$, more restricted assumptions have to be made on $B(u)$. Thirdly, it is illustrated that the entropy method can be also employed for proving the global boundedness of solutions for the case of $B(u) \equiv 0$ and $\sigma \neq 0$. Compared to existing results, even if more restricted conditions are needed on the reaction term f, the structure of diffusion tensor matrix $A(u)$ is allowed to take a more general form, which is usually assumed to be a diagonal constant matrix in the literatures such as [11–13,24,30,35]. We finally remark that our approach is also applicable for the case $\tau \neq 0$ and $\sigma \neq 0$ for both the cases $n = 1$ and $n \geq 2$.

The rest of this paper is organized as follows. In Section 2, the global boundedness of solutions of (1.1) is proved for the above three cases. A brief discussion is given in Section 3, and the proofs of some entropy inequalities are presented in Appendix A. Throughout the paper, we use $\mathbb{N}_0$ to represent $\mathbb{N} \cup \{0\}$. The notation $\lceil x \rceil$ represents the smallest integer that exceeds x and the notation $\lfloor x \rfloor$ represents the largest integer not exceeding x.

2. Global existence and boundedness

2.1. The case of a scalar equation

Let $n = 1$, $A(u) = d_1 > 0$, $B(u) = d_2 g(u)$ with $d_2 \in \mathbb{R}$, and $\sigma = 0$ in (1.1). Then, the model (1.1) becomes a scalar reaction-diffusion equation with memory-based diffusion:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + d_2 \nabla \cdot (g(u) \nabla u_\tau) + f(u), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,t) = \phi(x,t), & x \in \Omega,\ -\tau \leq t \leq 0. \end{cases} \tag{2.1}$$

In particular, if $g(u) = u$, then (2.1) is the scalar reaction-diffusion population model with memory-based diffusion proposed in [33]. Here, $d_2 \nabla \cdot (g(u) \nabla u_\tau)$ describes the spatial movement depending on the spatial gradient at a particular past time. For model (2.1) with $g(u) = u$, the principle of linearized stability is established in [33], showing that the local stability of a constant steady state $u = u_*$ only depends on the ratio of d_1 and $|d_2 u_*|$. However, the global boundedness of solutions for (2.1) is still left open, which will be considered in this subsection. Here we assume that

(**H0**) For $a > 0$, $f, g \in C^0[0, a]$, and there exists a nonnegative bounded convex C^2 function $h(u) : (0, a) \to \mathbb{R}^+$ such that its derivative $h'(u)$ is invertible, and

$$h''(u) \geq \frac{1}{u}, \quad \forall\, u \in (0, a); \tag{2.2}$$

Moreover, there exist constants $C_f, C_g > 0$ such that

$$f(u)h'(u) \leq C_f(1 + h(u)), \qquad |g(u)h''(u)| \leq C_g, \quad \forall\, u \in (0, a); \tag{2.3}$$

(**H1**) The initial function $\phi(x,t)$ satisfies $0 < \phi(x,t) < a$, and

$$\phi(x,t) \in C^{1,0}(\overline{\Omega} \times [-\tau, 0]), \ \ \frac{\partial \phi}{\partial n}(x,t) = 0\,, \ \ (x,t) \in \partial\Omega \times [-\tau, 0].$$

Remark 2.1. The function $h(u)$ in the assumption (**H0**) is called an entropy function, and it is usually chosen as

$$h(u) = u(\ln u - 1) + (a - u)(\ln(a - u) - 1) + H, \tag{2.4}$$

where H is a positive constant that makes $h(u) > 0$ for $0 < u < a$, see [19]. Note that

$$h'(u) = \ln \frac{u}{a - u}, \quad h''(u) = \frac{a}{u(a - u)}.$$

Therefore $h'(u)$ is invertible and (2.2) is satisfied for this $h(u)$. (2.3) is the technical assumption for *a prior* estimates of the solutions, and it is also satisfied for some typical choice of f and g with h given by (2.4), such as $f(u) = g(u) = u(a - u)$. In this case, (2.1) turns into

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + d_2 \nabla \cdot (u(a - u)\nabla u_\tau) + u(a - u), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,t) = \phi(x,t), & x \in \Omega,\ -\tau \leq t \leq 0. \end{cases} \tag{2.5}$$

In the context of population dynamics, the second term in (2.5) represents the directional movement of the population, influenced by the volume filling effect [1,14,25,29,40,41] and the gradient of its past density [33]. For (2.5), it is easy to verify that

$$f(u)h'(u) \leq C_f(1 + h(u)), \quad g(u)h''(u) = a = C_g, \quad \forall\, u \in [0, a],$$

with $C_f = \sup_{0<u<a} f(u)h'(u)$ and $C_g = a$. The assumption (**H1**) requires less regularity on initial functions than the ones in [33], since we focus on the weak solution for (2.1).

Definition 2.2. We call $u(x,t)$ a weak solution to (2.1) if for $T > 0$,

(1) $u(t,\cdot) \in L^2((0,T); H^1(\Omega;\mathbb{R}))$ and $\partial_t u(t,\cdot) \in L^2((0,T); H^1(\Omega;\mathbb{R})')$;
(2) for any $\varphi \in L^2((0,T); H^1(\Omega;\mathbb{R}))$,

$$\int_0^T \langle \partial_t u, \varphi\rangle dt + d_1 \int_0^T \int_\Omega \nabla\varphi \cdot \nabla u dxdt + d_2 \int_0^T \int_\Omega \nabla\varphi \cdot g(u)\nabla u_\tau dxdt = \int_0^T \int_\Omega f(u)\cdot \varphi dxdt;$$

(3) $u(x,t) = \phi(x,t)$ *, a.e. on* $\Omega \times [-\tau, 0]$.

Here, $\langle\cdot,\cdot\rangle$ denotes the dual product between $H^1(\Omega;\mathbb{R})'$ and $H^1(\Omega;\mathbb{R})$.

Our main result for (2.1) is as follows.

Theorem 2.3. *Assume $d_1 > 0$, $d_2 \in \mathbb{R}$, $\tau > 0$ and **(H0)**-**(H1)** hold. Then (2.1) possesses a bounded weak solution $u(x,t)$ such that $u(x,t) \in [0,a]$, for $x \in \Omega$ and $t > 0$.*

Proof. Step 1. We construct an approximated discrete problem of (2.1). Let $w = h'(u)$. Then, $u(w) = (h')^{-1}(w)$ and (2.1) can be rewritten as

$$\partial_t u(w) = d_1 \nabla([h''(u(w))]^{-1}\nabla w) + d_2 \nabla(g(u(w))\nabla u(w_\tau)) + f(u(w)). \tag{2.6}$$

Given any $T > 0$, choose $N \in \mathbb{N}$ and let $\delta = \tau/N$, $N_1 = \lceil T/\delta \rceil$ and $T_1 = N_1\delta \geq T$. Applying the implicit Euler discretization for the time variable to (2.6), we obtain

$$\begin{aligned}&\frac{1}{\delta}(u(w^k) - u(w^{k-1}))\\ =&d_1 \nabla([h''(u(w^k))]^{-1}\nabla w^k) + d_2 \nabla(g(u(w^k))\nabla u(w^{k-N})) + f(u(w^k)),\end{aligned} \tag{2.7}$$

for $k \in \mathbb{N}$ and $1 \leq k \leq N_1$, where $w^k(x) = h'(\phi(x,k\delta))$ for $k \in \mathbb{Z}$ and $-N \leq k \leq 0$. For sufficiently small $\varepsilon > 0$, we construct a weak version of the approximated problem (2.7) as follows

$$\begin{aligned}&\frac{1}{\delta}\int_\Omega (u(w^k) - u(w^{k-1}))\varphi dx + d_1 \int_\Omega \nabla\varphi \cdot [h''(u(w^k))]^{-1}\nabla w^k dx\\ &+ \varepsilon \int_\Omega (\sum_{|\alpha|=m} D^\alpha w^k D^\alpha \varphi + w^k\varphi)dx + d_2 \int_\Omega \nabla\varphi \cdot g(u(w^k))\nabla u(w^{k-N})dx\\ =&\int_\Omega f(u(w^k))\varphi dx, \qquad \forall \varphi \in H^m(\Omega;\mathbb{R}),\end{aligned} \tag{2.8}$$

where $m \in \mathbb{N}$ with $m > d/2$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \mathbb{N}_0^d$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d = m$ and $D^\alpha = \frac{\partial^m}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}$. Note that if $m > d/2$, then $H^m(\Omega;\mathbb{R}) \hookrightarrow L^\infty(\Omega;\mathbb{R})$ where $\hookrightarrow$ represents a compact embedding. From (**H0**), we know $w^k(x) \in L^\infty(\Omega;\mathbb{R}) \cap H^1(\Omega;\mathbb{R})$ for $-N \leq k \leq 0$. Suppose $w^{k-1}, \cdots, w^{k-N} \in L^\infty(\Omega;\mathbb{R}) \cap H^1(\Omega;\mathbb{R})$ are given for some $k \geq 1$, our aim is to find $w^k \in H^m(\Omega;\mathbb{R})$ satisfying (2.8), which will be accomplished in the following two steps.

Step 2. We prove the existence of the solution $w = w^k \in H^m(\Omega; \mathbb{R})$ of the following auxiliary equation:

$$a(w, \varphi) = F(\varphi), \quad \forall \varphi \in H^m(\Omega; \mathbb{R}) \tag{2.9}$$

where

$$a(w, \varphi) = d_1 \int_\Omega \nabla\varphi \cdot [h''(u(y))]^{-1} \nabla w dx + \varepsilon \int_\Omega (\sum_{|\alpha|=m} D^\alpha w D^\alpha \varphi + w\varphi) dx,$$

$$F(\varphi) = -\frac{\beta}{\delta} \int_\Omega (u(y) - u(w^{k-1}))\varphi dx - \beta d_2 \int_\Omega \nabla\varphi \cdot g(u(y)) \nabla u(w^{k-N}) dx + \beta \int_\Omega f(u(y))\varphi dx,$$

for $y \in L^\infty(\Omega; \mathbb{R})$ and $\beta \in [0, 1]$. Note that (2.9) is equivalent to (2.8) when $\beta = 1$.

Given $y \in L^\infty(\Omega; \mathbb{R})$, by the Hölder inequality, we know that $a(\cdot, \cdot)$ and $F(\cdot)$ are bounded bilinear and linear operators on $H^m(\Omega; \mathbb{R}^n)$ respectively. From the nonnegativity of h and the Poincaré inequality, we have

$$a(w, w) \geq \varepsilon \int_\Omega (\sum_{|\alpha|=m} (D^\alpha w)^2 + w^2) dx \geq \varepsilon C ||w||^2_{H^m(\Omega;\mathbb{R})}, \tag{2.10}$$

for all $w \in H^m(\Omega; \mathbb{R})$, where $C > 0$ is a constant depending on Ω. This implies the bilinear form $a(\cdot, \cdot)$ is coercive. It then follows from the Lax-Milgram Lemma (see Theorem 1 in Section 6.2 of [9]) that there exists a unique solution $w \in H^m(\Omega; \mathbb{R})$ to (2.9) for any $y \in L^\infty(\Omega; \mathbb{R})$ and $\beta \in [0, 1]$, such that

$$\|w\|_{H^m(\Omega;\mathbb{R})} \leq \frac{1}{\varepsilon C} \|F\|_{H^m(\Omega;\mathbb{R})'}. \tag{2.11}$$

This allows us to define an operator $S : L^\infty(\Omega; \mathbb{R}) \times [0, 1] \to L^\infty(\Omega; \mathbb{R})$ by

$$S(y, \beta) = w,$$

where w uniquely solves (2.9). Note that if the operator $S(\cdot, 1)$ has a fixed point w in $L^\infty(\Omega; \mathbb{R})$, then w is the solution of (2.8).

Step 3. We show that the operator $S(\cdot, 1)$ has a fixed point in $L^\infty(\Omega; \mathbb{R})$. Firstly, we will prove the continuity and compactness of $S(\cdot, \cdot)$. Choose $y_n \to y$ in $L^\infty(\Omega; \mathbb{R})$ and $\beta_n \to \beta$ in $[0, 1]$ as $n \to \infty$. Set $w_n = S(y_n, \beta_n)$. By the continuity of h, f and g, we have, as $n \to \infty$,

$$\begin{aligned} &u(y_n) \to u(y), \ g(u(y_n)) \to g(u(y)), \\ &f(u(y_n)) \to f(u(y)), \ [h''(u(y_n))]^{-1} \to [h''(u(y))]^{-1}. \end{aligned} \tag{2.12}$$

Thus F_n is bounded in $H^m(\Omega; \mathbb{R})'$, where

$$F_n(\varphi) = -\frac{\beta_n}{\delta}\int_\Omega (u(y_n) - u(w^{k-1}))\varphi dx - \beta_n d_2 \int_\Omega \nabla\varphi \cdot g(u(y_n))\nabla u(w^{k-N})dx$$
$$+ \beta_n \int_\Omega f(u(y_n))\varphi dx.$$

From (2.11), we further have that $\{w_n\}$ is bounded in $H^m(\Omega;\mathbb{R})$. Since $H^m(\Omega;\mathbb{R})$ is reflexive, there exists a subsequence of $\{w_n\}$, still denoted by $\{w_n\}$, such that $w_n \rightharpoonup w$ in $H^m(\Omega;\mathbb{R})$. Here $\rightharpoonup$ represents the weak convergence. This, together with (2.12), allows us taking the limit on both sides of

$$a_n(w_n,\varphi) = F_n(\varphi),$$

where

$$a_n(w_n,\varphi) = d_1 \int_\Omega \nabla\varphi \cdot [h''(u(y_n))]^{-1}\nabla w_n dx + \varepsilon \int_\Omega (\sum_{|\alpha|=m} D^\alpha w_n D^\alpha \varphi + w_n\varphi)dx,$$

to obtain that $w_n = S(y_n,\beta_n) \rightharpoonup S(y,\beta) = w$ in $H^m(\Omega;\mathbb{R})$. Recall that $H^m(\Omega;\mathbb{R}) \hookrightarrow L^\infty(\Omega;\mathbb{R})$. Therefore, there exists a subsequence of $\{w_n\}$ satisfying $w_n = S(y_n,\beta_n)$, still denoted by itself, such that $w_n = S(y_n,\beta_n) \to S(y,\beta) = w$ in $L^\infty(\Omega;\mathbb{R})$. This proves the continuity of $S(\cdot,\cdot)$, and the compactness of $S(\cdot,\cdot)$ follows directly from the compact embedding of $H^m(\Omega;\mathbb{R})$ into $L^\infty(\Omega;\mathbb{R})$.

Let

$$S_1 = \{w \in H^m(\Omega;\mathbb{R}) : S(w,\beta) = w,\ \beta \in [0,1]\}.$$

Next, we prove that S_1 is bounded in $L^\infty(\Omega;\mathbb{R})$, that is, all fixed points of the operator S are uniformly bounded with respect to β. Let

$$S_2 = \{w \in H^m(\Omega;\mathbb{R}) : a(w,w) = F(w) \text{ with } y = w,\ \beta \in [0,1]\}.$$

Obviously, $S_1 \subseteq S_2$. Thus, it suffices to show that S_2 is bounded in $L^\infty(\Omega;\mathbb{R})$. Substituting $y = w$ and $\varphi = w$ into (2.9), we have

$$\begin{aligned}
&\frac{\beta}{\delta}\int_\Omega (u(w) - u(w^{k-1}))w dx + d_1 \int_\Omega \nabla w \cdot [h''(u(w))]^{-1}\nabla w dx \\
&+ \varepsilon \int_\Omega (\sum_{|\alpha|=m} (D^\alpha w)^2 + w^2)dx + \beta d_2 \int_\Omega \nabla w \cdot g(u(w))\nabla u(w^{k-N})dx \\
=&\beta \int_\Omega f(u(w)) w dx.
\end{aligned} \tag{2.13}$$

By the convexity of h, i.e., $h(u_1) - h(u_2) \le h'(u_1)(u_1 - u_2)$ for all $u_1, u_2 \in (0,a)$ and the fact that $h'(u(w)) = w$, we have

$$\frac{\beta}{\delta}\int_{\Omega}(u(w)-u(w^{k-1}))w\,dx \geq \frac{\beta}{\delta}\int_{\Omega}[h(u(w))-h(u(w^{k-1}))]dx. \tag{2.14}$$

Combining (2.3), (2.13) and (2.14), we obtain

$$\begin{aligned}&\beta\int_{\Omega}h(u(w))dx+\delta d_1\int_{\Omega}\nabla w\cdot[h''(u(w))]^{-1}\nabla w\,dx\\&+\delta\varepsilon\int_{\Omega}(\sum_{|\alpha|=m}(D^{\alpha}w)^2+w^2)dx+\delta\beta d_2\int_{\Omega}\nabla w\cdot g(u(w))\nabla u(w^{k-N})dx\\&\leq C_f\delta\beta\int_{\Omega}(1+h(u(w)))dx+\beta\int_{\Omega}h(u(w^{k-1}))dx.\end{aligned} \tag{2.15}$$

Choose $\delta < 1/C_f$. Then, (2.15) implies

$$\begin{aligned}&\delta\varepsilon\int_{\Omega}(\sum_{|\alpha|=m}(D^{\alpha}w)^2+w^2)dx+\delta\beta d_2\int_{\Omega}\nabla w\cdot g(u(w))\nabla u(w^{k-N})dx\\&\leq\beta|\Omega|+\beta\int_{\Omega}h(u(w^{k-1}))dx.\end{aligned} \tag{2.16}$$

If $\delta\beta d_2\int_{\Omega}\nabla w\cdot g(u(w))\nabla u(w^{k-N})dx > 0$ for all $\beta \in [0,1]$, then from (2.10) and (2.16), we have

$$\delta\varepsilon C||w||^2_{H^m(\Omega;\mathbb{R})} \leq \delta\varepsilon\int_{\Omega}(\sum_{|\alpha|=m}(D^{\alpha}w)^2+w^2)dx \leq \beta|\Omega|+\beta\int_{\Omega}h(u(w^{k-1}))dx,$$

which implies that S_2 is bounded in $L^{\infty}(\Omega;\mathbb{R})$, because of the positivity and boundedness of $h(u)$.

Next we prove the boundedness of S_2 in the case that $\delta\beta d_2\int_{\Omega}\nabla w\cdot g(u(w))\nabla u(w^{k-N})dx$ is not always positive for some $\beta\in[0,1]$. By the Hölder inequality and (**H1**), there exists $\eta > 0$, independent of β, such that

$$\begin{aligned}&|\delta\beta d_2\int_{\Omega}\nabla w\cdot g(u(w))\nabla u(w^{k-N})dx|\\&\leq|\delta\beta d_2|\|\nabla w\|_{L^2(\Omega;\mathbb{R}^d)}\|g(u(w))\nabla u(w^{k-N})\|_{L^2(\Omega;\mathbb{R}^d)}\leq\eta\|\nabla w\|_{L^2(\Omega;\mathbb{R}^d)}.\end{aligned} \tag{2.17}$$

Using (2.10) and (2.17), we have

$$\begin{aligned}&\delta\varepsilon\int_{\Omega}(\sum_{|\alpha|=m}(D^{\alpha}w)^2+w^2)dx-\left|\delta\beta d_2\int_{\Omega}\nabla w\cdot g(u(w))\nabla u(w^{k-N})dx\right|\\&\geq\delta\varepsilon C\|w\|^2_{H^m(\Omega;\mathbb{R})}-\eta\|\nabla w\|_{L^2(\Omega;\mathbb{R}^d)}.\end{aligned} \tag{2.18}$$

Suppose that $\|w\|_{H^m(\Omega,\mathbb{R})}$ is sufficiently large. It then follows from (2.18) that

$$\begin{aligned}&\delta\varepsilon\int_\Omega \Big(\sum_{|\alpha|=m}(D^\alpha w)^2+w^2\Big)dx-\left|\delta\beta d_2\int_\Omega \nabla w\cdot g(u(w))\nabla u(w^{k-N})dx\right|\\ \gg&\beta|\Omega|+\beta\int_\Omega h(u(w^{k-1}))dx,\end{aligned}$$

which contradicts with (2.16). This also yields the boundedness of S_2 in $L^\infty(\Omega;\mathbb{R})$ for this case, which further implies that S_1 is bounded in $L^\infty(\Omega;\mathbb{R})$. Therefore, from Leray-Schauder Theorem (see Theorem A.4 in [20]), there exists a solution $w=w^k\in H^m(\Omega;\mathbb{R})$ such that $S(w,1)=w$ which solves (2.8), where $1\le k\le N_1$.

We remark that w^k (hence also the bound of w^k) depends on ε and δ. For later steps, we need a boundedness result for w^k with the bound independent of ε and δ. Given $j\in\mathbb{N}$, $1\le j\le N_1$, summing (2.15) with $w=w^k$ and $\beta=1$ over $k=1,\cdots,j$, and using (2.10), we find that

$$\begin{aligned}&(1-C_f\delta)\int_\Omega h(u(w^j))dx+\delta d_1\sum_{k=1}^{j}\int_\Omega \nabla w^k\cdot[h''(u(w^k))]^{-1}\nabla w^k dx\\ &+\varepsilon C\delta\sum_{k=1}^{j}\|w^k\|^2_{H^m(\Omega;\mathbb{R})}+\delta d_2\sum_{k=1}^{j}\int_\Omega \nabla w^k\cdot g(u(w^k))\nabla u(w^{k-N})dx\\ \le&C_f\delta j|\Omega|+C_f\delta\sum_{k=1}^{j-1}\int_\Omega h(u(w^k))dx+\int_\Omega h(u(w^0))dx.\end{aligned}\tag{2.19}$$

Equation (2.19) implies that there exists a constant $C_1>0$, independent of δ and ε, such that

$$\delta d_1\sum_{k=1}^{j}\int_\Omega \nabla w^k\cdot[h''(u(w^k))]^{-1}\nabla w^k dx+\varepsilon C\delta\sum_{k=1}^{j}\|w^k\|^2_{H^m(\Omega;\mathbb{R})}\le C_1.\tag{2.20}$$

The proof of (2.20) is postponed to the Appendix.

Step 4. We now use the time-discrete approximate solution w^k to define an approximate solution u^δ defined on a continuous time interval $[-\tau,T_1]$, and we also show a related uniform boundedness result for the approximate solution u^δ.

For $m\in\mathbb{N}_0$ and $0\le m\le N$, we define

$$(\sigma_m w^\delta)(x,t)=\begin{cases} w^{k-m}(x), & t\in((k-1)\delta,\ k\delta],\ k=1,2,\cdots,N_1,\\ h'(\phi(x,-m\delta)), & t=0,\end{cases}$$

and $(\sigma_m u^\delta)(x,t)=u((\sigma_m w^\delta)(x,t))$. We also define $u^\delta(x,t)=(\sigma_0 u^\delta)(x,t)$. Integrating (2.8) on $(0,T_1)$, we have

$$
\begin{aligned}
&\frac{1}{\delta}\int_0^{T_1}\int_\Omega (u^\delta-\sigma_1 u^\delta)\psi\, dxdt + d_1\int_0^{T_1}\int_\Omega \nabla\psi\cdot\nabla u^\delta dxdt\\
&+\varepsilon\int_0^{T_1}\int_\Omega \Big(\sum_{|\alpha|=m} D^\alpha w^\delta D^\alpha\psi + w^\delta\psi\Big)dxdt + d_2\int_0^{T_1}\int_\Omega \nabla\psi\cdot g(u^\delta)\nabla(\sigma_N u^\delta)dxdt\\
=&\int_0^{T_1}\int_\Omega f(u^\delta)\psi\, dxdt,
\end{aligned}
\tag{2.21}
$$

where $\psi:(0,T_1)\to H^m(\Omega;\mathbb{R})$ is any piecewise constant function, which is contained in a dense subset of $L^2((0,T_1);H^m(\Omega;\mathbb{R}))$ (see Proposition 1.36 in [31]). Applying the Hölder inequality to (2.21), we have

$$
\begin{aligned}
&\frac{1}{\delta}\left|\int_0^{T_1}\int_\Omega (u^\delta-\sigma_1 u^\delta)\psi\, dxdt\right|\\
\le& d_1\|\nabla\psi\|_{L^2((0,T_1);L^2(\Omega;\mathbb{R}^d))}\|\nabla u^\delta\|_{L^2((0,T_1);L^2(\Omega;\mathbb{R}^d))}\\
&+\varepsilon\|w^\delta\|_{L^2((0,T_1);H^m(\Omega;\mathbb{R}))}\|\psi\|_{L^2((0,T_1);H^m(\Omega;\mathbb{R}))}\\
&+|d_2|\|\nabla\psi\|_{L^2((0,T_1);L^2(\Omega;\mathbb{R}^d))}\|g(u^\delta)\nabla(\sigma_N u^\delta)\|_{L^2((0,T_1);L^2(\Omega;\mathbb{R}^d))}\\
&+\|f(u^\delta)\|_{L^2((0,T_1);L^2(\Omega;\mathbb{R}))}\|\psi\|_{L^2((0,T_1);L^2(\Omega;\mathbb{R}))}.
\end{aligned}
\tag{2.22}
$$

Applying (2.20) with $j=N_1$ and (A.5) with $N=N_1$, we obtain that

$$
\begin{aligned}
&\|u^\delta\|^2_{L^2((0,T_1);H^1(\Omega;\mathbb{R}))}+\varepsilon\|w^\delta\|^2_{L^2((0,T_1);H^m(\Omega;\mathbb{R}))}\\
=&\delta\sum_{k=1}^{N_1}\|u(w^k)\|^2_{H^1(\Omega;\mathbb{R})}+\varepsilon\delta\sum_{k=1}^{N_1}\|w^k\|^2_{H^m(\Omega;\mathbb{R})}\le C_3,
\end{aligned}
\tag{2.23}
$$

where $C_3>0$ is a constant independent of δ and ε. Taking into account the assumptions (**H0**)-(**H1**), it follows from (2.22) and (2.23) that

$$
\delta^{-1}\|u^\delta-\sigma_1 u^\delta\|_{L^2((0,T_1);H^m(\Omega;\mathbb{R})')}\le C_4,
\tag{2.24}
$$

where C_4 is a constant independent of δ and ε.

Step 5. We complete the proof by showing the approximate solution u^δ converges to a limit as $\delta\to 0$ and $\varepsilon\to 0$. The uniform estimates (2.23) and (2.24) allow us to apply the nonlinear Aubin-Lions Lemma (see Theorem A.5 in [20]) to obtain that there exists $u\in L^2((0,T_1);L^2(\Omega;\mathbb{R}))$ such that as $(\delta,\varepsilon)\to(0,0)$,

$$
u^\delta\to u \text{ in } L^2((0,T_1);L^2(\Omega;\mathbb{R})) \text{ and } a.e. \text{ in } \Omega\times(0,T_1).
$$

Recall that, when $t \le 0$, $u^\delta(x,t) = \phi(x, k\delta)$ for $t \in ((k-1)\delta,\ k\delta]$ for $k \in \mathbb{Z}$ and $-N \le k \le 0$, and $u^\delta(x, -\tau) = \phi(x, -\tau)$. Hence it follows from **(H1)** that $u^\delta \to \phi$ in $L^\infty([-\tau, 0]; L^\infty(\Omega; \mathbb{R}))$ as $\delta \to 0$. Therefore, if we define $u(x,t) = \phi(x,t)$ for $t \in [-\tau, 0]$ and $x \in \Omega$, then we have

$$u^\delta \to u \ \text{ in } L^2((-\tau, T_1); L^2(\Omega; \mathbb{R})).$$

In particular, $\sigma_N u^\delta \to u_\tau$ in $L^2((0,T_1); L^2(\Omega; \mathbb{R}))$. Furthermore, by (2.23), (2.24) and an argument of weak compactness, we have

$$\begin{aligned}
&\nabla u^\delta \rightharpoonup \nabla u \ \text{ in } L^2((0,T_1); L^2(\Omega; \mathbb{R}^d)),\\
&\varepsilon w^\delta \to 0 \ \text{ in } L^2((0,T_1); H^m(\Omega; \mathbb{R})),\\
&\delta^{-1}(u^\delta - \sigma_\delta u^\delta) \rightharpoonup \partial_t u \ \text{ in } L^2((0,T_1); H^m(\Omega; \mathbb{R})'),\\
&\nabla(\sigma_N u^\delta) \rightharpoonup \nabla u_\tau \ \text{ in } L^2((0,T_1); L^2(\Omega; \mathbb{R}^d)).
\end{aligned} \tag{2.25}$$

According to the dominated convergence theorem, we also have

$$\begin{aligned}
&g(u^\delta) \to g(u) \ \text{ in } L^2((0,T_1); L^2(\Omega; \mathbb{R}^d)),\\
&f(u^\delta) \to f(u) \ \text{ in } L^2((0,T_1); L^2(\Omega; \mathbb{R}^d)).
\end{aligned} \tag{2.26}$$

Therefore, we can pass to the limit $(\delta, \varepsilon) \to (0,0)$ in (2.21) to obtain

$$\int_0^{T_1} \langle \partial_t u, \psi \rangle dt + d_1 \int_0^{T_1}\int_\Omega \nabla\psi \cdot \nabla u dx dt + d_2 \int_0^{T_1}\int_\Omega \nabla\psi \cdot (g(u)\nabla u_\tau) dx dt = \int_0^{T_1}\int_\Omega f(u)\psi dx dt,$$

for all $\psi \in L^2((0,T_1); H^m(\Omega; \mathbb{R}))$, which is also valid for $\psi \in L^2((0,T_1); H^1(\Omega; \mathbb{R}))$ by the density of $L^2((0,T_1); H^m(\Omega; \mathbb{R}))$ in $L^2((0,T_1); H^1(\Omega; \mathbb{R}))$ (see Theorem 4.1 in [20]). Thus u is a weak solution to (2.1) on $(0, T_1)$. Since T_1 and T are arbitrary, then the weak solution exists for any $t > 0$, and $0 \le u(x,t) \le a$ for $x \in \Omega$ and $t > 0$. □

Corollary 2.4. *Assume $d_1 > 0$, $d_2 \in \mathbb{R}$, $\tau > 0$ and $a > 0$. Then (2.5) possesses a bounded weak solution $u(x,t)$ such that $u(x,t) \in [0,a]$, for $x \in \Omega$ and $t > 0$.*

Remark 2.5. We remark that Theorem 2.3 remains valid, when the time delay τ in (2.1) is a distributed one. For example, if u_τ in (2.1) is replaced by $\int_{t-\tau}^t u(s,x) ds$, then (2.19) now reads

$$\begin{aligned}
&(1 - C_f\delta)\int_\Omega h(u(w^j)) dx + \delta d_1 \sum_{k=1}^{j} \int_\Omega \nabla w^k \cdot [h''(u(w^k))]^{-1} \nabla w^k dx\\
&+ \varepsilon C\delta \sum_{k=1}^{j} \|w^k\|^2_{H^m(\Omega;\mathbb{R})} + \delta^2 d_2 \sum_{k=1}^{j}\sum_{i=1}^{N} \int_\Omega \nabla w^k \cdot g(u(w^k)) \nabla u(w^{k-i}) dx\\
&\le C_f \delta j |\Omega| + C_f\delta \sum_{k=1}^{j-1} \int_\Omega h(u(w^k)) dx + \int_\Omega h(u(w^0)) dx,
\end{aligned}$$

which still allows us to get the inequality (2.20). The remaining part of the proof of Theorem 2.3 can be accomplished by similar arguments. According to [33], it is known if $\phi \in C^{2+\alpha,\alpha}(\overline{\Omega} \times [-\tau, 0])$ for some $\alpha \in (0, 1)$, (2.1) has a unique classic solution for $t \in (0, \infty)$. It then can be concluded from Theorem 2.3 that the classic solution of (2.1) is also L^∞-bounded, by further assuming $\phi \in C^{2+\alpha,\alpha}(\overline{\Omega} \times [-\tau, 0])$.

2.2. *The case of a system of equations*

In this section, we consider (1.1) with $\sigma = 0$ and $n \geq 2$, that is,

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (A(u)\nabla u) + \nabla \cdot (B(u)\nabla u_\tau) + f(u), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,t) = \phi(x,t), & x \in \Omega,\ -\tau \leq t \leq 0. \end{cases} \tag{2.27}$$

Here $u \in \mathbb{R}^n$, $A(u), B(u) \in \mathbb{R}^{n\times n}$ are vector and matrices as described in the Introduction. It should be mentioned that in [39], the global boundedness of solutions to (2.27) was proved for a 2×2 diagonal constant matrix A and a specific choice of matrix B (a 2×2 matrix with only one nonzero off-diagonal element). In this subsection we show the global boundedness of solutions to (2.27) for more general matrices A and B. Given $a_i > 0$ $(i = 1, \cdots, n)$, let $\mathcal{D}$ be an open subset of the cube $(0, a_1) \times (0, a_2) \times \cdots \times (0, a_n)$ in $\mathbb{R}^n$. Assume

(A0) $f(u) \in C^0(\overline{\mathcal{D}}; \mathbb{R}^n)$; and there exists a nonnegative bounded convex C^2 function $h : \mathcal{D} \to \mathbb{R}^+$, such that its gradient $\nabla h : \mathcal{D} \to \mathbb{R}^n$ is invertible. Furthermore, there exists $C_f > 0$, such that

$$f(u) \cdot \nabla h(u) \leq C_f(1 + h(u)), \quad \forall\, u \in \mathcal{D}; \tag{2.28}$$

(A1) $A(u)$ and $B(u) \in C^0(\overline{\mathcal{D}}; \mathbb{R}^{n\times n})$. For any $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$, $u = (u_1, \cdots, u_n) \in \mathcal{D}$, it holds

$$z \cdot D^2h(u)A(u)z \geq \sum_{i=1}^{n} \frac{z_i^2}{u_i}, \tag{2.29}$$

and there exists $e_{ij} > 0$, $i, j = 1, \cdots, n$, with $\sum_{j=1}^{n} |e_{ij}| \leq \frac{1}{2a_i}$ and $\sum_{i=1}^{n} |e_{ij}| \leq \frac{1}{2a_j}$, $i, j = 1, \cdots, n$, such that for any $b = (b_1, \cdots, b_n) \in \mathbb{R}^n$, $c = (c_1, \cdots, c_n) \in \mathbb{R}^n$,

$$|b \cdot D^2h(u)B(u)c| \leq \sum_{i,j=1}^{n} |e_{ij} b_i c_j|; \tag{2.30}$$

(A2) The initial functions $\phi(x,t) \in \mathcal{D}$, and

$$\phi_i(x,t) \in C^{1,0}(\overline{\Omega} \times [-\tau, 0]), \ \frac{\partial \phi_i}{\partial n}(x,t) = 0\,, \ (x,t) \in \partial\Omega \times [-\tau, 0],\ i = 1, 2, \cdots, n.$$

Remark 2.6. When $n=2$, if we choose $f(u)=(u_1(a_1-u_1-m_1u_2), u_2(a_2-u_2-m_2u_1))$ and

$$A(u)=\begin{pmatrix} d_3 & 0 \\ 0 & d_4 \end{pmatrix}, \qquad B(u)=\begin{pmatrix} 0 & d_5u_1(a_1-u_1) \\ d_6u_2(a_2-u_2) & 0 \end{pmatrix},$$

where $a_1, a_2, m_1, m_2>0$, $d_3, d_4>0$ and $d_5, d_6\in\mathbb{R}$ are constants, then (2.27) becomes

$$\begin{cases} \frac{\partial u_1}{\partial t}=d_3\Delta u_1+d_5\nabla\cdot(u_1(a_1-u_1)\nabla u_{2\tau})+u_1(a_1-u_1-m_1u_2), & x\in\Omega,\ t>0, \\ \frac{\partial u_2}{\partial t}=d_4\Delta u_2+d_6\nabla\cdot(u_2(a_2-u_2)\nabla u_{1\tau})+u_2(a_2-u_2-m_2u_1), & x\in\Omega,\ t>0, \\ \frac{\partial u_1}{\partial \mathbf{n}}=\frac{\partial u_2}{\partial \mathbf{n}}=0, & x\in\partial\Omega,\ t>0, \\ u_1(x,0)=\phi_1(x,t),\ u_2(x,t)=\phi_2(x,t), & x\in\Omega,\ -\tau\le t\le 0. \end{cases} \tag{2.31}$$

In particular, when $\tau=0$, (2.31) can be regarded as a diffusive Lotka-Volterra competition model with a volume filling chemotactic effect. Choose $h(u)$ as

$$\begin{aligned} h(u)=&\frac{1}{d_3}[u_1(\ln u_1-1)+(a_1-u_1)(\ln(a_1-u_1)-1)]+ \\ &\frac{1}{d_4}[u_2(\ln u_2-1)+(a_2-u_2)(\ln(a_2-u_2)-1)]+H_1, \end{aligned} \tag{2.32}$$

where H_1 is a positive constant that makes $h(u)>0$ for $u\in(0,a_1)\times(0,a_2)$. From (2.32), we know

$$\nabla h(u)=\left(\frac{1}{d_3}\ln\frac{u_1}{a_1-u_1}, \frac{1}{d_4}\ln\frac{u_2}{a_2-u_2}\right), \qquad D^2h(u)=\begin{pmatrix} \frac{a_1}{d_3u_1(a_1-u_1)} & 0 \\ 0 & \frac{a_2}{d_4u_2(a_2-u_2)} \end{pmatrix}.$$

Define $F_1(u_1,u_2)=\frac{1}{d_3}u_1(a_1-u_1-m_1u_2)\ln\frac{u_1}{a_1-u_1}$. Then, $F_1(u_1,u_2)>0$ when $0<u_1<a_1/2$ and $a_1-u_1<m_1u_2<m_1a_2$, or $a_1/2<u_1<a_1$ and $0<m_1u_2<a_1-u_1$, otherwise $F(u_1,u_2)\le 0$. We also have $\lim\limits_{u_1\to a_1, u_2\to 0}F(u_1,u_2)=0=\lim\limits_{u_1\to 0, u_2\to a_2}F(u_1,u_2)$. By the continuity of $F_1(u_1,u_2)$, we know that there exists $C_{f,1}>0$ such that $F_1(u_1,u_2)\le C_{f,1}$ for $h(u)>0$ for $u\in(0,a_1)\times(0,a_2)$. Using the similar argument, we can also show that there exists $C_{f,2}>0$ such that $F_2(u_1,u_2):=\frac{1}{d_4}u_2(a_2-u_2-m_2u_1)\ln\frac{u_2}{a_2-u_2}\le C_{f,2}$ for $u\in(0,a_1)\times(0,a_2)$, which implies (2.28) holds. For $z=(z_1,z_2)\in\mathbb{R}^2$, $b=(b_1,b_2)\in\mathbb{R}^2$ and $c=(c_1,c_2)\in\mathbb{R}^2$, if

$$|d_5|\le\frac{d_3}{2a_1\bar{a}}, \quad |d_6|\le\frac{d_4}{2a_2\bar{a}},$$

where $\bar{a}=\max\{a_1,a_2\}$, then we have

$$z\cdot D^2h(u)A(u)z\ge\frac{z_1^2}{u_1}+\frac{z_2^2}{u_2}, \qquad |b\cdot D^2h(u)B(u)c|\le\left|\frac{1}{2\bar{a}}b_1c_2\right|+\left|\frac{1}{2\bar{a}}b_2c_1\right|.$$

Therefore, (**A0**) and (**A1**) are satisfied for (2.31).

Definition 2.7. We call $u(x,t)$ a weak solution to (2.27) if for $T>0$,

(1) $u(t,\cdot) \in L^2((0,T); H^1(\Omega;\mathbb{R}^n))$ and $\partial_t u(t,\cdot) \in L^2((0,T); H^1(\Omega;\mathbb{R}^n)')$;
(2) for all $\varphi \in L^2((0,T); H^1(\Omega;\mathbb{R}^n))$,

$$\int_0^T \langle \partial_t u, \varphi \rangle dt + \int_0^T \int_\Omega \nabla\varphi \cdot A(u)\nabla u dxdt + \int_0^T \int_\Omega \nabla\varphi \cdot B(u)\nabla u_\tau dxdt = \int_0^T \int_\Omega f(u)\cdot\varphi dxdt;$$

(3) $u(x,t) = \phi(x,t), a.e. \ on \ (x,t) \in \Omega \times [-\tau, 0]$.

Here $\langle\cdot,\cdot\rangle$ denotes the dual product between $H^1(\Omega;\mathbb{R}^n)'$ and $H^1(\Omega;\mathbb{R}^n)$.

Theorem 2.8. *Assume (**A0**)-(**A2**) hold. Then* (2.27) *has a bounded weak solution* $u(x,t)$ *such that* $u(x,t) \in \overline{\mathcal{D}}$ *for* $x \in \Omega$, $t > 0$.

Proof. The proof is analogous to the one of Theorem 2.3, so we only outline the proof here. Define $w = \nabla h(u)$, then $u(w) = (\nabla h)^{-1}u$. We could formulate similar equations as (2.6) and (2.7). Then a weak version of the approximated discrete problem for (2.27) can be formulated as

$$\begin{aligned}
&\frac{1}{\delta}\int_\Omega (u(w^k) - u(w^{k-1}))\cdot\varphi dx + \int_\Omega \nabla\varphi \cdot A(u(w^k))[D^2h(u(w^k))]^{-1}\nabla w^k dx \\
&+ \varepsilon\int_\Omega (\sum_{|\alpha|=m} D^\alpha w^k \cdot D^\alpha\varphi + w^k\cdot\varphi)dx + \int_\Omega \nabla\varphi\cdot B(u(w^k))\nabla u(w^{k-N})dx \\
=& \int_\Omega f(u(w^k))\cdot\varphi dx, \quad \forall \varphi \in H^m(\Omega;\mathbb{R}^n).
\end{aligned} \tag{2.33}$$

Next we show the existence of the solution $w^k \in H^m(\Omega;\mathbb{R}^n)$ of (2.33) by considering

$$a(w,\varphi) = F(\varphi), \qquad \forall\, \varphi \in H^m(\Omega;\mathbb{R}^n) \tag{2.34}$$

where

$$\begin{aligned}
a(w,\varphi) &= \int_\Omega \nabla\varphi\cdot A(u(y))[D^2h(u(y))]^{-1}\nabla w dx + \varepsilon\int_\Omega (\sum_{|\alpha|=m} D^\alpha w\cdot D^\alpha\varphi + w\cdot\varphi)dx, \\
F(\varphi) &= -\frac{\beta}{\delta}\int_\Omega (u(y) - u(w^{k-1}))\cdot\varphi dx - \beta\int_\Omega \nabla\varphi\cdot B(u(w^k))\nabla u(w^{k-N})dx \\
&\quad + \beta\int_\Omega f(u(y))\cdot\varphi dx,
\end{aligned}$$

for $y \in L^\infty(\Omega;\mathbb{R}^n)$ and $\beta \in [0,1]$. Again, from (**A0**)-(**A1**), we know that $a(\cdot,\cdot)$ and $F(\cdot)$ are bounded bilinear and linear operators on $H^m(\Omega;\mathbb{R}^n)$ respectively. From **(A1)**, we have

$$
\begin{aligned}
&\nabla w \cdot A(u(w))[D^2 h(u(w))]^{-1}\nabla w \\
=&([D^2 h(u(w))]^{-1}\nabla w)\cdot D^2 h(u(w))A(u(w))[D^2 h(u(w))]^{-1}\nabla w \geq 0,
\end{aligned}
\tag{2.35}
$$

which implies a vector version of (2.10) also holds for $w \in H^m(\Omega;\mathbb{R}^n)$, that is, $a(\cdot,\cdot)$ is coercive. So by the Lax-Milgram Lemma, we know that there exists a unique solution $w \in H^m(\Omega;\mathbb{R}^n)$ to (2.34) for any $y \in L^\infty(\Omega;\mathbb{R}^n)$ and $\beta \in [0,1]$. This allows us to define an operator $S: L^\infty(\Omega;\mathbb{R}^n)\times[0,1] \to L^\infty(\Omega;\mathbb{R}^n)$ by

$$
S(y,\beta) = w,
$$

where w solves (2.34). The continuity of $S(\cdot,\cdot)$ can be shown along the same lines as in **Step3** of the proof of Theorem 2.3. From **(A0)** and (2.34), we have

$$
\begin{aligned}
&\beta\int_\Omega h(u(w))dx + \delta\int_\Omega \nabla w\cdot A(u(w))[D^2h(u(w))]^{-1}\nabla w dx \\
&+\delta\varepsilon\int_\Omega (\sum_{|\alpha|=m}|D^\alpha w|^2 + |w|^2)dx + \delta\int_\Omega \nabla w\cdot B(u(w))\nabla u(w^{k-N})dx \\
\leq& C_f\delta\beta\int_\Omega (1+h(u(w)))dx + \beta\int_\Omega h(u(w^{k-1}))dx,
\end{aligned}
\tag{2.36}
$$

by setting $y = w$ and $\varphi = w$. Taking into account of (**A1**), we can get

$$
\begin{aligned}
&\left|\int_\Omega \nabla w\cdot B(u(w))\nabla u(w^{k-N})dx\right| \leq \|\nabla w\|_{L^2(\Omega;\mathbb{R}^n)}\|B(u(w))\nabla u(w^{k-N})\|_{L^2(\Omega;\mathbb{R}^n)} \\
\leq&\|\nabla w\|_{L^2(\Omega;\mathbb{R}^n)}\sum_{i=1}^n\sum_{j=1}^n\|b_{ij}(u(w))\|_{L^\infty(\Omega;\mathbb{R})}\|\nabla u_j(w^{k-N})\|_{L^2(\Omega;\mathbb{R}^d)} \\
\leq&\mu\|\nabla w\|_{L^2(\Omega;\mathbb{R}^n)},
\end{aligned}
\tag{2.37}
$$

where μ is a constant independent of β. So the boundedness of w with respect to β follows directly by combining (2.10), (2.35), (2.36), (2.37) and **(A0)**. Applying Leray-Schauder Theorem again, it can be concluded that the operator $S(\cdot,1)$ has a fixed point $w = w^k$, which is a solution of (2.33). Moreover, w^k is uniformly bounded with respect to β, and w^k (hence also the bound of w^k) depends on ε and δ.

Now we give a boundedness result for w^k with the bound independent of ε and δ. Summing (2.36) with $\beta = 1$ and $w = w^k$ and using (2.10), we find

$$
\begin{aligned}
&(1-C_f\delta)\int_\Omega h(u(w^j))dx + \delta\sum_{k=1}^j\int_\Omega \nabla w^k\cdot A(u(w^k))[D^2h(u(w^k))]^{-1}\nabla w^k dx \\
&+\varepsilon\delta C\sum_{k=1}^j\|w^k\|^2_{H^m(\Omega;R^n)} + \delta\sum_{k=1}^j\int_\Omega \nabla w^k\cdot B(u(w^k))\nabla u(w^{k-N})dx
\end{aligned}
\tag{2.38}
$$

$$\leq C_f \delta j |\Omega| + C_f \delta \sum_{k=1}^{j-1} \int_{\Omega} h(u(w^k))dx + \int_{\Omega} h(u(w^0))dx.$$

Same as (2.20) in the proof of Theorem 2.3, we need to show there exists $E_1 > 0$ independent of δ and ε, such that

$$\delta \sum_{k=1}^{j} \int_{\Omega} \nabla w^k \cdot A(u(w^k))[D^2 h(u(w^k))]^{-1} \nabla w^k dx + \varepsilon\delta \sum_{k=1}^{j} \|w^k\|^2_{H^m(\Omega;\mathbb{R}^n)} \leq E_1. \tag{2.39}$$

The proof of (2.39) is different from (2.20), and is also presented in Appendix.

Next we show a related uniform boundedness result for the approximate solution u^δ. Integrating (2.33) on $(0, T_1)$, we get

$$\begin{aligned}
&\frac{1}{\delta}\int_0^{T_1}\int_{\Omega} (u^\delta - \sigma_1 u^\delta) \cdot \varphi dxdt + \int_0^{T_1}\int_{\Omega} \nabla\varphi \cdot A(u^\delta)\nabla u^\delta dxdt \\
&+ \varepsilon \int_0^{T_1}\int_{\Omega} (\sum_{|\alpha|=m} D^\alpha w^\delta \cdot D^\alpha \varphi + w^\delta \cdot \varphi) dxdt + \int_0^{T_1}\int_{\Omega} \nabla\varphi \cdot B(u^\delta)\nabla(\sigma_N u^\delta) dxdt \\
&= \int_0^{T_1}\int_{\Omega} f(u^\delta)\cdot\varphi dxdt.
\end{aligned} \tag{2.40}$$

Here, u^δ, w^δ, σ_1 and σ_N are defined as in the proof of Theorem 2.3 by replacing $\mathbb{R}$ with $\mathbb{R}^n$. From (2.40), we have

$$\begin{aligned}
&\frac{1}{\delta}\left|\int_\delta^{T_1}\int_{\Omega} (u^\delta - \sigma_1 u^\delta)\cdot\varphi dxdt\right| \\
\leq& \|\nabla\varphi\|_{L^2((0,T_1);L^2(\Omega;\mathbb{R}^n))} \|A(u^\delta)\nabla u^\delta\|_{L^2((0,T_1);L^2(\Omega;\mathbb{R}^n))} \\
&+ \varepsilon \|w^\delta\|_{L^2((0,T_1);H^m(\Omega;\mathbb{R}^n))} \|\varphi\|_{L^2((0,T_1);H^m(\Omega;\mathbb{R}^n))} \\
&+ \|\nabla\varphi\|_{L^2((0,T_1);L^2(\Omega;\mathbb{R}^n))} \|B(u^\delta)\nabla(\sigma_N u^\delta)\|_{L^2((0,T_1);L^2(\Omega;\mathbb{R}^n))} \\
&+ \|f(u^\delta)\|_{L^2((0,T_1);L^2(\Omega;\mathbb{R}^n))} \|\varphi\|_{L^2((0,T_1);L^2(\Omega;\mathbb{R}^n))}.
\end{aligned} \tag{2.41}$$

On the other hand, we can also obtain a similar estimate as (2.23) from (2.39), which further implies by **(A1)** and **(A2)** that

$$\begin{aligned}
&\|(A(u^\delta)\nabla u^\delta)_i\|_{L^2((0,T_1);L^2(\Omega;\mathbb{R}))} \\
\leq& \sum_{j=1}^{n} \|a_{ij}(u^\delta)\|_{L^\infty((0,T_1);L^\infty(\Omega;\mathbb{R}))} \|\nabla u_j^\delta\|_{L^2((0,T_1);L^2(\Omega;\mathbb{R}^d))} \leq E_5,
\end{aligned} \tag{2.42}$$

$$\|(B(u^\delta)\nabla\sigma_N u^\delta)_i\|_{L^2((0,T_1);L^2(\Omega;\mathbb{R}))} \le E_6, \quad i=1,\cdots,n,$$

for some constants $E_5, E_6 > 0$ independent δ and ε. Combining (2.39), (2.41) and (2.42), we conclude that there exists a constant $E_7 > 0$ independent of δ and ε such that

$$\delta^{-1}\|u^\delta - \sigma_1 u^\delta\|_{L^2((0,T_1);H^m(\Omega;\mathbb{R}^n)')} \le E_7. \tag{2.43}$$

Finally, by taking the limit $(\delta,\varepsilon)\to 0$ and using (2.23) and (2.43), we know (2.25) also holds. Since $A(u^\delta)$ and $B(u^\delta) \in C^0(\overline{\mathcal{D}};\mathbb{R}^{n\times n})$, we have $a_{ij}(u^\delta)\to a_{ij}(u)$ and $b_{ij}(u^\delta)\to b_{ij}(u)$ in $L^2((0,T_1);L^p(\Omega;\mathbb{R}^n))$. This implies

$$(B(u^\delta)\nabla(\sigma_N u^\delta))_i = \sum_{j=1}^n b_{ij}(u^\delta)\nabla(\sigma_N u_j^\delta) \rightharpoonup \sum_{j=1}^n b_{ij}(u)(\nabla u_\tau)_j = (B(u)\nabla u_\tau)_i \ \ i=1,\cdots,n,$$

$$(A(u^\delta)\nabla u^\delta)_i = \sum_{j=1}^n a_{ij}(u^\delta)\nabla u^\delta \rightharpoonup \sum_{j=1}^n a_{ij}(u)\nabla u_j = (A(u)\nabla u)_i \ \ i=1,\cdots,n,$$

in $L^2((0,T_1);L^2(\Omega;\mathbb{R}))$. Therefore we have

$$\begin{aligned}&\int_0^{T_1}\langle\partial_t u,\varphi\rangle dt + \int_0^{T_1}\int_\Omega \nabla\varphi\cdot A(u)\nabla u dxdt + \int_0^{T_1}\int_\Omega \nabla\varphi\cdot(B(u)\nabla u_\tau)dxdt\\ &= \int_0^{T_1}\int_\Omega f(u)\cdot\varphi dxdt,\end{aligned}$$

for all $\varphi\in L^2((0,T_1);H^1(\Omega;\mathbb{R}^n))$. □

Corollary 2.9. *Assume $\tau>0$, $d_3,d_4>0$, $a_i,m_i>0$, $i=1,2$, $|d_5|\le\dfrac{d_3}{2a_1\bar a}$ and $|d_6|\le\dfrac{d_4}{2a_2\bar a}$, where $\bar a=\max\{a_1,a_2\}$. Then (2.31) possesses a bounded weak solution $u(x,t)$ such that $u(x,t)\in[0,a_1]\times[0,a_2]$, for $x\in\Omega$ and $t>0$.*

2.3. The case of partial functional differential equations

In this part, we show that the method of entropy can be also used to prove the global boundedness of the following system

$$\begin{cases}\frac{\partial u}{\partial t} = \nabla(A(u)\nabla u) + f(u,u_\sigma), & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial \mathbf{n}} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,t)=\phi(x,t), & x\in\Omega,\ -\tau\le t\le 0.\end{cases} \tag{2.44}$$

Compared to the classic partial functional differential equations considered in [11–13,24,30,35], the diffusion matrix $A(u)$ in (2.44) may take a more general form than a constant diagonal matrix. Given any $a>0$ and let $\mathcal{D}\subseteq(0,a)^n$. Assume

(P0) $f(u,v) \in C^0(\overline{\mathcal{D}} \times \overline{\mathcal{D}}; \mathbb{R}^n)$. There exists a nonnegative bounded convex C^2 function $h: \mathcal{D} \to \mathbb{R}^+$ such that its gradient $\nabla h : \mathcal{D} \to \mathbb{R}^n$ is invertible. In addition, there exists $C_f > 0$, such that

$$f(u,v) \cdot \nabla h(u) \le C_f(1 + h(u)), \quad \forall\, u, v \in \mathcal{D}. \tag{2.45}$$

(P1) $A(u) \in C^0(\overline{\mathcal{D}}; \mathbb{R}^{n\times n})$, and for $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$, $u = (u_1, \cdots, u_n) \in \mathcal{D}$, it holds

$$z \cdot D^2 h(u) A(u) z \ge \sum_{i=1}^{n} \alpha_i u_i^2 z_i^2,$$

for either $\alpha_i(u_i) = \beta_i u_i^{m_i - 1}$ or $\alpha_i(u_i) = \beta_i (a - u_i)^{m_i - 1}$ with some $\beta_i > 0$ and $m_i > 0$, $i = 1, \cdots, n$. Furthermore, $\exists \gamma > 0$ such that $|A_{ij}(u)| \le \gamma |\alpha_j(u_j)|$ for $i, j = 1, \cdots, n$;

(P2) The initial functions $\phi(x,t) \in \mathcal{D}$, and

$$\phi_i(x,t) \in C^{0,0}(\overline{\Omega} \times [-\tau, 0]), \ \ i = 1, 2, \cdots, n.$$

The hypotheses (**P0**) and (**P1**) on $h(u)$ and $A(u)$ are exactly the same as (**H1**), (**H2′**) and (**H2″**) in [19], except that the estimation of $f(u) \cdot \nabla h(u)$ is now replaced by (2.45). The assumption on $f(u,v)$ in (2.45) is about the condition on the delayed term in f. It can be verified that the functions $f(u) = u(a - u - u_\sigma)$ in [30] and $f(u) = u(a - u - \int_{-1}^{0} u(t + r(s), x) d\eta(s))$ in [17] all satisfy (2.45) with h given by (2.4). The weak solution of (2.27) can be similarly defined as in Definition 2.7 by letting $B = 0$.

Theorem 2.10. *Assume (**P0**)-(**P2**) hold. Then* (2.44) *has a bounded weak solution $u(x,t)$ such that $u(x,t) \in \overline{\mathcal{D}}$ for $x \in \Omega$, $t > 0$.*

Proof. From (**P0**), we know that the analogous version of (2.36) reads as

$$\begin{aligned}
&\beta \int_\Omega h(u(w^k))dx + \delta \int_\Omega \nabla w^k \cdot A(u(w^k))[D^2 h(u(w^k))]^{-1} \nabla w^k dx + \\
&\delta\varepsilon \int_\Omega \Big(\sum_{|\alpha| = m} |D^\alpha w^k|^2 + |w^k|^2\Big)dx \le C_f \delta\beta \int_\Omega (1 + h(u(w^k)))dx + \beta \int_\Omega h(u(w^{k-1}))dx.
\end{aligned}$$

The rest of the proof are the same as the one of Theorem 2.8. □

3. Discussion

In this paper, we studied a general partial differential equations involving time delays, i.e. (1.1), and proved the global boundedness of weak solutions in three different cases, namely (2.1), (2.27) and (2.44), using the entropy method proposed in [19]. This seems to be the first attempt to apply the entropy method to partial differential equation with time delay, particularly for the case that diffusions and time delays are strongly coupled (due to the term $\nabla \cdot (B(u)\nabla u_\tau)$).

In the first two cases, $\sigma = 0$ is assumed, but it can be easily seen that the proofs are also valid when $0 \ne \sigma = \tau$, as long as the assumptions of f in (2.3) and (2.28) are replaced by

(2.45). If $\sigma \neq \tau \neq 0$, without loss of generality, assume that $\tau > \sigma$. Let $\delta = \frac{\tau}{N}$. Then, the term u_σ usually not lies on the meshed points, when we apply implicit Euler discretization to (1.1). For this reason, one can use the linear interpolation $hu(w^{k-\lceil\frac{\sigma}{\delta}\rceil}) + (1-h)u(w^{k-\lfloor\frac{\sigma}{\delta}\rfloor})$ for any $h \in (0,1)$ to approximate u_σ, which makes the proof remain valid. In addition, all the results in Section 2 for the interval $(0, a_i)$ can be extended to any bounded interval (p_i, q_i) with $p_i, q_i > 0$ $(i = 1, \cdots, n)$. For instance, Theorem 2.8 also holds if in (**A1**) it is assumed that there exists some constant $\alpha_i > 0$ $(i = 1, \cdots, n)$ such that $z \cdot D^2h(u)A(u)z \geq \sum_{i=1}^n \frac{\alpha_i z_i^2}{q_i - u_i}$ or $z \cdot D^2h(u)A(u)z \geq \sum_{i=1}^n \frac{\alpha_i z_i^2}{u_i - p_i}$, and $\sum_{j=1}^n |e_{ij}| \leq \frac{\alpha_i}{2(q_i - p_i)}$, $i = 1, \cdots, n$, $\sum_{i=1}^n |e_{ij}| \leq \frac{\alpha_j}{2(q_j - p_j)}$, $j = 1, \cdots, n$.

For now, only the global boundedness of weak solution of (1.1) can be shown, as we know that even for the case of $B(u) \equiv 0$, the global boundedness of strong solution is obtained for some special cases of $A(u)$ [11–13,24,30,35]. Moreover, conditions on $B(u)$ imposed here also need to be improved, since they are not satisfied for the models with $g(u) = u$ in [33]. These will be considered in the future work.

Appendix A

Proof of (2.20). Denote $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|_{L^2(\Omega;\mathbb{R}^d)}$ for ease of notation, and suppose $\delta < \frac{1}{C_f}$ in the proof. By (**H0**) and the boundedness of domain Ω, we can infer that $\delta j < T_1$ and

$$C_f \delta j |\Omega| + C_f \delta \sum_{k=1}^{j-1} \int_\Omega h(u(w^k))dx + \int_\Omega h(u(w^0))dx \leq C_2, \tag{A.1}$$

for some $C_2 > 0$ which is independent of δ and ε. If $d_2\delta \sum_{k=1}^{j} \int_\Omega \nabla w^k \cdot g(u(w^k))\nabla u(w^{k-N})dx > 0$ for all small δ and ε, then (2.20) is obviously true by (2.19).

If $d_2\delta \sum_{k=1}^{j} \int_\Omega \nabla w^k \cdot g(u(w^k))\nabla u(w^{k-N})dx$ is negative for some small δ and ε, we will show that this term cannot go to negative infinity by an argument of contradiction. Suppose

$$d_2\delta \sum_{k=1}^{N} \int_\Omega \nabla w^k \cdot g(u(w^k))\nabla u(w^{k-N})dx \to -\infty, \tag{A.2}$$

as δ and ε tend to 0. For $k \leq N$, by (**H1**), we have $\|\nabla u(w^{k-N})\|_{L^2(\Omega)} = \|\nabla\phi(x, (k-N)\delta)\|_{L^2(\Omega)} < D$ for some $D > 0$ independent of δ and ε. Using (2.3), the fourth term in (2.19) can be estimated as:

$$\begin{aligned} &\left| d_2\delta \sum_{k=1}^{N} \int_\Omega \nabla w^k \cdot g(u(w^k))\nabla u(w^{k-N})dx \right| \\ &\leq d_2 C_g \delta \sum_{k=1}^{N} \|\nabla u(w^k)\|_{L^2(\Omega)} \|\nabla u(w^{k-N})\|_{L^2(\Omega)} \\ &\leq d_2 C_g D \delta \sum_{k=1}^{N} \|\nabla u(w^k)\|_{L^2(\Omega)}. \end{aligned} \tag{A.3}$$

This, together with (A.2), implies there exists a sufficiently large $G > D$ such that

$$\|\nabla u(w^k)\|_{L^2(\Omega)} > G, \tag{A.4}$$

for some $k \le N$. According to (2.2), the second term in (2.19) can be estimated as follows:

$$\begin{aligned}
&\delta d_1 \sum_{k=1}^{N} \int_{\Omega} \nabla w^k \cdot [h''(u(w^k))]^{-1} \nabla w^k dx = \delta d_1 \sum_{k=1}^{N} \int_{\Omega} \nabla u(w^k) \cdot h''(u(w^k)) \nabla u(w^k) dx \\
\ge& \delta d_1 \sum_{k=1}^{N} \int_{\Omega} \frac{|\nabla u(w^k)|^2}{u(w^k)} dx \ge \delta d_1 \sum_{k=1}^{N} \int_{\Omega} \frac{|\nabla u(w^k)|^2}{a} dx = \frac{\delta d_1}{a} \sum_{k=1}^{N} \|\nabla u(w^k)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{A.5}$$

Subtracting the right hand side of (A.5) and (A.3), and then using (A.4), we have

$$\begin{aligned}
&\sum_{\{k: \|\nabla u(w^k)\| \ge G\}} \left(\frac{\delta d_1}{a} \|\nabla u(w^k)\|^2_{L^2(\Omega)} - d_2 C_g \delta D \|\nabla u(w^k)\|_{L^2(\Omega)} \right) \\
> & \sum_{\{k: \|\nabla u(w^k)\| < G\}} \left| \frac{\delta d_1}{a} \|\nabla u(w^k)\|^2_{L^2(\Omega)} - d_2 C_g \delta D \|\nabla u(w^k)\|_{L^2(\Omega)} \right| + C_2,
\end{aligned} \tag{A.6}$$

for some small δ and ε. Combining (A.3), (A.5) and (A.6), we get

$$\begin{aligned}
&d_1 \delta \sum_{k=1}^{N} \int_{\Omega} \nabla w^k \cdot [h''(u(w^k))]^{-1} \nabla w^k dx + d_2 \delta \sum_{k=1}^{N} \int_{\Omega} \nabla w^k \cdot g(u(w^k)) \nabla u(w^{k-N}) dx \\
\ge& \frac{d_1 \delta}{a} \sum_{k=1}^{N} \|\nabla u(w^k)\|^2_{L^2(\Omega)} - d_2 C_g \delta D \sum_{k=1}^{N} \|\nabla u(w^k)\|_{L^2(\Omega)} > C_2,
\end{aligned} \tag{A.7}$$

which contradicts with (2.19) and (A.1) with $j = N$. Therefore, there exists $D_1 > 0$, independent of δ and ε, such that

$$\left| d_2 \delta \sum_{k=1}^{N} \int_{\Omega} \nabla w^k \cdot g(u(w^k)) \nabla u(w^{k-N}) dx \right| \le D_1.$$

This, together with (2.19), (A.1) and (A.5), will further imply

$$\delta \sum_{k=1}^{N} \|\nabla u(w^k)\|^2_{L^2(\Omega)} \le D_2, \tag{A.8}$$

for some large $D_2 > 0$, which is also independent of δ and ε.

For $N < k \le 2N$, from (2.3), (A.8), Hölder and Schwarz inequalities, we can infer that

$$\begin{aligned}
&\left| d_2\delta \sum_{k=N+1}^{2N} \int_\Omega \nabla w^k \cdot g(u(w^k))\nabla u(w^{k-N})dx \right| \\
\le & d_2 C_g \delta \sum_{k=N+1}^{2N} \int_\Omega |\nabla u(w^k) \cdot \nabla u(w^{k-N})|dx \\
\le & d_2 C_g \delta \left(\sum_{k=N+1}^{2N} \|\nabla u(w^k)\|^2_{L^2(\Omega)} \sum_{k=N+1}^{2N} \|\nabla u(w^{k-N})\|^2_{L^2(\Omega)} \right)^{1/2} \\
\le & d_2 C_g \left(D_2\delta \sum_{k=N+1}^{2N} \|\nabla u(w^k)\|^2_{L^2(\Omega)} \right)^{1/2}.
\end{aligned} \tag{A.9}$$

Similarly, from (A.5), we know

$$d_1\delta \sum_{k=N+1}^{2N} \int_\Omega \nabla w^k \cdot [h''(u(w^k))]^{-1} \nabla w^k dx \ge \frac{\delta d_1}{a} \sum_{k=N+1}^{2N} \|\nabla u(w^k)\|^2_{L^2(\Omega)}. \tag{A.10}$$

So, if

$$d_2\delta \sum_{k=N+1}^{2N} \int_\Omega \nabla w^k \cdot g(u(w^k))\nabla u(w^{k-N})dx \to -\infty,$$

then, by a similar argument as above, we can also deduce from (2.19), (A.9) and (A.10) that

$$\begin{aligned}
& d_1\delta \sum_{k=N+1}^{2N} \int_\Omega \nabla w^k \cdot [h''(u(w^k))]^{-1} \nabla w^k dx + d_2\delta \sum_{k=N+1}^{2N} \int_\Omega \nabla w^k \cdot g(u(w^k))\nabla u(w^{k-N})dx \\
\ge & d_1\delta \sum_{k=N+1}^{2N} \int_\Omega \frac{|\nabla u(w^k)|^2}{a} dx - d_2 C_g \left(D_2\delta \sum_{k=N+1}^{2N} \|\nabla u(w^k)\|^2_{L^2(\Omega)} \right)^{1/2} > C_2,
\end{aligned}$$

which is also a contradiction to (2.19) and (A.1) with $j = 2N$. Thus, there exists $D_3 > 0$, independent of δ and ε, such that

$$\left| d_2\delta \sum_{k=N+1}^{2N} \int_\Omega \nabla w^k \cdot g(u(w^k))\nabla u(w^{k-N})dx \right| \le D_3.$$

Repeating the above process for $k > 2N$, one can show that, for any $l \in \mathbb{N}$ with $2 \le l$ and $(l+1)N \le N_1$,

$$\left| d_2\delta \sum_{k=lN+1}^{(l+1)N} \int_\Omega \nabla w^k \cdot g(u(w^k))\nabla u(w^{k-N})dx \right| \le D_l,$$

for some $D_l > 0$. Hence, $d_2\delta \sum_{k=1}^{j} \int_\Omega \nabla w^k \cdot g(u(w^k))\nabla u(w^{k-N})dx$ is bounded from below for any δ and ε, and therefore (2.20) also follows directly from (2.19). □

Proof of (2.39). Note that $j\delta \le N_1\delta = T_1$ and Ω is bounded. Then, from assumption **(A0)**, there exists a constant $E_2 > 0$, independent of δ and ε, such that

$$C_f\delta j|\Omega| + C_f\delta \sum_{k=1}^{j-1} \int_\Omega h(u(w^k))dx + \int_\Omega h(u(w^0))dx \le E_2. \tag{A.11}$$

If $\delta \sum_{k=1}^{j} \int_\Omega \nabla w^k \cdot B(u(w^k))\nabla u(w^{k-N})dx > 0$ for all small $\delta > 0$ and $\varepsilon > 0$, then (2.39) is obviously true by (2.38). If $\delta \sum_{k=1}^{j} \int_\Omega \nabla w^k \cdot B(u(w^k))\nabla u(w^{k-N})dx$ is negative for some $\delta > 0$ and $\varepsilon > 0$, we will show that this term cannot go to negative infinity by an argument of contradiction.

Suppose

$$\delta \sum_{k=1}^{j} \int_\Omega \nabla w^k \cdot B(u(w^k))\nabla u(w^{k-N})dx \to -\infty, \tag{A.12}$$

as δ and ε vary. From **(A1)** and **(A2)**, the second and fourth terms with $j = N$ in (2.38) can be estimated as:

$$\begin{aligned} &\delta \sum_{k=1}^{N} \int_\Omega \nabla w^k \cdot A(u(w^k))[D^2h(u(w^k))]^{-1}\nabla w^k dx \\ =&\delta \sum_{k=1}^{N} \int_\Omega \nabla u(w^k) \cdot D^2h(u(w^k))A(u(w^k))\nabla u(w^k)dx \\ \ge&\delta \sum_{k=1}^{N}\sum_{i=1}^{n} \int_\Omega \frac{|\nabla u_i(w^k)|^2}{u_i(w^k)}dx \ge \frac{\delta}{a_i}\sum_{k=1}^{N}\sum_{i=1}^{n} \int_\Omega |\nabla u_i(w^k)|^2 dx, \end{aligned} \tag{A.13}$$

and

$$\begin{aligned} &|\delta \sum_{k=1}^{N} \int_\Omega \nabla w^k \cdot B(u(w^k))\nabla u(w^{k-N})dx| \\ \le&\delta \sum_{k=1}^{N}\sum_{i,l=1}^{n} |e_{il}| \int_\Omega |\nabla u_i(w^k)||\nabla u_l(w^{k-N})|dx \end{aligned} \tag{A.14}$$

$$\leq \delta \sum_{k=1}^{N} \sum_{i,l=1}^{n} D|e_{il}| \int_{\Omega} |\nabla u_i(w^k)| dx \leq \delta \sum_{k=1}^{N} \sum_{i=1}^{n} \frac{D}{2a_i} \int_{\Omega} |\nabla u_i(w^k)| dx,$$

where $D = \sup_{x \in \Omega, t \in [-\tau, 0]} [|\nabla \phi_1(x,t)|, ..., |\nabla \phi_n(x,t)|]$ is a constant. By a similar argument as the proof of (A.7), we can conclude that if $\delta \sum_{k=1}^{N} \int_{\Omega} \nabla w^k \cdot B(u(w^k)) \nabla u(w^{k-N}) dx \to -\infty$, it then follows from (A.13) and (A.14) that

$$\begin{aligned} &\delta \sum_{k=1}^{N} \int_{\Omega} \nabla w^k \cdot A(u(w^k))[D^2 h(u(w^k))]^{-1} \nabla w^k dx \\ &+ \delta \sum_{k=1}^{N} \int_{\Omega} \nabla w^k \cdot B(u(w^k)) \nabla u(w^{k-N}) dx \\ \geq &\frac{\delta}{2a_i} \sum_{k=1}^{N} \sum_{i=1}^{n} \int_{\Omega} (|\nabla u_i(w^k)|^2 - D|\nabla u_i(w^k)|) dx > E_2, \end{aligned} \tag{A.15}$$

which contradicts with (2.38) with $j = N$ and (A.11). Therefore, by (A.12), we must have

$$\delta \sum_{k=N+1}^{j} \int_{\Omega} \nabla w^k \cdot B(u(w^k)) \nabla u(w^{k-N}) dx \to -\infty. \tag{A.16}$$

Furthermore, combining (2.38) with $j = N$ and (A.13), we have

$$\frac{\delta}{2a_i} \sum_{k=1}^{N} \sum_{i=1}^{n} \int_{\Omega} |\nabla u_i(w^k)|^2 dx + \delta \sum_{k=1}^{N} \int_{\Omega} \nabla w^k \cdot B(u(w^k)) \nabla u(w^{k-N}) dx \geq E_3, \tag{A.17}$$

for some constant E_3, independent of δ and ε.

Using **(A1)** again, we obtain

$$\begin{aligned} &2 \left| \delta \sum_{k=N+1}^{j} \int_{\Omega} \nabla w^k \cdot B(u(w^k)) \nabla u(w^{k-N}) dx \right| \\ \leq &2\delta \sum_{k=N+1}^{j} \left| \int_{\Omega} \nabla u(w^k) \cdot D^2 h(u(w^k)) B(u(w^k)) \nabla u(w^{k-N}) dx \right| \\ \leq &2\delta \sum_{k=N+1}^{j} \sum_{i,l=1}^{n} |e_{il}| \int_{\Omega} |\nabla u_i(w^k)||\nabla u_l(w^{k-N})| dx \\ \leq &\frac{\delta}{2a_i} \left(\sum_{k=N+1}^{j} \sum_{i=1}^{n} \int_{\Omega} |\nabla u_i(w^k)|^2 dx + \sum_{k=1}^{j-N} \sum_{i=1}^{n} \int_{\Omega} |\nabla u_i(w^k)|^2 dx \right), \end{aligned} \tag{A.18}$$

for $N < j \le N_1$. This, together with (A.16), further implies that

$$\begin{aligned}&\frac{\delta}{2a_i}\left(\sum_{k=N+1}^{j}\sum_{i=1}^{n}\int_{\Omega}|\nabla u_i(w^k)|^2dx+\sum_{k=1}^{j-N}\sum_{i=1}^{n}\int_{\Omega}|\nabla u_i(w^k)|^2dx\right)\\&-\left|\delta\sum_{k=N+1}^{j}\int_{\Omega}\nabla w^k\cdot B(u(w^k))\nabla u(w^{k-N})dx\right|\gg E_2.\end{aligned}\tag{A.19}$$

On the other hand, from (A.13) with $N=j$, we have

$$\begin{aligned}&\delta\sum_{k=1}^{j}\int_{\Omega}\nabla w^k\cdot A(u(w^k))[D^2h(u(w^k))]^{-1}\nabla w^kdx\ge\frac{\delta}{2a_i}\sum_{k=1}^{N}\sum_{i=1}^{n}\int_{\Omega}|\nabla u_i(w^k)|^2dx\\&+\frac{\delta}{2a_i}\left(\sum_{k=N+1}^{j}\sum_{i=1}^{n}\int_{\Omega}|\nabla u_i(w^k)|^2dx+\sum_{k=1}^{j-N}\sum_{i=1}^{n}\int_{\Omega}|\nabla u_i(w^k)|^2dx\right).\end{aligned}\tag{A.20}$$

Thus, from (A.17), (A.19) and (A.20), we get

$$\begin{aligned}&\delta\sum_{k=1}^{j}\int_{\Omega}\nabla w^k\cdot A(u(w^k))[D^2h(u(w^k))]^{-1}\nabla w^kdx\\&+\delta\sum_{k=1}^{j}\int_{\Omega}\nabla w^k\cdot B(u(w^k))\nabla u(w^{k-N})dx>E_2,\end{aligned}\tag{A.21}$$

which also contradicts with (2.38) and (A.11). So, (A.12) is not valid, and therefore,

$$\left|\delta\sum_{k=1}^{j}\int_{\Omega}\nabla w^k\cdot B(u(w^k))\nabla u(w^{k-N})dx\right|\le E_4,\tag{A.22}$$

for some $E_4>0$, independent of δ and ε. As a consequence of (2.38), (A.11) and (A.22), the inequality (2.39) follows. □

Data availability

No data was used for the research described in the article.

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