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# Local and global bifurcation analysis of density-suppressed motility model ☆

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ABSTRACT

In this paper, we study a density-suppressed motility reaction-diffusion population model with Dirichlet boundary conditions in spatially heterogeneous environments. We establish the existence of local-in-time classical solutions and apply local bifurcation theory to identify a positive bifurcation point for steady-state solutions. The existence of non-constant positive steady-state solutions is obtained, and it is shown that the bifurcation direction of the bifurcation curve can be either forward or backward, which is determined by the density-suppressed diffusion term. Furthermore, the boundedness of non-constant positive steady-state solutions is obtained by the comparison principle, and the boundedness of solutions implies that the bifurcation branches from local bifurcation can be extended globally, hence a global bifurcation diagram is derived rigorously. Finally, numerical simulations verify our theoretical results and demonstrate the effect of spatial heterogeneity on pattern formation.



## 1. Introduction

Reaction-diffusion equations have been widely used in modeling movement and growth phenomena in biology, chemistry, physics and other fields. They are one of the types of mathematical models used to describe the formation of various complex spatio-temporal patterns. As early as 1952, Turing [50] creatively used reaction-diffusion models to describe the generation of spatial patterns in nature. In recent years, many variations of reaction-diffusion systems have also been proposed as mechanisms of spatial-temporal pattern formation. In addition to random diffusion, other more complex movement models such as chemotaxis

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motion, advection motion, cross-diffusion and memory-based diffusion have been proposed [9,24,34,35,43,46] and some of them are also able to drive the formation of spatial-temporal patterns. For memory-based diffusion, the diffusive movement of the population depends on its own density at present or past times, while in chemotaxis and cross-diffusion, the population movement also depends on the population concentration of other species.

Liu et al. [28] used the method of synthetic biology to introduce density-suppressed motility for cell population through a so-called "self-trapping" mechanism. *E. coli* cells (Escherichia coli cells) changes in run-and-tumble states of motion through the change of the concentration of signal molecule acyl-homoserine lactone (AHL), in which AHL is secreted by *E. coli* cells. At low AHL concentration, *E. coli* cells run, otherwise *E. coli* cells tumble. See [15,28] for more detailed biological explanation. In [28] a three-component reaction-diffusion system for bacterial density, AHL concentration and nutrient density was proposed to analyze the stripe patterns obtained in the experiment. However, it is more difficult to analyze the three-component model, and there are relatively few research literature in that direction [21]. Fu et al. [15] used the following simplified two-component model to analyze the essential characteristics of stripe formation:

$$\begin{cases} u_t = \Delta(\gamma(v)u) + \sigma u(1-u), & x \in \Omega,\ t > 0, \\ v_t = d\Delta v + u - v, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ represent the population density of *E. coli* cells and the concentration of AHL, respectively. The function $\gamma(v)$, characterizing the nonlinear diffusion, satisfies $\gamma'(v) < 0$, indicating that the diffusion of *E. coli* cells is suppressed by the concentration of AHL. Here, $d > 0$ denotes the random diffusion rate of signal molecule AHL; $\sigma > 0$ represents the intrinsic growth rate of *E. coli* cells, and *E. coli* cells grow in a logistic growth rate; $\Omega$ is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$, subject to Neumann (no-flux) boundary conditions ensuring a closed system.

Note that the first equation of (1.1) can be rewritten as

$$u_t = \nabla \cdot (\gamma(v)\nabla u) + \nabla \cdot (u\gamma'(v)\nabla v) + \sigma u(1-u),$$

so model (1.1) can be regarded as a Keller–Segel type model [24] with logistic growth for cells. When $\gamma(v) = c$ (where $c > 0$ is a positive constant), it reduces to the well-known Fisher-KPP equation ($\sigma > 0$). When $\sigma = 0$, the global existence of classical or weak solutions of Eq. (1.1) has been obtained when $\gamma(v)$ satisfies certain conditions, see [1,14,16,19,36,48,56] for more details, and in [17,22,49], the possible blow-up phenomenon of solutions was discussed under certain parameter conditions. When $\sigma > 0$, Jin et al. [20] derived the global existence of classical solutions of Eq. (1.1) with a uniform-in-time bound in a bounded smooth domain $\Omega \subset \mathbb{R}^2$ with homogeneous Neumann boundary conditions when $\gamma(v)$ satisfies the following assumptions:

**(A1)** $\gamma(v) \in C^3([0,\infty))$, $\gamma(v) > 0$, and $\gamma'(v) < 0$ on $[0,\infty)$, $\lim_{v\to\infty} \gamma(v) = 0$ and $\lim_{v\to\infty} \dfrac{\gamma'(v)}{\gamma(v)}$ exists.

Furthermore, pattern formation does not occur if $\sigma > \frac{K_0}{16}$ with $K_0 = \max_{0\leq v\leq\infty} \frac{|\gamma'(v)|^2}{\gamma(v)}$ and the constant steady-state of Eq. (1.1) is globally asymptotically stable [20]. In [51], the global existence and boundedness of the solution of Eq. (1.1) in an $n$-dimensional case with $n \geq 3$ were established. Pattern formation and traveling wave solutions in the density-suppressed motility model can be found in [37,54,23,47,26]. The results in [37] did not require the condition '$\lim_{v\to\infty} \frac{\gamma'(v)}{\gamma(v)}$ exists' and the authors proved the non-existence of non-constant steady-state solutions of Eq. (1.1) for large $\sigma d$ or when $\sigma d$ is sufficiently small and $\gamma(v)$

satisfying some additional conditions, and for moderate value of $\sigma d$, non-constant positive solutions may exist. Wang and Xu [54] explored the existence and asymptotic profiles of non-constant steady-state solutions under some conditions in one dimension with Neumann boundary conditions, and they also showed the monotonicity of solutions when $\sigma = 0$. Recently, Xiang and Zhou [55] showed that a chemotaxis-Navier-Stokes system with density-suppressed motility admits a global classical solution and removing the smallness assumption in [32].

The dynamics of density-suppressed motility model are less understood when the Neumann boundary conditions in Eq. (1.1) are replaced by the Dirichlet boundary conditions that represent a hostile external environment. To study the non-uniform distribution of bacterial population in the environment, we consider the following density-suppressed motility population model with Dirichlet boundary conditions in a spatially heterogeneous environment:

$$\begin{cases} u_t = \Delta(\gamma(v)u) + \sigma u(m(x) - u), & x \in \Omega,\ t > 0, \\ v_t = d\Delta v + u - v, & x \in \Omega,\ t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq (\not\equiv) 0, v(x,0) = v_0(x) \geq (\not\equiv) 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $u(x,t),\ v(x,t)$ represent the population density of *E. coli* cells and the concentration of AHL, respectively. The constant $d > 0$ is the random diffusion rate of signal molecule AHL. The parameter $\sigma > 0$ is a constant providing a scaling for the bacteria growth rate. The function $m(x)$ represents a spatially heterogeneous intrinsic growth rate of *E. coli* cells; and the function $\gamma(v)$ indicates that the diffusion motility of *E. coli* cells is affected by AHL concentration. $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$ with $n \geq 1$. With these definitions, we make the following assumptions:

**(A2)** $\gamma(v) \in C^3([0,\infty))$, $\gamma(v) > 0$ and $\gamma'(v) < 0$ for any $v \in [0,\infty)$, and $\gamma(0) = r_0 > \lim_{v\to\infty} \gamma(v) = r_\infty > 0$,
**(A3)** $m(x) \in C^\alpha(\overline{\Omega})$ where $\alpha \in (0,1)$, and $m(x) > 0$ on $\overline{\Omega}$.

Zuo and Shi [57] considered steady-state solutions of a reaction-diffusion equation with spatiotemporal delay and homogeneous Dirichlet boundary conditions, which are equivalent to a system of reaction-diffusion equations. Note that (1.2) can also be rewritten as a scalar equation with a spatiotemporal delay in the diffusion rate of $u$. Li and Ma [27] considered positive steady-state solutions of diffusive predator-prey systems with prey-taxis under homogeneous Dirichlet boundary conditions, and they established the existence of steady-states and their limiting behavior. Other related studies on Dirichlet boundary value problems can be found in [4,10,41,45].

Bifurcation phenomenon exist widely in nature and have attracted great attention by biologists, physicists, chemists and mathematicians. Numerous studies have addressed local and global bifurcation problems, see [11,12,30,31,38,42,40,44,53] and the references therein. The "bifurcation from Simple Eigenvalue" theorem in [11] is commonly used about local bifurcation theorem, and global and unilateral bifurcation theorems were established in [13,33,38,44]. In this paper we employ both local and global bifurcation theorems to establish the existence and boundedness of steady-state solutions.

In this paper, we consider cross-diffusion system (1.2) caused by nonlinear motion function $\gamma(v)$, employ the bifurcation theorem to obtain the existence of non-constant steady-state solutions of system (1.2), and establish that whether the direction of the bifurcating solution branch is forward or backward depends on the size of $|\gamma'(0)|$. For sufficiently large $|\gamma'(0)|$ values, the backward bifurcation is likely to occur. This phenomenon results from nonlinear diffusion, which differs from the forward bifurcation typically observed in standard reaction-diffusion population models. Such backward bifurcations have been more frequently documented in bifurcations occurred in epidemiological models [18,39]. Furthermore, by constructing appropriate auxiliary functions and using comparison theorem, the upper and lower bounds of the positive solutions are obtained, and then the global bifurcation structure of the system (1.2) is given in Fig. 2. The possible backward bifurcation shows that the nonlinear diffusion of *E. coli* cells facilitated by the AHL concentration induces an effect analogous to the weak Allee effect found in reaction-diffusion population model [42,52]. Such backward bifurcation was also found in a scalar population model with density-dependent nonlinear diffusion depends on the species' own density [6,7,25].

This paper is organized as follows. In Section 2, we prove the existence of local-in-time classical solutions. In Section 3, we establish the existence and local asymptotic stability of non-constant positive steady-state solutions using local bifurcation theory; the global bifurcation structure is derived in Section 4 using global bifurcation theorems; numerical results and the impact of spatial heterogeneity are presented in Section 5; and concluding remarks are given in Section 6. Throughout this paper, we use the notation $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for $p > n$ and $Y = L^p(\Omega)$. The kernel (null space) and range of operator $L$ are denoted by $\mathcal{N}(L)$ and $\mathcal{R}(L)$, respectively.

## 2. Local existence

First we use the abstract theory of quasilinear parabolic systems developed by Amann [2,3] to prove the existence of local-in-time classical solutions of Eq. (1.2).

**Theorem 2.1** *(Local existence). Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with smooth boundary and assumptions **(A2)** and **(A3)** hold. Assume that the initial data $(u_0, v_0) \in [W^{1,p}(\Omega)]^2$ for $p > n$, $u_0, v_0 \geq (\not\equiv) 0$. Then there exists the maximal existence time $T_{max} \in (0, \infty]$ such that the system (1.2) has a unique classical solution $(u, v) \in [C([0, T_{max}) \times \overline{\Omega}) \cap C^{2,1}((0, T_{max}) \times \overline{\Omega})]^2$ satisfying $u, v > 0$ for all $t > 0$. Moreover,*

$$\text{either } T_{max} = \infty, \text{ or } \limsup_{t \nearrow T_{max}} (\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty}) = \infty. \tag{2.1}$$

**Proof.** Let $\omega = (u, v)^{\mathrm{T}}$, then system (1.2) can be rewritten as

$$\begin{cases} \omega_t = \nabla \cdot (a(w) \nabla \omega) + \Phi(\omega), & x \in \Omega,\ t > 0, \\ \omega = 0, & x \in \partial\Omega,\ t > 0, \\ \omega(\cdot, 0) = (u_0, v_0), & x \in \partial\Omega, \end{cases} \tag{2.2}$$

where

$$a(\omega) = \begin{pmatrix} \gamma(v) & \gamma'(v)u \\ 0 & d \end{pmatrix}, \ \Phi(\omega) = \begin{pmatrix} \sigma u(m(x) - u) \\ u - v \end{pmatrix}.$$

Applying Theorem 7.3 in [2], the local existence of solution $(u(x, t), v(x, t))$ can be obtained. We rewrite the first equation of system (1.2) as

$$\begin{cases} u_t = \Delta(\gamma(v)u) + \sigma u(m(x) - u), & x \in \Omega,\ t > 0, \\ u(x, t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x) \geq (\not\equiv) 0, & x \in \partial\Omega. \end{cases} \tag{2.3}$$

Applying the strong maximum principle to Eq. (2.3), $u(x, t) > 0$ can be obtained due to $u_0 \geq (\not\equiv) 0$. Similarly, $v(x, t) > 0$ for all $(x, t) \in \Omega \times (0, T_{max})$. Using the similar arguments as in [22, Lemma 2.1], we obtain (2.1) directly. This completes the proof of Theorem 2.1. □

## 3. Local bifurcation

In this section, we use the local bifurcation theorem in [11] to study the steady-state bifurcation of Eq. (1.2), the steady-state equation corresponding to Eq. (1.2) is the following elliptic system

$$\begin{cases} \Delta(\gamma(v)u) + \sigma u(m(x) - u) = 0, & x \in \Omega, \\ d\Delta v + u - v = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

We assume that $d > 0$ is fixed in this paper, and take $\sigma$ as the bifurcation parameter. We define a nonlinear mapping $F : \mathbb{R}^+ \times X^2 \to Y^2$ by

$$F(\sigma, u, v) = \begin{pmatrix} \Delta(\gamma(v)u) + \sigma u(m(x) - u) \\ d\Delta v + u - v \end{pmatrix}. \tag{3.2}$$

Since $(u, v) = (0, 0)$ is a trivial solution of system (3.1) for any $\sigma > 0$, then $F(\sigma, 0, 0) = 0$ holds for any $\sigma > 0$. Let $F_{(u,v)}$ denote the Fréchet derivative of $F$ with respect to $(u, v)$,

$$F_{(u,v)}(\sigma, u, v) \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{pmatrix} \Delta(\gamma(v)\phi + \gamma'(v)u\psi) + \sigma m(x)\phi - 2\sigma u\phi \\ d\Delta\psi + \phi - \psi \end{pmatrix},$$

then

$$F_{(u,v)}(\sigma, 0, 0) \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{pmatrix} r_0\Delta\phi + \sigma m(x)\phi \\ d\Delta\psi + \phi - \psi \end{pmatrix}. \tag{3.3}$$

Let $\lambda_1$ be the principal eigenvalue of the following eigenvalue problem

$$\begin{cases} \Delta\phi + \lambda m(x)\phi = 0, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{3.4}$$

From **(A3)**, we have $m(x) > 0$, then the eigenvalue problem (3.4) has a unique positive eigenvalue $\lambda_1$ such that the corresponding eigenfunction $\phi_1$ satisfies $\phi_1(x) > 0$ for all $x \in \Omega$, see [5,8]. Then

$$\sigma = \sigma_1 \equiv r_0\lambda_1 \tag{3.5}$$

is the unique bifurcation point for (3.1) where positive solutions of (3.1) bifurcate from the line of trivial solutions $\Gamma_0 = \{(\sigma, 0, 0) : \sigma > 0\}$. At $\sigma = \sigma_1$,

$$\mathcal{N}(F_{(u,v)}(\sigma_1, 0, 0)) = \text{span}\left\{ \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} \right\} \neq \{\mathbf{0}\},$$

where $\phi_1 > 0$ satisfies Eq. (3.4) with $\lambda = \lambda_1$, and

$$\psi_1 = (-d\Delta + I)^{-1}\phi_1 > 0. \tag{3.6}$$

It is easy to verify that

$$\mathcal{R}(F_{(u,v)}(\sigma_1, 0, 0)) = \left\{ (h_1, h_2)^{\mathrm{T}} \in Y^2 : \int_\Omega h_1\phi_1 \mathrm{d}x = 0 \right\}.$$

Clearly, $\dim(\mathcal{N}(F_{(u,v)}(\sigma_1,0,0))) = \mathrm{codim}(\mathcal{R}(F_{(u,v)}(\sigma_1,0,0))) = 1$. Moreover,

$$F_{\sigma(u,v)}(\sigma_1,u,v)\begin{bmatrix}\phi\\ \psi\end{bmatrix} = \begin{pmatrix} m(x)\phi - 2u\phi \\ 0\end{pmatrix},$$

then

$$F_{\sigma(u,v)}(\sigma_1,0,0)\begin{bmatrix}\phi_1\\ \psi_1\end{bmatrix} = \begin{pmatrix} m(x)\phi_1 \\ 0\end{pmatrix} \notin \mathcal{R}(F_{(u,v)}(\sigma_1,0,0))$$

as $\int\limits_\Omega m(x)\phi_1^2 \mathrm{d}x \neq 0$. We apply the Crandall-Rabinowitz Bifurcation from Simple Eigenvalue Theorem [11, Theorem 1.17], the solutions of $F(\sigma,u,v)=0$ near $(\sigma_1,0,0)$ are in a form of

$$\Gamma_1 = \{(\sigma(s),u(s),v(s)) : -\delta < s < \delta\},$$

where $\delta$ is a positive constant, and

$$\begin{cases} \sigma(s) = \sigma_1 + \sigma'(0)s + o(s), \\ u(s) = s\phi_1 + o(s), \\ v(s) = s\psi_1 + o(s). \end{cases}$$

By using equation (4.5) of in [40], the direction of the bifurcation curve $\Gamma_1$ is determined by the sign of the following formula,

$$\sigma'(0) = -\frac{\langle h, F_{(u,v)(u,v)}(\sigma_1,0,0)[\phi_1,\psi_1]^2\rangle}{2\langle h, F_{\sigma(u,v)}(\sigma_1,0,0)[\phi_1,\psi_1]\rangle}, \tag{3.7}$$

where $h \in (Y^2)^*$ satisfying $\mathcal{N}(h) = \mathcal{R}(F_{(u,v)}(\sigma_1,0,0))$, and

$$F_{(u,v)(u,v)}(\sigma,u,v)\left(\begin{bmatrix}\phi\\ \psi\end{bmatrix}, \begin{bmatrix}\Phi\\ \Psi\end{bmatrix}\right) = \begin{pmatrix} \Delta(\gamma'(v)\phi\Psi + \gamma'(v)\psi\Phi) - 2\sigma\phi\Phi \\ 0\end{pmatrix},$$

so

$$\langle h, F_{(u,v)(u,v)}(\sigma_1,0,0)[\phi_1,\psi_1]^2\rangle = \int\limits_\Omega \Delta(2\gamma'(0)\phi_1\psi_1)\phi_1 \mathrm{d}x - 2\sigma_1 \int\limits_\Omega \phi_1^3 \mathrm{d}x. \tag{3.8}$$

By using the Green's identity and Dirichlet boundary condition, we have

$$\begin{aligned} \int\limits_\Omega \Delta(\phi_1\psi_1)\phi_1 \mathrm{d}x &= \int\limits_\Omega \phi_1\psi_1\Delta(\phi_1)\mathrm{d}x + \int\limits_{\partial\Omega} \left(\frac{\partial(\phi_1\psi_1)}{\partial \mathbf{n}}\phi_1 - \frac{\partial\phi_1}{\partial\mathbf{n}}\phi_1\psi_1\right)\mathrm{d}\mathbf{S} \\ &= \int\limits_\Omega \phi_1\psi_1\left(-\frac{\sigma_1}{r_0}\phi_1\right)\mathrm{d}x \\ &= -\lambda_1 \int\limits_\Omega m(x)\phi_1^2\psi_1 \mathrm{d}x. \end{aligned} \tag{3.9}$$

Substituting (3.9), (3.8) into (3.7), one obtains that

$$\sigma'(0) = \frac{\lambda_1 \gamma'(0) \int_\Omega m(x)\phi_1^2 \psi_1 \mathrm{d}x + \lambda_1 r_0 \int_\Omega \phi_1^3 \mathrm{d}x}{\int_\Omega m(x)\phi_1^2 \mathrm{d}x}. \tag{3.10}$$

Therefore the sign of $\sigma'(0)$ in (3.10) can either be positive or negative as $\gamma'(0) < 0$ and $r_0 > 0$ from **(A2)**, while $m, \phi_1, \psi_1 > 0$. A forward bifurcation occurs when $\sigma'(0) > 0$ and non-constant positive solutions of (3.1) exist for $\sigma \in (\sigma_1, \sigma_1 + \varepsilon)$, and a backward bifurcation occurs when $\sigma'(0) < 0$ so non-constant positive solutions of (3.1) exist for $\sigma \in (\sigma_1 - \varepsilon, \sigma_1))$ (see Fig. 2). Note that $r_0$ and $\gamma'(0)$ are independent of $m, \phi_1, \psi_1$, so for fixed $r_0 > 0$, the bifurcation is backward (forward) if $|\gamma'(0)|$ is larger (smaller).

To summarize what we have proved, we obtain the following local bifurcation theorem for system (3.1).

**Theorem 3.1.** *Suppose assumptions **(A2)** and **(A3)** hold. Let $\sigma_1, \phi_1, \psi_1$ be defined in Eq.* (3.5), (3.4) *and* (3.6) *respectively. Then*

1. *$\sigma = \sigma_1$ is the unique bifurcation point of system* (3.1) *at which positive solutions bifurcate from the line of trivial solutions $\Gamma_0 = \{(\sigma, 0, 0) : \sigma > 0\}$.*
2. *Let $Z$ be any complement of $\mathcal{N}(F_{(u,v)}(\sigma_1, 0, 0))$ in $X$, then there exist an open interval $I = (-\delta, \delta)$ and continuous functions $\sigma : I \to \mathbb{R}$, $z_1 : I \to Z$, $z_2 : I \to Z$ such that $\sigma(0) = \sigma_1$, $z_1(0) = 0$, $z_2(0) = 0$, and, if $u(s) = s\phi_1 + sz_1(s)$, $v(s) = s\psi_1 + sz_2(s)$, for $s \in I$, then $F(\sigma(s), u(s), v(s)) = 0$. Moreover, $F^{-1}(\{(0, 0)\})$ near $(\sigma_1, 0, 0)$ consists precisely of the curves $\Gamma_0$ and $\Gamma_1 = \{(\sigma(s), u(s), v(s)) : s \in I\}$.*
3. *The bifurcation is forward (backward) if*

$$\frac{|\gamma'(0)|}{\gamma(0)} < (>) \frac{\int_\Omega \phi_1^3 \mathrm{d}x}{\int_\Omega m(x)\phi_1^2 \psi_1 \mathrm{d}x}. \tag{3.11}$$

On the branch $\Gamma_1$ of non-constant solutions, $(u(s), v(s)) = s(\phi_1, \psi_1) + o(s)$ is positive when $s \in (0, \delta)$. Solutions with $s < 0$ are negative so they are not relevant for the physical model here. An example of the nonlinear diffusion function is $\gamma(v) = A(1 + e^{-Kv})$ for $A, k > 0$, and here $\dfrac{|\gamma'(0)|}{\gamma(0)} = \dfrac{k}{2}$. Hence the bifurcation is backward if $k$ is sufficiently large while it is a forward one if $k$ is sufficiently small. This shows that a backward bifurcation will occur when $\gamma(v)$ has a sharp decrease near $v = 0$ when the value of $r_0 = \gamma(0)$ remains the same. A backward bifurcation means that on a left hand side neighborhood of $\sigma = \sigma_1$, there are multiple positive steady-state solutions of (3.1), and a bistable dynamics would occur in that left hand side neighborhood of $\sigma_1$. The backward bifurcation often occurs in a system with a weak Allee effect growth rate [42,52], hence the possible backward bifurcation in (3.1) can be thought as a weak Allee effect induced by a nonlinear diffusion.

Finally it follows from [11,29] that the local stability of non-constant positive solutions bifurcating from the line of trivial solutions can be determined. We first recall the theorem of exchange of stability in [12].

**Definition 3.2.** [12, Definition 1.2] Let $B(X, Y)$ denote the set of bounded linear maps of $X$ into $Y$, let $T, K \in B(X, Y)$, then $\mu \in \mathbb{R}$ is a $K$-simple eigenvalue of $T$ if

$$\dim \mathcal{N}(T - \mu K) = \operatorname{codim} \mathcal{R}(T - \mu K) = 1,$$

and if $\mathcal{N}(T - \mu K) = \operatorname{span}\{x_0\}$, $Kx_0 \notin \mathcal{R}(T - \mu K)$.

The following theorem is the result for Eq. (3.1) corresponding to Corollary 1.13 and Theorem 1.16 in [12].

**Theorem 3.3.** *Suppose that all conditions in Theorem 3.1 are satisfied. Let*

$$\Gamma_1 = \{(\sigma(s), u(s), v(s)) : s \in I\}$$

*be the curve of non-trivial solutions in Theorem 3.1. Then there exist continuously differentiable functions* $r : (\sigma_1 - \varepsilon, \sigma_1 + \varepsilon) \to \mathbb{R}$, $z : (\sigma_1 - \varepsilon, \sigma_1 + \varepsilon) \to X^2$, $\mu : (-\delta, \delta) \to \mathbb{R}$, $w : (-\delta, \delta) \to X^2$, *such that*

$$\begin{aligned} F_{(u,v)}(\sigma, 0, 0)z(\sigma) &= r(\sigma)Kz(\sigma), \quad \sigma \in (\sigma_1 - \varepsilon, \sigma_1 + \varepsilon), \\ F_{(u,v)}(\sigma(s), u(s, \cdot), v(s, \cdot))w(s) &= \mu(s)Kw(s), \quad s \in (-\delta, \delta), \end{aligned} \tag{3.12}$$

*where* $r(\sigma_1) = \mu(0) = 0$, $z(\sigma_1) = w(0) = (\phi_1, \psi_1)$, $K : X^2 \to Y^2$ *is the inclusion map with* $K(u) = u$. *Moreover, near* $s = 0$ *the functions* $\mu(s)$ *and* $-s\sigma'(s)r'(\sigma_1)$ *have the same zeroes and, whenever* $\mu(s) \neq 0$ *the same sign and satisfy*

$$\lim_{s \to 0} \frac{-s\sigma'(s)r'(\sigma_1)}{\mu(s)} = 1. \tag{3.13}$$

Using Theorem 3.3, we have the following stability results for the bifurcating solutions.

**Theorem 3.4.** *Suppose that all conditions in Theorem 3.1 are satisfied. Then the bifurcating non-constant positive steady-state solutions in Theorem 3.1 are locally asymptotically stable for* $s \in (0, \delta)$ *if the bifurcation is forward, and the bifurcating non-constant positive steady-state solutions in Theorem 3.1 are unstable for* $s \in (0, \delta)$ *if the bifurcation is backward.*

**Proof.** Note that $r(\sigma)$ is the principal eigenvalue of

$$\begin{cases} r_0 \Delta\phi + \sigma m(x)\phi = r(\sigma)\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{3.14}$$

Differentiating (3.14) with respect to $\sigma$, we obtain

$$\begin{cases} r_0 \Delta\phi' + \sigma m(x)\phi' + m(x)\phi = r'(\sigma)\phi + r(\sigma)\phi', & x \in \Omega, \\ \phi' = 0, & x \in \partial\Omega, \end{cases} \tag{3.15}$$

where $\phi' = \partial\phi/\partial\sigma$. Multiplying (3.14) by $\phi'$, multiplying (3.15) by $\phi$, subtracting and integrating on $\Omega$, we obtain

$$r'(\sigma) \int_\Omega \phi^2(x)\mathrm{d}x = \int_\Omega m(x)\phi^2(x)\mathrm{d}x.$$

Hence $r'(\sigma_1) > 0$ by assumption **(A3)**. From Eq. (3.10), if $\sigma'(0) > 0$, then it follows from (3.13) that $\mu(s) < 0$ when $s \in (0, \delta)$, and the non-constant positive steady-state solutions are locally asymptotically stable; similarly if $\sigma'(0) < 0$, then $\mu(s) > 0$ when $s \in (0, \delta)$, and the non-constant positive steady-state solutions are unstable. □

## 4. Global bifurcation

In this section, we use the global bifurcation theorem to study the global bifurcation structure of the bifurcation branch of positive steady-state solutions obtained in Section 3. Since we are more interested in positive solutions, we mainly apply the unilateral global bifurcation theorem in [11,44].

To obtain the global structure of the set of steady-state solutions bifurcating from the trivial ones, we prove several preliminary results. First we prove that when $\sigma > 0$ is small, then (3.1) has no positive solutions.

**Lemma 4.1.** *Suppose assumptions* ***(A2)*** *and* ***(A3)*** *hold. Then system* (3.1) *has no positive solution when* $\sigma < \frac{\sigma_1 r_\infty}{r_0} = r_\infty \lambda_1$, *where* $r_\infty$ *and* $r_0$ *are defined in* ***(A2)***, *and* $\lambda_1$ *is the principal eigenvalue of problem* (3.4).

**Proof.** Suppose $(u, v)$ is a positive solution of (3.1). Multiplying both sides of the first equation of (3.1) by $\phi_1$ and integrating on $\Omega$, we get

$$\int_\Omega \frac{\sigma_1 r_\infty}{r_0} m(x) u \phi_1 \mathrm{d}x < \int_\Omega \gamma(v) \frac{\sigma_1}{r_0} m(x) u \phi_1 \mathrm{d}x = \int_\Omega \gamma(v) \lambda_1 m(x) u \phi_1 \mathrm{d}x$$
$$= -\int_\Omega \Delta(\gamma(v)u)\phi_1 \mathrm{d}x \le \int_\Omega \sigma u m(x) \phi_1 \mathrm{d}x,$$

which implies that

$$\int_\Omega \left(\frac{\sigma_1 r_\infty}{r_0} - \sigma\right) m(x) u \phi_1 \mathrm{d}x \le 0.$$

Hence when $\sigma < \frac{\sigma_1 r_\infty}{r_0} = r_\infty \lambda_1$, (3.1) has no positive solution. □

In the following, we use the comparison principle to obtain *a priori* bound of the steady-state solution $(u, v)$ of (3.1). Note that when $\gamma(v) \equiv r$ where $r$ is a positive constant, system (3.1) is a classical semilinear elliptic system with constant diffusion rates, and in that case the system is partially decoupled:

$$\begin{cases} \Delta(ru) + \sigma u(m(x) - u) = 0, & x \in \Omega, \\ d\Delta v + u - v = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

For the first equation of (4.1)

$$\begin{cases} \Delta(ru) + \sigma u(m(x) - u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4.2}$$

it is well-known that when $\sigma > r\lambda_1$, Eq. (4.2) has a unique positive solution $u_{\sigma,r}(x) > 0$, which is globally asymptotically stable with respect to the corresponding reaction-diffusion equation, see for instance [8]. The $v$ in the second equation of (4.1) can be uniquely solved for any $u > 0$, and the solution $v > 0$. Hence (4.1) has a unique positive solution $(u_{\sigma,r}(x), v_{\sigma,r}(x))$ for $\sigma > r_0\lambda_1$. When $\sigma \le r_0\lambda_1$, (4.1) has only the trivial solution $(0, 0)$. Making a change of variables $w = ru$, then $w(x)$ satisfies

$$\begin{cases} \Delta w + \frac{\sigma}{r} w(m(x) - \frac{w}{r}) = 0, & x \in \Omega, \\ w = 0, & x \in \partial\Omega. \end{cases} \tag{4.3}$$

Hence Eq. (4.3) has a unique positive solution $w_{\sigma,r}(x) > 0$ when $\sigma > r\lambda_1$. We define a more general problem

$$\begin{cases} \Delta w + \frac{\sigma}{p} w(m(x) - \frac{w}{q}) = 0, & x \in \Omega, \\ w = 0, & x \in \partial\Omega, \end{cases} \tag{4.4}$$

where $p, q > 0$. Then Eq. (4.4) possesses a unique positive solution $w_{\sigma,p,q}(x)$ when $\sigma > p\lambda_1$.

Let $(u_\sigma(x), v_\sigma(x))$ be a positive solution of Eq. (3.1), and let $w_\sigma(x) = \gamma(v_\sigma(x))u_\sigma(x)$. Then $(w_\sigma(x), v_\sigma(x))$ satisfies

$$\begin{cases} \Delta w_\sigma + \dfrac{\sigma}{\gamma(v_\sigma(x))} w_\sigma(m(x) - \dfrac{w_\sigma}{\gamma(v_\sigma(x))}) = 0, & x \in \Omega, \\ d\Delta v_\sigma + \dfrac{w_\sigma}{\gamma(v_\sigma(x))} - v_\sigma = 0, & x \in \Omega, \\ w_\sigma = v_\sigma = 0, & x \in \partial\Omega. \end{cases} \tag{4.5}$$

We have the following estimates of $w_\sigma$ when it exists.

**Lemma 4.2.** *Suppose assumptions **(A2)** and **(A3)** hold. Let $(w_\sigma(x), v_\sigma(x))$ be a positive solution of (4.5) for $\sigma > r_\infty\lambda_1$, and let $w_{\sigma,p,q}(x)$ be the unique positive solution of (4.4). Then*

$$w_{\sigma,r_0,r_\infty^2/r_0}(x) < w_\sigma(x) < w_{\sigma,r_\infty,r_0^2/r_\infty}(x), \quad x \in \Omega. \tag{4.6}$$

**Proof.** Suppose that $(w_\sigma(x), v_\sigma(x))$ is a positive solution of (4.5) for $\sigma > r_\infty\lambda_1$. Applying $r_\infty < \gamma(v_\sigma(x)) < r_0$ to the first equation of (4.5), we obtain

$$\begin{aligned} 0 &= \Delta w_\sigma + \frac{\sigma}{\gamma(v_\sigma(x))} w_\sigma(m(x) - \frac{w_\sigma}{\gamma(v_\sigma(x))}) \\ &< \Delta w_\sigma + \frac{\sigma}{r_\infty} w_\sigma m(x) - \frac{\sigma}{r_0^2} w_\sigma^2 \\ &= \Delta w_\sigma + \frac{\sigma}{r_\infty} w_\sigma(m(x) - \frac{r_\infty}{r_0^2} w_\sigma). \end{aligned} \tag{4.7}$$

This implies that $w_\sigma$ is a lower solution of

$$\begin{cases} \Delta w + \dfrac{\sigma}{r_\infty} w(m(x) - \dfrac{r_\infty}{r_0^2} w) = 0, & x \in \Omega, \\ w = 0, & x \in \partial\Omega. \end{cases} \tag{4.8}$$

Since the positive solution $w_{\sigma,r_\infty,r_0^2/r_\infty}$ of (4.8) is unique, then it follows from the comparison principle that $w_\sigma(x) < w_{\sigma,r_\infty,r_0^2/r_\infty}(x)$.

Similarly,

$$\begin{aligned} 0 &= \Delta w_\sigma + \frac{\sigma}{\gamma(v_\sigma(x))} w_\sigma(m(x) - \frac{w_\sigma}{\gamma(v_\sigma(x))}) \\ &> \Delta w_\sigma + \frac{\sigma}{r_0} w_\sigma m(x) - \frac{\sigma}{r_\infty^2} w_\sigma^2 \\ &= \Delta w_\sigma + \frac{\sigma}{r_0} w_\sigma(m(x) - \frac{r_0}{r_\infty^2} w_\sigma), \end{aligned} \tag{4.9}$$

which shows that $w_\sigma(x)$ is an upper solution of the following equation

$$\begin{cases} \Delta w + \dfrac{\sigma}{r_0} w(m(x) - \dfrac{r_0}{r_\infty^2} w) = 0, & x \in \Omega, \\ w = 0, & x \in \partial\Omega. \end{cases} \tag{4.10}$$

Again by using the comparison principle and the unique of $w_{\sigma,r_0,r_\infty^2/r_0}$, we conclude that $w_\sigma(x) > w_{\sigma,r_0,r_\infty^2/r_0}(x)$. Hence the estimates (4.6) hold, see Fig. 1 for an illustration. This completes the proof. □

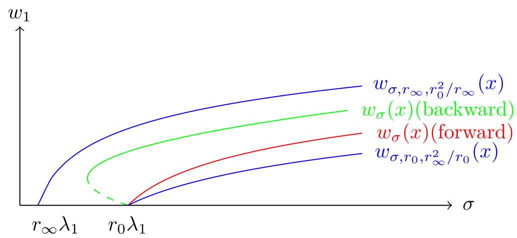



**Fig. 1.** Relationship between solution $w_\sigma(x)$ of Eq. (4.5) and solution $w_{\sigma,r}(x)$ of Eq. (4.3), where the bifurcation direction of $w_\sigma(x)$ can be forward or backward. Here, the solid line indicates that the solutions are locally asymptotically stable and the dashed line indicates that the solutions are unstable.

By using (4.6), we have the following boundedness of the positive solution $(u_\sigma(x), v_\sigma(x))$ of Eq. (3.1).

**Proposition 4.3.** *Suppose assumptions **(A2)** and **(A3)** hold. Then the positive solutions $(u_\sigma(x), v_\sigma(x))$ of Eq. (3.1) are bounded for all $\sigma > r_\infty\lambda_1$ and satisfy*

$$0 < u_\sigma(x) < \frac{r_0^2}{r_\infty^2}\max_{x\in\overline{\Omega}} m(x), \quad 0 < v_\sigma(x) < \frac{r_0^2}{r_\infty^2}\max_{x\in\overline{\Omega}} m(x), \quad x \in \Omega. \tag{4.11}$$

**Proof.** Let $(u_\sigma(x), v_\sigma(x))$ be a positive solution of (3.1). Then from $u_\sigma(x) = \dfrac{w_\sigma(x)}{\gamma(v_\sigma(x))}$, $r_\infty < \gamma(v_\sigma(x)) < r_0$ and Lemma 4.2, we have

$$u_\sigma(x) = \frac{w_\sigma(x)}{\gamma(v_\sigma(x))} < \frac{1}{r_\infty} w_\alpha(x) < \frac{1}{r_\infty} w_{\sigma,r_\infty,r_0^2/r_\infty}(x), \quad x \in \Omega. \tag{4.12}$$

Assume that $w_{\alpha,r_\infty,r_0^2/r_\infty}(x_0) = \max_{x\in\overline{\Omega}} w_{\alpha,r_\infty,r_0^2/r_\infty}(x)$. Then from the maximum principle of elliptic equations,

$$w_{\alpha,r_\infty,r_0^2/r_\infty}(x_0) \le \frac{r_0^2}{r_\infty} m(x_0). \tag{4.13}$$

Now combining (4.12) and (4.13), we obtain

$$\max_{x\in\overline{\Omega}} u_\sigma(x) < \frac{1}{r_\infty}\max_{x\in\overline{\Omega}} w_{\sigma,r_\infty,r_0^2/r_\infty}(x) = \frac{1}{r_\infty} w_{\alpha,r_\infty,r_0^2/r_\infty}(x_0) \le \frac{r_0^2}{r_\infty^2} m(x_0) \le \frac{r_0^2}{r_\infty^2}\max_{x\in\overline{\Omega}} m(x). \tag{4.14}$$

Next assume that $v_\sigma(x_1) = \max_{x\in\overline{\Omega}} v_\sigma(x)$. From the second equation of (4.2) and (4.14), we get

$$0 \le -d\Delta v_\sigma(x_1) = u_\sigma(x_1) - v_\sigma(x_1) \le \frac{r_0^2}{r_\infty^2}\max_{x\in\overline{\Omega}} m(x) - v_\sigma(x_1),$$

that is,

$$\max_{x\in\overline{\Omega}} u_\sigma(x) = v_\sigma(x_1) \le \frac{r_0^2}{r_\infty^2}\max_{x\in\overline{\Omega}} m(x). \quad \square$$

We are now in a position to describe a global bifurcation theorem for the set of positive solutions of (4.2). Let $E = C^\alpha(\overline{\Omega})$, $E^+ = \{u \in E : u(x) \ge 0, x \in \Omega\}$, and let the set of non-zero solutions of (3.1) to be

$$S = \{(\sigma, u, v) \in \mathbb{R}^+ \times E \times E : F(\sigma, u, v) = 0, (u, v) \ne (0, 0)\}. \tag{4.15}$$

We can apply the unilateral bifurcation theorem in [11,44] to obtain the following results.

**Theorem 4.4.** *Suppose assumptions **(A2)** and **(A3)** hold. Let $\Gamma_1 = \{(\sigma(s), u(s), v(s)) : s \in (-\delta, \delta)\}$ be the curve of non-zero solutions obtained in Theorem 3.1. Define $\Gamma_1^+ = \{(\sigma(s), u(s), v(s)) : 0 < s < \delta\}$ and $\Gamma_1^- = \{(\sigma(s), u(s), v(s)) : -\delta < s < 0\}$. Then there exists a connected component $\mathcal{C}$ of $\overline{S}$ such that $(\sigma_1, 0, 0) \in \mathcal{C}$. Let $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) be the connected component of $\mathcal{C} \setminus \Gamma_1^-$ which contains $\Gamma_1^+$ (resp. the connected component of $\mathcal{C} \setminus \Gamma_1^+$ which contains $\Gamma_1^-$). Then $\mathcal{C}^+$ is an unbounded subset of $\overline{S}$ consisting of positive solutions of (4.2), and the projection of $\mathcal{C}^+$ onto $\sigma$-axis satisfies $(r_\infty\lambda_1, \infty) \supset Proj_\sigma\mathcal{C}^+ \supset (\sigma_1, \infty)$. That is, for any $\sigma > \sigma_1$, Eq. (3.1) possesses at least one positive solution $(\sigma, u_\sigma, v_\sigma)$ that is on $\mathcal{C}^+$. Moreover all these positive solutions are uniformly bounded in $\sigma$ and satisfy (4.11).*

**Proof.** By using the same arguments as in [44] section 4.2, we can show that Theorems 4.3 and 4.4 in [44] can be applied to (4.2) or equivalently the function $F$, which is defined as in (3.2). Then there exists a connected component $\mathcal{C}$ of $\overline{S}$ such that $(\sigma_1, 0, 0) \in \mathcal{C}$. Let $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) be the connected component of $\mathcal{C} \setminus \Gamma_1^-$ which contains $\Gamma_1^+$ (resp. the connected component of $\mathcal{C} \setminus \Gamma_1^+$ which contains $\Gamma_1^-$). Moreover each of $\mathcal{C}^+$ and $\mathcal{C}^-$ satisfies one of the following:

(i) it is not compact;
(ii) it contains a point $(\sigma_*, 0, 0)$ with $\sigma_* \neq \sigma_1$;
(iii) it contains a point $(\sigma, u, v)$ where $(u, v) \in Z$ ($Z$ is a complement of $\mathcal{N}(F_{(u,v)}(\sigma_1, 0, 0))$ in $X$).

We only consider $\mathcal{C}^+$ since those on $\mathcal{C}^-$ are negative. We claim that $\mathcal{C}^+ \subseteq \mathbb{R}^+ \times E^+ \times E^+$. Since $\Gamma_1^+ \subset \mathcal{C}^+$, hence $\mathcal{C}^+ \cap (\mathbb{R}^+ \times E^+ \times E^+) \neq \emptyset$. If there exists $(\sigma, u, v) \in \mathcal{C}^+$ such that $(u, v)$ is not nonnegative, then there exists $(\tilde{\sigma}, \tilde{u}, \tilde{v}) \in \mathcal{C}^+$ such that $(\tilde{u}, \tilde{v}) \in \partial(E^+ \times E^+)$. From the maximum principle, $\tilde{u}$ and $\tilde{v}$ either $\equiv 0$ or $> 0$ (not both $> 0$). It is obvious that if one of $\tilde{u}$ or $\tilde{v}$ is zero, then the other is also zero. Hence $(\tilde{u}, \tilde{v}) = (0, 0)$. Then $\sigma = \tilde{\sigma}$ is a bifurcation point for (4.2) where positive solutions bifurcate. From Theorem 3.1, $\sigma_1$ is the unique bifurcation point for positive solutions, hence $\tilde{\sigma} = \sigma_1$. But $\Gamma_1^- \cap \mathcal{C}^+ = \emptyset$, hence $\mathcal{C}^+$ cannot connect to any $(\sigma, u, v)$ such that $(u, v)$ is not nonnegative. Therefore $\mathcal{C}^+ \subseteq \mathbb{R}^+ \times E^+ \times E^+$.

Now since $\mathcal{C}^+ \subseteq \mathbb{R}^+ \times E^+ \times E^+$, and from Theorem 3.1, $\sigma_1$ is the unique bifurcation point for positive solutions, so (ii) above cannot occur. Again since $\mathcal{C}^+ \subseteq \mathbb{R}^+ \times E^+ \times E^+$, (iii) cannot occur as $(u, v) \in Z$ must be sign-changing.

Therefore alternative (i) must occur. This could be either (iv) $\mathcal{C}^+$ contains a boundary point of $\mathbb{R}^+ \times E^+ \times E^+$, or (v) $\mathcal{C}^+$ is unbounded in $\mathbb{R}^+ \times E^+ \times E^+$. If (iv) occurs, we have shown that $\mathcal{C}^+ \cap \mathbb{R}^+ \times \partial(E^+ \times E^+) = \{(\sigma_1, 0, 0)\}$, and if $\overline{\mathcal{C}^+}$ intersects with $\{0\} \times E^+ \times E^+$, then there are positive solutions of (4.2) for all $\sigma \in (0, \sigma_1)$, which contradicts with Lemma 4.1 that there is no positive solution when $\sigma < r_\infty\lambda_1$. Hence (v) occurs.

From Lemma 4.1 and Proposition 4.3, all solutions of Eq. (3.1) are uniformly bounded and there is no solution when $\sigma < r_\infty\lambda_1$, then the projection of $\mathcal{C}^+$ to $\sigma$-axis $(r_\infty\lambda_1, \infty) \supset Proj_\sigma\mathcal{C}^+ \supset (\sigma_1, \infty)$. This proves the existence of positive solutions for any $\sigma > \sigma_1$. The proof is completed. □

According to the global bifurcation structure revealed in Theorem 4.4, a global bifurcation diagram for the positive solutions of (4.2) can be depicted as in Fig. 2. From Theorem 3.4, the local bifurcation at $(\sigma_1, 0, 0)$ can be either forward or backward. The forward bifurcated solutions are locally asymptotically stable near the bifurcation point, and the backward bifurcated solutions are unstable near the bifurcation point. However, there must be a turning point in the backward bifurcation situation where the bifurcation curve turns back since there are no positive solutions when $\sigma < r_\infty\lambda_1$ from Lemma 4.1, and the solutions on the upper bifurcation branch may be stable. Also from Proposition 4.3 in both the forward bifurcation and the backward bifurcation cases, all positive solutions are uniformly bounded for $\sigma > r_\infty\lambda_1$.

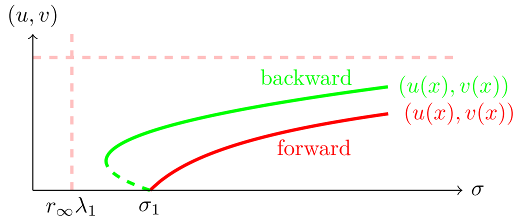



**Fig. 2.** The global bifurcation diagram of Eq. (1.2). Here, the solid line indicates that the solutions are locally asymptotically stable, the green dashed line indicates that the solutions are unstable and the pink dashed line represents the bound of the solutions. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

## 5. Numerical results

In this section, we perform numerical simulations to validate the theoretical results established in previous sections. Furthermore, we demonstrate how spatial heterogeneity affects the steady-state distributions of both *E. coli* cells and AHL concentration.

For all numerical simulations, we consider the domain $\Omega = (0, \pi)$ with fixed parameter $d = 0.2$ and varying $\sigma > 0$. The diffusion motility function and spatially heterogeneous growth rate function are specified as follows:

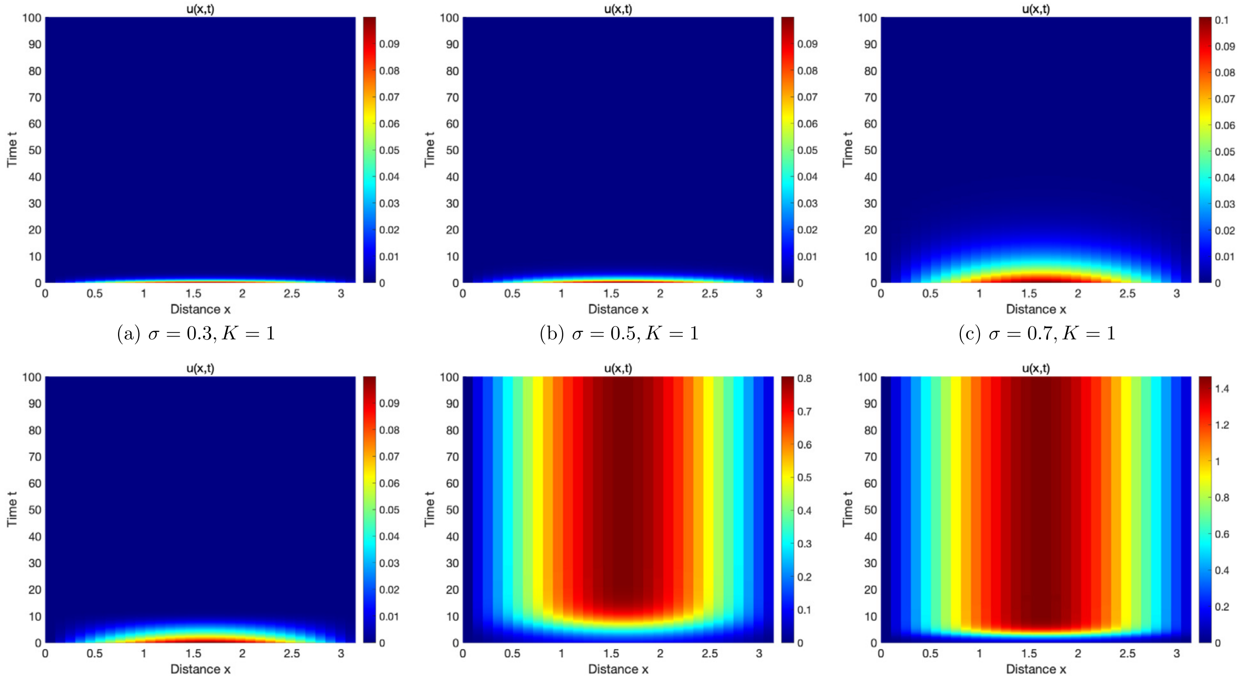
$$\gamma(v) = 1 + e^{-Kv}, \quad m(x) = 3\sin x + 0.1, \quad x \in (0, \pi). \tag{5.1}$$

Evidently, the functions in Eq. (5.1) satisfy assumptions **(A2)** and **(A3)**, with $r_0 = 2$, $r_\infty = 1$, and $\gamma'(0) = -K$. By Lemma 4.1, there is no positive solutions when $\sigma_1 < \lambda_1 \approx 0.375$, and $\sigma_1 = 2\lambda_1 \approx 0.75$ is the unique bifurcation point where positive solutions of system (3.1) bifurcate from the trivial ones.

From Theorems 3.1 and 4.4, a positive solution of system (3.1) always exists when $\sigma > \sigma_1 \approx 0.75$ regardless the bifurcation direction being forward or backward, while a positive solution of system (3.1) also exists when $\sigma < \sigma_1 \approx 0.75$ but close to $\sigma_1$ if the bifurcation at $\sigma_1$ is backward. From Theorem 3.1, the bifurcation direction is forward when $K$ is small, and it is backward when $K$ is large.

In Figs. 3 and 4, we compare the solutions of (1.2) for the cases of $K = 1$ (forward bifurcation) and $K = 100$ (backward bifurcation) with varying $\sigma$. Since the graphs of $v$-component are similar to the ones of $u$-component, here we only plot the graphs of the $u$-component. In Fig. 3, when $K = 1$ we can see that the solution of (1.2) always converges to zero and there is no positive steady-state solution for all $\sigma = 0.3$, $0.5$, $0.7 < \sigma_1$; when $K = 100$, the solution of (1.2) converges to zero for $\sigma = 0.3$, but it converges to a positive steady-state for $\sigma = 0.5$ and $0.7$, which verifies the existence of positive steady-state solutions for $\sigma < \sigma_1$ in the backward bifurcation scenario. In Fig. 4, the solution of (1.2) always converges to a positive non-constant steady-state solution, regardless of forward bifurcation ($K = 1$) or backward bifurcation ($K = 100$). These are consistent with our theoretical results for the global bifurcation in Theorem 4.4.

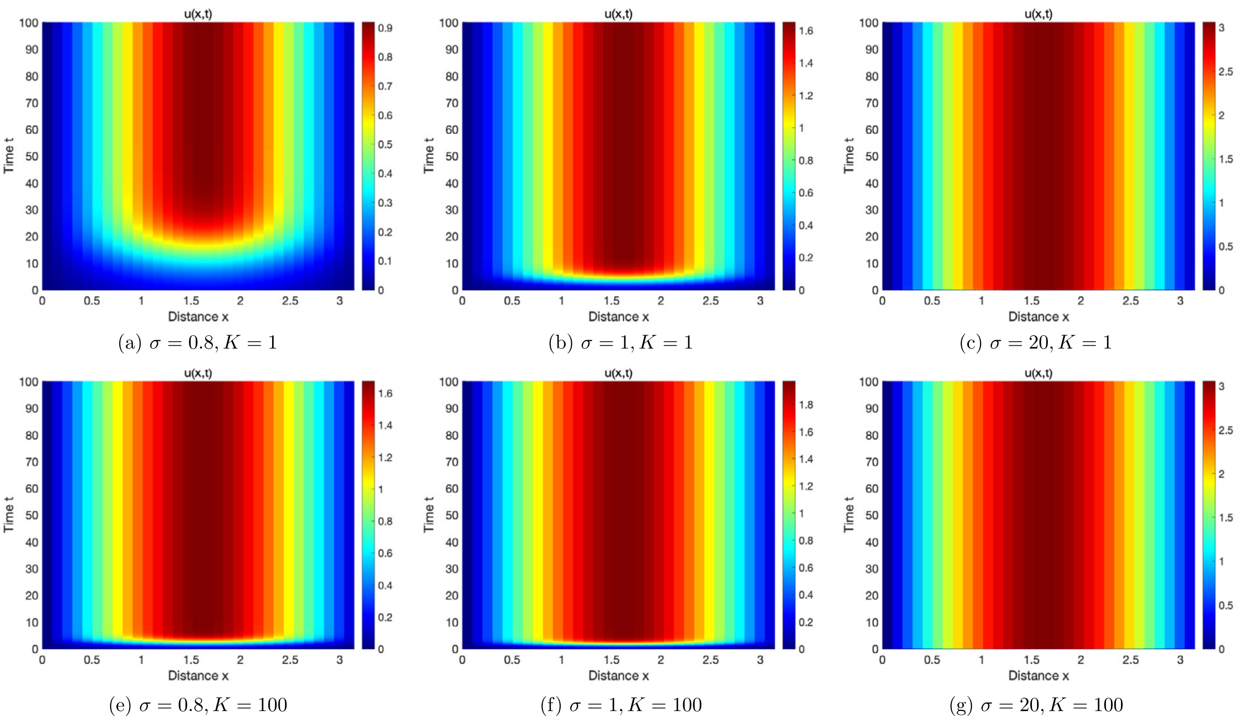
Numerical simulations of system (1.2) can be used to show the influence of the function $m(x)$ reflecting the uneven distribution of spatial resources on the density of *E. coli* cells and AHL concentration. In Fig. 5, the graphs of the function $m(x)$ and the steady-state solution $(u(x), v(x))$ are plotted. Panels (a)-(c) of Fig. 5 show that the spatial density distribution of *E. coli* cells and AHL concentration basically conforms to the distribution of resources. The populations have higher concentration where the resources are abundant, while the population is at lower concentration where the resources are scarce. The distribution of resources directly affects the density distribution of *E. coli* cells, while the change of *E. coli* cells density affects the density of AHL concentration. Therefore the variation of AHL concentration is smaller than that of *E. coli* cells. On the other hand since our model has Dirichlet boundary conditions (hostile boundary conditions), hence no matter how rich resources are on the boundary, the population cannot survive at the boundary, see Panels (d)-(f) of Fig. 5 as examples.




(a) $\sigma = 0.3, K = 1$ (b) $\sigma = 0.5, K = 1$ (c) $\sigma = 0.7, K = 1$

(e) $\sigma = 0.3, K = 100$ (f) $\sigma = 0.5, K = 100$ (g) $\sigma = 0.7, K = 100$

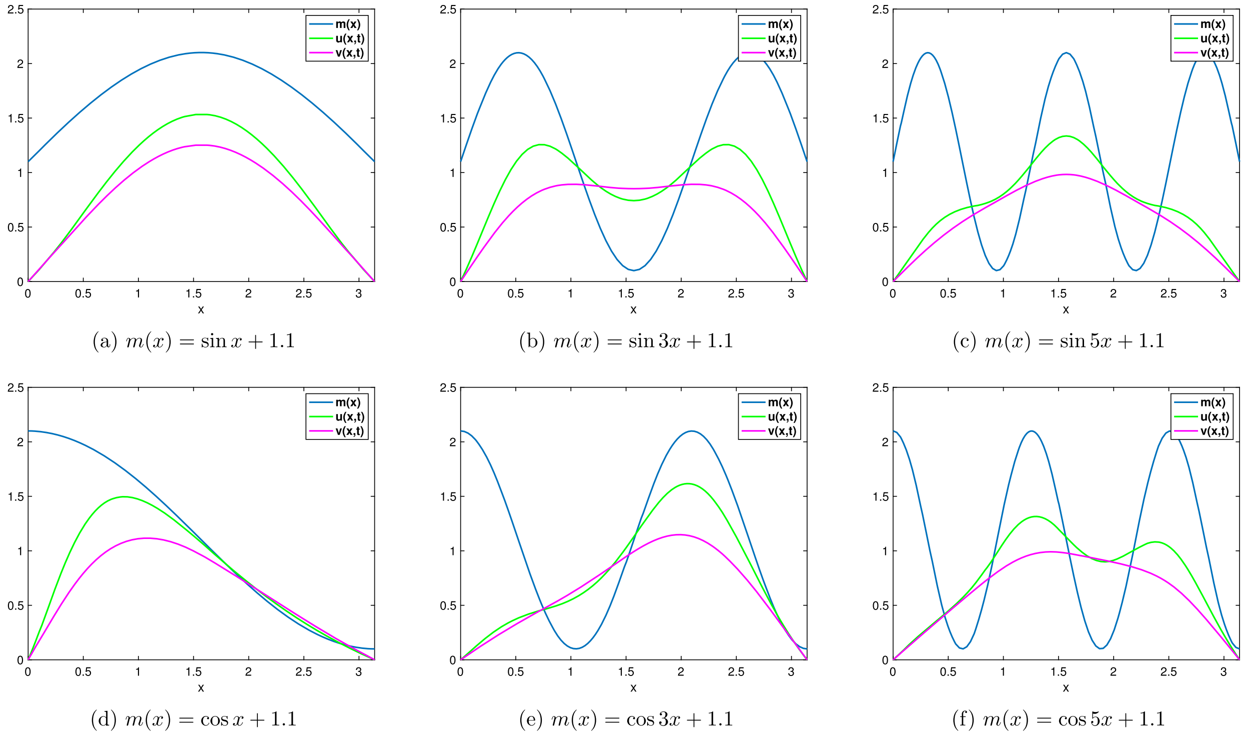
**Fig. 3.** $u$-component of solutions of (1.2) when $\sigma < \sigma_1$, and $K = 1$ or $K = 100$. Here the initial condition is $u(x, 0) = 0.1 \sin x$, $v(x, 0) = 0.1 \sin x$. The horizontal axis is $x$ $(0 \le x \le \pi)$, and the vertical axis is $t$ $(0 \le t \le 100)$.




(a) $\sigma = 0.8, K = 1$ (b) $\sigma = 1, K = 1$ (c) $\sigma = 20, K = 1$

(e) $\sigma = 0.8, K = 100$ (f) $\sigma = 1, K = 100$ (g) $\sigma = 20, K = 100$

**Fig. 4.** $u$-component of solutions of (1.2) when $\sigma > \sigma_1$, and $K = 1$ or $K = 100$. Here the initial condition is $u(x, 0) = 0.1 \sin x$, $v(x, 0) = 0.1 \sin x$. The horizontal axis is $x$ $(0 \le x \le \pi)$, and the vertical axis is $t$ $(0 \le t \le 100)$.




(a) $m(x) = \sin x + 1.1$ (b) $m(x) = \sin 3x + 1.1$ (c) $m(x) = \sin 5x + 1.1$

(d) $m(x) = \cos x + 1.1$ (e) $m(x) = \cos 3x + 1.1$ (f) $m(x) = \cos 5x + 1.1$

**Fig. 5.** The comparison between the resource function $m(x)$ and the solutions $u$ and $v$ of Eq. (3.1). Here the initial condition is $u(x, 0) = 0.1 \sin x$, $v(x, 0) = 0.1 \sin x$, $\gamma(x)$ is as (5.1) with $K = 1$, $d = 0.2$ and $\sigma = 8$. The horizontal axis is $x$ $(0 \leq x \leq \pi)$, and the vertical axis is the value of $m(x)$, $u(x, 100)$ and $v(x, 100)$.

## 6. Conclusions

In this paper, we investigate a density-suppressed motility model in a spatially heterogeneous environment under Dirichlet boundary conditions. We first employ the abstract theory of quasilinear parabolic systems from [2,3] to establish the existence of local-in-time classical solutions. Then, we apply bifurcation theory [11,12] to obtain the existence of small-amplitude non-constant steady-state solutions of Eq. (1.2), and we determine both the direction and local stability of bifurcating solutions. We show that the density-suppressed diffusion rate can cause the occurrence of the backward bifurcation, this bifurcation also appears in model with a weak Allee effect growth rate [42,52], which are different from the forward bifurcation that usually occur in a reaction-diffusion system with constant diffusion rate. Moreover, utilizing the global bifurcation theorem in [44,38], we describe the global bifurcation structure of the steady-state solutions of Eq. (1.2), and present the corresponding bifurcation diagram in Fig. 2. Furthermore, our numerical simulations demonstrate that solution patterns emerge from the joint influence of resource functions and boundary conditions.

Although the local existence of solutions for the model has been established, the global existence and boundedness of solutions require further investigation. In cases exhibiting backward bifurcation, multiple positive steady-state solutions coexist within at least a left neighborhood of the bifurcation point. When $\sigma$ exceeds the bifurcation value $\sigma_1$, numerical evidence suggests the positive steady-state solution becomes unique and asymptotically stable. However, rigorous proof of this solution's uniqueness and stability remains an open problem, representing a promising direction for future research.

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