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A diffusive predator–prey model with a protection zone ☆

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Abstract

In this paper we study the effects of a protection zone Ω_0 for the prey on a diffusive predator–prey model with Holling type II response and no-flux boundary condition. We show the existence of a critical patch size described by the principal eigenvalue $\lambda_1^D(\Omega_0)$ of the Laplacian operator over Ω_0 with homogeneous Dirichlet boundary conditions. If the protection zone is over the critical patch size, i.e., if $\lambda_1^D(\Omega_0)$ is less than the prey growth rate, then the dynamics of the model is fundamentally changed from the usual predator–prey dynamics; in such a case, the prey population persists regardless of the growth rate of its predator, and if the predator is strong, then the two populations stabilize at a unique coexistence state. If the protection zone is below the critical patch size, then the dynamics of the model is qualitatively similar to the case without protection zone, but the chances of survival of the prey species increase with the size of the protection zone, as generally expected. Our mathematical approach is based on bifurcation theory, topological degree theory, the comparison principles for elliptic and parabolic equations, and various elliptic estimates.

MSC: primary 35J55, 92D40; secondary 35J60, 35K50, 35K57, 92D25

Keywords: Reaction–diffusion system; Predator–prey model; Protection zone

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1. Introduction

Interaction between a pair of predator and prey influences the population growth of both species. This was observed from the population data of Canadian lynx and snowshoe hare from the 1840s, and the first differential equations of predator–prey type were formulated by Alfred James Lotka in 1925, and Vito Volterra in 1926. More complicated but realistic predator–prey systems have been used by ecologists and mathematicians since then. When the spatial distribution of the populations is also considered, a prototypical predator–prey system is of the form

$$\begin{cases} u_t - d_1 \Delta u = f(u) - b\phi(u)v, \\ v_t - d_2 \Delta v = g(v) + c\phi(u)v. \end{cases} \tag{1.1}$$

Here $f(u)$ and $g(v)$ represent, when the other species is absent, the growth of prey and predator populations respectively. In the earlier work of Volterra, he assumed $f(u) = \lambda u$ and $g(v) = -\mu v$, but it is more reasonable to use logistic growth for both species since the predator may have alternative food sources, and the prey usually has bounded growth even without predation. The function $\phi(u)$ represents the response of the predator, which was first examined by Holling [28]. The classical Lotka–Volterra model assumes $\phi(u) = u$. If the handling time of each prey is also considered, then a more reasonable response function is $\phi(u) = u/(1 + mu)$ $(m > 0)$ [28], called a Holling type II response.

In most predator–prey interactions, the prey population would extinguish if the growth rate of the predator is too large, or the predation rate is too high. Human interference is often needed to save the endangered prey species, and a natural idea is to set up one or several protection zones for the prey, where the prey species can enter and leave freely but the predator is kept out. Several related biological questions then arise: Are such protection zones effective to save an endangered prey population? What are the effects of such protection zones on the predator species? Could such protection zones induce unexpected new dynamics for the species involved?

In this paper, we attempt to address these questions by examining a diffusive predator–prey model with a single protection zone, where the prey is free to enter and leave, but the predator is blocked out. Our model is described by the following system of equations:

$$\begin{cases} u_t - d_1 \Delta u = \lambda(x)u - a(x)u^2 - \dfrac{b(x)uv}{1 + m(x)u} & \text{for } x \in \Omega,\ t > 0, \\ v_t - d_2 \Delta v = \mu(x)v - d(x)v^2 + \dfrac{c(x)uv}{1 + m(x)u} & \text{for } x \in \Omega \backslash \overline{\Omega}_0,\ t > 0, \\ \partial_\nu u = 0 \quad \text{for } x \in \partial\Omega,\ t > 0, & \partial_\nu v = 0 \quad \text{for } x \in \partial\Omega \cup \partial\Omega_0,\ t > 0, \\ u(x,0) = u_0(x) \geqslant 0 \quad \text{for } x \in \Omega, & v(x,0) = v_0(x) \geqslant 0 \quad \text{for } x \in \Omega \backslash \overline{\Omega}_0, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, and Ω_0 is a subdomain of Ω whose boundary $\partial\Omega_0$ is also smooth. The larger region Ω is the habitat of the prey, with Ω_0 its protection zone; thus the predator species can only exist in $\Omega \backslash \overline{\Omega}_0$. All the coefficient functions are non-negative in $\overline{\Omega}$. Logistic growth is assumed for both species, and Holling type II functional response is assumed for predation. The function $b(x)$ is zero when $x \in \Omega_0$, representing the assumption that the prey population enjoys predation-free growth in Ω_0; this also makes the interaction term in the equation for u well defined over Ω. On $\partial\Omega$, a no-flux boundary condition is assumed for both species, so the predator and prey live in a closed ecosystem. The boundary of the protection zone does not affect the dispersal of prey, but it works as a barrier to block the predator from entering Ω_0; thus a no-flux boundary condition should be imposed for the predator on $\partial\Omega_0$. For technical reasons, we assume further that $\overline{\Omega}_0 \subset \Omega$. Therefore, we may call Ω_0 an interior protection zone. If a portion of $\partial\Omega_0$ lies on the boundary of Ω (so the protection zone locates along part of $\partial\Omega$), then we speak of a boundary protection zone. This case is biologically important, but involves technically difficult mixed boundary value problems, and thus will not be treated in this paper. We will nevertheless briefly comment on the boundary protection zone case at the end of the paper.

To focus on the impact of the protection zone on the dynamics, we will assume that all the coefficient functions in (1.2) are constants, except $b(x)$ which is zero in Ω_0 but positive otherwise. Moreover, except for μ which may take negative values, all the other constant coefficients are positive. It turns out that the asymptotic dynamical behavior of our system is closely related to the well-known diffusive logistic equation with hostile boundary condition:

$$\begin{cases} u_t - d\Delta u = \lambda u - u^2, & x \in \Omega_0,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega_0,\ t > 0, \\ u(x,0) = u_0(x) \geqslant 0, & x \in \Omega_0. \end{cases} \tag{1.3}$$

It is known that there exists a unique $\lambda_* = \lambda_1^D(\Omega_0)/d > 0$ (where $\lambda_1^D(\Omega_0)$ denotes the principal eigenvalue of $-\Delta$ in Ω_0 with zero Dirichlet boundary conditions on $\partial\Omega_0$), such that when $\lambda \leqslant \lambda_*$, the population u would eventually extinguish, and when $\lambda > \lambda_*$, the population u will persist and settle at a unique positive steady state. Thus λ_* is the threshold growth rate for persistence/extinction. There are several different interpretations of this conclusion. If one regard the growth rate λ as fixed, then there is a maximal diffusion rate $d_* = \lambda_1^D(\Omega_0)/\lambda$ (or equivalently, when the diffusion rate is also fixed, a minimal patch size S), such that when d is over d_* (or the domain is smaller than S), the population will become extinct, but otherwise the population will persist. The minimal patch size is determined by the principal eigenvalue of the associated linear operator, and it is dependent on the geometry of the habitat (see [3]). (The concept of minimal patch size first appeared in the pioneering work [41] and [29].) It is interesting to note that, if a no-flux boundary condition is imposed for (1.3), then λ_* would be 0, independent of the size and shape of Ω_0.

Our study shows that there is a critical patch size for the protection zone, determined by the prey growth rate λ, so that above this size, the protection zone guarantees the survival of the endangered prey species regardless of the predator growth rate; while below this critical size, extinction of the prey species is possible, and whether it can survive or not is determined by a combination of factors including the protection zone and the growth rate μ of the predator.

To be more precise, when $\lambda_* > \lambda$ (small protection zone case), the dynamics of (1.2) is qualitatively similar to the case without protection zone considered in [24]: when the predator growth rate μ is small or negative, the prey-only steady state $(\lambda, 0)$ is globally asymptotically stable; when μ is large, the predator-only steady state $(0, \mu)$ is globally asymptotically stable; and when μ is in the intermediate range, one or more coexistence steady states exist which attract all initial values. Allee effect or bistability could exist in the last case (see details in [24]). However when $\lambda_* < \lambda$ (large protection zone case), the dynamics of (1.2) is fundamentally different from the case without protection zone; now the predator-only steady state $(0, \mu)$ is never a stable one even when the predator has strong growth rate. Instead, for all positive μ, the system is permanent in the sense that both the prey and predator populations persist. When μ is large, we are able to show that the global attractor is a single coexistence steady state (u_μ, v_μ), and the profile of the coexistence state can be described in terms of the unique positive steady state w_λ of (1.3): $u_\mu \approx w_\lambda$ in Ω_0, $u_\mu \approx 0$ outside of Ω_0, and $v_\mu \approx \mu$ outside of Ω_0.

The biological implication of this coexistence state is clear: when the predator is very strong, virtually every individual of the prey population will be consumed by the predator as long as they move out of the protection zone Ω_0, but the population of the prey inside Ω_0 is near the carrying capacity steady state with hostile boundary condition. The impact of the protection zone on the predator species is that its total population drops from $\mu \cdot |\Omega|$ to $\mu \cdot (|\Omega| - |\Omega_0|)$, where $|O|$ represents the area of the region O, but the density of the predator population outside Ω_0 is almost unchanged. Note that $\mu > 0$ implies that the predator has other food source apart from the prey under consideration. The impact of the protection zone on the prey population is more profound: Without the protection zone or if the size of the protection zone is too small, the prey is destined to extinction with strong predation; but with a protection zone whose size is larger than a certain critical one, the prey population at least persists in the protection zone. Another relevant biological question is the design of an optimal protection zone, which will be discussed in Section 5.

When the environment is homogeneous, the set of steady state solutions of competition and predator–prey systems was extensively studied, see, e.g., [1,4,5,7–9,20,21,30,33,34,36] and the references therein. In particular, if no protection zone is present, the corresponding homogeneous model of (1.2) was studied in [12] and [22]. More recently, the impact of heterogeneity and degeneracy of the environment has been studied in [2,13–16,25–27] for the Lotka–Volterra competition model, and in [10,19,24] for the diffusive predator–prey model. However, the effect of a protection zone does not appear to have been studied before. It is perhaps worthwhile to mention that in this paper we are able to show the existence, uniqueness and global asymptotical stability of the (non-constant) coexistence steady state when μ is large. This kind of result is rarely achieved for a diffusive predator–prey system; uniqueness and global stability of constant steady state was obtained in [12] by a Lyapunov function technique (not applicable here), uniqueness of (non-constant) coexistence state for certain predator–prey models was proved in [32] (see also [11] and [17]) in the one space dimension (or radially symmetric) case, but the stability is still unknown.

Much research and debates can be found in the literature on the establishment of marine reserves [37,38,42,43], which are protection zones where fishing activities are banned. We could not find in the literature any such research based on reaction–diffusion models, but a predator–prey model as the one discussed in this paper does not seem to fit well into the marine reserve situation. Nevertheless, our work might hopefully shed some lights on the research in that direction.

We organize the rest of the paper in the following way. In Section 2, we present some basic results on the set of steady-state solutions based on standard bifurcation analysis, and we also derive the critical patch size for the protection zone. Then in Sections 3 and 4, we study the dynamics of the predator–prey system with protection zones above and below the critical patch size respectively. However, the mathematical presentations in these two sections are very different. We provide a detailed analysis in Section 3 for the large protection zone case, where new mathematical techniques are developed to cope with the new dynamical behavior of the model. As the small protection zone case exhibits similar qualitative dynamical behavior as the case without protection zone, and this can be proved by slightly modifying the techniques in our recent work [24], we feel that it is reasonable to omit the detailed proofs while only state the results and explain their biological implications; the interested reader should have no problem to reconstruct the proofs for the results in Section 4 following [24]. We conclude with Section 5, where we briefly discuss the biological applications and the design of optimal protection zones, and also leave some open questions.

2. Preliminaries

In this section, we apply a standard bifurcation analysis to gain a basic understanding of the set of steady-state solutions of our model. Since the analysis is rather routine, our arguments here are sometimes less detailed than in other parts of the paper.

To simplify the notations and make our analysis more transparent, we will assume that all the parameter functions in (1.2) are constant except $b(x)$, and $d_1 = d_2 = 1$. Thus we have the following system:

$$\begin{cases} u_t - \Delta u = \lambda u - u^2 - \dfrac{b(x)uv}{1+mu} & \text{for } x \in \Omega, t > 0, \\ v_t - \Delta v = \mu v - v^2 + \dfrac{cuv}{1+mu} & \text{for } x \in \Omega \backslash \overline{\Omega}_0,\ t > 0, \\ \partial_\nu u = 0 \quad \text{for } x \in \partial\Omega,\ t > 0, & \partial_\nu v = 0 \quad \text{for } x \in \partial\Omega \cup \partial\Omega_0,\ t > 0, \\ u(x,0) = u_0(x) \geqslant 0 \quad \text{for } x \in \Omega, & v(x,0) = v_0(x) \geqslant 0 \quad \text{for } x \in \Omega \backslash \overline{\Omega}_0. \end{cases} \tag{2.1}$$

The steady state solutions satisfy

$$\begin{cases} -\Delta u = \lambda u - u^2 - \dfrac{b(x)uv}{1+mu} & \text{in } \Omega, \\ -\Delta v = \mu v - v^2 + \dfrac{cuv}{1+mu} & \text{in } \Omega \backslash \overline{\Omega}_0, \\ \partial_\nu u = 0 \quad \text{on } \partial\Omega, & \partial_\nu v = 0 \quad \text{on } \partial\Omega \cup \partial\Omega_0. \end{cases} \tag{2.2}$$

Here λ, μ, c, m are positive constants, $b(x) \in L^\infty(\Omega)$, $b(x) \geqslant 0$ in $\overline{\Omega}$, $b(x) \equiv 0$ on $\overline{\Omega}_0$ and for any compact subset A of $\overline{\Omega} \setminus \overline{\Omega}_0$, there exists $\delta_A > 0$ such that

$$b(x) \geqslant \delta_A, \quad \forall x \in A. \tag{2.3}$$

In the following, for simplicity of notation, we write $p(u) = u/(1+mu)$. It is easy to see that

$$p(u) = \frac{u}{1+mu}, \quad p'(u) = \frac{1}{(1+mu)^2}, \quad p''(u) = \frac{-2m}{(1+mu)^3}. \tag{2.4}$$

Linear eigenvalue problems will play important roles in our analysis. We denote by $\lambda_1^D(\phi, O)$ and $\lambda_1^N(\phi, O)$ the first eigenvalues of $-\Delta + \phi$ over a region O, with Dirichlet or Neumann boundary conditions respectively. If O is omitted from the notation, then we understand that $O = \Omega$. If the potential function ϕ is omitted, then we understand that $\phi = 0$. Some well-known properties of $\lambda_1^D(\phi, O)$ and $\lambda_1^N(\phi, O)$ are:

(1) $\lambda_1^D(\phi, O) > \lambda_1^N(\phi, O)$;
(2) $\lambda_1^B(\phi_1, O) > \lambda_1^B(\phi_2, O)$ if $\phi_1 \geqslant \phi_2$ and $\phi_1 \not\equiv \phi_2$, for $B = D, N$;
(3) $\lambda_1^D(\phi, O_1) \geqslant \lambda_1^D(\phi, O_2)$ if $O_1 \subset O_2$.

We start our analysis by a standard local bifurcation argument. We fix $\lambda, c, m > 0$, and take μ as the bifurcation parameter. For any $\mu > 0$, (2.2) has two semi-trivial solutions: $(\lambda, 0)$ and $(0, \mu)$. So we have two curves of these solutions in the space of (μ, u, v):

$$\Gamma_u = \{(\mu, \lambda, 0)\colon -\infty < \mu < \infty\}, \qquad \Gamma_v = \{(\mu, 0, \mu)\colon 0 < \mu < \infty\}. \tag{2.5}$$

From the strong maximum principle, any non-negative solution (u, v) of (2.2) is either $(0, 0)$, or semi-trivial, or positive.

Bifurcation could occur along the semi-trivial branches. We now set up the abstract framework for our bifurcation analysis. Let $\Omega_1 = \Omega \backslash \overline{\Omega}_0$. For $p > 1$, let $X_1 = \{u \in W^{2,p}(\Omega)\colon \partial_\nu u = 0$ on $\partial\Omega\}$, and let $Y_1 = L^p(\Omega)$. Similarly, let $X_2 = \{u \in W^{2,p}(\Omega_1)\colon \partial_\nu u = 0 \text{ on } \partial\Omega_1\}$, and let $Y_2 = L^p(\Omega_1)$. We first consider local bifurcation along Γ_u. We let $w = \lambda - u$, and define $G : \mathbf{R} \times X_1 \times X_2 \to Y_1 \times Y_2$ by

$$G(\mu, w, v) = \begin{pmatrix} \Delta w - \lambda w + w^2 + b(x)p(\lambda - w)v \\ \Delta v + \mu v - v^2 + cp(\lambda - w)v \end{pmatrix}^T. \tag{2.6}$$

It should be understood that in the first row, the function $b(x)p(\lambda - w)v$ is simply extended to zero in Ω_0 for $v \in X_2$. Similar convention shall be made in the following. From a simple calculation, we obtain

$$G_{(w,v)}(\mu, w, v)[\phi, \psi] = \begin{pmatrix} \Delta\phi - \lambda\phi + 2w\phi - b(x)p'(\lambda - w)v\phi + b(x)p(\lambda - w)\psi \\ \Delta\psi + \mu\psi - 2v\psi - cp'(\lambda - w)v\phi + cp(\lambda - w)\psi \end{pmatrix}^T,$$

$$G_\mu(\mu, w, v) = (0, v), \qquad G_{\mu(w,v)}(\mu, u, w)[\phi, \psi] = (0, \psi),$$

$$G_{(w,v)(w,v)}(\mu, w, v)[\phi, \psi]^2 = \begin{pmatrix} 2\phi^2 - 2b(x)p'(\lambda - w)\phi\psi + b(x)p''(\lambda - w)v\phi^2 \\ -2\psi^2 - 2cp'(\lambda - w)\phi\psi + cp''(\lambda - w)v\psi^2 \end{pmatrix}^T.$$

By letting $(w, v) = (0, 0)$, we can find that only when $\mu = -cp(\lambda)$ that $G_{(w,v)}(\mu, 0, 0)[\phi, \psi] = 0$ has a solution with $\psi > 0$; thus $\mu_1 := -cp(\lambda) = -c\lambda/(1 + m\lambda)$ is the only bifurcation point along Γ_u where positive solutions of (2.2) bifurcates. At $(\mu, u, w) = (\mu_1, 0, 0)$, it is easy to verify that the kernel $\mathcal{N}(G_{(w,v)}(\mu_1, 0, 0)) = \operatorname{span}\{(\varphi_1, \varphi_2)\}$, where $(\varphi_1, \varphi_2) \neq (0, 0)$ satisfies

$$\begin{cases} \Delta\phi - \lambda\phi + b(x)p(\lambda)\psi = 0 & \text{in } \Omega, \\ \Delta\psi + \mu_1\psi + cp(\lambda)\psi = 0 & \text{in } \Omega_1, \\ \partial_\nu\phi = 0 \quad \text{on } \partial\Omega, & \partial_\nu\psi = 0 \quad \text{on } \partial\Omega_1. \end{cases} \tag{2.7}$$

Since $\mu_1 = \lambda_1^N(-cp(\lambda)) = -cp(\lambda)$, we can choose $\varphi_2 = 1$, and then

$$\varphi_1 = (-\Delta + \lambda I)^{-1}[b(x)p(\lambda)] > 0.$$

The range of the operator is given by

$$\mathcal{R}(G_{(w,v)}(\mu_1, 0, 0)) = \left\{(f, g) \in Y_1 \times Y_2\colon \int_{\Omega_1} g(x)\,dx = 0\right\},$$

which is of co-dimension one, and

$$G_{\mu(w,v)}(\mu_1,0,0)[\varphi_1,\varphi_2]=(0,1)\notin \mathcal{R}\big(G_{(w,v)}(\mu_1,0,0)\big)$$

since $\int_{\Omega_1} 1\,dx>0$. Thus we can apply the result of [6] to conclude that the set of positive solutions to (2.2) near $(\mu_1,\lambda,0)$ is a smooth curve

$$\Gamma_1=\big\{\big(\mu_1(s),\lambda-u_1(s),v_1(s)\big)\colon s\in[0,\delta)\big\}, \tag{2.8}$$

such that $\mu_1(0)=-cp(\lambda)$, $u_1(s)=s\varphi_1(x)+o(|s|)$, $v_1(s)=s+o(|s|)$. Moreover $\mu_1'(0)$ can be calculated (see, for example, [39])

$$\mu_1'(0)=-\frac{\langle G_{(w,v)(w,v)}(\mu_1,0,0)[\varphi_1,\varphi_2]^2,l_1\rangle}{2\langle G_{\mu(w,v)}(\mu_1,0,0)[\varphi_1,\varphi_2],l_1\rangle}=1+\frac{cp'(\lambda)}{|\Omega|}\int\limits_{\Omega_1}\varphi_1(x)\,dx>0, \tag{2.9}$$

where l_1 is the linear functional on $Y_1\times Y_2$ defined by $\langle[f,g],l_1\rangle=\int_{\Omega_1} g(x)\,dx$. Therefore the bifurcation at $(\mu_1,\lambda,0)$ is always supercritical.

Next we consider the bifurcation along Γ_v. We use the change of variable $v=\mu+w$ (thus $(u,w)=(0,0)$ corresponds to the semi-trivial solution). Define $F:\mathbf{R}\times X_1\times X_2\to Y_1\times Y_2$ by

$$F(\mu,u,w)=\begin{pmatrix}\Delta u+\lambda u-u^2-b(x)p(u)(\mu+w)\\ \Delta w-\mu w-w^2+cp(u)(\mu+w)\end{pmatrix}^T. \tag{2.10}$$

A simple calculation shows

$$F_{(u,w)}(\mu,u,w)[\phi,\psi]=\begin{pmatrix}\Delta\phi+\lambda\phi-2u\phi-b(x)p'(u)(\mu+w)\phi-b(x)p(u)\psi\\ \Delta\psi-\mu\psi-2w\psi+cp'(u)(\mu+w)\phi+cp(u)\psi\end{pmatrix}^T,$$

$$F_\mu(\mu,u,w)=\begin{pmatrix}-b(x)p(u)\\ -w+cp(u)\end{pmatrix}^T,\qquad F_{\mu(u,w)}(\mu,u,w)[\phi,\psi]=\begin{pmatrix}-b(x)p'(u)\phi\\ -\psi+cp'(u)\phi\end{pmatrix}^T,$$

$$F_{(u,w)(u,w)}(\mu,u,w)[\phi,\psi]^2=\begin{pmatrix}-2\phi^2-2b(x)p'(u)\phi\psi-b(x)p''(u)(\mu+w)\phi^2\\ -2\psi^2+2cp'(u)\phi\psi+cp''(u)(\mu+w)\psi^2\end{pmatrix}^T.$$

The equation $F_{(u,w)}(\mu,0,0)[\phi,\psi]=0$ is equivalent to

$$\begin{cases}\Delta\phi+\lambda\phi-b(x)\mu\phi=0 & \text{in } \Omega,\\ \Delta\psi-\mu\psi+c\mu\phi=0 & \text{in } \Omega_1,\\ \partial_\nu\phi=0 \quad \text{on } \partial\Omega, & \partial_\nu\psi=0 \quad \text{on } \partial\Omega_1,\end{cases} \tag{2.11}$$

which has a solution with $\phi>0$ if and only if

$$\lambda=\lambda_1^N\big(b(x)\mu,\Omega\big). \tag{2.12}$$

Equation (2.12) will play a central role in our analysis to come.

Theorem 2.1. *We have the following results*:

(1) *If* $\lambda \geqslant \lambda_1^D(\Omega_0)$, *then*
   (a) $\mu_1 = -c\lambda/(1+m\lambda)$ *is a bifurcation point where an unbounded continuum* Γ_1 *of positive solutions to* (2.2) *bifurcates from* Γ_u *at* $(\mu, u, v) = (\mu_1, \lambda, 0)$;
   (b) *near* $(\mu_1, \lambda, 0)$, Γ_1 *is a smooth curve* $(\mu(s), u(s), v(s))$ *with* $s \in (0, \delta)$, *such that* $(\mu(0), u(0), v(0)) = (\mu_1, \lambda, 0)$ *and* $\mu'(0) > 0$;
   (c) $\mathrm{Proj}_\mu \Gamma_1 = (\mu_1, \infty)$, *and so* (2.2) *has at least one positive solution for any* $\mu > \mu_1$, *but* (2.2) *has no positive solution for* $\mu \leqslant \mu_1$;
   (d) $(0, \mu)$ *is an unstable steady state of* (2.1) *for any* $\mu > 0$ (*it is neutrally stable when* $\lambda = \lambda_1^D(\Omega_0)$), *and there is no bifurcation of positive solutions occurring along* Γ_v.
(2) *If* $0 < \lambda < \lambda_1^D(\Omega_0)$, *then*
   (a) *there exists a unique* $\mu_2 = \mu_2(\lambda) > 0$ *determined by* (2.12), *and a continuum* Γ_2 *of positive solutions to* (2.2) *bifurcating from* Γ_v *at* $(\mu_2, 0, \mu_2)$;
   (b) $\mu_2(\lambda)$ *is strictly increasing with respect to* λ, *and*

$$\lim_{\lambda \to 0^+} \mu_2(\lambda) = 0, \qquad \lim_{\lambda \to [\lambda_1^D(\Omega_0)]^-} \mu_2(\lambda) = \infty;$$

   (c) Γ_2 *is a smooth curve near the bifurcation point* $(\mu_2, 0, \mu_2)$, *the bifurcation is subcritical if* $0 < m < m_0$, *and supercritical if* $m > m_0$, *where* $m_0 = m_0(\lambda)$ *is defined by*

$$m_0(\lambda) = \frac{\int_\Omega \varphi_1^3\,dx + \int_\Omega b(x)\varphi_1^2\varphi_2\,dx}{\mu_2(\lambda)\int_\Omega b(x)\varphi_1^3\,dx}, \tag{2.13}$$

   and (φ_1, φ_2) *is a positive solution of* (2.11);
   (d) Γ_2 *can be extended to a bounded global continuum of positive solutions to* (2.2), *which meets* Γ_u *at* $(\mu_1, \lambda, 0)$, *where* $\mu_1 = -cp(\lambda) < 0$. *Therefore,* (2.2) *has at least one positive solution for* $\mu \in (\mu_1, \mu_2)$. *Moreover, it has no positive solution if* $\mu \leqslant \mu_1$ *or* $\mu \geqslant \mu_2(1 + m\lambda)$.

Proof. Suppose that (u, v) is a positive solution of (2.2). From the equation for u we obtain $-\Delta u \leqslant \lambda u - u^2$ in Ω, which implies, by a standard comparison argument,

$$0 \leqslant u \leqslant \lambda \quad \text{in } \Omega. \tag{2.14}$$

Then, from the equation for v, we find

$$-\Delta v \geqslant \mu v - v^2, \qquad -\Delta v \leqslant \left(\mu + \frac{c\lambda}{1+m\lambda}\right)v - v^2 \quad \text{in } \Omega_1.$$

Therefore a similar comparison argument yields

$$\max\{\mu, 0\} \leqslant v \leqslant \mu + \frac{c\lambda}{1+m\lambda} \quad \text{in } \Omega_1. \tag{2.15}$$

Moreover, we also have

$$\mu = \lambda_1^N\left(v - \frac{cu}{1+mu}, \Omega_1\right) > \lambda_1^N\left(-\frac{c\lambda}{1+m\lambda}, \Omega_1\right) = \mu_1.$$

Therefore there is no positive solution when $\mu \leqslant \mu_1$.

We now define a function $h(\mu)=\lambda_1^N(b(x)\mu,\Omega)$. Then $h(\mu)$ is a continuous increasing function from the properties of the first eigenvalue, and $h(0)=\lambda_1^N(\Omega)=0$. From the variational characterization of the eigenvalue,

$$\lambda_1^N\big(b(x)\mu,\Omega\big)=\inf_{\phi\in H^1(\Omega)}\frac{\int_\Omega|\nabla\phi|^2\,dx+\mu\int_\Omega b(x)\phi^2\,dx}{\int_\Omega\phi^2\,dx}\leqslant\lambda_1^D(\Omega_0). \tag{2.16}$$

Here we obtain the inequality by letting ϕ to be the eigenfunction associated with $\lambda_1^D(\Omega_0)$ when $x\in\Omega_0$ and $\phi=0$ when $x\in\Omega\backslash\Omega_0$. We must have $\lambda_1^N(b(x)\mu,\Omega)<\lambda_1^D(\Omega_0)$ for any $\mu>0$ since $h(\mu)$ is strictly increasing. Next we show that $h(\mu)\to\lambda_1^D(\Omega_0)$ as $\mu\to\infty$. Let $\phi_n>0$ satisfy

$$-\Delta\phi_n+b(x)\mu_n\phi_n=\lambda_1^N\big(b(x)\mu_n,\Omega\big)\phi_n\quad\text{in }\Omega,\qquad\partial_\nu\phi_n=0\quad\text{on }\partial\Omega, \tag{2.17}$$

where $\{\mu_n\}$ is a sequence satisfying $\mu_n\to\infty$, and we also assume that $\|\phi_n\|_\infty=1$. From (2.16) and $b(x)\geqslant 0$, we have $-\Delta\phi_n\leqslant\lambda_1^D(\Omega_0)\phi_n$, and

$$\int_\Omega|\nabla\phi_n|^2\,dx+\int_\Omega\phi_n^2\,dx\leqslant\big[\lambda_1^D(\Omega_0)+1\big]\int_\Omega\phi_n^2\,dx\leqslant\big[\lambda_1^D(\Omega_0)+1\big]|\Omega|. \tag{2.18}$$

Hence $\{\phi_n\}$ is bounded in $H^1(\Omega)$, and there is a subsequence (which we still denote by $\{\phi_n\}$) converging to some ϕ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Since $\|\phi_n\|_\infty=1$, we can also assume that $\phi_n\to\phi$ in $L^p(\Omega)$ for any $p>1$. From Lemma 2.2 of [10], we have $\|\phi\|_\infty=1$. On the other hand, since $\{\lambda_1^N(b(x)\mu_n,\Omega)\}$ is bounded, we may assume $h(\mu_n)=\lambda_1^N(b(x)\mu_n,\Omega)\to\lambda_\infty\in(0,\lambda_1^D(\Omega_0)]$. We multiply (2.17) by ϕ_n, and integrate over Ω, to find

$$\int_\Omega|\nabla\phi_n|^2\,dx-\lambda_1^N\big(b(x)\mu_n,\Omega\big)\int_\Omega\phi_n^2\,dx=-\mu_n\int_{\Omega_1}b(x)\phi_n^2\,dx. \tag{2.19}$$

As $n\to\infty$, the left-hand side of (2.19) is bounded due to (2.18), but $\mu_n\to\infty$, so it is necessary that $\int_{\Omega_1}b(x)\phi_n^2\,dx\to 0$, which implies

$$\int_{\Omega_1}b(x)\phi^2\,dx=0. \tag{2.20}$$

Since $b(x)>0$ in Ω_1, we must have $\phi(x)=0$ almost everywhere in Ω_1. Since $\partial\Omega_0$ is smooth, this implies that $\phi|_{\Omega_0}\in H_0^1(\Omega_0)$. Inside Ω_0, $b(x)\equiv 0$, thus by letting $n\to\infty$ in (2.17), and use weak formulation, we find that $\phi|_{\Omega_0}$ is a weak non-negative solution of

$$-\Delta\phi=\lambda_\infty\phi\quad\text{in }\Omega_0,\qquad\phi(x)=0\quad\text{on }\partial\Omega_0. \tag{2.21}$$

Then λ_∞ must equal to $\lambda_1^D(\Omega_0)$ for otherwise $\phi=0$ in Ω_0 and hence $\phi=0$ in Ω, which contradicts with $\|\phi\|_\infty=1$. Since $h(\mu)$ is strictly increasing, we conclude $\lim_{\mu\to\infty}h(\mu)=\lambda_\infty=\lambda_1^D(\Omega_0)$.

Now if $\lambda\geqslant\lambda_1^D(\Omega_0)$, then for any $\mu>0$, $\lambda>\lambda_1^N(b(x)\mu,\Omega)$. Hence, by (2.12), no bifurcation of positive solutions can occur along Γ_v, and it also implies that $(0,\mu)$ is unstable since

$\lambda_1^N(-\lambda + b(x)\mu, \Omega) < 0$. On the other hand, we already know from earlier arguments that a branch of positive solutions Γ_1 bifurcates from Γ_u at $(\mu_1, \lambda, 0)$, which is supercritical. Combining this with a standard global bifurcation argument, as in [1], and the bound obtained above for possible positive solutions (μ, u, v) of (2.2), and the fact that no bifurcation of positive solutions occurs along Γ_v, we find that Γ_1 can be extended to an unbounded continuum of positive solutions of (2.2) and $\mathrm{Proj}_\mu \Gamma_1 = (\mu_1, \infty)$. This finishes the proof for part (1).

When $0 < \lambda < \lambda_1^D(\Omega_0)$, there exists a unique $\mu_2(\lambda)$ such that $\lambda = \lambda_1^N(b(x)\mu_2, \Omega)$ due to the monotonicity of $h(\mu)$, and the continuity and monotonicity of $\mu_2(\lambda)$ also follow. Since $h(0) = 0$, $h(\infty) = \lambda_1^D(\Omega_0)$ and $h(\mu)$ is strictly increasing, we easily see that $\mu_2(\lambda) \to 0$ as λ decreases to 0, and $\mu_2(\lambda) \to \infty$ as λ increases to $\lambda_1^D(\Omega_0)$.

We now show that (2.2) has no positive solution if $\mu \geqslant (1 + m\lambda)\mu_2$. Indeed, if (u, v) is a positive solution of (2.2), then since $v \geqslant \mu$ and $u \leqslant \lambda$,

$$\lambda_1^N\big(b(x)\mu_2\big) = \lambda = \lambda_1^N\left(u + \frac{b(x)v}{1+mu}\right) > \lambda_1^N\left(\frac{b(x)\mu}{1+m\lambda}\right),$$

which implies $\mu_2 > \mu/(1 + m\lambda)$.

At $(\mu, u, w) = (\mu_2, 0, 0)$, $\mathcal{N}(F_{(u,w)}(\mu_2, 0, 0)) = \mathrm{span}\{(\varphi_1, \varphi_2)\}$, and we can choose $\varphi_1 > 0$, and $\varphi_2 = (-\Delta + \mu I)^{-1}[c\mu\varphi_1] > 0$. The range

$$\mathcal{R}\big(F_{(u,w)}(\mu_2, 0, 0)\big) = \left\{(f, g) \in Y_1 \times Y_2\colon \int_\Omega f(x)\varphi_1(x)\,dx = 0\right\},$$

and

$$F_{\mu(u,w)}(\mu_2, 0, 0)[\varphi_1, \varphi_2] = (-b\varphi_1, -\varphi_2 + c\varphi_1) \notin \mathcal{R}\big(F_{(u,w)}(\mu_2, 0, 0)\big)$$

since $-\int_\Omega b(x)\varphi_1^2\,dx \neq 0$. Thus we can apply the result of [6] to conclude that the set of positive solutions to (2.2) near $(\mu_2, 0, \mu_2)$ is a smooth curve

$$\Gamma_2 = \big\{\big(\mu_2(s), u_2(s), \mu_2 + w_2(s)\big)\colon s \in (0, \delta)\big\}, \tag{2.22}$$

such that $\mu_2(0) = \mu_2$, $u_2(s) = \varphi_1 s + o(|s|)$, $w_2(s) = \varphi_2 s + o(|s|)$. Moreover $\mu_2'(0)$ can be calculated:

$$\begin{aligned}\mu_2'(0) &= -\frac{\langle F_{(u,w)(u,w)}(\mu_2, 0, 0)[\varphi_1, \varphi_2]^2, l_2\rangle}{2\langle F_{\mu(u,w)}(\mu_2, 0, 0)[\varphi_1, \varphi_2], l_2\rangle} \\ &= \frac{-\int_\Omega \varphi_1^3\,dx - \int_\Omega b(x)\varphi_1^2\varphi_2\,dx + \mu_2 m\int_\Omega b(x)\varphi_1^3\,dx}{\int_\Omega b(x)\varphi_1^2\,dx},\end{aligned} \tag{2.23}$$

where l_2 is a linear functional on $Y_1 \times Y_2$ defined by $\langle (f, g), l_2\rangle = \int_\Omega f(x)\varphi_1(x)\,dx$.

Due to the bound obtained for positive solutions (μ, u, v) of (2.2), conclusion (d) in part (2) follows from a standard global bifurcation consideration, as in [1]; we omit the details. □

The above result shows that for fixed prey growth rate λ, the value of $\lambda_1^D(\Omega_0)$ determines the bifurcation structure of (2.2). As will become clear later, it also plays a crucial role in determining the dynamics of (2.1). Accordingly, we will divide our discussions below into two cases:

(a) small protection zone: $\lambda < \lambda_1^D(\Omega_0)$, and
(b) large protection zone: $\lambda > \lambda_1^D(\Omega_0)$.

3. The large protection zone case

In this section we consider the set of positive steady state solutions and related dynamical behavior of (2.1) when the protection zone Ω_0 is large so that $\lambda_1^D(\Omega_0) < \lambda$. Let us recall that we assume $\overline{\Omega}_0 \subset \Omega$.

3.1. Steady state solutions

In this subsection, we consider the existence, uniqueness and asymptotic profile of the positive steady state solutions of (2.2). The existence problem was already considered in Theorem 2.1, which shows that (2.2) has no positive solution if $\mu \leqslant \mu_1 = -c\lambda/(1 + m\lambda)$, and there is at least one positive solution if $\mu > \mu_1$. Moreover, there is an unbounded continuum Γ_1 of positive solutions to (2.2) emanating from $(\mu, u, v) = (\mu_1, \lambda, 0) \in \Gamma_u$, and $\mathrm{Proj}_\mu \Gamma_1 = (\mu_1, \infty)$.

In order to obtain more information on the positive solutions, we study the scalar equation

$$-\Delta u = \lambda u - u^2 - b(x)\mu \frac{u}{1 + mu} \quad \text{in } \Omega, \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega. \tag{3.1}$$

The following lemma will be useful in our later discussions.

Lemma 3.1. *Suppose that $f : \Omega \times \mathbf{R}^+ \to \mathbf{R}$ is a continuous function such that $f(x, s)$ is decreasing for $s > 0$ at almost all $x \in \overline{\Omega}$. Let $w, v \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfy*:

(1) $\Delta w + wf(x, w) \leqslant 0 \leqslant \Delta v + vf(x, v)$ *in* Ω,
(2) $w, v > 0$ *in* Ω *and* $w \geqslant v$ *on* $\partial\Omega$,
(3) $\Delta v \in L^1(\Omega)$.

Then $w \geqslant v$ *in* $\overline{\Omega}$.

This lemma is well known, and can be found in [40] (see Lemma 2.3); a more general version can be found in [23] (see Lemma 2.1). We have the following results on (3.1).

Proposition 3.2. *Suppose that* $\lambda > \lambda_1^D(\Omega_0)$. *Then*:

(1) *For each* $\mu \leqslant 0$, (3.1) *has a unique positive solution* U_μ, *which is strictly decreasing in* μ, *and* $\{(\mu, U_\mu)\colon\ \mu \leqslant 0\}$ *is a smooth curve.*
(2) *For each* $\mu > 0$, (3.1) *has a minimal positive solution* $\underline{U}_\mu$ *and a maximal positive solution* $\overline{U}_\mu$, *and they satisfy*

$$W_\lambda(x) < \underline{U}_\mu(x) \leqslant \overline{U}_\mu(x) < \lambda, \quad \forall x \in \Omega, \tag{3.2}$$

where W_λ *is defined by* $W_\lambda(x) = 0$ *in* $\Omega \backslash \overline{\Omega}_0$, $W_\lambda = w_\lambda$ *in* Ω_0, *where* w_λ *denotes the unique positive solution of*

$$-\Delta w = \lambda w - w^2 \quad \textit{in } \Omega_0, \qquad w = 0 \quad \textit{on } \partial\Omega_0. \tag{3.3}$$

(3) *There exists* $\overline{\mu}^* > 0$ *such that for* $\mu > \overline{\mu}^*$, $\overline{U}_\mu = \underline{U}_\mu$, *and* (3.1) *has a unique positive solution, which we denote by* U_μ, *and* $\{(\mu, U_\mu)\colon\ \mu > \overline{\mu}^*\}$ *is a smooth curve. Moreover, as* $\mu \to \infty$, $U_\mu \to W_\lambda$ *in* $L^p(\Omega)$ *for any* $p > 1$.

Proof. For $\mu \leqslant 0$, $f(x,u) = \lambda u - u^2 - b(x)\mu u/(1+mu)$ is a concave function such that $f(x,0) = 0$ and $f_u(x,0) > 0$; thus (3.1) is a logistic type equation. Existence and uniqueness of positive solutions is well known in that case. It is also easy to see and well known that the unique solution U_μ is globally asymptotically stable for the corresponding parabolic equation when $\mu \leqslant 0$. In particular, U_μ is non-degenerate so $\{(\mu, U_\mu)\colon\ \mu \leqslant 0\}$ is a smooth curve. Note that when $\mu = 0$, $U_0 \equiv \lambda$.

For $\mu > 0$, if u is a positive solution of (3.1), then a simple comparison argument shows $u < \lambda$. On the other hand, for $x \in \Omega_0$, $-\Delta u = \lambda u - u^2$, and $u > 0$ for $x \in \partial\Omega_0$; therefore $u > w_\lambda$ from Lemma 3.1. Hence $u > W_\lambda$ for $x \in \overline{\Omega}$. Therefore any positive solution u must belong to the order interval $\{v \in C(\overline{\Omega})\colon\ W_\lambda \leqslant v \leqslant \lambda\}$. On the other hand, λ is a supersolution of (3.1), and W_λ is a weak subsolution of (3.1). Thus the existence of the minimal and maximal solutions follows from the weak sub- and supersolution method.

To study the asymptotic behavior of positive solutions when $\mu \to \infty$, we take a sequence of positive solutions of (3.1), $\{(\mu_n, u_n)\}$, with $\mu_n \to \infty$ as $n \to \infty$. Since $-\Delta u_n \leqslant \lambda u_n$, from the same proof in that of Theorem 2.1, we can assume that $u_n \to \hat{u}$ weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$ for any $p > 1$. On the other hand, from (3.2), we have $\hat{u} \geqslant W_\lambda$. We multiply (3.1) by u_n, integrate over Ω, to obtain

$$\int_\Omega |\nabla u_n|^2\,dx - \lambda \int_\Omega u_n^2\,dx + \int_\Omega u_n^3\,dx = -\mu_n \int_{\Omega_1} \frac{b(x)u_n^2}{1+mu_n}\,dx. \tag{3.4}$$

Each integral on the left-hand side of (3.4) is bounded, thus when $n \to \infty$, we must have $\int_{\Omega_1} b(x)u_n^2(1+mu_n)^{-1}\,dx \to 0$, which implies

$$\int_{\Omega_1} \frac{b(x)\hat{u}^2}{1+m\hat{u}}\,dx = 0. \tag{3.5}$$

But $b(x) > 0$ in Ω_1; hence $\hat{u} = 0$ almost everywhere in Ω_1, and due to the smoothness of $\partial\Omega_0$, we conclude that $\hat{u}|_{\Omega_0} \in H_0^1(\Omega_0)$. In Ω_0, u_n satisfies a logistic equation, and $u_n > w_\lambda$. It follows that $\hat{u}|_{\Omega_0}$ is a positive solution of (3.3), and hence $\hat{u}|_{\Omega_0} = w_\lambda$ due to uniqueness. Thus the entire sequence u_n converges to W_λ in $L^p(\Omega)$ for any $p > 1$.

Next we show that for all large n, u_n is linearly stable, that is, if the eigenvalue problem

$$-\Delta\psi_n = \lambda\psi_n - 2u_n\psi_n - \mu_n b(x)p'(u_n)\psi_n + \eta_n\psi_n \quad \text{in } \Omega, \qquad \partial_\nu\psi_n = 0 \quad \text{on } \partial\Omega \tag{3.6}$$

has a solution (η_n, ψ_n) with $\psi_n > 0$ and $\|\psi_n\|_{L^2(\Omega)} = 1$, then $\eta_n > 0$. We first claim that $\{\eta_n\}$ is bounded. Indeed, from the variational characterization of η_n, we easily see that $\eta_n \geqslant -\lambda$, and

$$\eta_n = \inf_{\psi \in H^1(\Omega),\ \|\psi\|_{L^2(\Omega)}=1} \int_\Omega \left[|\nabla\psi|^2 - \lambda\psi^2 + 2u_n\psi^2 + \mu_n b(x)p'(u_n)\psi^2\right] dx$$

$$\leqslant \int_{\Omega_0} \left[|\nabla \psi^*|^2 - \lambda (\psi^*)^2 + 2u_n (\psi^*)^2 \right] dx = \eta^* + 2 \int_{\Omega_0} |u_n - w_\lambda| (\psi^*)^2 \, dx \to \eta^*, \quad (3.7)$$

as $n \to \infty$, where (η^*, ψ^*) is the principal eigen-pair of

$$-\Delta \psi = \lambda \psi - 2w_\lambda \psi + \eta \psi \quad \text{in } \Omega_0, \qquad \psi(x) = 0 \quad \text{on } \partial \Omega_0, \quad (3.8)$$

with $\psi^* \geqslant 0$ extended by 0 outside Ω_0, and $\|\psi^*\|_{L^2(\Omega)} = 1$. Hence $-\lambda \leqslant \eta_n \leqslant \eta^* + 1$ for n large, and we may assume that $\eta_n \to \hat{\eta}$ as $n \to \infty$. Then from (3.6) we find that $\int_\Omega |\nabla \psi_n|^2 \, dx$ stays bounded, and hence subject to a subsequence, we can assume that $\psi_n \to \hat{\psi}$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Multiplying (3.6) by an arbitrary function $\phi \in C_0^\infty(\Omega)$ and integrating over Ω, we find that $\mu_n \int_\Omega b(x) p'(u_n) \psi_n \phi \, dx$ stays bounded as $n \to \infty$. It follows that $\int_\Omega b(x) p'(W_\lambda) \hat{\psi} \phi \, dx = 0$, which implies that $\hat{\psi} = 0$ almost everywhere in Ω_1, and $\hat{\psi}|_{\Omega_0} \in H_0^1(\Omega_0)$. We then easily deduce that $\hat{\psi}|_{\Omega_0} \geqslant 0$ is a solution of (3.8) with $\eta = \hat{\eta}$. Since $\|\psi_n\|_{L^2(\Omega)} = 1$, we have $\|\hat{\psi}\|_{L^2(\Omega)} = 1$ and therefore $\hat{\psi}|_{\Omega_0} \neq 0$. This implies that $\hat{\eta}$ is the principal eigenvalue of (3.8), which is well known to be positive. Thus $\eta_n > 0$ when n is sufficiently large, and we have proved that for large μ, any positive solution u of (3.1) is linearly stable.

We can now apply a standard fixed point index argument to show that for all large μ, (3.1) has a unique positive solution. Since the argument is rather well known, we only indicate the main steps. Firstly, Eq. (3.1) can be transformed into an equivalent fixed point equation of the form $A_\mu u = u$ with A_μ a completely continuous operator on $C(\overline{\Omega})$ for each $\mu > 0$, and A_μ maps $B_\mu := \{u \in C(\overline{\Omega}): 0 \leqslant u \leqslant \lambda\}$ into the cone K of non-negative functions in $C(\overline{\Omega})$. Secondly, the fixed point index $i_K(A_\mu, B_\mu)$ is well defined, and by homotopy invariance, it is easy to see that $i_K(A_\mu, B_\mu) = 1$. On the other hand, for large μ, $u = 0$ is an isolated fixed point of A_μ with fixed point index 0, and by the above proved linear stability for positive solutions of (3.1), each positive fixed point of A_μ is isolated and has fixed point index 1. By the compactness of A_μ, only finitely many isolated positive fixed points can exist; let them be $u_1, \ldots, u_i$. Then from the additivity property of the fixed point index, we obtain

$$1 = i_K(A_\mu, B_\mu) = i_K(A_\mu, 0) + i_K(A_\mu, u_1) + \cdots + i_K(A_\mu, u_i) = i.$$

Therefore there is a unique positive fixed point for all large μ.

For large μ, the linear stability implies the non-degeneracy of the unique positive solution, and thus the solutions form a smooth curve $\{(\mu, u)\}$ in $\mathbf{R} \times C(\overline{\Omega})$ when μ is large. This finishes the proof. □

The following result improves the asymptotic behavior described in Proposition 3.2 for U_μ as $\mu \to \infty$; it also plays an important role in our later analysis.

Proposition 3.3. *$U_\mu \to W_\lambda$ in $C(\overline{\Omega})$ as $\mu \to \infty$.*

Proof. For clarity, we divide the proof into three steps.

Step 1. As $\mu \to \infty$, $U_\mu \to W_\lambda$ uniformly on compact subsets of Ω_0.

On Ω_0, we have $-\Delta U_\mu = \lambda U_\mu - U_\mu^2$. Since $0 < U_\mu < \lambda$, we find that $-\Delta U_\mu$ has an L^∞-bound over Ω_0 which is independent of μ. Therefore, by standard interior L^p-estimates, for any compact subset K of Ω_0 and any $p > 1$, the $W^{2,p}(K)$-norm of $U_\mu|_K$ has a bound independent of μ. Using the Sobolev imbedding theorem we now easily see that for any sequence $\mu_n \to \infty$, U_{μ_n} has a subsequence which converges in $C^1(K)$. But we already know from Proposition 3.2 that $U_\mu \to W_\lambda$ in $L^2(\Omega)$. Therefore, necessarily $U_{\mu_n} \to W_\lambda$ uniformly on K. This implies that $U_\mu \to W_\lambda$ as $\mu \to \infty$ uniformly on any compact subset of Ω_0.

Step 2. As $\mu \to \infty$, $U_\mu \to W_\lambda$ uniformly on compact subsets of $\overline{\Omega} \setminus \overline{\Omega}_0$.

Let $\mu_n \to \infty$ be an arbitrary increasing sequence. It suffices to show that $U_n := U_{\mu_n} \to 0$ uniformly on compact subsets of $\overline{\Omega} \setminus \overline{\Omega}_0$, where W_λ is identically zero by definition.

For any point $x_0 \in \Omega \setminus \overline{\Omega}_0$, we can find $\delta > 0$ small so that the closed ball $\overline{B}_\delta(x_0)$ is contained in $\Omega \setminus \overline{\Omega}_0$, and $b(x) \geqslant \sigma_\delta > 0$ in this ball. It follows that, for $x \in B_\delta(x_0)$,

$$\lambda - \mu_n \frac{b(x)}{1 + mU_n} \leqslant \lambda - \mu_n \frac{\sigma_\delta}{1 + m\lambda} =: \lambda_n \to -\infty.$$

Consider the auxiliary problem

$$-\Delta V = \lambda_n V - V^2 \quad \text{in } B_\delta(x_0), \qquad V|_{\partial B_\delta(x_0)} = \lambda. \tag{3.9}$$

By Lemma 3.1 and a simple sub- and supersolution argument, we find that (3.9) has a unique positive solution V_n and $U_n \leqslant V_n \leqslant \lambda$ in $B_\delta(x_0)$. The uniqueness implies that V_n is radially symmetric; so we may write $V_n(x) = Z_n(r)$, $r = |x - x_0|$, and we have

$$\left(r^{N-1} Z_n'\right)' = r^{N-1}\left(-\lambda_n Z_n + Z_n^2\right) > 0, \quad Z_n'(0) = 0.$$

It follows that $r^{N-1} Z_n'(r) > 0$ for $r \in (0, \delta)$ and hence $Z_n(r)$ is increasing in r. Since λ_n is decreasing, we can use Lemma 3.1 to conclude that Z_n decreases in n and therefore $V_\infty(x) = Z_\infty(|x - x_0|) = \lim_{n\to\infty} Z_n(|x - x_0|)$ is a well-defined non-negative function. Moreover, $V_n \to V_\infty$ weakly in $L^2(B_\delta(x_0))$. We must have $V_\infty \equiv 0$, since for any $\phi \in C_0^\infty(B_\delta(x_0))$,

$$\int\limits_{B_\delta(x_0)} V_\infty \phi\, dx = \lim_{n\to\infty} \int\limits_{B_\delta(x_0)} V_n \phi\, dx = \lim_{n\to\infty} \lambda_n^{-1} \int\limits_{B_\delta(x_0)} \left(V_n(-\Delta\phi) + V_n^2 \phi\right) dx = 0.$$

Due to the monotonicity of $Z_n(r)$, for any $r_0 \in (0, \delta)$, we deduce from $Z_n(r_0) \to 0$ that $Z_n(r) \to 0$ uniformly for $r \in [0, r_0]$. Therefore $V_n(x) \to 0$ uniformly in $B_{\delta/2}(x_0)$. Since $0 \leqslant U_n \leqslant V_n$ in $B_\delta(x_0)$, the above conclusion also holds for U_n.

If $x_0 \in \partial\Omega$, the above argument needs to be modified. We choose $y_0 \in \Omega \setminus \overline{\Omega}_0$ and $\delta > 0$ such that $B_\delta(y_0) \cap \overline{\Omega}_0 = \emptyset$, $x_0 \in \Gamma := B_\delta(y_0) \cap \partial\Omega$ and $\nu(x) \cdot (x - y_0) > 0$ for $x \in \Gamma$, where $\nu(x)$ denotes the outward unit normal of $\partial\Omega$ at x. Let $\sigma_\delta > 0$ be such that $b(x) \geqslant \sigma_\delta$ in $B_\delta(y_0) \cap \Omega$. We can now define λ_n, V_n and Z_n as before except that x_0 is replaced by y_0. We find from $Z_n'(r) > 0$ that $\partial_\nu V_n > 0$ on Γ. We can now apply Lemma 3.1 in [18] to conclude that $U_n \leqslant V_n$ in $B_\delta(y_0) \cap \Omega$. (Note that the proof of Lemma 3.1 in [18] is not affected if the solutions are C^1 instead of C^2 as assumed there.) Therefore if $\delta' \in (0, \delta)$ is such that $\overline{B}_{\delta'}(x_0) \subset B_\delta(y_0)$, then $U_n \to 0$ uniformly on $B_{\delta'}(x_0) \cap \Omega$.

If K is a compact subset of $\overline{\Omega} \setminus \overline{\Omega}_0$, then it can be covered by finitely many balls of the form $B_{\delta/2}(x_0)$ and $B_{\delta'}(x_0)$ as obtained above. It follows that $U_n \to 0$ uniformly on A. This proves Step 2.

Step 3. As $\mu \to \infty$, $U_\mu \to W_\lambda$ uniformly on $\overline{\Omega}$.

For $\delta > 0$, denote

$$\Omega_\delta = \{x \in \Omega\colon d(x, \Omega_0) < \delta\}, \qquad b_\delta(x) = b(x)(1 - \chi_{\Omega_\delta}).$$

Since $\lambda > \lambda_1^D(\Omega_0)$, by choosing $\delta > 0$ small enough, we may assume that $\lambda > \lambda_1^D(\Omega_\delta)$. By Proposition 3.2, we know that for all large μ, (3.1) with $b(x)$ replaced by $b_\delta(x)$ has a unique positive solution U_μ^δ and $U_\mu^\delta \to W_\lambda^\delta$ in $L^p(\Omega)$ for any $p > 1$, where W_λ^δ is the obvious variation of W_λ. From Steps 1 and 2 above, we also know that $U_\mu^\delta \to W_\lambda^\delta$ uniformly on any compact subsets of $\overline{\Omega} \setminus \partial\Omega_\delta$. Since $b_\delta \leqslant b$ in Ω, a simple comparison argument shows that $U_\mu^\delta \geqslant U_\mu$ for all large μ.

We now note that w_λ, the unique positive solution of (3.3), varies in a continuous way with smooth changes of Ω_0. This fact is folklore and can be proved by classical perturbation arguments. As a result, we find that $W_\lambda^\delta \to W_\lambda$ in $C(\overline{\Omega})$ as $\delta \to 0$. Now, for any given $\epsilon > 0$, we can find $\delta > 0$ small so that $W_\lambda, W_\lambda^\delta < \epsilon$ when $d(x, \partial\Omega_0) < \delta$. Applying Steps 1 and 2 for U_μ^δ, we can find $\mu_1^* = \mu_1^*(\delta) > 0$ large enough so that

$$\left|U_\mu^\delta(x) - W_\lambda^\delta(x)\right| < \epsilon \quad \text{when } d(x, \partial\Omega_0) < \delta/2.$$

Hence,

$$0 < U_\mu(x) \leqslant U_\mu^\delta(x) \leqslant W_\lambda^\delta(x) + \epsilon < 2\epsilon \quad \text{when } d(x, \partial\Omega_0) < \delta/2.$$

It follows that

$$\left|U_\mu(x) - W_\lambda(x)\right| \leqslant 3\epsilon \quad \text{when } d(x, \partial\Omega_0) < \delta/2.$$

By Steps 1 and 2, we can find $\mu_2^* = \mu_2^*(\delta) > 0$ large such that the above inequality holds for $\mu \geqslant \mu_2^*$ and all $x \in \overline{\Omega}$ such that $d(x, \partial\Omega_0) > \delta/3$. Therefore,

$$\left|U_\mu(x) - W_\lambda(x)\right| \leqslant 3\epsilon, \quad \forall x \in \overline{\Omega},\ \forall \mu > \max\{\mu_1^*, \mu_2^*\}.$$

This completes the proof for Step 3 and hence finishes the proof. □

Making use of Propositions 3.2 and 3.3, we can now have a better characterization of the set of positive solutions of (2.2), especially when μ is large.

Theorem 3.4. *Suppose that $\lambda > \lambda_1^D(\Omega_0)$. Then*:

(1) *There exists $\delta > 0$ such that* (2.2) *has a unique positive solution when $\mu \in (\mu_1, \mu_1 + \delta)$.*
(2) *For any $\mu > \mu_1$, if (u, v) is a positive solution of* (2.2)*, then*

$$\underline{U}_{\mu+c/m}(x) \leqslant u(x) \leqslant \overline{U}_\mu(x), \qquad \max\{\mu, 0\} \leqslant v(x) \leqslant \mu + c/m, \tag{3.10}$$

where $\underline{U}_\mu$ and $\overline{U}_\mu$ are the minimal and maximal solutions of (3.1)*, respectively.*

(3) *There exists* $\mu^* > 0$ *such that* (2.2) *has a unique positive solution* (u_μ, v_μ) *when* $\mu \geqslant \mu^*$, *and* (u_μ, v_μ) *is linearly stable in the sense that* $\mathrm{Re}(\eta) > 0$ *if* η *is an eigenvalue of the linearized eigenvalue problem at* (u_μ, v_μ). *Moreover, when* $\mu \to \infty$, $u_\mu \to W_\lambda$ *uniformly in* $\overline{\Omega}$, *and* $v_\mu - \mu \to 0$ *uniformly in* $\overline{\Omega}_1$.

Proof. By (2.15), we find that if (u, v) is a positive solution of (2.2) with μ close to μ_1, then $\|v\|_\infty$ is small and hence u must be close to λ. The uniqueness for $\mu \in (\mu_1, \mu_1 + \delta)$ with small enough $\delta > 0$ now follows from our earlier local bifurcation analysis near $(\mu_1, \lambda, 0)$.

If (u, v) is any positive solution of (2.2), then from (2.15), $\max\{\mu, 0\} \leqslant v(x) \leqslant \mu + c/m$, and the estimate of u in (3.10) can be obtained by substituting this estimate of v into the equation for u and using the minimality and maximality of the solutions to (3.1).

Now we consider a positive solution (u, v) of (2.2) with large μ. Recall that the principal eigenvalue η^* of (3.8) is positive. The linearized eigenvalue problem of (2.2) at (u, v) is given by

$$\begin{cases} -\Delta\phi = \lambda\phi - 2u\phi - b(x)p'(u)v\phi - b(x)p(u)\psi + \eta\phi & \text{in } \Omega, \\ -\Delta\psi = \mu\psi - 2v\psi + cp'(u)v\phi + cp(u)\psi + \eta\psi & \text{in } \Omega_1, \\ \partial_\nu\phi = 0 \quad \text{on } \partial\Omega, \qquad \partial_\nu\psi = 0 & \text{on } \partial\Omega_1. \end{cases} \tag{3.11}$$

Here, unlike elsewhere in the paper, ϕ, ψ and η may be complex-valued. We claim that for any $\delta > 0$, there is $\mu_\delta > 0$ such that when $\mu \geqslant \mu_\delta$, any eigenvalue η of (3.11) satisfies $\mathrm{Re}(\eta) \geqslant \eta^* - \delta$. Otherwise there exists some small $\delta > 0$ and a sequence of solutions $(\mu_n, \eta_n, u_n, v_n, \phi_n, \psi_n)$ of (3.11) such that $\mu_n \to \infty$, $\mathrm{Re}(\eta_n) < \eta^* - \delta$, and $\|\psi_n\|_2 + \|\phi_n\|_2 = 1$. From Kato's inequality, we have

$$\begin{aligned} -\Delta|\phi_n| &\leqslant -\mathrm{Re}\left(\frac{\overline{\phi_n}}{|\phi_n|}\Delta\phi_n\right) \\ &\leqslant \lambda|\phi_n| - 2u_n|\phi_n| - b(x)p'(u_n)v_n|\phi_n| + b(x)p(u_n)|\psi_n| + (\eta^* - \delta)|\phi_n| \\ &\leqslant \left[\lambda - 2u_n - b(x)p'(u_n)\mu_n + \eta^* - \delta\right]|\phi_n| + b(x)p(u_n)|\psi_n|. \end{aligned}$$

Thus we have

$$\mathcal{L}_n|\phi_n| := \left(-\Delta - \lambda + 2u_n + b(x)p'(u_n)\mu_n - \eta^* + \delta\right)|\phi_n| \leqslant b(x)p(u_n)|\psi_n|. \tag{3.12}$$

From (3.10) and the uniqueness of positive solutions to (3.1), we have $U_{\mu_n + c/m} \leqslant u_n \leqslant U_{\mu_n}$. Since $U_\mu \to W_\lambda$ uniformly in $\overline{\Omega}$ as $\mu \to \infty$, so is u_n. From the same proof as in that of Proposition 3.2, one can show that the principal eigenvalue of $-\Delta - \lambda + 2u_n + b(x)p'(u_n)\mu_n$ approaches η^* as $n \to \infty$. Therefore the principal eigenvalue of the operator $\mathcal{L}_n$ is bounded from below by $\delta/2$ for large n. It follows that

$$\frac{\delta}{2}\int_\Omega |\phi_n|^2 \leqslant \int_\Omega \left(\mathcal{L}_n|\phi_n|\right)|\phi_n|\,dx \leqslant \int_{\Omega_1} b(x)p(u_n)|\psi_n||\phi_n|\,dx. \tag{3.13}$$

But the right-hand side of (3.13) converges to 0 since by Proposition 3.3, $p(u_n) \to 0$ uniformly for $x \in \Omega_1$. It follows that $\|\phi_n\|_2 \to 0$ and $\|\psi_n\|_2 \to 1$ as $n \to \infty$. From Kato's inequality again, we have

$$
\begin{aligned}
-\Delta|\psi_n| &\leqslant -\mathrm{Re}\left(\frac{\overline{\psi_n}}{|\psi_n|}\Delta\psi_n\right)\\
&\leqslant \mu_n|\psi_n| - 2v_n|\psi_n| + cp(u_n)|\psi_n| + cp'(u_n)v_n|\phi_n| + (\eta^* - \delta)|\psi_n|\\
&\leqslant \left[-\mu_n + \frac{c\lambda}{1+m\lambda} + \eta^* - \delta\right]|\psi_n| + cp'(u_n)v_n|\phi_n|. \qquad (3.14)
\end{aligned}
$$

We multiply (3.14) by $|\psi_n|/\mu_n$, and integrate over Ω_1, to obtain

$$
\begin{aligned}
\int_{\Omega_1} |\psi_n|^2\,dx &\leqslant \frac{c\lambda(1+m\lambda)^{-1} + \eta^* - \delta}{\mu_n}\int_{\Omega_1}|\psi_n|^2\,dx + \frac{c}{\mu_n}\int_{\Omega_1} p'(u_n)v_n|\phi_n|\cdot|\psi_n|\,dx\\
&\leqslant \frac{c\lambda(1+m\lambda)^{-1} + \eta^* - \delta}{\mu_n}\int_{\Omega_1}|\psi_n|^2\,dx + \frac{c(\mu_n + c/m)}{\mu_n}\|\phi_n\|_2\|\psi_n\|_2. \qquad (3.15)
\end{aligned}
$$

But as $n \to \infty$, the left-hand side of (3.15) approaches 1 while the right-hand side of (3.15) approaches 0, which is a contradiction. Therefore there exists $\mu_\delta > 0$ such that $\mathrm{Re}(\eta) \geqslant \eta^* - \delta$ for any positive solution (u, v) as long as $\mu \geqslant \mu_\delta$.

Since any positive solution (u, v) is linearly stable when $\mu \geqslant \mu_\delta$, we can use a fixed point index argument, similar to that in the proof of Proposition 3.2, to show that there is only one positive solution when μ is large. We omit the details since the presentation is long but standard.

The asymptotic behavior of the unique positive solution (u_μ, v_μ) for large μ follows easily from (3.10) and Proposition 3.3. Indeed, since $U_{\mu+c/m} \leqslant u_\mu \leqslant U_\mu$ and as $\mu \to \infty$, $U_\mu \to W_\lambda$ uniformly in $\overline{\Omega}$, so is u_μ. Now for any given $\epsilon > 0$, we can find $\mu(\epsilon)$ large so that for $\mu > \mu(\epsilon)$, $u_\mu < \epsilon$ in Ω_1 (since $W_\lambda = 0$ in Ω_1). It follows that, for $\mu > \mu(\epsilon)$,

$$
\mu v_\mu - v_\mu^2 \leqslant -\Delta v_\mu \leqslant (\mu + c\epsilon)v_\mu - v_\mu^2 \quad \text{in } \Omega_1.
$$

This implies that $\mu \leqslant v_\mu \leqslant \mu + c\epsilon$ for $\mu > \mu(\epsilon)$. □

3.2. Dynamical behavior

First we consider the dynamics of the auxiliary equation

$$
\begin{cases}
u_t - \Delta u = \lambda u - u^2 - b(x)\mu\dfrac{u}{1+mu}, & x \in \Omega,\ t > 0,\\
\partial_\nu u = 0, \quad x \in \partial\Omega,\ t > 0,\\
u(x, 0) = u_0(x) \geqslant, \not\equiv 0, \quad x \in \Omega.
\end{cases}
\qquad (3.16)
$$

Recall that, by Proposition 3.2, (3.16) has a unique positive steady-state solution U_μ if $\mu \notin (0, \overline{\mu}^*)$, and for each $\mu \in (0, \overline{\mu}^*)$, it has a maximal positive steady-state solution $\overline{U}_\mu$ and a minimal positive steady-state solution $\underline{U}_\mu$. The following result shows that the dynamics of (3.16) is largely determined by these steady-state solutions.

Proposition 3.5. *Suppose $\lambda > \lambda_1^D(\Omega_0)$ and let $u(x, t)$ be a solution of* (3.16)*.*

(1) *If $\mu \leqslant 0$ or $\mu \geqslant \overline{\mu}^*$, then $u(x, t) \to U_\mu(x)$ uniformly for $x \in \Omega$ as $t \to \infty$.*

(2) *If* $0 < \mu < \overline{\mu}^*$, *then*

$$\underline{U}_\mu(x) \leqslant \varliminf_{t\to\infty} u(x,t) \leqslant \varlimsup_{t\to\infty} u(x,t) \leqslant \overline{U}_\mu(x), \tag{3.17}$$

uniformly for $x \in \Omega$.

Proof. It is well known that the solution of (3.16) exists globally and the ω-limit set of $\{u(\cdot,t)\}$ is contained in the union of the non-negative steady state solutions. On the other hand, due to its linear instability, the steady state $u = 0$ is unstable in the sense that for $u_0 \geqslant, \not\equiv 0$ in a $C(\overline{\Omega})$ neighborhood of 0, 0 is not in the ω-limit set of $\{u(\cdot,t)\}$. When $\mu \leqslant 0$ or $\mu \geqslant \overline{\mu}^*$, (3.1) has a unique positive solution U_μ, which is the only candidate for the ω-limit set of $\{u(\cdot,t)\}$. When $0 < \mu < \overline{\mu}^*$, any positive steady state of (3.16) is between the minimal and maximal solutions of (3.1), thus the ω-limit set of $\{u(\cdot,t)\}$ is also between them, which proves (3.17). □

From Proposition 3.5, it is not hard to derive the following results on the asymptotic behavior of (2.1).

Theorem 3.6. *Suppose* $\lambda > \lambda_1^D(\Omega_0)$ *and let* $(u(x,t), v(x,t))$ *be a solution of* (2.1).

(1) *If* $\mu < -c\lambda/(1+m\lambda)$, *then* $\lim_{t\to\infty} u(x,t) = \lambda$ *uniformly for* $x \in \overline{\Omega}$, *and* $\lim_{t\to\infty} v(x,t) = 0$ *uniformly for* $x \in \overline{\Omega}_1$.
(2) *If* $\mu > -c\lambda/(1+m\lambda)$, *then*

$$\underline{U}_{\mu+c/m}(x) \leqslant \varliminf_{t\to\infty} u(x,t) \leqslant \varlimsup_{t\to\infty} u(x,t) \leqslant \overline{U}_\mu(x) \quad \textit{uniformly in } \overline{\Omega},$$

$$\max\{\mu,0\} \leqslant \varliminf_{t\to\infty} v(x,t) \leqslant \varlimsup_{t\to\infty} v(x,t) \leqslant \mu + \frac{c}{m} \quad \textit{uniformly in } \overline{\Omega}_1. \tag{3.18}$$

Proof. The proof of part (1) is similar to the proof of Theorem 3.8 of [24], so we omit the details. For the proof of part (2), the estimate for $v(x,t)$ can be obtained from a simple application of the comparison principle, since

$$\mu v - v^2 < \mu v - v^2 + \frac{cuv}{1+mu} < \mu v - v^2 + \frac{c}{m}, \tag{3.19}$$

for all positive u, v. Then for any $\delta > 0$, there exists $T > 0$ such that for $t > T$

$$\max\{\mu,0\} - \delta < v(x,t) < \mu + \frac{c}{m} + \delta. \tag{3.20}$$

The estimate for $u(x,t)$ can then be obtained by using part (2) of Proposition 3.5 and letting $\delta \to 0$. □

Theorem 3.6 shows that for any $\mu > \mu_1 = -c\lambda/(1+m\lambda)$, there is an attracting region defined by (3.18). From a quite general result in [2], we can easily deduce that (2.1) is permanent, but the estimates in (3.18) give a more specific attracting region. By Theorem 3.4, (2.1) has at least one positive steady state solution in the attracting region, but it is possible that for certain μ (2.1) can have multiple steady state solutions or even periodic solutions in the attracting region. However when μ is large, we have shown in Theorem 3.4 that (2.1) has a unique locally asymptotically stable positive steady state solution (u_μ, v_μ). In the following we show that (u_μ, v_μ) is actually globally asymptotically stable.

Theorem 3.7. *Suppose that $\lambda > \lambda_1^D(\Omega_0)$. Then there exists $\mu_* > 0$ such that if $\mu \geqslant \mu_*$, and if $(u(x,t), v(x,t))$ is a solution of (2.1), then $\lim_{t\to\infty} u(x,t) = u_\mu(x)$ and $\lim_{t\to\infty} v(x,t) = v_\mu(x)$ uniformly for $x \in \Omega$ and $x \in \Omega_1$, respectively.*

Proof. The global existence of the solution follows from a simple comparison argument. By standard parabolic regularity, it suffices to prove that $\|u(\cdot,t) - u_\mu\|_{L^2(\Omega)} \to 0$ and $\|v(\cdot,t) - v_\mu\|_{L^2(\Omega_1)} \to 0$ as $t \to \infty$ for any initial condition $u_0 \geqslant, \not\equiv 0$ and $v_0 \geqslant, \not\equiv 0$. We divide the proof into several steps.

Step 1. We prove that for any given small $\delta_1 > 0$, we can find $\mu_1 = \mu_1(\delta_1) > 0$ so that for each $\mu > \mu_1$, there exists $T_1 = T_1(\mu) = T_1(\mu, u_0, v_0) > 0$ such that when $t > T_1$,

$$\left|u(x,t) - u_\mu\right| \leqslant \delta_1, \quad \forall x \in \overline{\Omega}, \qquad \left|v(x,t) - v_\mu\right| \leqslant \delta_1, \quad \forall x \in \overline{\Omega}_1. \tag{3.21}$$

By Theorem 3.6, for any $\varepsilon_1 > 0$, there exists $T_a > 0$ such that $u(x,t) \leqslant U_{\mu-\varepsilon_1}(x)$ for $t > T_a$. On the other hand, by Proposition 3.3 and Theorem 3.4, $U_\mu \to W_\lambda$ uniformly on $\overline{\Omega}$ and $v_\mu - \mu \to 0$ uniformly on $\overline{\Omega}_1$ as $\mu \to \infty$. Thus there exists $\mu_0 = \mu_0(\varepsilon_1, \delta_1) > 0$ such that when $\mu > \mu_0$, $U_{\mu-\varepsilon_1} < \varepsilon_1$ in Ω_1 (note that $W_\lambda = 0$ on Ω_1) and $|v_\mu - \mu| < \delta_1/2$. Now we select ε_1 so that $c\varepsilon_1/(1 + m\varepsilon_1) < \delta_1/3$. Then for $\mu > \mu_0$ and $t > T_a$, we have

$$v_t - \Delta v \leqslant \mu v - v^2 + (\delta_1/3)v. \tag{3.22}$$

Hence $\overline{\lim}_{t\to\infty} v(x,t) \leqslant \mu + \delta_1/3$. In particular, there exists $T_b \geqslant T_a$ such that when $t > T_b$, $v(x,t) \leqslant \mu + \delta_1/2$. Similarly we can show that $v(x,t) \geqslant \mu - \delta_1/2$ when $t > T_c \geqslant T_b$. Therefore the estimate for v in (3.21) holds when $t > T_c$ and $\mu > \mu_0$. Substituting the estimate of v into the equation of u, and making use of the comparison principle, we obtain

$$U_{\mu+\delta_1}(x) \leqslant \underline{\lim}_{t\to\infty} u(x,t) \leqslant \overline{\lim}_{t\to\infty} u(x,t) \leqslant U_{\mu-\delta_1}(x). \tag{3.23}$$

Since both u_μ and U_μ converge to W_λ uniformly on $\overline{\Omega}$ as $\mu \to \infty$, there exists $T_d \geqslant T_c$ and $\mu_1 \geqslant \mu_0$ such that the estimate of u in (3.21) holds for $t > T_d$ and $\mu > \mu_1$. We choose $T_1 = T_d$ and the conclusion in Step 1 is proved.

Step 2. Let $\Phi(x,t) = u(x,t) - u_\mu(x)$ and $\Psi(x,t) = v(x,t) - v_\mu(x)$. We prove that for all small $\delta_1 > 0$, if $\mu_1(\delta_1)$ and $T_1(\mu)$ are defined as in Step 1, then the following inequalities hold for $\mu > \mu_1$ and $t > T_1$:

$$\begin{aligned}(\Phi_t - \Delta\Phi)\Phi \leqslant{}& \big[\lambda - 2u_\mu - b(x)\mu p'(u_\mu + \theta_1\Phi) + \delta_1\big(b(x)+1\big)\big]\Phi^2 \\ &+ b(x)p(u_\mu)|\Phi||\Psi|,\end{aligned} \tag{3.24}$$

$$(\Psi_t - \Delta\Psi)\Psi \leqslant \left[-\mu + \left(3\delta_1 + \frac{c}{m}\right)\right]\Psi^2 + (c + \delta_1 cm)(\mu + \delta_1)|\Phi||\Psi|, \tag{3.25}$$

where $\theta_1(x)$ satisfies $0 \leqslant \theta_1(x) \leqslant 1$.

To prove the above inequalities, we first use (2.1) to obtain

$$\Phi_t - \Delta\Phi = (\lambda - 2u_\mu - \Phi)\Phi - b(x)\big[p(u_\mu + \Phi)(v_\mu + \Psi) - p(u_\mu)v_\mu\big], \tag{3.26}$$

$$\Psi_t - \Delta\Psi = (\mu - 2v_\mu - \Psi)\Psi + c\big[p(u_\mu + \Phi)(v_\mu + \Psi) - p(u_\mu)v_\mu\big]. \tag{3.27}$$

By Taylor's expansion formula, for some $0 \leqslant \theta_2(x) \leqslant 1$, we have

$$\begin{aligned} &p(u_\mu + \Phi)(v_\mu + \Psi) - p(u_\mu)v_\mu \\ &\quad = p(u_\mu)\Psi + p'(u_\mu)(v_\mu + \Psi)\Phi + \frac{1}{2}p''(u_\mu + \theta_2\Phi)(v_\mu + \Psi)\Phi^2. \end{aligned} \tag{3.28}$$

From (3.27) and (3.28), we obtain

$$\begin{aligned} (\Psi_t - \Delta\Psi)\Psi &= (\mu - 2v_\mu)\Psi^2 - \Psi^3 + cp(u_\mu)\Psi^2 + cp'(u_\mu)(v_\mu + \Psi)\Phi\Psi \\ &\quad + \frac{c}{2}p''(u_\mu + \theta_2\Phi)(v_\mu + \Psi)\Phi^2\Psi \\ &= -\mu\Psi^2 + \big[2(\mu - v_\mu) - \Psi + cp(u_\mu)\big]\Psi^2 \\ &\quad + \left[cp'(u_\mu)(v_\mu + \Psi) + \frac{c}{2}p''(u_\mu + \theta_2\Phi)(v_\mu + \Psi)\Phi\right]\Phi\Psi. \end{aligned} \tag{3.29}$$

Since $|v_\mu - \mu|, |\Phi|, |\Psi| \leqslant \delta_1$, $p(u_\mu) \leqslant 1/m$, $|v_\mu + \Psi| \leqslant \mu + \delta_1$, $p'(u_\mu) \leqslant 1$ and $|p''(u_\mu + \theta_2\Phi)| \leqslant 2m$, (3.25) follows from (3.29). Similarly, since

$$p(u_\mu + \Phi)(v_\mu + \Psi) - p(u_\mu)v_\mu = p(u_\mu)\Psi + p'(u_\mu + \theta_1\Phi)(v_\mu + \Psi)\Phi, \tag{3.30}$$

and $|v_\mu + \Psi - \mu|, |\Phi| \leqslant \delta_1$, we obtain (3.24) from (3.26).

Step 3. For any small $\delta_2 > 0$, there exists $\mu_2 = \mu_2(\delta_2) > 0$ and $\sigma = \sigma(\delta_2) > 0$ such that if $\mu > \mu_2$ and if $W_1, W_2 \in L^\infty(\Omega)$ satisfies

$$\|W_i - W_\lambda\|_\infty < \sigma, \quad i = 1, 2,$$

then the principal eigenvalue $\eta(W) = \eta(W_1, W_2)$ of

$$-\Delta\varphi = \lambda\varphi - 2W_1(x)\varphi - \mu b(x)p'\big(W_2(x)\big)\varphi + \eta\varphi \quad \text{in } \Omega, \qquad \partial_\nu\varphi = 0 \quad \text{on } \partial\Omega \tag{3.31}$$

satisfies $\eta(W) \geqslant \eta^* - \delta_2$, where $\eta^* > 0$ is the principal eigenvalue of (3.8).

The proof of Step 3 is similar to the proof of the stability of u_n in Proposition 3.2. We omit the details.

Step 4. Denote

$$f(t)=\int_{\Omega}\Phi^2(x,t)\,dx,\qquad g(t)=\int_{\Omega_1}\Psi^2(x,t)\,dx.$$

We prove that for any given $\epsilon>0$, there exists $\mu_*(\varepsilon)>0$ so that for each $\mu>\mu_*(\varepsilon)$ and $t>T_1(\mu)$,

$$\frac{1}{2}f'(t)\leqslant-\frac{\eta^*}{2}f(t)+\varepsilon g(t),\qquad \frac{1}{2}g'(t)\leqslant-\frac{\mu}{4}g(t)+(c+1)^2\mu f(t). \tag{3.32}$$

Fix $\delta_1>0$ small so that $\delta_1 cm<1$ and suppose $\mu>\mu_1$, $t>T_1(\mu)$. We integrate (3.24) and (3.25) over Ω and Ω_1 respectively, to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega}\Phi^2\,dx\leqslant{}&-\int_{\Omega}|\nabla\Phi|^2\,dx+\int_{\Omega_1}b(x)p(u_\mu)|\Phi|\cdot|\Psi|\,dx\\&+\int_{\Omega}\bigl[\lambda-2u_\mu-b(x)\mu p'(u_\mu+\theta_1\Phi)+\delta_1\bigl(b(x)+1\bigr)\bigr]\Phi^2\,dx,\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_1}\Psi^2\,dx&\leqslant\frac{1}{2}\frac{d}{dt}\int_{\Omega_1}\Psi^2\,dx+\int_{\Omega_1}|\nabla\Psi|^2\,dx\\&\leqslant\left(-\mu+3\delta_1+\frac{c}{m}\right)\int_{\Omega_1}\Psi^2\,dx+(c+1)(\mu+\delta_1)\int_{\Omega_1}|\Phi|\cdot|\Psi|\,dx.\end{aligned} \tag{3.34}$$

We may assume that $\eta^*>8\|b+1\|_\infty\delta_1$. Then choose $\delta_2=\|b+1\|_\infty\delta_1$ in Step 3 and let $\mu_3>\max\{\mu_1,\mu_2\}$ be such that for $\mu>\mu_3$ and $\sigma=\sigma(\delta_2)$ determined in Step 3,

$$|u_\mu-W_\lambda|<\sigma,\quad |U_{\mu\pm2\delta_1}-W_\lambda|<\sigma\quad\text{in }\overline{\Omega}.$$

Therefore by Step 1, for $\mu>\mu_3$ and $t>T_1(\mu)$, we have

$$|u_\mu+\theta_1\Phi-W_\lambda|<\sigma\quad\text{in }\overline{\Omega}.$$

We can now apply the conclusion in Step 3 to obtain, for $\mu>\mu_3$ and $t>T_1(\mu)$,

$$\begin{aligned}&\int_{\Omega}|\nabla\Phi|^2\,dx-\int_{\Omega}\bigl[\lambda-2u_\mu-b(x)\mu p'(u_\mu+\theta_1\Phi)+\delta_1\bigl(b(x)+1\bigr)\bigr]\Phi^2\,dx\\&\quad\geqslant\bigl(\eta^*-2\|b+1\|_\infty\delta_1\bigr)\int_{\Omega}\Phi^2\,dx.\end{aligned} \tag{3.35}$$

Combining (3.33) and (3.35), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}\Phi^2\,dx \leqslant \bigl(-\eta^* + 2\|b+1\|_\infty\delta_1\bigr)\int_{\Omega}\Phi^2\,dx + \int_{\Omega_1} b(x)p(u_\mu)|\Phi|\cdot|\Psi|\,dx. \quad (3.36)$$

We now choose $\mu_* = \mu_*(\epsilon) > \max\{\mu_3, 4(\eta^* + c/m)\}$ so that for $\mu > \mu_*$, $\|b\|_\infty\|u_\mu\|_{L^\infty(\Omega_1)} \leqslant \min\{\|b+1\|_\infty\delta_1, \varepsilon\}$; this is possible since $u_\mu \to W_\lambda$ uniformly in $\overline{\Omega}$ as $\mu \to \infty$ and $W_\lambda = 0$ on Ω_1. Then, for $\mu > \mu_*$ and $t > T_1$, we have

$$\begin{aligned}\int_{\Omega_1} b(x)p(u_\mu)|\Phi|\cdot|\Psi|\,dx &\leqslant \min\bigl\{\|b+1\|_\infty\delta_1, \varepsilon\bigr\}\int_{\Omega_1}|\Phi|\cdot|\Psi|\,dx \\ &\leqslant \|b+1\|_\infty\delta_1\int_{\Omega}\Phi^2\,dx + \varepsilon\int_{\Omega_1}\Psi^2\,dx. \quad (3.37)\end{aligned}$$

It follows from (3.36) and (3.37) that

$$\frac{1}{2}f'(t) \leqslant \bigl(-\eta^* + 3\|b+1\|_\infty\delta_1\bigr)f(t) + \varepsilon g(t) \leqslant -\frac{\eta^*}{2}f(t) + \varepsilon g(t).$$

Similarly, using

$$\int_{\Omega_1}|\Phi|\cdot|\Psi|\,dx \leqslant \frac{1}{2(c+1)}\int_{\Omega}\Phi^2\,dx + \frac{c+1}{2}\int_{\Omega_1}\Psi^2\,dx,$$

we obtain from (3.34) that, for $\mu > \mu_*$ and $t > T_1(\mu)$,

$$\frac{1}{2}g'(t) \leqslant \left(-\frac{\mu}{2} + 4\delta_1 + \frac{c}{m}\right)g(t) + \frac{(c+1)^2}{2}(\mu + \delta_1)f(t) \leqslant -\frac{\mu}{4}g(t) + (c+1)^2\mu f(t).$$

Step 5. Let $\varepsilon_0 = \eta^*/(32(c+1)^2)$ and let $\mu_*(\epsilon_0)$ be determined by Step 4. We show that for each $\mu > \mu_*(\varepsilon_0)$, $\lim_{t\to\infty} g(t) = 0$.

For $\mu > \mu_*(\varepsilon_0)$ and $t > T_1(\mu)$, from (3.32) we obtain

$$\bigl(e^{\eta^* t}f(t)\bigr)' \leqslant 2\varepsilon_0 e^{\eta^* t}g(t).$$

It follows that

$$f(t) \leqslant e^{-\eta^*(t-T)}f(T) + 2\varepsilon_0 e^{-\eta^* t}\int_T^t e^{\eta^* s}g(s)\,ds, \quad \forall t > T > T_1, \quad (3.38)$$

and

$$g'(t) + \frac{\mu}{2}g(t) \leqslant 2(c+1)^2\mu e^{-\eta^*(t-T)}f(T) + 4(c+1)^2\mu\varepsilon_0 e^{-\eta^* t}\int_T^t e^{\eta^* s}g(s)\,ds, \quad \forall t > T > T_1. \tag{3.39}$$

Suppose by way of contradiction that $\lim_{t\to\infty} g(t) = 0$ does not hold. Then, due to (3.21), $\gamma := \overline{\lim}_{t\to\infty} g(t)$ must be a finite positive number. By elementary analysis, there exist $T_* > T_1(\mu)$ and a sequence $t_n > T_*$ satisfying $t_n \to \infty$ such that

$$g(t) < 2\gamma, \quad \forall t > T_*, \qquad \lim_{n\to\infty} g'(t_n) = 0, \quad \lim_{n\to\infty} g(t_n) = \gamma.$$

We now take $t = t_n$ and $T = T_*$ in (3.39) to obtain

$$\begin{aligned} g'(t_n) + \frac{\mu}{2}g(t_n) &\leqslant 2(c+1)^2\mu e^{-\eta^*(t_n-T_*)}f(T_*) + 8(c+1)^2\mu\varepsilon_0 e^{-\eta^* t_n}\sigma\int_{T_*}^{t_n} e^{\eta^* s}\,ds \\ &\leqslant 2(c+1)^2\mu e^{-\eta^*(t_n-T_*)}f(T_*) + \frac{1}{4}\mu\gamma. \end{aligned}$$

Letting $n \to \infty$, we obtain

$$\frac{1}{2}\mu\gamma \leqslant \frac{1}{4}\mu\gamma.$$

This contradiction proves Step 5.

Step 6. For $\mu > \mu_*(\epsilon_0)$, $\lim_{t\to\infty} f(t) = 0$.

Let $\mu > \mu_*(\epsilon_0)$. By Step 5, for any given $\delta > 0$, we can find $T^* > T_1(\mu)$ so that $g(t) < \delta\eta^*/(2\varepsilon_0)$ for $t > T^*$. Now taking $T = T^*$ in (3.38) we obtain

$$f(t) \leqslant e^{-\eta^*(t-T^*)}f(T^*) + 2\varepsilon_0 e^{-\eta^* t}\frac{\delta\eta^*}{2\varepsilon_0}\int_{T^*}^t e^{\eta^* s}\,ds \leqslant e^{-\eta^*(t-T^*)}f(T^*) + \delta.$$

It follows that $\overline{\lim}_{t\to\infty} f(t) \leqslant \delta$. Since $\delta > 0$ is arbitrary, this implies $\overline{\lim}_{t\to\infty} f(t) \leqslant 0$. Therefore $f(t) \to 0$ as $t \to \infty$. This completes the proof. □

4. The small protection zone case

In this section we consider the set of positive steady state solutions and related dynamical behavior of (2.1) when the protection zone Ω_0 satisfies $\lambda_1^D(\Omega_0) > \lambda$. Our results in this section are qualitatively similar to those in Section 3 of our earlier work [24], where no protection zone is present. We notice that (2.1) and (2.2) are different from the equations in [24] in three aspects:

(a) the prey crowding effect is a function $a(x) \geqslant 0$ with possible degeneracy in [24], while in our case $a(x) \equiv 1$;
(b) $b(x) \equiv 1$ in [24] but is degenerate in Ω_0 in this paper;
(c) the equation of v is over a smaller spatial domain Ω_1 here.

However a careful examination of the proofs in [24] shows that they can be carried over here without essential changes. Therefore mostly we will only state the results, and refer the proofs to [24]. We would like to remark that the dynamics of the model here is fundamentally different from the large protection zone case considered in the previous section; the effect of the protection zone is now better described in quantitative terms. Let us recall that we assume throughout this section

$$\overline{\Omega}_0 \subset \Omega, \qquad \lambda_1^D(\Omega_0) > \lambda.$$

4.1. Steady state solutions

In Theorem 2.1, we have shown that there is a bounded continuum Γ_2 consisting of positive solutions (μ, u, v) of (2.2) that joins Γ_u at $(\mu_1, \lambda, 0)$ and joins Γ_v at $(\mu_2, 0, \mu_2)$. Therefore there is at least one positive solution for (2.2) if $\mu \in (\mu_1, \mu_2)$. Moreover, we have also shown that (2.2) has no positive solution if $\mu \leqslant \mu_1$ or $\mu \geqslant (1 + m\lambda)\mu_2$.

As in [24], a more detailed description of the continuum Γ_2 can be gained from the studies of the scalar equation (3.1). For the steady state solutions of (3.1), we have the following result which is almost identical to Proposition 3.3 of [24].

Proposition 4.1. *Suppose that* $0 < \lambda < \lambda_1^D(\Omega_0)$. *Then* $\mu = \mu_2(\lambda)$ *is a bifurcation point for* (3.1) *such that a global unbounded continuum* Σ *of positive solutions of* (3.1) *emanates from* $(\mu, u) = (\mu_2, 0)$, *and*

$$\operatorname{proj}_\mu \Sigma = (-\infty, \hat{\mu}^*] \text{ or } (-\infty, \hat{\mu}^*), \tag{4.1}$$

where $\hat{\mu}^* = \sup\{\mu > 0\colon (3.1) \text{ has a positive solution}\} \geqslant \mu_2$. *Moreover* Σ *has the following properties*:

(1) *Near* $(\mu, u) = (\mu_2, 0)$, Σ *is a curve.*
(2) *When* $\mu \leqslant 0$, (3.1) *has a unique positive solution* $\overline{U}_\mu(x)$, *and* $\{(\mu, \overline{U}_\mu)\colon \mu \leqslant 0\}$ *is a smooth curve.*
(3) *For* $\mu \in (-\infty, \hat{\mu}^*)$, (3.1) *has a maximal positive solution* $\overline{U}_\mu(x)$, *and* $\overline{U}_\mu$ *is strictly decreasing with respect to* μ.
(4) *For* $\mu \in (-\infty, \mu_2)$, (3.1) *has a minimal positive solution* $\underline{U}_\mu(x)$, $\underline{U}_\mu = \overline{U}_\mu$ *when* $\mu \leqslant 0$, *and* $\underline{U}_\mu$ *is strictly decreasing with respect to* μ.
(5) *If* $\hat{\mu}^* > \mu_2$, *then* (3.1) *has a maximal positive solution for* $\mu = \hat{\mu}^*$, *and has at least two positive solutions for* $\mu \in (\mu_2, \hat{\mu}^*)$. *A sufficient condition for* $\hat{\mu}^* > \mu_2$ *is* $m > m_0$, *where* m_0 *is given by* (2.13) *with* $\varphi_2 = 0$.
(6) *If* $\hat{\mu}^* > \mu_2$ *and* $0 < m < m_0$, *then there exists* $\hat{\mu}_* \in (0, \mu_2)$ *such that* (3.1) *has at least three positive solutions for* $\mu \in (\hat{\mu}_*, \mu_2)$, *and* $\underline{U}_\mu < \overline{U}_\mu$ *for* $\mu \in (\hat{\mu}_*, \mu_2)$. *Moreover* $\lim_{\mu \to (\mu_2)^-} \underline{U}_\mu = 0$ *uniformly for* $x \in \overline{\Omega}$.

All these solutions mentioned above can be chosen from the unbounded continuum Σ.

The system (2.2) lacks an order preserving property, which makes it more difficult to analyze. However, making use of the comparison principle and our estimates in (2.15), it is possible to relate (2.2) to the scalar problem (3.1), which enjoys the order preserving property. In fact, several conclusions in Proposition 4.1 are consequences of this order preserving property. As in [24], we can explore the relationship between (2.2) and (3.1) and use topological method such as fixed point index and Schauder's fixed point theorem to obtain some multiplicity results for (2.2) if (3.1) has multiple positive solutions.

Theorem 4.2. *Suppose that* $0 < \lambda < \lambda_1^D(\Omega_0)$. *Let* $\hat{\mu}^*$ *and* $\hat{\mu}_*$ *be defined as in Proposition* 4.1. *Then the following conclusions hold*:

(1) *Define*

$$\overline{\mu}^* = \sup\{\mu > 0\colon\ (2.2)\ \text{has a positive solution}\},$$

and

$$\mu_* = \inf\{\mu > 0\colon\ (2.2)\ \text{has a positive solution } (u, v)\text{, and } u \ngeq \overline{U}_{\hat{\mu}^*}\}.$$

Then $0 \leqslant \hat{\mu}^* - \overline{\mu}^* \leqslant c/m$, $0 \leqslant \mu_* - \hat{\mu}_* \leqslant c/m$.

(2) *If* $\mu_1 < \hat{\mu}^* - c/m$, *then for* $\mu \in (\mu_1, \hat{\mu}^* - c/m]$, (2.2) *has a positive solution* (u_μ^1, v_μ^1) *satisfying*

$$\begin{aligned} &\min\{\overline{U}_\mu(x), \overline{U}_0(x)\} > u_\mu^1(x) > \overline{U}_{\mu+c/m}(x), \quad x \in \Omega,\\ &\mu + \frac{c}{m} > v_\mu^1(x) > \max\{\mu, 0\}, \quad x \in \Omega_1. \end{aligned} \tag{4.2}$$

(3) *If* $\hat{\mu}_* + c/m < \mu_2$, *then for* $\mu \in [\hat{\mu}_* + c/m, \mu_2)$, (2.2) *has a positive solution* (u_μ^2, v_μ^2) *satisfying*

$$\begin{aligned} &\underline{U}_\mu(x) > u_\mu^2(x) > \max\{\underline{U}_{\mu+c/m}(x), 0\}, \quad x \in \Omega,\\ &\mu + \frac{c}{m} > v_\mu^2(x) > \mu, \quad x \in \Omega_1. \end{aligned} \tag{4.3}$$

(4) *If* $\hat{\mu}^* > \mu_2 + c/m$, *then* (2.2) *has at least two positive solutions for* $\mu_2 < \mu \leqslant \hat{\mu}^* - c/m$.

(5) *If* $\hat{\mu}^* > \mu_2 + c/m$ *and* $\hat{\mu}_* < \lambda/b - c/m$, *then* (2.2) *has at least three positive solutions for* $\hat{\mu}_* + c/m < \mu < \mu_2$.

All these solutions above can be chosen from the continuum Γ_2.

(Note that $\underline{U}_{\mu+c/m}$ is not always defined in part (3). In case it is not defined we assume $\underline{U}_{\mu+c/m} = 0$. Similarly, if $\overline{U}_{\hat{\mu}^*}$ is not defined, we understand that it equals zero.)

We omit the proof of Theorem 4.2, as it is similar to that of Theorem 3.5 in [24].

4.2. Dynamical behavior

First we consider the dynamics of the auxiliary equation (3.16). A rather complete description of the dynamical behavior of (3.16) is give in the following theorem, which is essentially the same as Theorem 3.6 of [24].

Proposition 4.3. *Suppose that* $0 < \lambda < \lambda_1^D(\Omega_0)$. *Then all solutions* $u(x,t)$ *of* (3.16) *are globally bounded, and the following hold*:

(1) *If* $\mu \leqslant 0$, *then* $\overline{U}_\mu$ *is globally asymptotically stable.*
(2) *If* $\mu > \hat{\mu}^*$, *then* 0 *is globally asymptotically stable.*
(3) *If* $0 < \mu < \mu_2$, *then for any* u_0, $\underline{\lim}_{t\to\infty} u(x,t) \geqslant \underline{U}_\mu$.
(4) *If* $0 < \mu \leqslant \hat{\mu}^*$, *then for any* u_0, $\overline{\lim}_{t\to\infty} u(x,t) \leqslant \overline{U}_\mu$. *Moreover,*
   (a) *if* $u_0(x) \geqslant \overline{U}_\mu(x)$, *then* $\lim_{t\to\infty} u(x,t) = \overline{U}_\mu(x)$;
   (b) *if* $u_0(x) \geqslant \overline{U}_{\hat{\mu}^*}(x)$, *then*

$$V_{\mu,1}(x) \leqslant \underline{\lim}_{t\to\infty} u(x,t) \leqslant \overline{\lim}_{t\to\infty} u(x,t) \leqslant \overline{U}_\mu(x), \tag{4.4}$$

   where $V_{\mu,1}(x)$ *is the unique solution of*

$$-\Delta u = \left(\lambda - \frac{b(x)\mu}{1+mU}\right)u - u^2, \quad x \in \Omega, \qquad \partial_\nu u = 0, \quad x \in \partial\Omega, \tag{4.5}$$

   with $U = \overline{U}_{\hat{\mu}^*}$.
(5) *If* $\hat{\mu}_* < \mu_2$ *and* $\hat{\mu}_* < \mu < \mu_2$, *then* $u_0(x) \leqslant \underline{U}_\mu(x)$ *implies* $\lim_{t\to\infty} u(x,t) = \underline{U}_\mu(x)$; $u_0(x) \leqslant \underline{U}_{\hat{\mu}_*}(x)$ *implies*

$$\underline{U}_\mu(x) \leqslant \underline{\lim}_{t\to\infty} u(x,t) \leqslant \overline{\lim}_{t\to\infty} u(x,t) \leqslant V_{\mu,2}(x), \tag{4.6}$$

   where $V_{\mu,2}$ *is the unique solution of* (4.5) *with* $U = \underline{U}_{\hat{\mu}_*}$.

Finally we obtain the main results on the dynamical behavior of the full system (2.1):

Theorem 4.4. *Suppose that* $0 < \lambda < \lambda_1^D(\Omega_0)$. *Then all solutions* $(u(x,t), v(x,t))$ *of* (2.1) *are globally bounded, and* $v(x,t)$ *satisfies*

$$\max\{\mu, 0\} \leqslant \underline{\lim}_{t\to\infty} v(x,t) \leqslant \overline{\lim}_{t\to\infty} v(x,t) \leqslant \max\left\{\mu + \frac{c}{m}, 0\right\}. \tag{4.7}$$

Moreover, the following conclusions hold:

(1) *If* $\mu < \mu_1$, *then* $\lim_{t\to\infty} u(x,t) = \lambda$ *uniformly for* $x \in \overline{\Omega}$ *and* $\lim_{t\to\infty} v(x,t) = 0$ *uniformly for* $x \in \overline{\Omega}_1$.
(2) *If* $\mu > \hat{\mu}^*$, *then* $\lim_{t\to\infty} u(x,t) = 0$ *uniformly for* $x \in \overline{\Omega}$ *and* $\lim_{t\to\infty} v(x,t) = \mu$ *uniformly for* $x \in \overline{\Omega}_1$.
(3) *If* $\mu_1 < \mu < \hat{\mu}^* - c/m$, *then* (3.18) *holds.*

The proof of parts (1) and (2) of Theorem 4.4 is rather standard, while part (3) can be proved as in Theorem 3.6. Therefore we omit the details. It is possible to make further use of (3.16), as in [24], to obtain more detailed information on the dynamical behavior of (2.1), and we leave this to the interested reader.

To end this section, we briefly comment on the dynamical behavior of (2.1) as described by Theorem 4.4. Let us note that $\hat{\mu}^* \in [\mu_2, \mu_2(1+m\lambda)]$, and $\mu_2 = \mu_2(\lambda)$ is close to 0 when λ is close to 0, and $\mu_2(\lambda)$ is close to ∞ if λ is close to $\lambda_1^D(\Omega_0)$. Therefore, for fixed $\lambda > 0$, μ_2 increases as Ω_0 is enlarged, and it becomes very large if $\lambda_1^D(\Omega_0)$ is close to λ. Part (3) in Theorem 4.4 shows that the prey species can survive a predator with growth rate in the range $\mu \in (\mu_1, \hat{\mu}^* - c/m)$, which increases when Ω_0 is enlarged, and when Ω_0 is about to reach the minimal patch size (i.e., $\lambda_1^D(\Omega_0)$ is close to λ), the range $(\mu_1, \hat{\mu}^* - c/m)$ is close to (μ_1, ∞). Once Ω_0 is enlarged over the minimal patch size, we are in the situation considered in Section 3, and the survival of the prey species is guaranteed no matter how strong the predator is.

5. Discussions

In this paper we have shown that establishing a protection zone for the prey in its habitat can save an otherwise extinguishing prey species. The most significant feature of our study is the existence of a critical patch size described by the principal eigenvalue $\lambda_1^D(\Omega_0)$ for the protection zone Ω_0. If the protection zone if above that size (i.e., if $\lambda_1^D(\Omega_0)$ is less than λ, the prey growth rate), then the dynamics of the model is fundamentally changed from the usual predator–prey dynamics; in such a case, the prey population can survive regardless of the level of predation, and if the predator is strong, then the two populations stabilize at a unique coexistence state. If the protection zone is below the critical patch size, then the dynamics of the model is qualitatively similar to the usual case without protection zone, but the chances of survival of the prey species increase with the size of the protection zone, as generally expected.

Mathematically, the value of $\lambda_1^D(\Omega_0)$ depends on the size as well as the shape of Ω_0. The smaller the value of $\lambda_1^D(\Omega_0)$, the more protection Ω_0 provides to the prey species. For the interior protection zone case discussed in this paper, if the volume of the zone is fixed, then it is well-known that a spherical protection zone has the smallest eigenvalue $\lambda_1^D(\Omega_0)$ from the classical Rayleigh–Faber–Krahn inequality (see [35]). Thus if a ball of the given size can be inscribed into Ω, then the optimal interior protection zone should be a ball. If fencing is needed to create the protection zone (for example, nets with suitable mesh sizes could be used if the prey has considerably smaller body size than its predator), then a ball shaped protection zone also uses the least fencing material as a ball has the least surface area among all regions of the same volume. This also suggests that having one big protection zone is usually better than having several protection zones that add to the same size (but this is not always so as the shape of the protection zones matters).

It is natural to have a boundary protection zone which is built along part or all the boundary of Ω. If Ω_0 is a ring shaped domain which has $\partial\Omega$ as its outer boundary, that is, $\partial\Omega \subset \partial\Omega_0$ and $\Gamma := \partial\Omega_0 \setminus \partial\Omega$ is nonempty and is contained in Ω, then the techniques and results in this paper carry over easily. The critical patch size for this case is determined by $\lambda_1^M(\Omega_0) = \lambda$, where $\lambda_1^M(\Omega_0)$ denotes the principal eigenvalue of the problem

$$-\Delta\phi = \lambda\phi \quad \text{in } \Omega_0, \qquad \partial_\nu\phi = 0 \quad \text{on } \partial\Omega, \qquad \phi = 0 \quad \text{on } \Gamma.$$

From the variational characterization of eigenvalues, we have

$$\lambda_1^D(\Omega_0) > \lambda_1^M(\Omega_0) > \lambda_1^N(\Omega_0),$$

for any given region Ω_0. Therefore, a boundary protection zone is better than a same shaped interior protection zone. We should note, however, that there are more choices for the shape of interior zones than boundary zones.

A more natural boundary protection zone is one where $\partial\Omega_0$ splits into two parts Γ_1 and Γ_2, with $\Gamma_1 \subset \partial\Omega$, $\mathrm{int}(\Gamma_2) \subset \Omega$, and $\Gamma_2 \cap \partial\Omega$ is an $(N-2)$-dimensional manifold. This is a mathematically challenging case, and great technical difficulties will be involved to extend our results here to this case. But we believe that similar results hold, and the critical patch size is determined by $\lambda_1^{M'}(\Omega_0) = \lambda$, where $\lambda_1^{M'}(\Omega_0)$ denotes the principal eigenvalue of the problem

$$-\Delta\phi = \lambda\phi \quad \text{in } \Omega_0, \qquad \partial_\nu\phi = 0 \quad \text{on } \Gamma_1, \qquad \phi = 0 \quad \text{on } \Gamma_2.$$

Again we have

$$\lambda_1^D(\Omega_0) > \lambda_1^{M'}(\Omega_0) > \lambda_1^N(\Omega_0).$$

If the boundary has a flat part, then a half ball H along the flat part of the boundary has the same principal eigenvalue $\lambda_1^{M'}(H)$ as that of a whole ball of the same radius in the interior. We conjecture that if the area is fixed, then the optimal protection zone is always achieved by a boundary one. This is apparently true for some special domains. A related optimization problem is considered in [31].

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