JOURNAL OF DIFFERENTIAL EQUATIONS 158, 94–151 (1999)
Article ID jdeq.1999.3644, available online at http://www.idealibrary.com on IDEAL®

Exact Multiplicity of Positive Solutions for a Class of Semilinear Problem, II

Tiancheng Ouyang and Junping Shi[1]

Department of Mathematics, Brigham Young University, Provo, Utah 84602

Received May 11, 1998

We consider the positive solutions to the semilinear problem:

$$\begin{cases} \Delta u + \lambda f(u) = 0, & \text{in } B^n, \\ u = 0, & \text{on } \partial B^n. \end{cases}$$

where B^n is the unit ball in $\mathbf{R}^n$, $n \geqslant 1$, and λ is a positive parameter. It is well known that if f is a smooth function, then any positive solution to the equation is radially symmetric, and all solutions can be parameterized by their maximum values. We develop a unified approach to obtain the exact multiplicity of the positive solutions for a wide class of nonlinear functions $f(u)$, and the precise shape of the global bifurcation diagrams are rigorously proved. Our technique combines the bifurcation analysis, stability analysis, and topological methods. We show that the shape of the bifurcation curve depends on the shape of the graph of function $f(u)/u$ as well as the growth rate of f.

1. INTRODUCTION

This paper is a continuation of [41]. We continue to study the exact multiplicity (including uniqueness) of positive solutions to a semilinear elliptic equation

$$\begin{cases} \Delta u + \lambda f(u) = 0, & \text{in } B^n, \\ u = 0, & \text{on } \partial B^n. \end{cases} \tag{1.1}$$

where B^n is the unit ball in $\mathbf{R}^n$, $n \geqslant 1$, and λ is a positive parameter.

The existence of solutions to (1.1) for the general smooth bounded domains has been studied extensively in recent years. (See surveys [1, 36].) But a full description of the solution set of (1.1) for most nonlinearities f remains open, even for the domain being the simplest one, the unit ball. Gidas *et al.* [24] show that all the positive solutions to (1.1) are radially

[1] Current address: Department of Mathematics, Tulane University, New Orleans, Louisiana 70118.

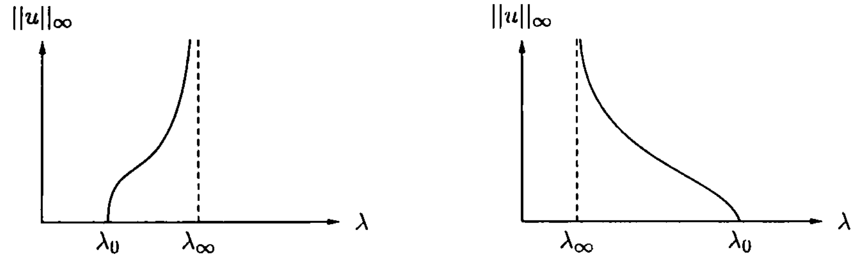
0022-0396/99 $30.00

symmetric if f is a local lipschitz continuous function in $\mathbf{R}^+ = [0, \infty)$, and the solutions can be parameterized by their maximum values. The goal of [41] and the present paper is to give a full description of the solution set of (1.1) for certain classes of nonlinearities f, and to determine the exact number of solutions for *any* $\lambda > 0$. In [41], we only study a special class of nonlinearities (f'' changes from positive to negative). In this paper, we investigate a much wider class of nonlinearities, and we also improve the method in [41] to a more general approach.

Through numerous studies in last thirty years, it is clear now that the complexity of the solution set of (1.1) with the general bounded domains depends on both the complexity of the domain and the structure of the nonlinearity $f(u)$. In the present paper, we consider the simplest space domain: the unit ball, and focus on the relation between the structure of the solution set and the structure of f. The results in our paper interpret a principle: if the structure of f is simple, then the structure of the solution set is also relatively simple. By saying a nonlinearity f is simple, we mean the monotonicity and the convexity of f is not too complicated. In fact, throughout the paper, we assume that $(f(u)/u)'$ and $f''(u)$ change sign at most once. We discover that the number of solutions to (1.1) usually is closely related to the number of sign-changes of $(f(u)/u)'$ and the growth rate of f. In all our results, if $(f(u)/u)'$ does not change sign, then (1.1) has at most one solution; and if $(f(u)/u)'$ changes sign exactly once, then (1.1) has at most two solutions. The growth rate of f also plays a role here. We define $K_f(u) = uf'(u)/f(u)$ as an indicator of the growth rate of a smooth function f. (For example, if $f(u) = u^p$, then $K_f(u) \equiv p$.) If the growth rate of f is greater than some critical exponents and the space dimension is higher (for example, $f(u) = e^u$ for $n \geqslant 3$), then the bifurcation diagram can be very complicated even for the balls [26]. But in this paper, we show that, under some boundedness or monotonicity conditions on K_f, the bifurcation diagram of (1.1) can be determined by $(f(u)/u)'$ and $f''(u)$.

We illustrate our results by some special examples. For instance, if $(f(u)/u)' \leqslant 0$ for $u > 0$, then (1.1) has at most one solution for any $\lambda > 0$ and the solution exists only when $\lambda \in (\lambda_0, \lambda_\infty)$. Here λ_0 (or λ_∞) is the point where a bifurcation from the trivial solutions (or from infinity) occurs. (Details are explained in Subsection 3.1; see Fig. 1(a).) If $(f(u)/u)' \geqslant 0$ for $u > 0, f(u) > 0$ for $u > 0$, and $K_f(u) \leqslant n/(n-2)$ or $K_f(u)$ is nonincreasing for $u > 0$, then (1.1) has at most one solution for any $\lambda > 0$ and the solution exists only when $\lambda \in (\lambda_\infty, \lambda_0)$. (See Fig. 1(b).) More general uniqueness results are presented in Subsections 6.1–6.3.

In the case $(f(u)/u)'$ changes sign, (1.1) may have multiple solutions, therefore the structure of the solution set is more interesting. For example, if $f(0) = 0, f(u) > 0$ for $u > 0$, $(f(u)/u)'$ and $f''(u)$ changes sign from positive to negative, and $K_f(u)$ is nonincreasing for $u > 0$, then all solutions to (1.1)



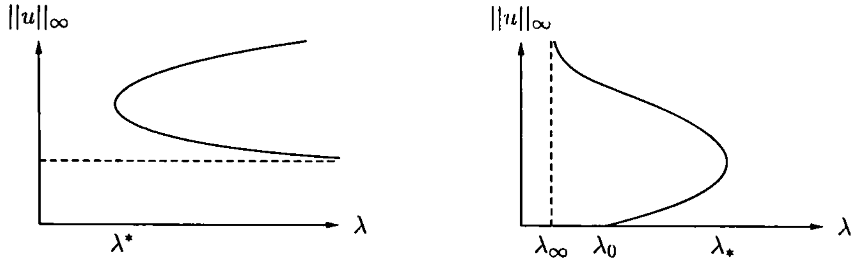
(a) Bifurcation diagram in the case $(f(u)/u)' \leq 0$

(b) Bifurcation diagram in the case $(f(u)/u)' \geq 0$

FIG. 1. Monotone Bifurcation Curves.

lie on a single curve, and there is only one turning point on the curve where the curve bends to the right. (See Fig. 2(a).) Another example is, if $n \geqslant 4$, $f(0)=0$, $f(u)>0$ for $u>0$, $(f(u)/u)'$ and $f''(u)$ changes sign from negative to positive, and $-(n-4)/(n-2) \leqslant K_f(u) \leqslant n/(n-2)$ for $u>0$, then all solutions of (1.1) lie on a single curve, and there is only one turning point on the curve where the curve bends to the left. (See Fig. 2(b).) More general exact multiplicity results are presented in Subsections 6.4–6.5.

Our results partially answer an open question asked by P. L. Lions [36] in the early 1980's. At the end of his survey paper [36], Lions conjectured that the structure of the set $\{(\lambda, u)\}$ of the positive solutions to (1.1) is similar to the structure of the solution set $\{(\lambda, u)\}$ of an algebraic equation $\lambda_1 u = \lambda f(u)$, where λ_1 is the first eigenvalue of $-\Delta$ on $H_1^0(\Omega)$. As Lions [36] pointed out, this was only a formal way of guessing the bifurcation diagrams at that time. In general, the bifurcation diagram can be much more complicated if the domain is complicated or the nonlinearity f grows supercritically. However, in the present paper, we rigorously prove that for all nonlinearities which we consider here, the exact shape of the bifurcation diagram of (1.1) is the same as that of the graph of $\lambda_1 u = \lambda f(u)$, and the



(a) Bifurcation diagram in the case $(f(u)/u)'$ from + to −

(b) Bifurcation diagram in the case $(f(u)/u)'$ from − to +

FIG. 2. Bifurcation Curves with One Turning Point.

bifurcation diagram of (1.1) is just a re-orientation of the graph of $u/f(u)$. (See remarks and the figures in Section 8.)

The uniqueness of solution to (1.1) has been studied extensively in the last decade. There is a large literature on the subject, which includes, for example, [1, 10, 12, 14, 24, 28, 32, 33, 34, 37, 38, 39, 40, 42, 43, 46]. In some of these papers, the uniqueness of the ground state solution was also studied. The ground state solution is the solution to

$$\begin{cases} \Delta u + f(u) = 0, & \text{in } \mathbf{R}^n, \\ u > 0, & \text{in } \mathbf{R}^n, \\ \max u(x) = u(0), & \\ u \to 0. & |x| \to \infty. \end{cases} \tag{1.2}$$

On the other hand, the uniqueness or the exact multiplicity of solutions for λ near infinity can be proved even for more general domains and certain f's. The literature on this direction includes [19, 20, 23]. There are also few results which give exact multiplicity (more than 1) of solutions for *all* $\lambda > 0$ and some special f's, see [2, 8, 26, 51]. The references here are only a very partial collection of works, and more references can be found in those papers. Our main results will be given in Subsections 6.1–6.5, and we will give more historical remarks at the end of each subsection of Section 6.

In the paper, we mostly use the bifurcation approach. The key of our method is to study the local behavior of the solution curve near a bifurcation point. A bifurcation point can be a turning point, a point where a bifurcation from the trivial solutions occurs, or a point where a bifurcation from infinity occurs. Another important part of our approach is the oscillatory property of the solutions of the linearized equation at the turning points, which essentially is an estimate of the upper bound of the Morse indices of the solutions. For all solutions to (1.1) which we consider in this paper, it is proved the Morse indices is either 0 or 1, that is the most common situation in the application. Combining these information, we can count the number of turning points and the number of the solution curve, thus determine the whole solution set and the exact global bifurcation diagram.

The bifurcation analysis near turning points was first developed by Korman *et al.* [29, 30] for $n = 1, 2$ and f being a generalized cubic function, and was generalized by Ouyang and Shi in [41] for any dimension and more general nonlinearities. In [29], among other things, Korman *et al.* gave a new proof to the result of [51]. It was extended in [30] to $n = 2$ and more general nonlinearities. Ouyang and Shi [41] generalized the result in [29, 30] to the ball of any dimension and much more general nonlinearities. On the other hand, the technique of studying the linearized equation is not new, and some of them can be traced to early works by

Kolodner [27] and Coffman [15]. Since then, a lot of improvement has been made in, for example, [12, 19, 20, 34, 37, 41].

We organize the paper in the following way: In Section 2, we introduce some classes of functions which we will study and the definition of bifurcation diagram for (1.1). The bifurcation method is described in detail in Section 3. We will first review some well-known local bifurcation results in Subsection 3.1, and we study the turning directions of bifurcation curves at bifurcation points in Subsection 3.2. The technique of studying the linearized equation is summarized in Section 4 in a uniform treatment involving Sturm comparison lemma. In Section 5, we study some properties of radially symmetric solutions of (1.1) for general f's. Our main results on the exact multiplicity will be given in Section 6. In Section 7, we discuss the uniqueness of the ground state solution, or the solution of the over-determined problem

$$\begin{cases} \Delta u + \lambda f(u) = 0, & \text{in } B^n, \\ u > 0, & \text{in } B^n, \\ u = \partial u/\partial n = 0, & \text{on } \partial B^n. \end{cases} \tag{1.3}$$

In Section 8, some remarks are given. In Appendix, we collect a few existence results for (1.1) which are used in the proofs of theorems, and we describe a bifurcation approach for a class of Hölder continuous nonlinearities.

Throughout the paper, we use u' or u_r to denote the derivative of the function or operator. $\mathbf{R}^+$ is the interval $[0, \infty)$, f is a function on $\mathbf{R}^+$ at least C^1, $F(u) = \int_0^u f(t)dt$, $\partial u/\partial n$ is the outer normal derivative of u, and *resp.* is the abbreviation of *respectively.*

2. PRELIMINARIES

2.1. *Some Function Classes*

For all exact multiplicity results here, the nonlinearity $f(u)$ satisfies certain convexity condition. In this section, we define several function classes.

DEFINITION 2.1. Let $f \in C^1[a, b]$. f is said to be *superlinear* (resp. *sublinear*) in $[a, b]$ if $(f(u)/u) \leqslant f'(u)$ (resp. $(f(u)/u) \geqslant f'(u)$) in $[a, b]$. And f is said to be *convex* (resp. *concave*) in $[a, b]$ if $f \in C^2[a, b]$, and $f''(u) \geqslant 0$(resp. $f''(u) \leqslant 0$) in $[a, b]$.

DEFINITION 2.2. Let $f \in C^1[a, b]$. f is said to be of *sup-sub* (resp. *sub-sup*) in $[a, b]$ if there exists $c \in (a, b)$ such that $f(u)$ is superlinear (resp.

sublinear) in $[a, c]$, and sublinear (resp. superlinear) in $[c, b]$. And f is said to be of *convex-concave* (resp. *concave-convex*) if $f \in C^2[a, b]$ and there exists $c \in (a, b)$ such that f is convex (resp. concave) in $[a, c]$, and f is concave (resp. convex) in $[c, b]$.

When $u = 0$ in above definition, we should understand $f(u)/u$ as $\lim_{u \to 0}(f(u)/u)$. When $a = -\infty$ or $b = \infty$, we should understand that the close interval in the definition above is open in that end. For sublinear (*resp.* superlinear) f, we do not regard it as a sub-sup or sup-sub type. Therefore, when we say a function f is sub-sup or sup-sub type, $(f(u)/u) - f'(u)$ must be strictly positive and strictly negative somewhere. Similar for concave and convex f.

In this paper, we will always assume that $f \in C^1(\mathbf{R}^+)$ and f belongs to one of the classes: superlinear, sublinear, concave, convex or sup-sub, sub-sup, convex-concave, concave-convex in $\mathbf{R}^+$, if nothing else is specified. Thus, with no confusion, we will only say, for example, f is "superlinear" instead of f "superlinear in $\mathbf{R}^+$."

The definitions of eight function classes above are not mutually exclusive. In fact, by some elementary calculations, we can clarify the relations between those function classes as following.

PROPOSITION 2.3. *Let* $f \in C^2(\mathbf{R}^+)$.

(1) *If* $f(0) = 0$, *then* f *being convex* (*resp. concave*) *implies* f *is superlinear* (*resp. sublinear*), *and* f *being convex-concave* (*resp. concave-convex*) *implies* f *is either sup-sub or superlinear* (*resp. either sub-sup or sublinear*).

(2) *If* $f(0) > 0$, *then* f *being concave implies* f *is sublinear*, f *being convex implies* f *is either sublinear or sub-sup, and* f *is concave-convex implies* f *is either sub-sup or sublinear.*

(3) *If* $f(0) < 0$, *then* f *being convex implies* f *is superlinear*, f *being concave implies* f *is either superlinear or sup-sub, and* f *is convex-concave implies* f *is sup-sub or superlinear.*

(4) *If* f *is sublinear or sub-sup, then* $f(0) \geqslant 0$; *if* f *is superlinear or sup-sub, then* $f(0) \leqslant 0$.

Note if $f(0) > 0$ and f is convex-concave, or $f(0) < 0$ and f is concave-convex, then the monotonicity of $f(u)/u$ can change at least twice. We will not consider such cases in this paper.

For f belonging to one of the classes we defined above, $f(u)/u$ is asymptotically monotone. Hence we could define $f'(\infty) = \lim_{u \to \infty}(f(u)/u)$, which is possibly $\pm\infty$. Let λ_1 be the first eigenvalue of $-\Delta$ on $H_0^1(B^n)$. We define two numbers λ_0 and λ_∞ as

$$\lambda_0 = \begin{cases} 0 & \text{if } f(0) > 0, \\ \lambda_1/f'(0) & \text{if } f(0) = 0, \\ \infty & \text{if } f(0) < 0, \end{cases}$$

and (2.1)

$$\lambda_\infty = \begin{cases} \infty & \text{if } f'(\infty) \leqslant 0, \\ \lambda_1/f'(\infty) & \text{if } 0 < f'(\infty) < \infty, \\ 0 & \text{if } f'(\infty) = \infty. \end{cases}$$

λ_∞ and λ_0 usually are the bifurcation points of Eq. (1.1) by Theorem 3.1 in the next section.

We conclude this section with the following definition:

DEFINITION 2.4. Let $f \in C^1(\mathbf{R}^+)$, and belong to one of the function classes defined in Definitions 2.1 and 2.2. f is said to be *asymptotic sublinear* if $f'(\infty) \leqslant 0$, *asymptotic linear* if $0 < f'(\infty) < \infty$, and *asymptotic superlinear* if $f'(\infty) = \infty$.

It is easy to verify the relations between these asymptotic conditions and the function classes we defined in Definitions 2.1 and 2.2.

PROPOSITION 2.5. (1) *If f is sublinear or sup-sub, then f is asymptotic sublinear or asymptotic linear.*

(2) *If f is superlinear or sub-sup, then f is asymptotic superlinear or asymptotic linear.*

2.2. *Bifurcation Diagrams*

One remarkable result regarding (1.1) was proved by Gidas *et al.* [24] in 1979: if f is local lipschitz continuous in $[0, \infty)$, then all positive solutions of (1.1) are radially symmetric. This result sets the foundation of our analysis of positive solutions to (1.1). We summarize some basic facts on (1.1) here:

LEMMA 2.6. (1) *If f is local lipschitz continuous in $[0, \infty)$, then all positive solutions of* (1.1) *are radially symmetric and satisfy*

$$\begin{cases} u'' + \dfrac{n-1}{r} u' + \lambda f(u) = 0, & r \in (0, 1), \\ u'(0) = u(1) = 0. \end{cases} \tag{2.2}$$

(2) *If u is a positive solution to* (2.2), $u'(1) \neq 0$, *and w is a solution to the linearized problem* (*if it exists*)

$$\begin{cases} \Delta w + \lambda f'(u) w = 0, & \text{in } B^n, \\ w = 0, & \text{on } \partial B^n, \end{cases} \tag{2.3}$$

then w is also radially symmetric and satisfies

$$\begin{cases} w'' + \dfrac{n-1}{r} w' + \lambda f'(u) w = 0, & r \in (0, 1), \\ w'(0) = w(1) = 0. \end{cases} \tag{2.4}$$

(3) *For any $d > 0$, there is at most one $\lambda_d > 0$ such that* (2.2) *has a positive solution u with $\lambda = \lambda_d$ and $u(0) = d$. Let $T = \{d > 0$: (2.2) has a positive solution with $u(0) = d\}$, then $\lambda(d) = \lambda_d$ is a well-defined continuous function from T to $\mathbf{R}^+$. If $f \in C^k(\mathbf{R}^+)$, then $\lambda(\cdot) \in C^k(T)$.*

Property (1) is the classical result of Gidas *et al.* [24]; property (2) is due to Lin and Ni [35]. Property (3) is well known; see for example [9, 28]. Because of (3), we call $\mathbf{R}^+ \times \mathbf{R}^+ = \{(\lambda, d) : \lambda > 0, d > 0\}$ the *phase space*, $\{(\lambda(d), d)$: (2.2) has a solution with $u(0) = d$, $\lambda = \lambda(d)\}$ the *bifurcation curve*, and the phase space with bifurcation curve the *bifurcation diagram*.

For a solution of (2.2), $u(\lambda, \cdot)$ is a *degenerate* if (2.4) has a nontrivial solution, otherwise it is *nondegenerate*. Sometimes we also call a degenerate solution a *turning point*. We also define the *Morse index* $M(u)$ of a solution u to (2.2) to be the number of negative eigenvalues μ's of

$$\begin{cases} \phi'' + \dfrac{n-1}{r} \phi' + \lambda f'(u) \phi = -\mu \phi, & r \in (0, 1), \\ \phi'(0) = \phi(1) = 0. \end{cases} \tag{2.5}$$

If all eigenvalues of (2.5) are positive, then the solution u is *stable*, otherwise it is *unstable*.

Note that in Lemma 2.2(2), we need $u'(1) \neq 0$. In general, since u is positive, we have $u'(1) \leq 0$. If $u'(1) = 0$, then (2.3) has non-radial solutions but may not have radially symmetric solutions.

3. BIFURCATION ANALYSIS

3.1. *Elementary Bifurcation Theorems*

In this section, we review some well-known local bifurcation theorems. In the original papers where they were proved, all the bifurcation theorems were stated in an abstract setting, but the applications of those theorems to nonlinear elliptic equations like (1.1) are standard, we refer to standard references like [1, 22]. So we state them in the context of Eq. (1.1) to ease the applications.

THEOREM 3.1. *Let* $f \in C^1(\mathbf{R}^+)$. (1) (*Implicit Function Theorem*) *Let* (λ_*, u_*) *be a positive solution to* (1.1) *which satisfies*

$$\frac{\partial u_*}{\partial n}(x) < 0 \qquad \text{for all} \quad x \in \partial B^n, \tag{3.1}$$

and the linearized equation (2.3) *has no nontrivial solution. Then all positive solutions of* (1.1) *near* (λ_*, u_*) *has form of* $(\lambda(s), u_* + sw + z(s))$ *for* $s \in (-\delta, \delta)$ *for some* $\delta > 0$, *where* w *is the solution of*

$$\begin{cases} \Delta w + \lambda_* f'(u_*) w = -f(u_*), & \text{in } B^n, \\ w = 0, & \text{on } \partial B^n, \end{cases} \tag{3.2}$$

and $\lambda(0) = \lambda_*$, $\lambda'(0) \neq 0$, $z(0) = z'(0) = 0$.

(2) (*Bifurcation from the trivial solutions*) *If* $f(0) = 0$ *and* $f'(0) > 0$, $\lambda_0 = \lambda_1 / f'(0)$, *then all positive solutions of* (1.1) *near* $(\lambda_0, 0)$ *has form of* $(\lambda(s), u(s)) = (\lambda(s), sw + z(s))$ *for* $s \in (0, \delta)$ *and some* $\delta > 0$, *where* w *is a positive solution of*

$$\begin{cases} \Delta w + \lambda_1 w = 0, & \text{in } B^n, \\ w = 0, & \text{on } \partial B^n, \end{cases} \tag{3.3}$$

and $\lambda(0) = \lambda_0$, $z(0) = z'(0) = 0$.

(3) (*Bifurcation from infinity*) *Let* $f'(\infty) = \lim_{u \to \infty} (f(u)/u) \in (0, \infty)$ *and* $\lambda_\infty = \lambda_1 / f'(\infty)$. *Then all positive solutions of* (1.1) *near* (λ_∞, ∞) *has form of* $(\lambda(s), sw + z(s))$ *for* $s \in (\delta, \infty)$ *and some* $\delta > 0$, *where* w *is a positive solution of* (3.3), $\lim_{s \to \infty} \lambda(s) = \lambda_\infty$, *and* $\|z(s)\|_{C^{2,\alpha}(\overline{B^n})} = o(s)$ *as* $s \to \infty$.

(4) (*Bifurcation at a turning point*) *Let* (λ_*, u_*) *be a positive solution of* (1.1) *which satisfies* (3.1), *and the linearized equation* (2.3) *have a one-dimensional solution space* $\operatorname{span}\{w\}$, *which satisfies*

$$\int_{B^n} f(u_*) w \, dx \neq 0. \tag{3.4}$$

Then all the positive solutions of (1.1) *near* (λ_*, u_*) *has form of* $(\lambda(s), u_* + sw + z(s))$ *for* $s \in (-\delta, \delta)$ *for some* $\delta > 0$, *where* $\lambda(0) = \lambda_*$, $\lambda'(0) = 0$, $z(0) = z'(0) = 0$.

The implicit function theorem can be found in any standard analysis textbook, for example, [22]. The bifurcation from the trivial solutions result was due to Crandall and Rabinowitz [16]; see also [44]. The bifurcation from infinity result was due to Rabinowitz [45], and the bifurcation at a turning point result was proved by Crandall and Rabinowitz [17]. We should remark that for asymptotic superlinear f, bifurcation from some (λ, ∞) is still possible, though $\lambda \neq \lambda_\infty$. For example, for $f(u) = e^u$ and $n \geqslant 3$, $\lambda = n(n-2)$ is a bifurcation point where a bifurcation from infinity occurs (see [26]). But these two types bifurcation from infinity are different. In this paper, if we say λ is a bifurcation point where a bifurcation from infinity occurs, it refers to the type in Theorem 3.1(3), and $\lambda = \lambda_\infty$. For future reference, we introduce another result on bifurcation from the trivial solutions when $f(0) > 0$. (A proof can be found in [5].)

THEOREM 3.2. *Let $f \in C^1(\mathbf{R})$ and $f(0) > 0$. Then all positive solutions of* (1.1) *near* (0, 0) *has form of* $(\lambda(s), sw + z(s))$ *for* $s \in (0, \delta)$ *for some* $\delta > 0$, *where* w *is the solution of*

$$\begin{cases} \Delta w = -f(0), & \text{in } B^n, \\ w = 0, & \text{on } \partial B^n, \end{cases} \tag{3.5}$$

and $\lambda(0) = 0$, $z(0) = z'(0) = 0$.

We remark that the results in this section are also true if we replace B^n by a general smooth bounded domain Ω, but we do not need that in this paper.

3.2. *Turning Directions at Bifurcation Points*

To use the information from local bifurcation in determining the global bifurcation curve, it is important to know the turning direction of the solution curve at a bifurcation point. Let $(\lambda(s), u(s))$, $s \in I$, be a solution curve of (1.1), where $I = (0, \delta)$, (δ, ∞) or $(-\delta, \delta)$ as in Theorem 3.1(2), (3), or (4). And the bifurcation point is (λ_*, u_*).

DEFINITION 3.3. If there is $\delta_0 > 0$, such that $\lambda(s) \geqslant \lambda_*$ for $s \in I$, then we say a *supercritical bifurcation* occurs at (λ_*, u_*); similarly, if $\lambda(s) \leqslant \lambda_*$ for $s \in I$, then we say a *subcritical bifurcation* occurs at (λ_*, u_*).

First we consider the turning direction at λ_* where a bifurcation from the trivial solutions or from infinity occurs.

PROPOSITION 3.4. (1) *Suppose that* $(\lambda_*, 0)$ *is a point where a bifurcation from the trivial solutions occurs, and* $(\lambda(s), u(s))$, $s \in (0, \delta)$, *is the positive solution curve in Theorem* 3.1(2). *We assume that, there exists* $\delta_1 > 0$ *such* *that* $(f(u)/u) \geqslant f'(0)$ *(resp.* $\leqslant f'(0)$*) in* $[0, \delta_1]$, *then* $(\lambda(s), u(s))$ *is subcritical (resp. supercritical).*

(2) *Suppose that that* (λ_*, ∞) *is a point where a bifurcation from infinity occurs, and* $(\lambda(s), u(s))$, $s \in (\delta, \infty)$, *is the positive solution curve in Theorem* 3.1(3). *We assume that,* $(f(u)/u) \leqslant f'(\infty)$ *(resp.* $\geqslant f'(\infty)$*) for* $u \in (0, \infty)$, *then* $(\lambda(s), u(s))$ *is supercritical (resp. subcritical).*

Proof. (1) Let ϕ_1 be the normalized positive eigenfunction corresponding to $\lambda_1 = \lambda_0 f'(0)$. Then ϕ_1 and $u(s)$ satisfy

$$\Delta\phi_1 + \lambda_0 f'(0)\phi_1 = 0 \tag{3.6}$$

and

$$\Delta u(s) + \lambda_0 f'(0)\, u(s) + (\lambda(s) - \lambda_0) f'(0)\, u(s) + \lambda(s)[f(u(s)) - f'(0)\, u(s)] = 0. \tag{3.7}$$

By integration, we get

$$(\lambda(s) - \lambda_0) f'(0) \int_{B^n} u(s)\, \phi_1\, dx + \lambda(s) \int_{B^n} \left[\frac{f(u(s))}{u(s)} - f'(0) \right] u(s)\, \phi_1\, dx = 0. \tag{3.8}$$

By the regularity theory of elliptic equation, since $f \in C^1$, $u(s) \in C^{2,\alpha}(\overline{B^n})$, then for $\delta_1 > 0$, $\|u(s)\|_{C^{2,\alpha}(\overline{B^n})} \leqslant \delta_1$ when $s > 0$ is small enough. If $(f(u)/u) \geqslant f'(0)$ for $u \in [0, \delta_1]$, then for $s > 0$ is small enough, the second integral in (3.8) is positive, hence $\lambda(s) < \lambda_0$ for small $s > 0$. The case of $(f(u)/u) \leqslant f'(0)$ is similar.

(2) Similar to (3.8), we have

$$(\lambda(s) - \lambda_\infty) f'(\infty) \int_{B^n} u(s)\, \phi_1\, dx + \lambda(s) \int_{B^n} \left[\frac{f(u(s))}{u(s)} - f'(\infty) \right] u(s)\, \phi_1\, dx = 0. \tag{3.9}$$

If $(f(u)/u) \leqslant f'(\infty)$, the second integral in (3.9) is negative, hence $\lambda(s) > \lambda_0$ for all $s > 0$. The case of $(f(u)/u) \geqslant f'(\infty)$ is similar. ▮

COROLLARY 3.5. (1) *Suppose that* $(\lambda_*, 0)$ *is a point where a bifurcation from the trivial solutions occurs, and* $(\lambda(s), u(s))$, $s \in (0, \delta)$, *is the positive solution curve in Theorem* 3.1(2). *We assume that, there exists* $\delta_1 > 0$ *such that f is superlinear (resp. sublinear) in* $[0, \delta_1]$, *then* $(\lambda(s), u(s))$ *is subcritical (resp. supercritical).*

(2) *Suppose that* (λ_*, ∞) *is a point where a bifurcation from infinity occurs, and* $(\lambda(s), u(s))$, $s \in (\delta, \infty)$, *is the positive solution curve in Theorem*

3.1(3). *We assume that, f is superlinear (resp. sublinear) in* $(0, \infty)$, *then* $(\lambda(s), u(s))$ *is supercritical (resp. subcritical).*

In (2) of Proposition 3.4, we require $(f(u)/u) \leqslant f'(\infty)$ for all $u > 0$, which is a very strong condition. In the future applications, we need a result which only imposes conditions on f near ∞. In fact, Ambrosetti and Hess [5] provided such a result, which we review here.

PROPOSITION 3.6. *Suppose that* (λ_*, ∞) *is a point where a bifurcation from infinity occurs, and* $(\lambda(s), u(s))$, $s \in (\delta, \infty)$, *is the positive solution curve in Theorem* 3.1(3). *We assume that*

$$\liminf_{u \to \infty} [f(u) - f'(\infty)u] < 0, \qquad (\text{resp. } \limsup_{u \to \infty} [f(u) - f'(\infty)u] > 0). \tag{3.10}$$

Then $(\lambda(s), u(s))$ *is supercritical (resp. subcritical).*

Proof. Again we use (3.9). Since $u(s)/\|u(s)\|_\infty \to \phi_1$ almost everywhere in B^n as $\lambda \to \lambda_*$, thus $u(s)(x) \to \infty$ almost everywhere in B^n. Let $c = \liminf_{u \to \infty} [f(u) - f'(\infty)u] < 0$, then by Fatou's Lemma,

$$\lim_{\lambda \to \lambda_*} \int_{B^n} [f(u) - f'(\infty)u] \phi_1 \, dx \leqslant c \int_{B^n} \phi_1 \, dx < 0. \tag{3.11}$$

Thus $(\lambda(s), u(s))$ is supercritical. ∎

LEMMA 3.7. *Suppose* $f \in C^2(\mathbf{R}^+)$ *and* $0 < f'(\infty) < \infty$. *If there is a* $\delta_1 > 0$ *such that f is superlinear in* $[\delta_1, \infty)$ *and* $f''(u) \geqslant (\not\equiv) 0$ *for* $u > \delta_1$. *Then* (3.10) *is satisfied. Similar result holds if f is sublinear in* $[\delta_1, \infty)$ *and* $f''(u) \leqslant (\not\equiv) 0$ *for* $u > \delta_1$.

Proof. We assume that f is superlinear for $u > \delta_1$ and $f'' \geqslant (\not\equiv) 0$ for $u > \delta_1$. Let $p(u) = f'(u)u - f(u)$, then $(f(u)/u)' = p(u)/u^2$ and $p'(u) = f''(u)u$. Since $f''(u) \not\equiv 0$, then there exists $\delta_2 \geqslant \delta_1$ such that $p(\delta_2) > 0$ and $p(u) \geqslant p(\delta_2) > 0$ for $u \geqslant \delta_2$. On the other hand, since $f''(u) \geqslant (\not\equiv) 0$, then $f'(\infty) = \lim_{u \to \infty} f'(u) \geqslant f'(u)$ for $u > \delta_1$. Hence for $u \geqslant \delta_2$

$$f(u) - f'(\infty)u \leqslant f(u) - f'(u)u = -p(u) \leqslant -p(\delta_2) < 0, \tag{3.12}$$

and (3.10) holds. ∎

COROLLARY 3.8. *Suppose* $f \in C^2(\mathbf{R}^+)$ *and* $0 < f'(\infty) < \infty$. *If there is a* $\delta_1 > 0$ *such that f is superlinear in* $[\delta_1, \infty)$ *and* $f''(u) \geqslant (\not\equiv) 0$ *for* $u > \delta_1$. *Then* $(\lambda(s), u(s))$ *is supercritical. Similar result holds if f is sublinear in*

$[\delta_1, \infty)$ and $f''(u) \leqslant (\not\equiv) 0$ for $u > \delta_1$. In particular, if f is sub-sup and concave-convex(or convex), then the bifurcation from infinity is subcritical; if f is sup-sub and convex-concave(or concave), then the bifurcation from infinity is supercritical.

We also need a stability result for the solutions bifurcating from the trivial solutions or from infinity:

PROPOSITION 3.9. (1) *Suppose that $(\lambda_*, 0)$ is a point where a bifurcation from the trivial solutions occurs, and $(\lambda(s), u(s))$, $s \in (0, \delta)$, is the solution curve in Theorem 3.1(2). Then for $s \in (0, \delta)$, $M(u(s)) = 1$ if and only if $\lambda'(s) < 0$, and $M(u(s)) = 0$ if and only if $\lambda'(s) \geqslant 0$.*

(2) *Suppose that (λ_*, ∞) is a point where a bifurcation from infinity occurs, $(\lambda(s), u(s))$, $s \geqslant \delta$, is the solution curve in Theorem 3.1(3). Then for $s \geqslant \delta$, $M(u(s)) = 1$ if and only if $\lambda'(s) < 0$, and $M(u(s)) = 0$ if and only if $\lambda'(s) \geqslant 0$.*

This proposition can be followed from Lemma 1.3, Corollary 1.13 and Theorem 1.16 of [17], we omit the proof here, and a detailed proof is given in [47]. (See also [48, and 49].)

Next we consider the turning directions at turning points. We notice that in this case, $\lambda'(0) = 0$ at turning point (λ_*, u_*). Thus a simple way to determine the turning direction is to examine the second derivative $\lambda''(0)$ if it exists. We assume that f is C^2, then $(\lambda(s), u(s))$ is also a C^2 curve. By the result of [41], we have

$$\lambda''(0) = -\frac{\lambda_* \int_{B^n} f''(u_*) w^3 \, dx}{\int_{B^n} f(u_*) w \, dx} = -\frac{\lambda_* \int_0^1 r^{n-1} f''(u_*) w^3 \, dr}{\int_0^1 r^{n-1} f(u_*) w \, dr}, \tag{3.13}$$

if (3.4) is satisfied, where w is a solution of (2.3). We proved in [41] that if u and w are the solutions of (2.2) and (2.4) respectively, then

$$\int_0^1 f(u) \, w r^{n-1} \, dr = \int_0^1 f'(u) \, u w r^{n-1} \, dr = \frac{1}{2\lambda} w'(1) \, u'(1). \tag{3.14}$$

So if $u > 0$ and $w > 0$ in $(0, 1)$, and $u'(1) < 0$, then $\int_0^1 r^{n-1} f(u) w \, dr > 0$. In fact, at a turning point, $u'(1) \neq 0$ is always true by Lemma 5.3 for the case of $f(0) < 0$ and by the maximal principle for the cases of $f(0) \geqslant 0$.

Let F: $H_0^1(B^n) \cap H^2(B^n) \to L^2(B^n)$ be defined as $F(\lambda, u) = \Delta u + \lambda f_1(u)$, where $f_1 : L^2(B^n) \to L^2(B^n)$ is the Nemyski operator of $f(u)$. We study the turning direction of solution curve at a turning point under the following assumption: if (λ_*, u_*) is a degenerate solution to (1.1), then

$$\dim Kernel(F_u(\lambda_*, u_*)) = 1, \qquad Kernel(F_u(\lambda_*, u_*)) = span\{w\}, \quad \text{and } w \text{ can be chosen as positive.} \tag{3.15}$$

In this section, we give some criteria of turning direction at a turning point under this assumption, and in Section 4 and applications, we will prove (3.15) is true for some given f's. We notice that, at (λ_*, u_*), 0 is always an eigenvalue of the problem (2.5). Assumption (3.15) is equivalent to saying that 0 is the first eigenvalue of (2.5), thus the eigenfunction w can be chosen as positive. This is a common case, since one of the solutions of (1.1) is usually stable, then at the turning points on the branch of the stable solutions, 0 must be the first eigenvalue. For example, the maximal solution for sup-sub f and the minimal solution for sub-sup f are stable.

By formula (3.13), we have following results:

THEOREM 3.10. *If $f \in C^1(\mathbf{R}^+)$, f is either superlinear or sublinear, and we assume at all degenerate solutions,* (3.15) *holds. Then* (1.1) *has no degenerate solution.*

Proof. Suppose that (λ_*, u_*) is a degenerate solution. Multiplying (1.1) by w, (2.3) by u_*, subtracting and integrating, we have

$$0 = \int_{B^n} (\Delta u_* \cdot w - \Delta w \cdot u_*)\, dx = \lambda \int_{B^n} [f'(u_*)u_* - f(u_*)]\, w\, dx.$$

But $f'(u)u - f(u)$ does not change sign in $[0, \infty)$, and w is positive. Therefore we reach a contradiction. ∎

COROLLARY 3.11. *If $f \in C^2(\mathbf{R}^+)$, and satisfies one of following:*

(1) $f(0) \leqslant 0$, *and f is convex.*

(2) $f(0) \geqslant 0$, *and f is concave.*

and we assume at all degenerate soltitions, (3.15) *holds. Then* (1.1) *has no degenerate solution.*

Proof. By Proposition 2.3, if f is convex and $f(0) \leqslant 0$, then it is superlinear. Similarly, if f is concave and $f(0) \geqslant 0$, then it is sublinear. Then the result follows from Theorem 3.10. ∎

THEOREM 3.12. *If $f \in C^2(\mathbf{R}^+)$, and satisfies one of following:*

(1) *$f(0) > 0$, and f is convex.*

(2) *$f(0) < 0$, and f is concave.*

At a degenerate solution (λ_*, u_*) of (1.1), if (3.15) holds, then $\lambda''(0) \neq 0$, and

(1) *$\lambda''(0) < 0$ if f is convex and $f(0) > 0$.*

(2) *$\lambda''(0) > 0$ if f is concave and $f(0) < 0$.*

Proof. It is obvious by using the formula (3.13) and (3.14). ▮

For f being concave-convex or convex-concave, we proved the following result in [41] (Theorem 2.2 and the Remark after the proof of Theorem 2.2):

THEOREM 3.13. *If $f \in C^2(\mathbf{R}^+)$ and satisfies one of following:*

(1) *$f(0) \geqslant 0$, and f is concave-convex,*

(2) *$f(0) \leqslant 0$, and f is convex-concave.*

At a degenerate solution (λ_*, u_*) of (1.1), if (3.15) holds, then $\lambda''(0) \neq 0$, and

(1) *$\lambda''(0) > 0$ if f is convex-concave,*

(2) *$\lambda''(0) < 0$ if f is concave-convex.*

The idea for proving Theorem 3.10 can also be used to prove the following fact:

PROPOSITION 3.14. *If $f \in C^1(\mathbf{R}^+)$, f is superlinear, then any solution u_* of (1.1) is unstable; if f is sublinear, then any solution u_* of (1.1) is stable.*

Proof. Let ϕ_1 be the positive eigenfunction of first eigenvalue μ_1 of (2.5). We multiply (1.1) by ϕ_1, the PDE version of (2.5) by u_*, subtract and integrate, then we have

$$\begin{aligned} -\mu_1 \int_{B^n} \phi_1 u_* \, dx &= \int_{B^n} (\Delta u_* \cdot \phi_1 - \Delta \phi_1 \cdot u_*) \, dx \\ &= \lambda \int_{B^n} [f'(u_*) u_* - f(u_*)] \, \phi_1 \, dx. \end{aligned}$$

Therefore $\mu_1 < 0$ if f is superlinear, while $\mu_1 > 0$ if f is sublinear. ▮

We notice that Theorems 3.10, 3.12, and 3.13 do not cover the example $f(u)=u^q$ with $0<q<1$, since it is not C^1 at $u=0$. Here we claim that the bifurcation theory for $f\in C^i(\mathbf{R}^+)$, $i=1,2$, can be extended to

$$f\in C^i(0,\infty) \quad \text{and} \quad \lim_{u\to 0^-}\frac{f(u)}{u^q}=b, \text{ for some } q\in(0,1) \text{ and } b>0. \tag{3.16}$$

(So $f(0)=0$ and $f\in C^i(0,\infty)\cap C^{0,q}(\mathbf{R}^+)$.) We will sketch a modification of bifurcation approach in Appendix to accommodate such nonlinearities.

4. DISCONJUGACY

An important tool to study (2.2) and (2.4) is the classical Sturm comparison lemma. We rewrite (2.2) and (2.4) to self-adjoint form.

$$\begin{cases}(r^{n-1}u')'+\lambda r^{n-1}f(u)=0, & r\in(0,1),\\ u'(0)=u(1)=0,\end{cases} \tag{4.1}$$

and

$$\begin{cases}(r^{n-1}w')'+\lambda r^{n-1}f'(u)w=0, & r\in(0,1),\\ w'(0)=w(1)=0.\end{cases} \tag{4.2}$$

We start with a weak form of Sturm comparison lemma. (The proof can be found in [41].)

LEMMA 4.1. *Let* $Lu(t)=[p(t)u'(t)]'+q(t)u(t)$, *where* $p(t)$ *and* $q(t)$ *are continuous in* $[a,b]$ *and* $p(t)\geqslant 0$, $t\in[a,b]$. *Suppose* $Lw(t)=0$, $w\not\equiv 0$. *If there exists* $v\in C^2[a,b]$ *such that* $Lv(t)\cdot v(t)\leqslant(\not\equiv)0$, *then* w *has at most one zero in* $[a,b]$. *In addition if* $w'(a)=0$ *or* $p(a)=0$, *then* w *does not have any zero in* $[a,b]$.

The zeros of w are related to the concept of disconjugacy.

DEFINITION 4.2. Let $Lu(t)=[p(t)u'(t)]'+q(t)u(t)$, where $p(t)$ and $q(t)$ are continuous in $[a,b]$, then the equation $Lu(t)=0$ is said to be *disconjugate* on $[a,b]$ if every nontrivial solution has at most one zero on $[a,b]$.

LEMMA 4.3. *Following the notation in Definition 4.2, in addition if* $p(t)>0$ *in* (a,b), *then* $Lw(t)=0$ *is disconjugate on* $[a,b]$ *if and only if* $Lw(t)=0$ *has a solution* $w(t)>0$ *in* (a,b).

Definition 4.2 and Lemma 4.3 are taken from [25]. By Lemma 4.3, to prove $w>0$, it is equivalent to prove $Lw=0$ is disconjugate on $[0, 1]$.

Next we prove $Lw=0$ is disconjugate on an interval for various nonlinearities f's. (In the remaining part of this section, we always assume that L is defined as $Lv=(r^{n-1}v_r)_r+\lambda r^{n-1}f'(u)v$.) First we notice that disconjugacy is *always* true when $n=1$. In fact w and u_r both satisfy the equation $v''+\lambda f'(u)v=0$ and $w'(0)=w(1)=0$, $u_r(0)=0$, $u_r(r)<0$ for $r\in(0, 1)$. Suppose that $w(r_0)=0$ for some $r_0\in(0, 1)$. Then by the integral argument, we obtain

$$w'(1)\,u_r(1)-w'(r_0)\,u_r(r_0)=0,$$

$w'(1)$, $w'(r_0)\neq 0$ and have opposite sign while $u_r(1)\leqslant 0$ and $u_r(r_0)<0$, which is a contradiction. Therefore $w(r)>0$ for $r\in(0, 1)$. Consequently, we can classify the solution set of (1.1) when $n=1$ and f belongs the function classes defined in Theorems 3.10, 3.12, and 3.13. (The detail classification will appear somewhere else).

PROPOSITION 4.4. *Let $n=1$ and f being one of superlinear, sublinear, convex, concave, convex-concave ($f(0)\leqslant 0$) and concave-convex ($f(0)\geqslant 0$). Then the solution set of (1.1) can be precisely determined.*

So in the remaining part of this section, we concentrate on the disconjugacy problem for $n\geqslant 2$ only. Recall that, for given $f\in C^1(\mathbf{R}^+)$, the definition of K_f is $K_f(u):\{u\geqslant 0: f(u)\neq 0\}\to\mathbf{R}$, $K_f(u)=uf'(u)/f(u)$.

We first prove that if f is sublinear, then $Lw=0$ is disconjugate. In fact we have

LEMMA 4.5. *If $f\in C^1(\mathbf{R}^+)$, $f(u)\geqslant 0$ in $[u_1, u_0]$, and $f(u)/(u-u_1)$ is decreasing in $[u_1, u_0]$. Let u and w be solutions of (4.1) and (4.2) respectively, and $[r_0, r_1]\subset[0, 1]$ such that $u(r_i)=u_i$ for $i=0, 1$. Then (4.2) is disconjugate on $[r_0, r_1]$.*

Proof. Since f is C^1, then $f(u)/(u-u_1)$ is decreasing implies $(u-u_1)f'(u)-f(u)\leqslant 0$ in $[u_1, u_0]$. It is easy to show, for $v(r)=u(r)-u_1$,

$$Lv(r)=-\lambda r^{n-1}[f(u(r))-f'(u(r))(u(r)-u_1)]. \tag{4.3}$$

Then $Lv(r)\leqslant 0$ on $[r_0, r_1]$, and $v(r)\geqslant 0$ in $[r_0, r_1]$. By Lemma 4.1, $Lu=0$ is disconjugate on $[r_0, r_1]$. ∎

For superlinear f or similar kind, we have the following two results:

LEMMA 4.6. *If $n\geqslant 3$, $f\in C^1(\mathbf{R}^+)$, $f(u)>0$ in $[u_1, u_0]$, and $K_f(u)$ satisfies $1\leqslant K_f(u)\leqslant n/(n-2)$ in $[u_1, u_0]$. Let u and w be solutions of (4.1)* *and* (4.2) *respectively, and* $[r_0, r_1] \subset [0, 1]$ *such that* $u(r_i) = u_i$ *for* $i = 0, 1$. *Then* w *has at most two zeros in* $[r_0, r_1]$. *Moreover, if* $r_0 = 0$, *then* (4.2) *is disconjugate on* $[r_0, r_1]$.

Proof. We choose the comparison function to be $v_1(r) = ru_r(r) + (n-2)\,u(r)$ and $v_2(r) = r^{-1}v_1(r)$. It is easy to calculate that

$$Lv_1(r) = \lambda r^{n-1}[(n-2)\,f'(u)u - nf(u)] \leqslant 0, \tag{4.4}$$

$$Lv_2(r) = \lambda r^{n-1}\left[\frac{(n-2)(f'(u)u - f(u))}{r} - (n-3)\,\frac{v_1(r)}{r^2}\right]. \tag{4.5}$$

Since $v_1'(r) = (n-1)\,u_r(r) + ru_{rr}(r) = -\lambda rf(u(r)) \leqslant 0$ for $r \in (0, 1)$, and $v_1(0) = (n-2)\,u(0) > 0$, $v_1(1) = u_r(1) < 0$. Then there is a unique zero $r_2 \in (0, 1)$ such that $v_1(r_2) = 0$, and

$$v_1(r) \geqslant 0 \quad \text{for} \quad r \in (0, r_2), \qquad v_1(r) \leqslant 0 \quad \text{for} \quad r \in (r_2, 1). \tag{4.6}$$

In $(0, r_2)$ we can use v_2 as the test function in Lemma 4.1, and in $(r_2, 1)$ we can use v_1 as the test function. So if $r_2 \in (r_0, r_1)$, by Lemma 4.1, (4.2) is disconjugate on $[r_0, r_2]$ and (4.2) is disconjugate on $[r_2, r_1]$. Hence there are at most two zeros of w on $[r_0, r_1]$. The second part of lemma can be obtained the same way as that of Lemma 4.1. ∎

LEMMA 4.7. *If* $n \geqslant 4$, $f \in C^1(\mathbf{R}^+)$, $f(u) > 0$ *in* $[u_1, u_0]$, *and* $K_f(u)$ *satisfies* $-(n-4)/(n-2) \leqslant K_f(u) \leqslant n/(n-2)$ *in* $[u_1, u_0]$. *Let* u *and* w *be solutions of* (4.1) *and* (4.2) *respectively, and* $[r_0, r_1] \subset [0, 1]$ *such that* $u(r_i) = u_i$ *for* $i = 0, 1$. *Then* w *has at most two zeros in* $[r_0, r_1]$. *Moreover, if* $r_0 = 0$, *then* (4.2) *is disconjugate on* $[r_0, r_1]$.

Proof. We choose the comparison function to be $v_1(r) = ru_r(r) + (n-2)\,u(r)$ and $v_2(r) = r^{2-n}v_1(r)$. It is easy to calculate that

$$Lv_1(r) = \lambda r^{n-1}[(n-2)\,f'(u)u - nf(u)] \leqslant 0, \tag{4.7}$$

$$Lv_2(r) = \lambda r[(n-2)\,f'(u)u + (n-4)\,f(u)] \geqslant 0. \tag{4.8}$$

Then using a similar argument as Lemma 4.6, we can show there are at most two zeros of w on $[r_0, r_1]$. The second part of lemma can be obtained the same way as that of Lemma 4.1. ∎

The next result involves Pohozaev's identity. We define

$$H(r) = \tfrac{1}{2}[ru_r^2(r) + (n-2)\,u_r(r)\,u(r)] + \lambda rF(u(r)), \tag{4.9}$$

where $F(u)=\int_0^u f(s)\,ds$. Pohozaev's identity is

$$r_2^{n-1}H(r_2)-r_1^{n-1}H(r_1)=\int_{r_1}^{r_2}\lambda r^{n-1}\left[nF(u(r))-\frac{n-2}{2}f(u(r))\,u(r)\right]dr, \tag{4.10}$$

where $0\leqslant r_1<r_2\leqslant 1$.

LEMMA 4.8. *If $f\in C^1(\mathbf{R}^+)$, $f(u)>0$ in $[u_1,u_0]$. In addition, we assume that*

(1) *$K_f(u)$ is decreasing in $[u_1,u_0]$;*

(2) *$f(u)u-2F(u)\geqslant 0$ in $[u_1,u_0]$.*

Let u and w be solutions of (4.1) and (4.2), respectively, $[r_0,r_1]\subset[0,1]$ such that $u(r_i)=u_i$ for $i=0,1$, and $H(r)\geqslant 0$ in $[r_0,r_1]$; Then w has at most two zeros in $[r_0,r_1]$. Moreover, if $r_0=0$, then (4.2) is disconjugate on $[r_0,r_1]$.

Proof. We choose the comparison function to be $v(r)=ru_r(r)+\mu u(r)$, where $\mu>0$ is a constant to be specified later. Then

$$Lv(r)=\lambda r^{n-1}\{\mu[f'(u)u-f(u)]-2f(u)\}=\lambda r^{n-1}g(u(r)), \tag{4.11}$$

where $g(u)=\mu[f'(u)u-f(u)]-2f(u)$. Define

$$h(r)=-\frac{ru_r(r)}{u(r)},\qquad \text{in } (r_0,r_1), \tag{4.12}$$

$$\mu(r)=\frac{2f(u(r))}{f'(u(r))\,u(r)-f(u(r))},\qquad \text{in } (r_0,r_1). \tag{4.13}$$

Then

$$\begin{aligned}h'(r)&=\frac{(n-2)\,uu_r+ru_r^2+\lambda fru}{u^2}\\&=\frac{2H(r)-2\lambda rF(u(r))+\lambda rf(u(r))\,u(r)}{u^2(r)}\\&=\frac{2H(r)}{u^2(r)}+\lambda r\frac{f(u(r))\,u(r)-2F(u(r))}{u^2(r)}.\end{aligned} \tag{4.14}$$

Here, in the second equality, we use Pohozaev's identity. By the conditions in the lemma, we have $h'(r)\geqslant 0$ in $[r_0,r_1]$. On the other hand,

$$\mu(r)=\frac{2}{K_f(u(r))-1}. \tag{4.15}$$

Since $K_f(u)$ is decreasing in $[u(r_1), u(r_0)]$, then $\mu(r)$ is decreasing in $[r_0, r_1]$. Then one of following is true:

(1) There exists $\mu_0 > 0$ and $r_2 \in [r_0, r_1]$ such that $v(r) \geqslant 0$ and $g(u(r)) \leqslant 0$ in $[r_0, r_2]$, $v(r) \leqslant 0$ and $g(u(r)) \geqslant 0$ in $[r_2, r_1]$;

(2) There exists $\mu_0 > 0$ such that $v(r) \geqslant 0$ and $g(u(r)) \leqslant 0$ in $[r_0, r_1]$;

(3) There exists $\mu_0 > 0$ such that $v(r) \leqslant 0$ and $g(u(r)) \geqslant 0$ in $[r_0, r_1]$.

Then by Lemma 4.1, w has at most two zeros on $[r_0, r_1]$. And the second part of lemma can also follow from the second part of Lemma 4.1. ∎

Finally, we include an important result due to Kwong and Zhang [34], which used the idea of Peletier and Serrin [42]. For the proof of this lemma, we refer to [34, Lemmas 13–16].

LEMMA 4.9. *If $f \in C^1(\mathbf{R}^+)$, and there exists $u_2 < u_1 < u_0$ such that:*

(1) $f(u) < 0$, $u \in (u_2, u_1)$; $f(u_1) = 0$; $f(u) > 0$, $u \in (u_1, u_0)$;

(2) $\int_{u_2}^{u_0} f(u)\, du = 0$.

Let u and w be solutions of (4.1) and (4.2), respectively, and $[r_0, r_2] \subset [0, 1]$ such that $u(r_i) = u_i$ for $i = 0, 1, 2$. Then (4.2) is disconjugate on $[r_0, r_2]$.

Remark 4.10. Lemma 4.5 was due to Peletier and Serrin [42], see also [20, 41]; Lemma 4.6 was originated from [40], our proof is based on [31]; Lemma 4.7 is new, and Lemma 4.8 is adapted from [41]; see also [12].

5. PROPERTIES OF RADIALLY SYMMETRIC SOLUTIONS

In this section, we prove some generic properties of solution set of (2.2) which do not depend on the specific form of f.

To study (2.2), it is often helpful to consider the non-parameter version of (2.2):

$$\begin{cases} u'' + \dfrac{n-1}{r} u' + f(u) = 0, & r \in (0, R(d)), \\ u'(0) = u(R(d)) = 0, \end{cases} \tag{5.1}$$

where $R(d) = \sqrt{\lambda(d)}$, $d = u(0)$, or from another angle, the initial value problem:

$$\begin{cases} u'' + \dfrac{n-1}{r} u' + f(u) = 0, & r > 0, \\ u'(0) = 0, \quad u(0) = d. \end{cases} \tag{5.2}$$

Problem (5.2) has a unique smooth solution $u(r, d)$ for $r \in [0, \infty)$ if f is lipshitz continuous. The existence, uniqueness and continuous dependence of $u(r, d)$ on d are standard, and have been proved in, for example, [40, 42]. It is easy to see that, via a change of variable $t = R(d)r$, a solution $u(\lambda, r)$ to (2.2) becomes a solution to (5.1) $u(t, d) = u(\lambda(d), r)$ with $u(0) = d$. (Though we have a little notation abuse here, but we will keep use $u(r, d)$ instead of $u(t, d)$. So keep in mind that, $u(\lambda, r)$ is a solution of (2.2), with $r \in [0, 1]$, while $u(r, d)$ is a solution of (5.1), with $r \in [0, R(d)]$.)

We apply a standard shooting procedure to (5.2). We define, for any $d > 0$,

$$R(d) = \sup\{r : u(s, d) > 0 \text{ and } u_r(s, d) < 0, \ s \in (0, r)\},$$

if $u_r(s, d) < 0$ for $0 < s \ll 1$, and $R(d) = 0$ if $u_r(s, d) \geqslant 0$ for $0 < s \ll 1$. If $R(d) = 0$, then either $u_r(s, d) \equiv 0$ for $s > 0$, then $f(d) = 0$, or $f(d) < 0$ and $u_r(s, d) > 0$ for $0 < s \ll 1$. In the latter case, $u(s, d) > d$ for $s > 0$ by Lemma 3 of [42]. According to the behavior of $R(d)$, we can classify each $d > 0$ as

$$\begin{aligned} N &= \{d > 0 : R(d) < \infty, u(R(d), d) = 0, u_r(R(d), d) < 0\}, \\ P &= \{d > 0 : R(d) < \infty, u(R(d), d) > 0, u_r(R(d), d) = 0\}, \\ G &= \{d > 0 : R(d) = \infty\}, \\ B &= \{d > 0 : R(d) < \infty, u(R(d), d) = 0, u_r(R(d), d) = 0\}, \\ C &= \{d > 0 : f(d) = 0\}, \\ E &= \{d > 0 : f(d) < 0\}. \end{aligned} \tag{5.3}$$

These sets are disjoint subsets of $\mathbf{R}^+$. It is easy to see that N, P and E are open subsets of $(0, \infty)$, G, B and C are closed subsets of $(0, \infty)$, and $(0, \infty) = N \cup P \cup G \cup B \cup C \cup E$.

By the result of [21], $P \supset \{d > 0 : f(d) > 0, F(d) < 0\}$. We recall $T = \{d > 0 : (2.2) \text{ has a positive solution with } u(0) = d\}$. If $d \in N \cup B$, then

$u(R(d)r, d)$ is a solution of (2.2) with $\lambda = R(d)^2$. So $T = N \cup B$. If $d \in B$, then $u(\cdot, d)$ is a solution of an overdetermined problem

$$\begin{cases} u'' + \dfrac{n-1}{r} u' + f(u) = 0, & r \in (0, R(d)), \\ u'(0) = u(R(d)) = u'(R(d)) = 0, & u(0) = d. \end{cases} \tag{5.4}$$

If $d \in G$, then $u(r, d) > 0$, $u_r(r, d) < 0$ for all $r > 0$ and $u(\cdot, d)$ is a solution of

$$\begin{cases} u'' + \dfrac{n-1}{r} u' + f(u) = 0, & r \in (0, \infty), \\ u(r) > 0, \quad u'(r) < 0, & r \in (0, \infty), \\ u'(0) = 0, \quad u(0) = d, & \lim_{r \to \infty} u(r) = c \geqslant 0. \end{cases} \tag{5.5}$$

If $c = 0$ in (5.5), then $u(\cdot, d)$ is a radially symmetric ground state solution. In general, we have the following characterization of set G:

LEMMA 5.1. *Let $d \in G$. Then $u(\cdot, d)$ is a solution of* (5.5). *Moreover, c satisfies $f(c) = 0$ and $f'(c) \leqslant 0$.*

Proof. We multiply (5.5) by u_r and integrate over $(0, r)$. Then we obtain

$$\frac{1}{2}[u_r(r, d)]^2 + (n-1) \int_0^r \frac{[u_r(s, d)]^2}{s} ds = F(u(0, d)) - F(u(r, d)). \tag{5.6}$$

Therefore $\int_0^\infty s^{-1}[u_r(s, d)]^2 ds < \infty$, and $\lim_{r \to \infty} u_r(r, d) = 0$, $\lim_{r \to \infty} u_{rr}(r, d) = 0$. Hence by the equation, $\lim_{r \to \infty} f(u(r, d)) = 0$.

So c is a zero of f. Suppose $f'(c) > 0$. We rewrite Eq. (5.5) as

$$(r^{n-1}u_r)_r + r^{n-1}f(u) = 0. \tag{5.7}$$

Let $p(r) = r^{n-1}u_r(r, d)$ and $w(r) = p'(r)/p(r)$, then we can check that $w'(r) = -w^2 - f'(u(r, d)) + (n-1)r^{-1}w$. So if $r > 1$, $w_r(r) < -w^2 + (n-1)w$, that implies $w(r) < n-1$. Since $u(r, d) > c$ for all $r > 0$, $f'(u(r, d)) > \delta > 0$ for r large. Hence $w'(r) \leqslant -w^2 - f'(u(r, d)) + (n-1)^2 r^{-1} \leqslant -\delta/2$ for r large enough. So there exists a $r_0 > 0$ such that $w(r_0) = 0$, which implies $f(u(r_0, d)) = 0$ and $u(r_0, d) = c$. That is a contradiction. Hence $f'(c) \leqslant 0$. ∎

Next we want to obtain a clearer characterization of B. We first introduce another auxiliary equation,

$$\begin{cases} w'' + \dfrac{n-1}{r} w' + \lambda f'(u(\lambda, r)) w = 0, & \text{in} \quad (0, 1), \\ w'(0) = 0, \qquad w(0) = 1, \end{cases} \tag{5.8}$$

where $u(\lambda, r)$ is a solution to (2.2). Let $w(\lambda, \cdot)$ be the solution of (5.8). Then $w(\lambda, \cdot)$ has the following relation with the Morse index of $u(\lambda, \cdot)$:

LEMMA 5.2. *Suppose that* $u(\lambda, \cdot)$ *is a solution of* (2.2), *and* $w(\lambda, \cdot)$ *is the solution of* (5.8). *Then* $M(u(\lambda, \cdot)) = k$ *if and only if* $w(\lambda, \cdot)$ *has exactly* k *zeros in* (0, 1).

Proof. Let φ_k be the eigenfunction corresponding to the kth eigenvalue η_k of (2.5). Then $w(r) = w(\lambda, r)$ and $\varphi_k(r)$ satisfy the following equations respectively:

$$(r^{n-1} w')' + \mu r^{n-1} f'(u) w = 0,$$

$$(r^{n-1} \varphi_k')' + \mu r^{n-1} f'(u) \varphi_k = -\eta_k \varphi_k .$$

Suppose that $M(u(\lambda, \cdot)) = k$, then $\eta_k < 0$ and $\eta_{k+1} > 0$. By Sturm comparison lemma, between any two consecutive zeros of φ_k, there exists at least one zero of w. We extend φ_k and w to $(-1, 0)$ evenly, then φ_k has $2k$ zeros in $[-1, 1]$. Hence w has at least $2k - 1$ zeros in $(-1, 1)$. But w is even and 0 is not a zero of w, therefore w has at least $2k$ zeros in $(-1, 1)$ and at least k zeros in (0, 1). Similarly, by comparing w with φ_{k+1}, we can prove that w has at most k zeros in (0, 1). Hence w has exactly k zeros in (0, 1).

On the other hand, suppose that w has exactly k zeros in (0, 1). Assume $M(u(\lambda, \cdot)) > k$, then $\eta_{k+1} < 0$. By the same argument as the last paragraph, we can show that w has at least $k + 1$ zeros in (0, 1), which is a contradiction. So $M(u(\lambda, \cdot)) \leqslant k$. Similarly, we can show that $M(u(\lambda, \cdot)) \geqslant k$. Therefore $M(u(\lambda, \cdot)) = k$. ∎

By the maximal principle, if $f(0) \geqslant 0$, then $u_r(\lambda, 1) < 0$. Even for the case of $f(0) < 0$, we still have:

LEMMA 5.3. *Let* $u(\lambda, \cdot)$ *and* $w(\lambda, \cdot)$ *be the solutions of* (2.2) *and* (5.8), *respectively*, $f(0) < 0$, *and* $u(\lambda, r) > 0$, $w(\lambda, r) > 0$ *in* $[0, 1)$, *then* $u_r(\lambda, 1) \neq 0$.

Proof. Let L be defined as $Lv = (r^{n-1}v_r)_r + \lambda r^{n-1}f'(u(\lambda, \cdot))v$, and $u(\cdot) = u(\lambda, \cdot)$, $w(\cdot) = w(\lambda, \cdot)$. It is straightforward to verify that $Lu_r(r) = (n-1)\, r^{n-3}u_r(r)$, and by Lemma 2.6, $u_r(r) < 0$ for all $r \in (0, 1)$. Let $r_0 \in (0, 1)$. Integrating

$$[r^{n-1}(wu_r' - w'u_r)]' = wLu_r - u_r Lw = wLu_r$$

over $(0, r_0)$, we obtain

$$r_0^{n-1}[w(r_0)\, u_r'(r_0) - w'(r_0)\, u_r(r_0)] = \int_0^{r_0} (n-1)\, r^{n-3}u_r(r)\, w(r)\, dr. \tag{5.9}$$

Since $w > 0$ and $u_r < 0$ in $(0, 1)$, then the right hand side of (5.9) is negative. On the other hand, if $r_0 \to 1^-$, $w(r_0) > 0$, $u_r'(r_0) > 0$ (recall $f(0) < 0$). Therefore, if $r_0 \to 1^-$, $w'(r_0)u_r(r_0) > 0$. In particular, $\lim_{r \to 1^-} u_r(r) < 0$ and $u_r(1) < 0$ since u_r is C^1 up to the boundary. ∎

COROLLARY 5.4. *$B \neq \emptyset$ only if $f(0) < 0$. Moreover, if $d \in B$, then $M(u(\cdot, d)) \geq 1$.*

Let $w(r, d) = \partial u(r, d)/\partial d$, then $w(\cdot, d)$ satisfies

$$\begin{cases} w'' + \dfrac{n-1}{r} w' + f'(u(r, d))\, w = 0, & r \in (0, R(d)), \\ w'(0) = 0, \quad w(0) = 1. \end{cases} \tag{5.10}$$

The function $w(r, d)$ played an important role in the study of uniqueness and/or bifurcation problem of (2.2) or (5.1). It was first introduced by Kolodner [27] and Coffman [15]. Later it was also the main tool of [31, 40]. Here we also use $w(\cdot, d)$ to obtain some useful information.

LEMMA 5.5. *Let $u(\cdot, d)$ and $w(\cdot, d)$ be the solutions to* (5.1) *and* (5.10), *respectively, and $d \in N \cup B$. Then following statements hold:*

(1) *$M(u(\cdot, d)) = k$ if and only if $w(\cdot, d)$ has exactly k zeros in $(0, R(d))$. In particular, $M(u(\cdot, d))$ is even if $w(R(d), d) > 0$, and $M(u(\cdot, d))$ is odd if $w(R(d), d) < 0$.*

(2) *If $d \in N$, then $sign(R'(d)) = sign(w(R(d), d))$.*

Proof. The proof of (1) is the same as that of Lemma 5.2, since the equations are different only by a rescaling. For (2), we differentiate $u(R(d), d) = 0$, and we have $u_r(R(d), d)R'(d) + u_d(R(d), d) = 0$. Since $u_d = w$ and $u_r(R(d), d) < 0$, we obtain (2). ∎

COROLLARY 5.6. *Let $u(\lambda(d), \cdot)$ be a positive solution to* (2.2), *and $w(\lambda(d), \cdot)$ be the solution of* (5.8). *If $d \in N$, then $sign(\lambda'(d)) = sign(w(\lambda(d), 1))$. Thus if $\lambda'(d) > 0$, then $M(u(\lambda(d), \cdot))$ is even; if $\lambda'(d) < 0$, then $M(u(\lambda(d), \cdot))$ is odd.*

Proof. Since $\lambda(d) = [R(d)]^2$, then $\lambda'(d) = 2R(d)\, R'(d)$. ∎

Let $d_0 \in N$, and we consider the solutions $u(r, d)$ of (5.2) for $d \in (d_0 - \varepsilon, d_0 + \varepsilon)$. If $w(r, d_0) > 0$ in $(0, r_0)$, then $u(r, d)$ is increasing with respect to d. But if $w(\lambda(d_0), r) > 0$ in $(0, 1)$, we cannot get that $u(\lambda(d), r)$ is increasing with respect to d or λ, since $u_d(\lambda(d_0), r)$ is not a rescaling of $w(r, d_0)$, though $u(\lambda(d), r)$ is a rescaling of $u(r, d)$. In fact, $u_d(\lambda(d), r) = \lambda'(d) u_\lambda(\lambda, r)$, and it satisfies

$$\begin{cases} u_d'' + \dfrac{n-1}{r}\, u_d' + \lambda(d)\, f'(u(\lambda(d), r))\, u_d + \lambda'(d)\, f(u(\lambda(d), r)) = 0, & r \in (0, 1), \\ u_d'(\lambda(d), 0) = u_d(\lambda(d), 1) = 0, \quad u_d(\lambda(d), 0) = 1. \end{cases} \tag{5.11}$$

Whether $u(\lambda, r)$ is increasing with respect to λ or d is an important question in some bifurcation analysis. We have the following result:

LEMMA 5.7. *Let $u(\lambda, \cdot)$ be a positive solution to* (2.2), *and $w(\lambda, r)$ be the solution to* (5.8). *Assume $w(\lambda, r) > 0$ for $r \in [0, 1]$, then $u_\lambda(\lambda, r) > 0$ and $u_d(\lambda, r) > 0$ for $r \in [0, 1)$ if one of the following conditions is also satisfied:*

(1) *$f(u) \geqslant 0$ in $[0, u(\lambda, 0)]$;*

(2) *$n = 1$;*

(3) *$n \geqslant 2$, there exists $b > 0$ such that $f(u) \leqslant 0$ in $[0, b]$ and $f(u) \geqslant 0$ in $[b, u(\lambda, 0)]$.*

Proof. By Corollary 5.6, $\lambda'(d) > 0$, so we only need to prove $u_\lambda(\lambda, r) > 0$.

(1) By Lemma 5.2, $w(\lambda, r) > 0$ implies that $M(u(\lambda, \cdot)) = 0$ and $\mu_1[u(\lambda, \cdot)] > 0$ (the first eigenvalue of (2.5)). That implies the operator $-L$ ($Lv = (r^{n-1} v_r)_r + \lambda r^{n-1} f'(u(\lambda, \cdot)) v$) is invertible, and the inverse $(-L)^{-1}$: $C^0[0, 1] \to C^2[0, 1]$ is a positive operator. By (5.11), $u_\lambda(\lambda, r)$ satisfies $(-L) u_\lambda(\lambda, r) = r^{n-1} f(u(\lambda, r))$ and $u_\lambda'(\lambda, 0) = u_\lambda(\lambda, 1) = 0$. Thus $u_\lambda(\lambda, r) = (-L)^{-1} r^{n-1} f(u(\lambda, r)) > 0$ in $[0, 1)$.

(2) and (3) From (5.11), we have $u_\lambda(\lambda, 0) > 0$ and $L u_\lambda(\lambda, r) = -r^{n-1} f(u(\lambda, r))$. On the other hand, we have $L(r u_r) = -2\lambda r^{n-1} f(u(r))$, and $r u_r(\lambda, r) < 0$ in $(0, 1)$. Let $v_1(r) = 2\lambda u_\lambda(\lambda, r) - r u_r(\lambda, r)$, then $L v_1(r) = 0$ and $v_1(0) > 0$, $v_1(1) \geqslant 0$. Thus if v changes sign in $(0, 1)$, then v_1 changes sign at least twice. But on the other hand, $L w(\lambda, r) = 0$ and $w(\lambda, \cdot)$ does not

change sign in (0, 1), which contradicts Sturm comparison lemma. Therefore $v_1(r)>0$ in $[0, 1]$. ($v_1(1)>0$ by Lemma 5.3.)

To complete the proof, we use a homotopy argument and another argument in [29, 30, 41]. Let $v_k(r)=2\lambda u_\lambda(\lambda, r)-kru_r(\lambda, r)$, where $k\in[0, 1]$. We claim that $v_k>0$ for all $k\in[0, 1]$. Suppose not, denote by k_0 the first k where $v_k>0$ is violated. Then $v_{k_0}(r)\geqslant 0$ in $[0, 1]$ and $v_{k_0}(r_0)=0$ for some $r_0\in[0, 1]$. Obviously, $r_0\neq 0$ since $v_k(0)=2\lambda u_\lambda(\lambda, 0)>0$. If $r_0=1$, then $k_0=0$, otherwise $v_k(1)=-ku_r(\lambda, 1)>0$. Hence we only need to consider the case of $r_0\in(0, 1)$. We apply the intergral procedure to u_r and v_k in $[0, r_0]$, then

$$\begin{aligned}
&r_0^{n-1}[u_r(\lambda, r_0)v'_{k_0}(r_0)-u'_r(\lambda, r_0)\, v_{k_0}(r_0)] \\
&\quad=-\int_0^{r_0}[2\lambda(1-k_0)\, r^{n-1}f(u(\lambda, r))\, u_r(\lambda, r) \\
&\qquad+(n-1)\, r^{n-3}v_{k_0}(r)\, u_r(\lambda, r)]\, dr. \qquad (5.12)
\end{aligned}$$

The left-hand side of (5.12) is 0 since r_0 is a local minimum of v_{k_0}. For the right-hand side, $\int_0^{r_0} r^{n-3}v_k(r)\, u_r(\lambda, r)\, dr<0$ since $v_k\geqslant 0$ and $u_r<0$ in $(0, r_0)$. So, if $\int_0^{r_0} r^{n-1}f(u(\lambda, r))\, u_r(\lambda, r)\, dr\leqslant 0$, we will reach a contradiction. If $n=1$, $\int_0^{r_0} f(u(\lambda, r))\, u_r(\lambda, r)\, dr=F(u(\lambda, r_0))-F(u(\lambda, 0))<0$ by (5.6). So (2) is proved. For (3), multiplying (4.1) by $r^{n-1}u_r$ and integrating from 0 to r_0, we have $0=(1/2)\, r_0^{2n-2}u_r^2(\lambda, r_0)=-\lambda\int_0^{r_0} r^{2n-2}f(u(\lambda, r))\, u_r(\lambda, r)\, dr$. Since $f(u)$ changes only once in $[0, u(\lambda, 0)]$, so there exists $r_1\in(0, 1)$ such that $f(u(\lambda, r_1))\geqslant 0$ in $[0, r_1]$ and $f(u(\lambda, r_1))\leqslant 0$ in $[r_1, 1]$. And $2\lambda(1-k)\, f(u(\lambda, r_0))=-u''_\lambda(\lambda, r_0)\leqslant 0$, so $r_1\leqslant r_0$. It follows that

$$\begin{aligned}
0&=\int_0^{r_0} r^{2n-2}f(u(\lambda,r))\, u_r(\lambda,r)\, dr \\
&>r_1^{n-1}\int_0^{r_1} r^{n-1}f(u(\lambda,r))\, u_r(\lambda,r)\, dr+r_1^{n-1}\int_{r_1}^{r_0} r^{n-1}f(u(\lambda,r))\, u_r(\lambda,r)\, dr \\
&=r_1^{n-1}\int_0^{r_0} r^{n-1}f(u(\lambda,r))\, u_r(\lambda,r)\, dr.
\end{aligned}$$

This proves (3). ∎

At last, we discuss the horizontal asympotes of the curve of positive solutions. Since $\lambda(d)$ is a continuous function on T, we define: $d=d_*$ is a *horizontal asymptote* of $\lambda(d)$ if

$$\lim_{d\to d_*,\, d\in T}\lambda(d)=\infty.$$

PROPOSITION 5.8. *Suppose that* $f \in C^1(\mathbf{R}^+)$, *and* $\lambda = \lambda(d)$ *is the bifurcation curve of positive solutions of* (2.2). *If* $d = d_*$ *is a horizontal asymptote of* $\lambda(d)$, *then either* $d_* \in G \cup C$. *Moreover,* $u(r, d) \to u(r, d_*)$ *in* $C^2_{loc}(\mathbf{R}^+)$ *as* $d \to d_*(\lambda \to \infty)$ *and* $d \in T$.

Proof. Since $d = d_*$ is a horizontal asymptote, then there exists a sequence $\{d_n\}$ such that $d_n \to d_*$ and $\lambda(d_n) \to \infty$ as $n \to \infty$, so $R(d_n) \to \infty$. Then $d_* \notin P \cup N \cup B$, since for any $d_* \in P \cup N \cup B$, $R(d) < \infty$ for d near d_*. Thus $d_* \in G \cup E \cup C$. If $d_* \in E$, then $u''(0, d_*) > 0$ since $u \in E$, $u'(r, d_*) \geqslant 0$. Thus $u'(r, d_*) > 0$ for $r \in (0, \varepsilon)$ for some $\varepsilon > 0$. By the continuous dependence of $u(r, d)$ on d, $u'(r, d) > 0$ for $r \in (0, \varepsilon)$ and $|d - d_*|$ small. But this is impossible since $d_n \in N \cup B$. Therefore $d_* \in G \cup C$.

By (5.6), if $d_* \in G$, $\int_0^\infty r^{-1} u_r^2(r, d_*)\, dr < \infty$ and $u_r(r, d_*) < 0$ for all $r > 0$, then $u(r, d_*) \to c$, $u_r(r, d_*) \to 0$, $u_{rr}(r, d_*) \to 0$ as $r \to \infty$. Similarly, by differentiating (5.1), $u_{rrr}(r, d_*) \to 0$ as $r \to \infty$. In particular, $u(r, d_*)$, $u_r(r, d_*)$, $u_{rr}(r, d_*)$, $u_{rrr}(r, d_*)$ are all uniformly bounded in $\mathbf{R}^+$. This is also true for $d_* \in C$, since all derivatives of $u(r, d_*)$ are identically zero. For both cases and any compact subset K of $\mathbf{R}^+$ and some $\varepsilon > 0$, for $d \in [d_* - \varepsilon, d_* + \varepsilon]$, $u(r, d)$, $u_r(r, d)$, $u_{rr}(r, d)$, $u_{rrr}(r, d)$ are all uniformly bounded for $r \in K$. Then by Ascoli-Arzela Theorem, for any sequence $d_n \to d_*$ and $d_n \in T$, there is a subsequence d_{n_k} such that $u(r, d_{n_k}) \to v(r)$ in $C^2(K)$. By a diagonal procedure, we can assume (by choosing a further subsequence) $u(r, d_{n_k}) \to v(r)$ in $C^2_{loc}(\mathbf{R}^+)$. Obviously, $v(r)$ satisfies (5.1) with $v(0) = d_*$. By the uniqueness of solution to (5.1), we have $v(r) = u(r, d_*)$. ∎

PROPOSITION 5.9. *Suppose that* $f \in C^1(\mathbf{R}^+)$, *and* $\lambda = \lambda(d)$ *is the bifurcation curve of positive solutions of* (2.2). *Suppose* $d = d_*$ *is a horizontal asymptote of* $\lambda(d)$. *If* $d_* \in G$, *then* $\limsup_{d \to d_*} u(\lambda(d), r) \leqslant c$ *uniformly for* r *in any compact subset* K *of* $(0, 1)$, *where* c *is defined in* (5.5). *In particular, if* $c = 0$, *then* $\lim_{d \to d_*} u(\lambda(d), r) = 0$ *uniformly for* $r \in K$.

Proof. For any $b \in (c, d_*)$, there exists a unique $r_0 > 0$ such that $u(r_0, d_*) = b$, and for $r > r_0$, $u(r, d_*) < b$. By the continuous dependence of $u(r, d)$ on d, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $u(r_0, d) < b + \varepsilon$ for $d \in [d_* - \delta, d_* + \delta]$. Then for $d \in [d_* - \delta, d_* + \delta]$, $u(r, d) < b + \varepsilon$ for all $r \in (r_0, R(d))$. On the other hand, for any compact $K \subset (0, 1)$, $K \subset [r_1, r_2]$ for some $0 < r_1 < r_2 < 1$, and there exists $\delta_1 > 0$ such that $d \in [d_* - \delta_1, d_* + \delta_1]$, then $[r_1 R(d), r_2 R(d)] \subset [r_0, R(d)]$ since $R(d) \to \infty$ as $d \to d_*$. Then for $d \in [d_* - \delta_1, d_* + \delta_1]$, $u(\lambda(d), r) < b + \varepsilon$ for $r \in [r_1, r_2]$, which implies that $\limsup_{d \to d_*} u(\lambda(d), r) \leqslant b + \varepsilon$. Since b and ε are chosen arbitrarily, then $\limsup_{d \to d_*} u(\lambda(d), r) \leqslant c$. If $c = 0$, we have $u(\lambda(d), r) > 0$ for any $r \in [0, 1)$, hence $\lim_{d \to d_*} u(\lambda(d), r) = 0$ uniformly for $r \in K$. ∎

PROPOSITION 5.10. *Suppose that* $f \in C^1(\mathbf{R}^+)$, *and* $\lambda = \lambda(d)$ *is the bifurcation curve of positive solutions of* (2.2). *Suppose* $d = d_*$ *is a horizontal asymptote of* $\lambda(d)$, *there exists* $\varepsilon > 0$, *such that* $(d_* - \varepsilon, d_*) \subset T$, $\lambda'(d) > 0$ *and* $u_\lambda(\lambda(d), r) > 0$ *for* $d \in (d_* - \varepsilon, d_*)$ *and* $r \in [0, 1)$, *then* $d_* \in C$.

Proof. By Proposition 5.8, $d_* \in G \cup C$. But if $d \in G$, by Proposition 5.9, $\limsup_{d \to d_*} u(\lambda(d), r) \leqslant c < d_*$, which is impossible if $u_\lambda(\lambda(d), r) > 0$ for $d \in (d_* - \varepsilon, d_*)$. Thus $d_* \in C$. ∎

6. EXACT MULTIPLICITY

This section is the central part of the whole paper. The exact multiplicity and global bifurcation diagrams of positive solutions to (2.2) will be obtained. The results here are still far away from a complete classification of bifurcation diagrams for all f's, but we have a very interesting exhibit of various bifurcation diagrams which are all rigorously verified.

In this section, we only consider the positive solutions to (2.2). So when we say a solution to (2.2), we actually mean a *positive* solution to (2.2). Recall that λ_0 and λ_∞ are defined as in (2.1), and the subsets P, N, G, B, C, and E of $\mathbf{R}^+$ are the same as in (5.3). In this section, the following proposition gives the stability information of the solutions:

PROPOSITION 6.1. *If any degenerate solution u to* (2.2) *satisfies* (3.15) *for a given f, then any solution* $u(\lambda, \cdot)$ *of* (4.1) *has Morse index* 0 *or* 1. *Moreover, if* $\lambda'(d) > 0$, *then* $u(\lambda, \cdot)$ *is stable with Morse index* 0; *if* $\lambda'(d) = 0$, *then* $u(\lambda, \cdot)$ *is degenerate, unstable with Morse index* 0; *and if* $\lambda'(d) < 0$, *then* $u(\lambda, \cdot)$ *is unstable with Morse index* 1.

Proof. In all the proofs of any solution w of (2.4) can be chosen as positive, we use the assumption that $w(1) = 0$. In fact, if we remove this assumption, using the same proof, we can prove that the solution $w(\lambda, r)$ has at most one zero in $[0, 1]$ for any solution $u(\lambda, \cdot)$ of (2.2), since our proofs do not depend on the fact that $u(\lambda, \cdot)$ is degenerate. Thus if (2.2) has a solution $u(\lambda, \cdot)$ which has Morse index $M(u(\lambda, \cdot)) \geqslant 2$, then by Lemma 5.2, the corresponding $w(\lambda, \cdot)$ has at least 2 zeros in $[0, 1]$, which is a contradiction. The last part can be derived from Corollary 5.6. ∎

Our proofs of all exact multiplicity results in this section follow a similar pattern, which we briefly describe here. The proof includes four steps:

(1) The problem has a curve of positive solutions.

(2) For any degenerate solution, the solution w to (2.4) can be chosen as positive.

(3) By (2) and the results on turning direction at turning points, there is at most one turning point in each component of solution curve.

(4) There is only one solution curve (*i.e.*, the solution set is connected).

On the other hand, we also discuss uniqueness of ground state solution or the solution to overdetermined problem (5.4), or more generally, the property of shooting problem (5.2).

6.1. *f Is Sublinear*

Equation (1.1) has at most one positive solution when f is sublinear. This is even true for the general bounded smooth domains. (For example, see [1, 50].)

THEOREM 6.2. *Let $f \in C^1(0, \infty) \cap C^{0,\alpha}[0, \infty)$. Assume $f(0) \geqslant 0$ and f is sublinear, then*

(1) *Equation (2.2) has no solution for $0 < \lambda \leqslant \lambda_0$ or $\lambda \geqslant \lambda_\infty$, and has exactly one solution for $\lambda_0 < \lambda < \lambda_\infty$.*

(2) *All solutions lie on a single smooth solution curve, which starts from $(\lambda_0, 0)$ and continues to the right up to (λ_∞, ∞) (resp. (∞, c) if there exists $c > 0$ such that $f(c) = 0$), there is no turning point on the curve, and $M(u) = 0$ for any solution u. (See Figs 3, 4, and 5.)*

(3) *$N = (0, \infty)$ if $f(u) > 0$ for $u > 0$; $N = (0, c)$, $C = \{c\}$ and $E = (c, \infty)$ if $f(u) > 0$ in $(0, c)$ and $f(u) < 0$ in (c, ∞).*

Remark 6.3. The asymptotic behavior of f determines the asymptotic behavior of bifurcation curve as $\lambda \to \infty$ or $\|u\|_\infty \to \infty$. If $\lambda_\infty < \infty$, f is asymptotic linear, then the bifurcation curve blows up at λ_∞ (see Fig. 5); if $\lambda_\infty = \infty$, f is asymptotic sublinear, then the bifurcation curve continues to ∞ in the λ direction (see Fig. 3). Theorem 6.2 is also true for $f(u)$ which is negative for $u > c$ and $f(c) = 0$. In that case, the bifurcation curve continues to ∞, and is bounded by $\|u\|_\infty = c$. (See Fig. 4). There is no solution with $\|u\|_\infty > c$ by the maximal principle. For the other nonlinearities in the

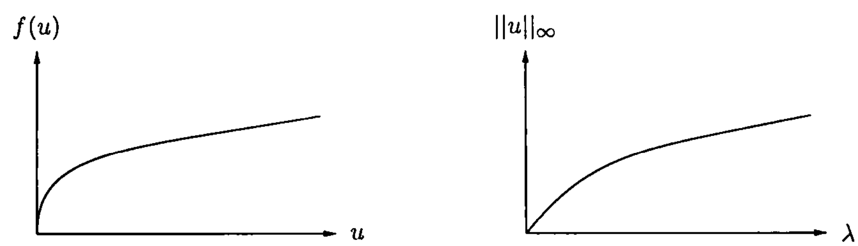


FIG. 3. f sublinear, and $\lambda_0 = 0$, $\lambda_\infty = \infty$.

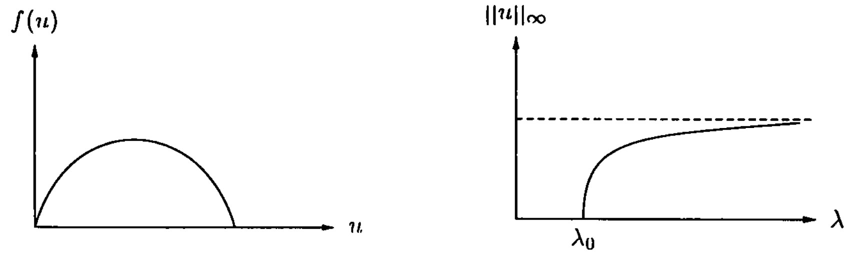


FIG. 4. f sublinear, and $\lambda_0 > 0$, $\lambda_\infty = \infty$.

following sections, we can discuss the asymptotic behavior of bifurcation curve in the same manner. So we will not repeat the same argument in this remark for other cases. $\lambda_0 > 0$ if $f(0) = 0$ and $f \in C^1(\mathbf{R}^+)$, $f'(0) > 0$ (see Figs. 4 and 5), and $\lambda_0 = 0$ if $f(0) > 0$, or $f(0) = 0$ and satisfies (3.16). (See Fig. 3).

Proof of Theorem 6.2. Let $v(r) = u(r) \geqslant 0$, then $Lv(r) = \lambda(f'u - f) \leqslant 0$. By Lemma 4.1, $Lw = 0$ is disconjugate on $[0, 1]$. By Theorem 3.10, (2.2) has no degenerate solution. By Proposition 2.3 (4), $f(0) \geqslant 0$. If $f(0) = 0$ and $f'(0) \leqslant 0$, then $f(u) < 0$ for all $u > 0$, (2.2) has no solution for any $\lambda > 0$ (in that case, $\lambda_0 = \lambda_\infty = \infty$). On the other hand, if $f(0) = 0$ and $f'(0) > 0$, from Theorem 3.1 (2) and Proposition 3.4 (1), there is a solution curve Σ_0 bifurcating from $(\lambda_0, 0)$ and the bifurcation is supercritical. Similarly, if $f(0) > 0$, there is also a solution curve Σ_0 bifurcating from $(0, 0)$ by Theorem (3.2). Since there is no degenerate solution on Σ_0, then Σ_0 can continue to $\lambda_s = sup\{\lambda > 0 : (2.2)$ has a solution $u(\lambda)$ on $\Sigma_0\} > 0$.

If $\lambda_s < \infty$, then $\|u(\lambda)\|_\infty \to \infty$ as $\lambda \to \lambda_s^-$. In fact, if $\limsup_{\lambda \to \lambda_s^-} \|u(\lambda)\|_\infty < \infty$, then (2.2) has a solution $(\lambda_s, u(\lambda_s))$, where $u(\lambda_s) = \lim_{\lambda \to \lambda_s^-} u(\lambda)$ and $u(\lambda_s)$ is not degenerate. Thus Implicit Function Theorem (Theorem 3.1 (1)) implies that we can extend Σ_0 beyond λ_s, which contradicts the definition of λ_s. Recall that $T = \{d > 0$: (2.2) has a positive solution with $u(0) = d\}$, then $T = N = (0, \infty)$. By Lemma 2.6 (3), all solutions

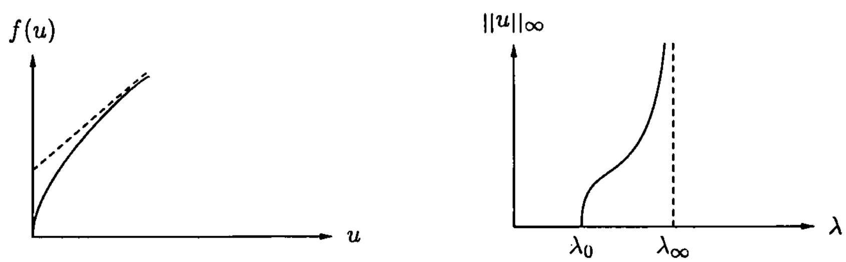


FIG. 5. f sublinear, and $\lambda_0 > 0$, $\lambda_\infty < \infty$.

of (2.2) are on Σ_0, and there is no solution for $\lambda \geqslant \lambda_s$. By Theorem A.1, (2.2) has a solution for any large λ if $f'(\infty)=0$. Therefore $\lambda_s < \infty$ only if $0 < f'(\infty) < \infty$. Hence λ_s is the point where a bifurcation from infinity occurs, so $\lambda_s = \lambda_\infty$.

If $\lambda_s = \infty$, then $f'(\infty)=0$, and Σ_0 continues to $\lambda = \infty$. If there exists $c>0$ such that $f(c)=0$, then $u(\lambda)(0) < c$ for any solution $u(\lambda)$ on Σ_0. So Σ_0 is bounded by a horizontal asymptote not greater than c. Since $\lambda'(d) > 0$, then $M(u(\lambda))=0$ by Proposition 6.1 and $u(\lambda)$ is increasing with respect to λ (or d) by Lemma 5.7. It follows that $\lim_{\lambda \to \infty} u(\lambda)(0) = c$ by Proposition 5.10, and there is no solution to (2.2) satisfying $u(0) \geqslant c$ since $f(u) < 0$ for $u > c$. If $f(u) > 0$ for all $u > 0$, then $u(\lambda)(0) \to \infty$ as $\lambda \to \infty$ by Proposition 5.10. From Lemma 2.6 (3), there is no any other solution of (2.2). (3) easily follows from the above argument, and by Proposition 6.1, $M(u)=0$ for any solution u. ∎

EXAMPLE 6.4. (1) $f(u)=u^q$ *for* $0<q<1$; (*asymptotic sublinear*)

(2) $f(u)=\log(u+1)$; (*asymptotic sublinear*)

(3) $f(u)=u-u^p$ for $p>1$; (*asymptotic negative*)

(4) $f(u)=\sqrt{u^2+2u}$; (*asymptotic linear*)

(5) $f(u)=2u+1-\sqrt{u^2+1}$; (*asymptotic linear*)

(6) $g(u)=f(u)+c$, *where* f *is any example above and* $c>0$.

6.2. *f Is Superlinear and Positive*

In this subsection, we assume that $n \geqslant 3$, f is superlinear and $f(u)>0$ for $u>0$. By Proposition 2.3, $f(0)=0$.

THEOREM 6.5. *Let* $f \in C^1(\mathbf{R}^+)$. *Assume* $n \geqslant 3$, $f(0)=0$, f *is superlinear, and* $f(u)>0$ *for* $u>0$. *In addition,* $K_f(u) \leqslant n/(n-2)$ *for all* $u>0$. *Then*

(1) *Equation* (2.2) *has no solution for* $0<\lambda \leqslant \lambda_\infty$ *or* $\lambda \geqslant \lambda_0$, *and has exactly one solution for* $\lambda_\infty < \lambda < \lambda_0$.

(2) *All solutions lie on a single smooth solution curve, which starts from* $(\lambda_0, 0)$ *and continues to the left up to* (λ_∞, ∞), *there is no turning point on the curve, and* $M(u)=1$ *for any solution* u. (*See Fig.* 6, *note that* λ_0 *can be* ∞.)

(3) $N=(0,\infty)$.

Proof. By Proposition 2.3, f is either asymptotic linear or asymptotic superlinear. We first consider that f is asymptotic linear. By Theorem 3.1 (3) and Proposition 3.4 (2), a solution curve Σ_∞ bifurcates from (λ_∞, ∞) and the bifurcation is supercritical. On the other hand, by Lemma 4.6, any solution w of (2.4) can be chosen as positive. Hence by Theorem 3.10, there

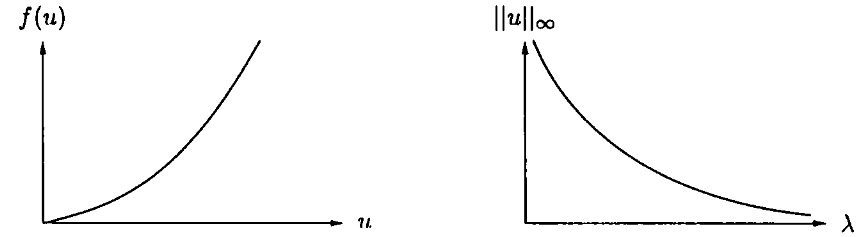


FIG. 6. f is superlinear, positive and subcritical near 0

is no turning point on Σ_∞. So Σ_∞ can continue to the right without any turning. As λ increases, Σ_∞ either goes to ∞ in the λ direction or intersects with $u=0$, while in the latter case, a bifurcation from the trivial solutions must occur at $(\lambda_0, 0)$. (The curve cannot be broken since $B=\emptyset$ by Corollary 5.4.)

Next we consider that f is asymptotic superlinear. Since $1 \leqslant K_f(u) \leqslant n/(n-2)$, (A.1) is satisfied, then by Theorem A.2, (2.2) has a solution $u(\lambda)$ for any $\lambda > 0$ small. On the other hand, by Proposition 3.14, $u(\lambda)$ is unstable. Hence $M(u(\lambda)) = 1$ and $\lambda'(d) < 0$ by Proposition 6.1. Thus there exists a solution curve Σ_∞ starts from $(0, \infty)$ continuing to the right, and there is no turning point on Σ_∞. Similar to the proof in last paragraph, as λ increases, Σ_∞ either goes to ∞ in the λ direction or intersects with $u=0$.

We claim there is no another curve. Suppose there is another one, say Σ_1, then Σ_1 is entirely below Σ_∞ from Lemma 2.6. Since $B=\emptyset$, then $T=N$ is open. If Σ_1 bifurcates from $u=0$, then the bifurcation is subcritical by Proposition 3.4 (1), and Σ_1 is bounded by some $u(0)=k>0$ and $\lambda=0$, thus it must have a turning point, which is impossible. If Σ_1 does not bifurcate from $u=0$, it also has a turning point, which is again a contradiction. Thus there is only one solution curve of (2.2). Proof of (3) can be followed from Proposition 6.6 below. ▌

PROPOSITION 6.6. *Let* $f \in C^1(\mathbf{R}^+)$. *Assume* $n \geqslant 3$, $f(0)=0$, $f(u)>0$ *in* $(0, \infty)$ (*resp. there exists* $c>0$ *such that* $f(u)>0$ *in* $(0, c)$ *and* $f(u)<0$ *in* (c, ∞)). *If* $K_f(u) < (n+2)/(n-2)$ *for all* $u>0$ (*resp. for* $u \in (0, c)$), *then* $T=N=(0, \infty)$ (*resp.* $T=N=(0, c)$); *if* $K_f(u) \geqslant (n+2)/(n-2)$ *for* $u \in (0, \delta)$ *for some* $\delta > 0$, *then* $G \supset (0, \delta)$.

Proof. We first claim that $P=B=\emptyset$. $B=\emptyset$ by Corollary 5.4. Suppose that $d \in P$; we multiply (5.1) by r^{n-1} and integrate it from 0 to $R(d)$. Then we obtain that $\int_0^{R(d)} r^{n-1} f(u(r))\, dr = 0$, which contradicts $f>0$. Therefore $(0, \infty) = N \cup G$ (*resp.* $(0, c) = N \cup G$).

If $K_f(u) < (n+2)/(n-2)$ for all $u > 0$ (*resp.* $u \in (0, c)$). Since f is positive in $(0, \infty)$ (*resp.* in $(0, c)$), then

$$f'(u)u - \frac{n+2}{n-2} f(u) < 0 \tag{6.1}$$

in $(0, \infty)$ (*resp.* in $(0, c)$). By Theorem A.2, for $\lambda > 0$ small, (2.2) has a solution $u(\lambda)$, hence $N \neq \emptyset$. We first prove that $N \supset (0, \delta)$ for some $\delta > 0$. Suppose it is not true, we can assume that there exists $d_0 > 0$ such that $d_0 \in G$ and $(d_0, d_0 + \delta) \subset N$, then $\lim_{d \to d_0^+} \lambda(d) = \infty$. For $d > d_0$, and $|d - d_0|$ small, we have

$$\begin{aligned}\int_{B(R(d))} |\nabla_x u(|x|, d)|^2 \, dx &= \frac{1}{\lambda(d)} \int_{B^n} |\nabla u(\lambda(d))|^2 \, dy \\ &= \int_{B^n} u(\lambda(d)) \, f(u(\lambda(d))) \, dy \leqslant C. \end{aligned} \tag{6.2}$$

Therefore, $\int_{\mathbf{R}^n} |\nabla_x u(|x|, d_0)|^2 \, dx \leqslant C$ and $\nabla_x u(|x|, d_0) \in L^2(\mathbf{R}^n)$. By [6, Proposition 1] and the equation ($u(|x|, d_0)$ is a ground state solution), we have

$$\int_{\mathbf{R}^n} |\nabla_x u(|x|, d_0)|^2 \, dx = \frac{2n}{n-2} \int_{\mathbf{R}^n} F(u(|x|, d_0)) dx, \tag{6.3}$$

and

$$\int_{\mathbf{R}^n} |\nabla_x u(|x|, d_0)|^2 \, dx = \int_{\mathbf{R}^n} u(|x|, d_0) \, f(u(|x|, d_0)) \, dx.$$

Therefore

$$\int_{\mathbf{R}^n} \left[\frac{2n}{n-2} F(u(|x|, d_0)) - u(|x|, d_0) \, f(u(|x|, d_0)) \right] dx = 0. \tag{6.4}$$

But $[(2n)/(n-2)F(u) - uf(u)]' = (n+2)/(n-2)f(u) - uf'(u) \geqslant 0$ by (6.1), so $[(2n)/(n-2)F(u) - uf(u)] \geqslant 0$ for all $u \geqslant 0$ (*resp.* $u \in [0, c)$), which contradicts (6.4). Therefore $G = \emptyset$ and $T = N = (0, \infty)$ (*resp.* $T = N = (0, c)$).

On the other hand, if $K_f(u) \geqslant (n+2)/(n-2)$ for all $u \in (0, \delta)$ for some $\delta > 0$, then

$$f'(u)u - \frac{n+2}{n-2} f(u) \geqslant 0, \qquad \text{and} \qquad \frac{2n}{n-2} F(u) - uf(u) \leqslant 0, \tag{6.5}$$

for $u \in (0, \delta)$. By Pohozaev's identity (see for example, [6]), if u is a solution of (2.2), then

$$0 < \int_{\partial B^n} \left| \frac{\partial u}{\partial n} \right|^2 ds = \lambda \int_{B^n} [2nF(u) - (n-2)\, uf(u)]\, dx \leqslant 0, \tag{6.6}$$

which is a contradiction. Then $G \supset (0, \delta)$. ∎

Proposition 6.6 is essentially an application of Pohozaev's identity. But if $K_f(u) \geqslant (n+2)/(n-2)$ only holds for a part of $\mathbf{R}^+$, and in the other part f is subcritical, then (2.2) can still have a solution. Moreover, it still can be unique as shown in the next theorem.

THEOREM 6.7. *Let $f \in C^1(\mathbf{R}^+)$. Assume $n \geqslant 3$, $f(0) = 0$, f is superlinear, and $f(u) > 0$ for $u > 0$. In addition, $K_f(u)$ is decreasing for all $u > 0$ and $K_\infty = \lim_{u \to \infty} K_f(u) \in [1, (n+2)/(n-2))$. Then*

(1) *Equation (2.2) has no solution for $0 < \lambda \leqslant \lambda_\infty$, and has exactly one solution for $\lambda > \lambda_\infty$.*

(2) *All solutions lie on a single smooth solution curve, which starts from (∞, g) for some $g \geqslant 0$, and continues to the left up to (λ_∞, ∞), there is no turning point on the curve, and $M(u) = 1$ for any solution u. (See Figs. 6 and 7.)*

(3) *if $K_0 = \lim_{u \to 0^+} K_f(u) > (n+2)/(n-2)$, then $N = (g, \infty)$ and $G = (0, g]$ for some $g > 0$; if $K_0 \leqslant (n+2)/(n-2)$, then $N = (0, \infty)$.*

Proof. We first prove any solution w of (4.2) can be chosen as positive. In fact we only need to check the conditions in Lemma 4.8. Let $Q(u) = uf(u) - 2F(u)$, then $Q'(u) = uf'(u) - f(u)$. Since $f(u)$ is superlinear, then $Q'(u) \geqslant 0$ for all $u > 0$, and $Q(0) = 0$. Therefore, $Q(u) \geqslant 0$ for all $u > 0$.

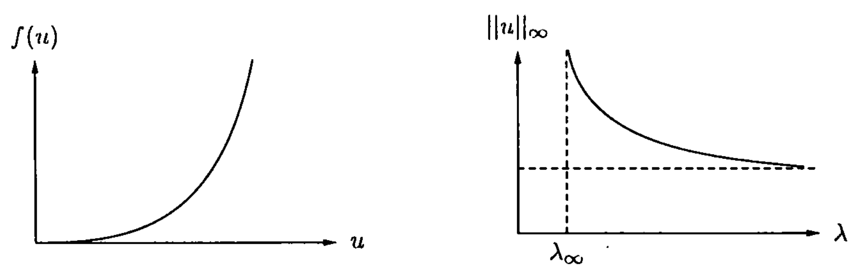


FIG. 7. f is superlinear, positive and supercritical near 0.

On the other hand, Let $G_f(u)=nF(u)-(n-2)/2f(u)u$, then $(r^{n-1}H(r))'=\lambda r^{n-1}G_f(u(r))$,

$$G_f'(u)=\frac{n+2}{2}f(u)-\frac{n-2}{2}f'(u)u=\frac{n-2}{2}f(u)\left[\frac{n+2}{n-2}-K_f(u)\right].$$

Since $K_f(u)$ is decreasing, then $G_f(u)$ has at most one critical point. If $K_0=\lim_{u\to 0^+}K_f(u)>(n+2)/(n-2)$, then there exists $u_1>0$ such that $G_f(u)<0$ for $u\in[0,u_1)$ and $G_f(u)>0$ for $u\in(u_1,\infty)$. Since $H(1)>0$, then $H(r)>0$ for all $r\in(0,1)$. If $K_0\leqslant(n+2)/(n-2)$, then $G_f'(u)>0$ for all $u>0$, and $G_f(u)>0$ for all $u>0$. So, we also have $H(r)>0$ for all $r\in(0,1)$. Therefore, by Lemma 4.8, $w>0$ for all $r\in(0,1)$. By Theorem 3.10, there is no turning point on any solution curve.

If f is asymptotic linear, then by Theorem 3.1 and Proposition 3.4 (2), a solution curve Σ_∞ bifurcates from (λ_∞,∞), and the bifurcation is supercritical. If f is asymptotic superlinear, since $K_f(u)<(n+2)/(n-2)$ for u large, (A.1) is satisfied, then by Theorem A.2, (2.2) has a solution for any $\lambda>0$ small. The other parts of the proof are the same as that of Theorem 6.5, so we omit them. Note that K_0 must be greater than 1, so $\lambda_0=\infty$. Proof of (3) can be followed from Proposition 6.6. ∎

The next result is essentially the case (1) of Theorem 1.3 in [41], and we refer the reader to [41] for the proof.

THEOREM 6.8. *Let $f\in C^1(\mathbf{R}^+)$. Assume $f(0)=0$, f is superlinear, and $f(u)>0$ for $u>0$. In addition, there exists $\rho<\infty$ such that $f(u)/(u-\rho)$ is nonincreasing for $u\geqslant\rho$, and K_f satisfies*

(a) *$K_f(u)$ is decreasing in $(0,\rho]$;*

(b) *In $[\rho,\infty)$, $K_f(u)\leqslant K_f(\rho)$.*

Then the conclusions in Theorem 6.7 hold. (See Figs. 6 and 7.)

The examples of f's satisfying the conditions in Theorems 6.5, 6.7, and 6.8 are summarized here.

EXAMPLE 6.9. (1) $f(u)=u^p$ for $1<p<(n+2)/(n-2)$; (*Theorem* 6.7)

(2) $f(u)=\sum_i a_i u^{p_i}$ *for* $a_i>0$ *and* $1\leqslant p_i\leqslant n/(n-2)$; (*Theorem* 6.5),

(3) $f(u)=\sqrt{u^2+1}-1$; (*Theorems* 6.7 *and* 6.8)

(4) $f(u)=(u^2+ku)/(u+1)$ *for* $(n-4)/n\leqslant k<1$ *and* $n\geqslant 4$. (*Theorem* 6.5).

Remark 6.10. (1) In Theorem 6.5, a solution of (2.2) exists for any "height" $d\in(0,\infty)$. This is also true for Theorems 6.7 and 6.8 if $K_0\leqslant(n+2)/(n-2)$. But if $K_0>(n+2)/(n-2)$, then there is no solution for small d due to Pohozaev's identity. For instance, in Example 6.9, for example, (1), a solution of (2.2) exists for any $d>0$; but, for example, (3), $K_0=2$, so (2.2) has no solution for small $d>0$ if $n \geqslant 7$. We also notice that in Theorem 6.8, we do not require $n \geqslant 3$.

(2) The uniqueness result in Theorem 6.5 was originally proved by Ni and Nussbaum [40], and later Kwong [31], Chen and Lin [12] gave simpler proofs. Our proof here is based on that of [31], but we also discuss the existence and asymptotic behavior of solution curve here. Theorem 6.7 and Theorem 6.8 appear to be new.

6.3. *f Is Superlinear and Has a Negative Part*

In this section, we consider f which is superlinear and has a negative part. First we revisit a uniqueness problem which has been studied by many authors (see [14, 12, 34, 37]). The model function here is $f(u) = -u + u^p$ for $1 < p < (n+2)/(n-2)$. The following theorem extends the uniqueness theorem in Kwong and Zhang [34].

THEOREM 6.11. *Let $f \in C^1(\mathbf{R}^+)$. Assume $n \geqslant 3$, $f(0) \leqslant 0$, $f(u) < 0$ for $u \in (0, b)$, $f(u) > 0$ for $u \in (b, \infty)$. In addition,*

$$\text{there exists } \theta > 0, \text{ such that } f(\theta) > 0,\ F(\theta) = \int_0^\theta f(u)\, du = 0, \tag{6.7}$$

and K_f satisfies

(a) *$K_f(u)$ is decreasing in $[\theta, \infty)$ and converges to $K_\infty \in [1, (n+2)/(n-2))$ as $u \to \infty$;*

(b) *In $(b, \theta]$, $K_f(u) \geqslant K_f(\theta)$;*

(c) *In $(0, b)$, $K_f(u) \leqslant K_\infty$.*

If $f(0) = 0$, then

(1) *Equation (2.2) has no solution for $0 < \lambda \leqslant \lambda_\infty$, and has exactly one solution for $\lambda > \lambda_\infty$.*

(2) *All solutions lie on a single smooth solution curve, which starts from (∞, g) for some $g \geqslant \theta$, and continues to the left up to (λ_∞, ∞), there is no turning point on the curve, and $M(u) = 1$ for any solution u. (See Fig. 8.)*

(3) *If in addition*

$$f'(u) \leqslant 0 \qquad \text{in } (0, \delta) \text{ for some } \delta > 0, \text{ and } f'(b) > 0, \tag{6.8}$$

then $N = (g, \infty)$, $G = \{g\}$, $P = (b, g)$, $C = \{b\}$ and $E = (0, b)$.

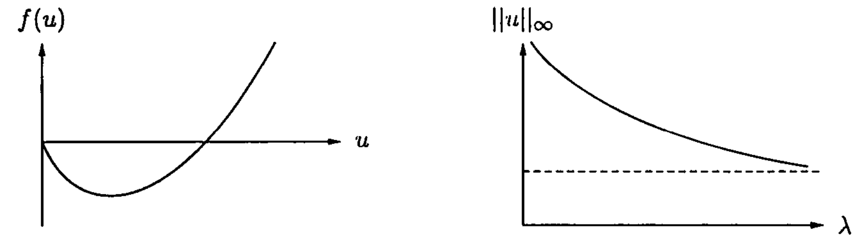


FIG. 8. f superlinear and has a negative part, $f(0)=0$.

If $f(0)<0$, we also assume that $f'(b)>0$, and f satisfies either $f'(u)\geqslant 0$ in $\mathbf{R}^+$ *and* (A.3), *or* (A.4) *when it is asymptotic superlinear. Then*

(1) *There exists $\bar{\lambda}>\lambda_\infty$ such that* (2.2) *has no solution for $0<\lambda\leqslant\lambda_\infty$ or $\lambda>\bar{\lambda}$, and has exactly one solution for $\lambda_\infty<\lambda\leqslant\bar{\lambda}$.*

(2) *All solutions lie on a single smooth solution curve, which starts from $(\bar{\lambda}, g)$ for some $g\geqslant\theta$, and continues to the left up to (λ_∞, ∞), there is no turning point on the curve, and $M(u)=1$ for any solution u.* (*See Fig.* 9.)

(3) $N=(g,\infty)$, $B=\{g\}$, $P=(b,g)$ *and* $C=\{b\}$ *and* $E=(0,b)$.

The key to the proof of Theorem 6.11 is the following proposition:

PROPOSITION 6.12. *Let f be as in Theorem* 6.11 *and u be a solution of* (2.2). *Then u is nondegenerate.*

Proof. Let $0<r_2<r_1<1$ such that $u(r_1)=b$ and $u(r_2)=\theta$. Suppose that u is degenerate, and w is a nontrivial solution of (2.4). We prove such w does not exist by four steps:

(A) w has no zero in $[r_2, 1)$;

(B) w has at most one zero in $[0, r_2]$;

(C) w cannot have exactly one zero in $[0, 1)$;

(D) w must change sign in $[0, 1)$.

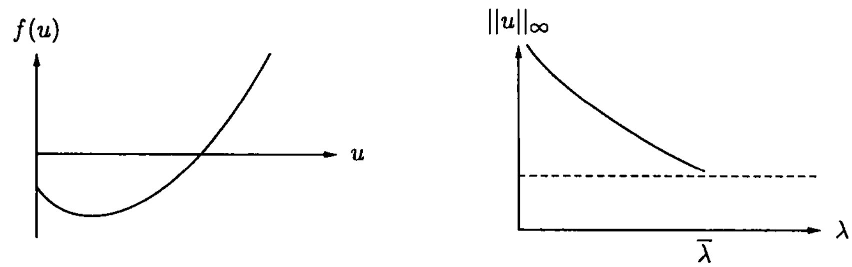


FIG. 9. f superlinear and has a negative part, $f(0)<0$.

The first three statements imply $w \neq 0$ in $[0, 1)$, that contradicts (D). So there does not exist such degenerate solution.

(A) This is by Lemma 4.9.

(B) We use Lemma 4.8. First, $K_f(u)$ is decreasing in $[\theta, u(0)]$, f is superlinear in $[\theta, u(0)]$, and $f(\theta)u(\theta) - 2F(\theta) > 0$, so $f(u)u - 2F(u) > 0$ in $[\theta, u(0)]$. Next, let $G_f(u)$ be same as in Theorem 6.7, since $K_f(u)$ is decreasing, then $G_f(u)$ has at most one critical point. By a similar argument as the proof of Lemma 3.6 in [41], we can show that $H(r) > 0$ for all $r \in (0, 1)$. Then, it follows from Lemma 4.8 that w has at most one zero in $[0, r_2)$.

(C) We notice that for any $u_0 > \theta$, there exists $\gamma > 1$ such that

$$\begin{aligned} \gamma f(u) - uf'(u) &\leqslant 0 \qquad \text{in } [0, u_0], \\ \gamma f(u) - uf'(u) &\geqslant 0 \qquad \text{in } [u_0, \infty). \end{aligned} \tag{6.9}$$

If w has exactly one zero, say, $w(r_0) = 0$ for $r_0 \in (0, r_2]$, we choose $u_0 = u(r_0)$, then $w \geqslant 0$ in $[0, r_0]$ and $w \leqslant 0$ in $[r_0, 1]$, and $\int_0^1 (\gamma f(u) - uf'(u))\, wr^{n-1}\, dr > 0$. But on the other hand, by (3.14), $\int_0^1 (\gamma f(u) - uf'(u))\, wr^{n-1}\, dr = (2\lambda)^{-1} (\gamma - 1)\, u'(1)\, w'(1) \leqslant 0$, that is a contradiction.

(D) If $K_\infty = 1$, f is superlinear, then by Theorem 3.10, w must change sign. If $K_\infty > 1$, and we assume that $w > 0$ in $(0, 1)$, then $\int_0^1 r^{n-1}[f'(u)u - K_\infty f(u)] w\, dr > 0$ since $f'(u)u - K_\infty f(u) \geqslant 0$ for all $u \geqslant 0$. But by (3.14), $\int_0^1 r^{n-1}[f'(u)u - K_\infty f(u)]\, w\, dr = (2\lambda)^{-1}(1 - K_\infty)\, u'(1)\ w'(1) \leqslant 0$, that is a contradiction. ∎

Proof of Theorem 6.11. The proof of (3) will be given in Propositions 7.1 and 7.2. By Proposition 6.12, (2.4) has no degenerate solution. By a similar argument as the proof of Theorem 6.5, there is a solution curve Σ_∞ bifurcating from (λ_∞, ∞). (This is also true when $f(0) < 0$, see Theorem A.3.) If $f(0) = 0$, the solution curve Σ_∞ can continue to the right up to $\lambda = \infty$, because that $B = \varnothing$ and Σ_∞ is bounded from below by $d = \theta$. $\lim_{\lambda \to \infty} u(\lambda)(0) = g \geqslant \theta$ exists and g is a horizontal asymptote of $\lambda(d)$. If $f(0) < 0$, we claim that (2.2) has no positive solution when λ is large. In fact, we continue Σ_∞ to the right as long as $u(\lambda)$ is positive. If Σ_∞ can continue to ∞, then $\lim_{\lambda \to \infty} u(\lambda)(0) = g \geqslant \theta$ exists and g is a horizontal asymptote of $\lambda(d)$. By Proposition 5.8, $g \in G \cup C$. Since $g \geqslant \theta > b$, then $g \notin C$. If $g \in G$, then by Lemma 5.1, $u(r, g) \to c$ as $r \to \infty$ for some c such that $f(c) = 0$ and $f'(c) \leqslant 0$, that is impossible since the only zero b of f satisfies $f'(b) > 0$. Hence Σ_∞ cannot continue to ∞, but stop at $\lambda = \bar{\lambda}$. $u(\bar{\lambda})$ must be a solution of (2.2) with $u_r(\bar{\lambda})(1) = 0$, otherwise Σ_∞ can go further by the Implicit Function Theorem. Hence $g \in B$ in this case.

Finally, we exclude the possibility of more than one solution curves. Suppose that there exists another curve $\Sigma_1 \neq \Sigma_\infty$, then Σ_1 must be entirely below Σ_∞. If $f(0)=0$, Σ_1 is homormophic to $\mathbf{R}$ since $T=N$ is open, and is bounded by $d=g$, $d=\theta$ and $\lambda=0$, then there must be a turning point on Σ_1, which is a contradiction. If $f(0)<0$, since $\lambda(d)$ does not have a horizontal asymptote, then $\Sigma_1=\{(\lambda(d), d): d\in[d_1, d_2]\}$ and $d_1, d_2 \in B$, where we can assume that $d_2=\sup\{d\in N\cup B: d<g\}$. But from the proof of Proposition 7.2, for any $d\in B$, $(d-\varepsilon, d)\subset N$ and $(d, d+\varepsilon)\subset P$. (This is even true without the condition $f'(b)>0$.) Thus such a d_2 does not exist. Hence, there is only one solution curve. ∎

The nondegeneracy condition (6.8) is *not* needed if we only want to prove the uniqueness of the solution in finite ball. But it is necessary for the uniqueness of the element in G. In fact, even when $f'(b)=0$, and $f'\leqslant 0$ near 0, we still can prove uniqueness of the ground state solution, but it is possible that there exists $d\in G$ such that $u(r,d)\to b$ as $r\to\infty$ just like in Theorem 6.7. So if we assume that

$$f'(u)\leqslant 0 \text{ in } (0,\delta) \text{ for some } \delta>0,$$

$$\text{and } \frac{(u-b)\,f'(u)}{f(u)}\leqslant\frac{n+2}{n-2} \quad \text{for} \quad u>b, \tag{6.10}$$

then the conclusions in Theorem 6.11 (3) are still true. Similar for Theorem 6.13 (3). On the other hand, if $f'(u)\leqslant 0$ near 0 is not true, then the uniqueness of ground state will be in question. In the case of $f(0)<0$, if $f'(b)>0$ is not true, it is possible that there exists $d\in G$ such that $u(r,d)\to b$ as $r\to\infty$, and the solution curve Σ_∞ converges to such a horizontal asymptote $d\in G$.

The uniqueness result for $f(0)=0$ in Theorem 6.11 is known by [34], and our result here also discuss the properties of bifurcation curves. On the other hand, Smoller and Wasserman [52] proved that, $f(0)<0$, f is superlinear and concave, then the solution of (2.2) is unique and only exists for $\lambda\in(\lambda_a, \lambda_b]$ for $\lambda_a, \lambda_b>0$. It is easy to verify that if f is superlinear and concave, then it satisfies the assumptions in Theorem 6.11. Hence Theorem 6.11 extends this result in [52]. For f asymptotic superlinear and $f(0)<0$, Theorem 6.11 extends a uniqueness result by Castro *et al.* [9]. Our next theorem is a variant of Theorem 6.8, and we refer the readers to case (1) of Theorem 1.2 in [41] for the proof.

THEOREM 6.13. *Let $f\in C^1(\mathbf{R}^+)$. Assume $f(0)\leqslant 0$, f is superlinear, $f(u)<0$ for $u\in(0,b)$, $f(u)>0$ for $u\in(b,\infty)$. In addition,* (6.7) *holds, and*

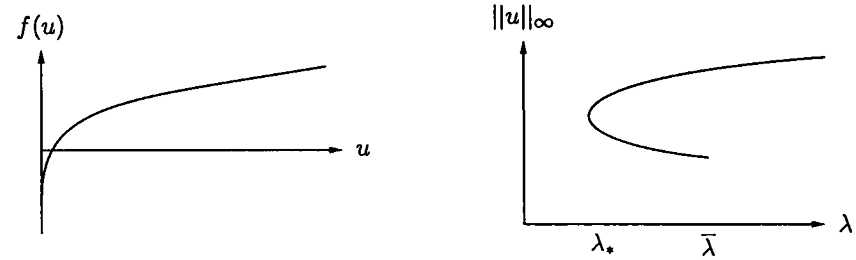


FIG. 10. f concave, $f(0)<0$ and $\lambda_\infty = \infty$.

there exists $\rho < \infty$ such that $f(u)/(u-\rho)$ is nonincreasing for $u \geqslant \rho$. If $\rho > \theta$, we assume K_f satisfies

(a) *$K_f(u)$ is decreasing in $[\theta, \rho]$;*

(b) *In $(b, \theta]$, $K_f(u) \geqslant K_f(\theta)$;*

(c) *In $[\rho, \infty)$, $K_f(u) \leqslant K_f(\rho)$;*

Then the same conclusions in Theorem 6.11 *hold.* (*See Figs.* 8 *and* 9.)

Remark 6.14. (1) Compared with Theorem 6.11, the conditions on K_f in Theorem 6.13 are weaker in $[\theta, \infty)$, but the existence of ρ implies f must be asymptotic linear, while f could be asymptotic superlinear in Theorem 6.11.

(2) The function f in Theorem 6.11 is not necessarily superlinear, and we did not apply Theorem 3.10. In fact, we can extend Theorem 3.10 in the following way: replacing f is superlinear (*resp.* sublinear) by $f(u)/u^p$ is increasing (*resp.* decreasing) for $p \geqslant 1$ (*resp.* $0 < p \leqslant 1$), then w must change sign. Such conditions on f have been discussed in [38, 34].

EXAMPLE 6.15. (1) $f(u) = -u + u^p - c$ for $c \geqslant 0$ and $1 < p < (n+2)/(n-2)$; (*Theorem* 6.11)

(2) $f(u) = -\sum_i a_i u^{p_i} + u^p$ for $1 < p_i < p < (n+2)/(n-2)$; (*Theorem* 6.11)

(3) $f(u) = (u^2 + ku)/(u+1)$ *for* $k \leqslant 0$. (*Theorems* 6.11 *and* 6.13).

6.4. *f Is Sup-sub*

Here we begin to explore the bifurcation diagrams with one turning point. We first consider concave f. By Proposition 2.3, if f is sup-sub and concave, then $f(0) < 0$.

THEOREM 6.16. *Let $f \in C^2(\mathbf{R}^+)$. Assume $f(0) < 0$, $f(u) < 0$ for $u \in [0, b)$, f is concave, sup-sub, and* (6.7) *holds. Then*

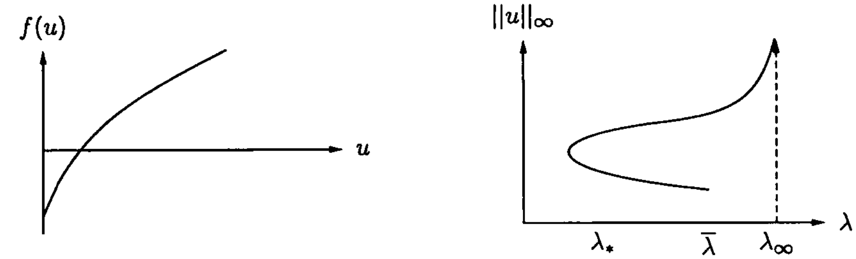


FIG. 11. f concave, $f(0)<0$ and $\lambda_\infty<\infty$.

(1) *There exist* $\bar{\lambda}>\lambda_*>0$ *such that* (2.2) *has no solution for* $\lambda<\lambda_*$ *or* $\lambda>\max(\lambda_\infty,\bar{\lambda})$, *has exactly one solution for* $\min(\lambda_\infty,\bar{\lambda})<\lambda<\max(\lambda_\infty,\bar{\lambda})$ *or* $\lambda=\lambda_*$, *and has exactly two solutions for* $\lambda_*<\lambda<\min(\lambda_\infty,\bar{\lambda})$.

(2) *All solutions lie on a single smooth solution curve* Σ, *which for* $\lambda>\lambda_*$ *has two branches denoted by* Σ^+ (*the upper branch*) *and* Σ^- (*the lower branch*); Σ^+ *continues to the right up to* (λ_∞,∞) *if* $f(u)>0$ *for all* u *large, or to* (∞,c) *if there exists* $c>b$ *such that* $f(c)=0$ *and* $f(u)<0$ *for* $u>c$; Σ^- *continues to the right down to* $(\bar{\lambda},g)$ *for some* $g\geqslant\theta$; *there is a unique turning point on the curve, the curve bends to the right at the turning point, and* $M(u)=0$ *for* u *on* Σ^+, $M(u)=1$ *for* u *on* Σ^-. (*See Figs.* 10 *and* 11.)

(3) $N=(g,\infty)$, $B=\{g\}$, $P=(b,g)$, $C=\{b\}$ *and* $E=(0,b)$ *if* $f(u)>0$ *for* $u>b$, $N=(g,c)$, $B=\{g\}$, $P=(b,g)$, $C=\{b,c\}$ *and* $E=(0,b)\cup(c,\infty)$ *if* $f(u)>0$ *for* $u\in(b,c)$ *and* $f(u)<0$ *in* (c,∞).

To prove Theorem 6.16, we first prove a lemma which is of interest by itself:

LEMMA 6.17. *If there exists* $a>0$ *such that* $f(u)\geqslant au$ *for* $u\geqslant 0$, *then there exists* $\Lambda>0$, *such that for* $\lambda>\Lambda$, (1.1) *has no positive solution. Similarly, if there exists* $a>0$ *such that* $f(u)\leqslant au$ *for* $u\geqslant 0$, *then there exists* $\Lambda>0$, *such that for* $\lambda<\Lambda$, (1.1) *has no positive solution.*

Proof. Suppose that $f(u)\geqslant au$ for $u\geqslant 0$. Let ϕ_1 be the positive eigenfunction of first eigenvalue λ_1, multiply (1.1) by ϕ_1, and integrate over Ω. We have

$$\lambda_1\int_{B^n}\phi_1\cdot u\,dx=\lambda\int_{B^n}\phi_1\cdot f(u)\,dx\geqslant\lambda a\int_{B^n}\phi_1\cdot u\,dx. \tag{6.11}$$

Thus if (1.1) has a positive solution, then $\lambda \leqslant \lambda_1/a$. Suppose that $f(u) \leqslant au$ for $u \geqslant 0$. If u is a positive solution of (1.1), then

$$\lambda_1 \int_{B^n} u^2\,dx \leqslant \int_{B^n} |\nabla u|^2\,dx = \lambda \int_{B^n} uf(u)\,dx \leqslant \lambda a \int_{B^n} u^2\,dx. \tag{6.12}$$

Thus if (1.1) has a solution, then $\lambda \geqslant \lambda_1/a$. ∎

Proof of Theorem 6.16. By Lemma 4.5 (choose $\rho = b$) and Lemma 4.9, any solution w of (2.4) can be chosen as positive. Thus by Theorem 3.12, at turning points, the solution curve bends to the right. So there is at most one turning point on each curve.

By Proposition 2.5, f is either asymptotic linear or asymptotic sublinear. If f is asymptotic linear, by Theorem 3.1 (3), there is a solution curve Σ_∞ bifurcating from (λ_∞, ∞). The solution in Σ_∞ is positive if $|\lambda - \lambda_\infty|$ small enough since $u(\lambda)/\|u(\lambda)\|_\infty \to \phi_1$ as $\lambda \to \lambda_\infty$, where ϕ_1 is the normalized eigenfunction of λ_1. Moreover, the bifurcation at λ_∞ is subcritical, Σ_∞ bifurcates to the left by Corollary 3.8, and by Proposition 3.9, the solutions on Σ_∞ are stable when $\|u(\lambda)\|_\infty$ is large enough. If f is asymptotic sublinear, we can apply Theorem A.1 to show that there exists a positive solution u_λ of (1.1) for any $\lambda > 0$ large enough and the solution is stable, satisfying $\lambda_1[u_\lambda] > 0$. Again, we denote the solution curve by Σ_∞. We continue Σ_∞ to the left by the Implicit Function Theorem. By Corollary 5.4, if the solution is stable, then the normal derivative at the boundary does not vanish, which implies the solutions nearby are still positive and stable. Therefore, we can continue Σ_∞ to $\lambda_* = inf\{\lambda > 0\colon (1.1)$ has a stable solution with this λ on $\Sigma_\infty\}$. From Lemma 6.17, $\lambda_* > 0$ since $f(u) \leqslant au$ for $u \geqslant 0$. Σ_∞ either bends back at λ_* or blows up at λ_* from the right side of λ_*. (Blow up means $\|u\| \to \infty$ as $\lambda \to \lambda_*$.)

If Σ_∞ blows up at λ_*, then λ_* becomes a point where a bifurcation from infinity occurs. From Corollary 3.8, the curve bifurcating from infinity can only be subcritical, which is a contradiction. Thus Σ_∞ must bend back at λ_*, and λ_* is a turning point. At the turning point, we still have $(\partial u(\lambda)/\partial n)(x) < 0$ for any $x \in \partial B^n$ by Corollary 5.4. Therefore, the lower branch of Σ_∞ still consists of positive solutions near turning point. The lower branch cannot be continued to $\lambda = \infty$ as in the proof of Theorem 6.11, and Σ_∞ must stop at some $\bar{\lambda}(d)$ for some $d \in B$.

Again, for the proof of (3), we prove it independently in Proposition 7.2. We claim that Σ_∞ is the unique solution curve of (2.2). In fact, if there is another solution curve Σ_1, then it must be entirely below Σ_∞. By the proof of Proposition 7.2, for each $d \in B$, $(d, d+\varepsilon) \subset N$. Thus if $\Sigma_1 = [d_1, d_2]$, then $d_2 \notin \Sigma_1$ and d_2 must be a horizontal asymptote. This is impossible if $f(u) > 0$ for $u > b$ since $G = \varnothing$ (notice $f'(b) > 0$) and $C = \{b\}$, then there is no horizontal asymptote. If $f(u) > 0$ in (b, c) and $f(u) < 0$ in (c, ∞), then c is the horizontal asymptote of Σ_∞, and it cannot be the horizontal asymptote of Σ_1 too. Hence Σ_∞ is the unique solution curve. ▌

Castro and Gadam [8] proved a similar result as Theorem 6.11. The proof here is based on our unified approach, and it seems to be more general.

Next we consider convex-concave f. The following results are extensions of Theorems 1.1 and 1.2 in [41]. The proof is the same as [41] except here we also include the case of $f(0) < 0$, but the proofs for $f(0) < 0$ can be easily modified from the proof of Theorems 1.1 and 1.2 in [41] and the proof of Theorem 6.16, so we omit them.

THEOREM 6.18. *Let $f \in C^2(\mathbf{R}^+)$. Assume $f(0) \leqslant 0$, f is convex-concave and sup-sub, $f(u) < 0$ in $(0, b)$, either $f(u) > 0$ in (b, ∞), (or there exists $c > 0$ such that $f(u) > 0$ in (b, c) and $f(u) < 0$ in (c, ∞),) and (6.7) holds. In addition, let α be the point where f'' changes sign, $\rho = \alpha - (f(\alpha)/f'(\alpha))$, and we assume that if $\rho > \theta$, then*

(a) *$K_f(u) \geqslant K_f(\theta)$ for $u \in (b, \theta]$;*

(b) *$K_f(u)$ is nonincreasing in $[\theta, \rho]$;*

(c) *$K_f(u) \leqslant K_f(\rho)$ for $u \in [\rho, \alpha]$.*

If $f(0) = 0$, then

(1) *There exists $\lambda_* > 0$ such that (2.2) has no solution for $\lambda < \lambda_*$, has exactly one solution for $\lambda_\infty \leqslant \lambda < \infty$ or $\lambda = \lambda_*$, and has exactly two solutions for $\lambda_* < \lambda < \lambda_\infty$.*

(2) *All solutions lie on a single smooth solution curve Σ, which for $\lambda > \lambda_*$ has two branches denoted by Σ^+ (the upper branch) and Σ^- (the lower branch); Σ^+ continues to the right up to (λ_∞, ∞) if $f(u) > 0$ for all u large, or to (∞, c) if there exists $c > 0$ such that $f(c) = 0$; Σ^- continues to the left up to (∞, g) for some $g \geqslant \theta$; There is a unique turning point on the*

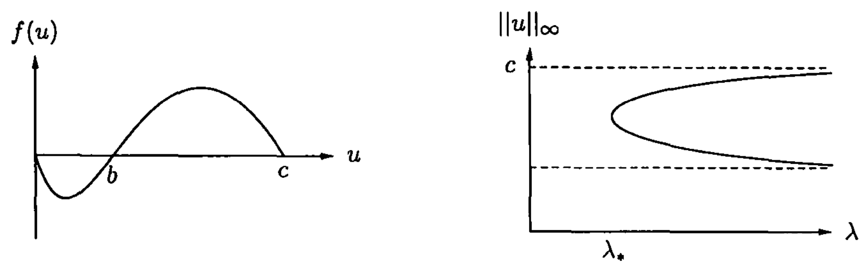


FIG. 12. f convex-concave, $f'(0) < 0$ and $\lambda_\infty = \infty$.

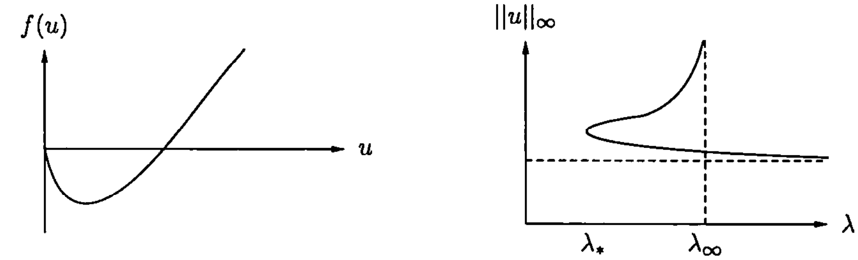


FIG. 13. f convex-concave, $f'(0)<0$ and $\lambda_\infty > 0$.

curve, the curve bends to the right at the turning point, and $M(u)=0$ *for* u *on* Σ^+, $M(u)=1$ *for* u *on* Σ^-. (*See Figs.* 12 *and* 13)

(3) $N=(g,\infty)$, $G=\{g\}$, $P=(b,g)$, $C=\{b\}$ *and* $E=(0,b)$ *if* $f(u)>0$ *for* $u>b$, $N=(g,c)$, $G=\{g\}$, $P=(b,g)$, $C=\{b,c\}$ *and* $E=(0,b)\cup(c,\infty)$ *if* $f(u)>0$ *for* $u\in(b,c)$ *and* $f(u)<0$ *in* (c,∞).

If $f(0)<0$, *then the conclusions in Theorem* 6.16 *hold.* (*See Figs.* 10 *and* 11.)

THEOREM 6.19. *Let* $f\in C^2(\mathbf{R}^+)$. *Assume* $f(0)=0$, f *is convex-concave and sup-sub, either* $f(u)>0$ *in* $(0,\infty)$ *or there exists* $c>0$ *such that* $f(u)>0$ *in* $(0,c)$ *and* $f(u)<0$ *in* (c,∞). *In addition, let* α *and* ρ *be defined as in Theorem* 6.18, *and we assume that*

(a) $K_f(u)$ *is nonincreasing in* $[0,\rho]$;

(b) $K_f(u)\leqslant K_f(\rho)$ *for* $u\in[\rho,\alpha]$.

Then

(1) *There exists* $\lambda_*>0$ *such that* (2.2) *has no solution for* $\lambda<\lambda_*$, *has exactly one solution for* $\lambda_\infty\leqslant\lambda<\infty$ *or* $\lambda=\lambda_*$, *and has exactly two solutions for* $\lambda_*<\lambda<\lambda_\infty$.

(2) *All solutions lie on a single smooth solution curve* Σ, *which for* $\lambda>\lambda_*$ *has two branches denoted by* Σ^+ (*the upper branch*) *and* Σ^- (*the lower branch*); Σ^+ *continues to the right up to* (λ_∞,∞) *if* $f(u)>0$ *for all* u *large, or to* (∞,c) *if there exists* $c>0$ *such that* $f(c)=0$; Σ^- *continues to the left up to* (∞,g) *for some* $g\geqslant 0$; *There is a unique turning point on the curve, the curve bends to the right at the turning point, and* $M(u)=0$ *for* u *on* Σ^+, $M(u)=1$ *for* u *on* Σ^-. (*See Figs.* 14 *and* 15.)

(3) $N=(g,\infty)$ *and* $G=(0,g]$, *if* $f(u)>0$ *for* $u>0$; $N=(g,c)$, $G=(0,g]$, $C=\{c\}$, *and* $E=(c,\infty)$ *if* $f(u)>0$ *for* $u\in(b,c)$ *and* $f(u)<0$ *in* (c,∞).

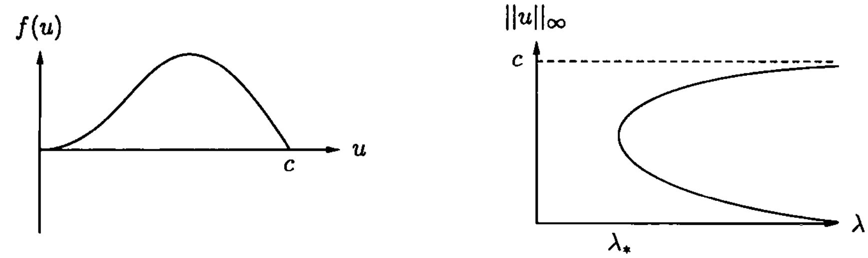


FIG. 14. f convex-concave, $f'(0)=0$ and $\lambda_\infty = \infty$.

EXAMPLE 6.20. (1) $f(u)=u(u-b)(c-u)$, *where* $0<2b<c$; (*Theorem* 6.18)

(2) $f(u)=u^p-u^q$, *where* $0<p<q$; (*Theorem* 6.19)

(3) $f(u)=u(u-b)/(1+au^p)$, *where* $a>0$, $b\geqslant 0$, $1<p\leqslant 2$; (*Theorem* 6.19)

(4) $f(u)=(u+c)^p-c^p-a$, *where* $0<p<1$ *and* $a, c>0$. (*Theorem* 6.16)

For $f(u)=u(u-b)(c-u)$, $0<2b<c$ and $n=1$, Smoller and Wasserman [51] used time-mapping technique to prove the exact multiplicity and showed the exact $\subset$-shape of the bifurcation curve. For n-dimensional ball, Gardner and Peletier [23] proved that (2.2) has exactly two solutions when λ is large, for a class of f including $f(u)=u(u-b)(c-u)$, $0<2b<c$ and $2b$ close to c. Korman *et al.* [29] developed a new global bifurcation approach, and they showed the exact $\subset$-shape of the bifurcation curve for a generalizing cubic nonlinearity for $n=1$ in [29] and $n=2$ in [30]. Theorems 6.18 and 6.19 are much more general not only for the dimension of the space but also the scope of the nonlinearities. Dancer [19, 20] proved (2.2) has exactly two solutions when λ is large, for a class of f similar to those in Theorem 6.18 and a class of symmetric domains which includes ellipses.

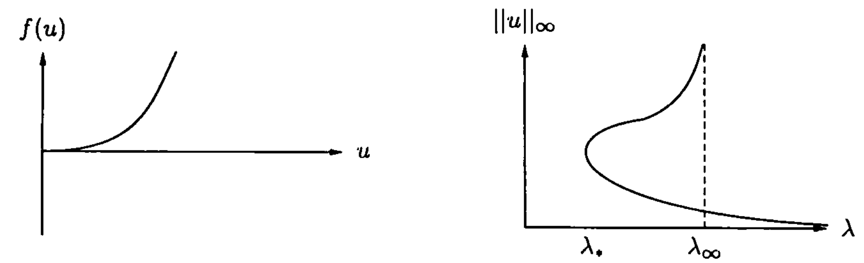


FIG. 15. f convex-concave, $f'(0)=0$ and $\lambda_\infty > 0$.

For the nonlinearity like $f(u) = u^p - u^q$, $1 < q < p$, our result on the exact multiplicity seems to be the first one to give the exact multiplicity of solutions on balls, though Kwong *et al.* [33] studied uniqueness of the ground state solution. Our result here include more general nonlinearities and our proof seems to be simpler. Note that, the ground state solution exists only when $K_0 = \lim_{u \to 0^+} K_f(u) > (n+2)/(n-2)$, which is equivalent to $q > (n+2)/(n-2)$ for $f(u) = u^p - u^q$, $1 < q < p$. Whether g in Theorem 6.19 is positive or zero can be determined by Proposition 6.6.

6.5. *f Is Sub-sup*

In this subsection, we will prove for some sub-sup f, the bifurcation diagram of (2.2) is exactly $\supset$-shape.

THEOREM 6.21. *Let $f \in C^2(\mathbf{R}^+)$. Assume f is sub-sup, $f(u) > 0$ for $u > 0$, either $f(0) > 0$ and f is convex, or $f(0) \geqslant 0$ and f is concave-convex. In addition, we assume $n \geqslant 4$ and, $-(n-4)/(n-2) \leqslant K_f(u) \leqslant n/(n-2)$ for $u > 0$. Then*

(1) *There is $\lambda_* > 0$ such that (2.2) has no solution for $0 \leqslant \lambda \leqslant \min(\lambda_\infty, \lambda_0)$ or $\lambda > \lambda_*$, has exactly one solution for $\min(\lambda_\infty, \lambda_0) < \lambda \leqslant \max(\lambda_\infty, \lambda_0)$ or $\lambda = \lambda_*$, and has exactly two solutions for $\max(\lambda_\infty, \lambda_0) < \lambda < \lambda_*$.*

(2) *All solutions lie on a single smooth solution curve Σ, which for $\lambda < \lambda_*$ has two branches denoted by Σ^+ (the upper branch) and Σ^- (the lower branch); Σ^+ continues to the left up to (λ_∞, ∞); Σ^- continues to the left down to $(\lambda_0, 0)$; There is a unique turning point on the curve, the curve bends to the left at the turning point, and $M(u) = 1$ for u on Σ^+, $M(u) = 0$ for u on Σ^-. (See Figs. 16 and 17.)*

(3) $N = (0, \infty)$.

The same conclusions hold if $f \in C^2(0, \infty)$, satisfies (3.16) and the same convexity condition and growth condition.

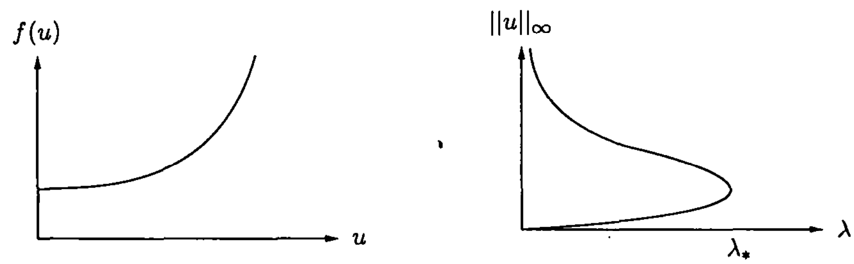


FIG. 16. f convex, sub-sup, $\lambda_0 = 0$ and $\lambda_\infty = 0$.

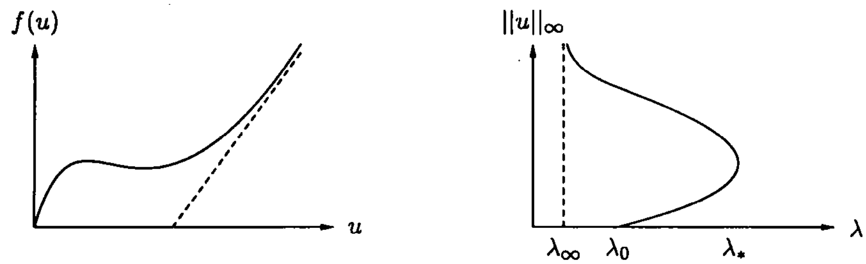


FIG. 17. f concave-convex, sub-sup, $\lambda_0 > 0$ and $\lambda_\infty > 0$..

Proof. For sub-sup f, by Proposition 2.3 (4), $f(0) \geqslant 0$. If $f(0) > 0$, then a solution curve bifurcates from $(0, 0)$ by Theorem 3.2 and the curve bifurcates to the right. If $f(0) = 0$, then $f'(0)$ must be positive. Otherwise $f < 0$ near 0. Then $\lambda_0 > 0$, and it is a point where a bifurcation from the trivial solutions occurs. Moreover, by Proposition 3.6, the curve bifurcates to the right. We denote this curve bifurcating from the trivial solutions by Σ_0. We continue Σ_0 to the right as long as it is possible. But, on the other hand, there is no solution of (1.1) for large $\lambda > 0$ since $f(u) \geqslant au$ for $u \geqslant 0$ and Proposition 6.17. We define $\lambda_* = sup\{\lambda > 0 : (1.1)$ has a stable solution with this λ on $\Sigma_0\} > 0$. Therefore Σ_0 either bends back at λ_* or blows up at λ_* from the left side of λ_*.

If Σ_0 blows up at λ_*, then λ_* becomes a point where a bifurcation from infinity occurs. By Corollary 3.8, the curve bifurcating from infinity can only be supercritical, which is a contradiction. Thus Σ_0 must bend back at λ_*, and λ_* is a turning point.

On the other hand, by Lemma 4.7, any solution w of (4.2) can be chosen as positive. Hence by Theorems 3.12 and 3.13, $\lambda''(0) < 0$ at any turning point. Hence (λ_*, u_*) is the only turning point on Σ_0. After turning to the left, Σ_0 continues to the left, and it must go unbounded in $\|u\|_\infty$ direction. Thus $T = N = (0, \infty)$ and Σ_0 is the only solution curve of (2.2). If f is asymptotic linear, then Σ_0 must terminate at (λ_∞, ∞). If f is asymptotic superlinear, $K_f(u) \leqslant n/(n-2)$, (2.2) has two solutions for each $\lambda \in (0, \lambda_*)$ (by using Mountain Pass Lemma in the same way as [4] or Theorem A.2,) then Σ_0 can be extended to $(0, \infty)$. ∎

This theorem is new, though the conditions on K_f are very similar to the conditions of Theorem 2.47 in [40], where a uniqueness result was proved. For f satisfies $f, f', f'' > 0$ in $[0, \infty)$, Crandall and Rabinowitz [18] showed the existence of the solution curve from the trivial solutions and the first turning point for general domains, and Amann [2] proved the exact multiplicity for f is asymptotic linear and $\lambda_0/\lambda_\infty \leqslant \lambda_2(\Omega)/\lambda_1(\Omega)$. Joseph and Lundgren [26] proved the exact multiplicity for special cases

$f(u)=(1+u)^p$ and $f(u)=e^u$ for ball domains. Recently, Ambrosetti *et al.* [4] studied Eq. (2.2) with general bounded domain and $f(u)=u^q+u^p$ for $0<q<1<p\leqslant(n+2)/(n-2)$. They proved the existence of two positive solutions for $\lambda\in(0,\lambda_*)$. Theorem 6.21 shows that, for the nonlinearities in [4] with $p\in(1,n/(n-2)]$ and ball domains with $n\geqslant 4$, there are exactly two solutions for $\lambda\in(0,\lambda_*)$.

EXAMPLE 6.22. (1) *$f(u)=u^p+c$, for $1<p\leqslant n/(n-2)$ and $c>0$;*

(2) *$f(u)=(1+u)^p$, for $1<p\leqslant n/(n-2)$;*

(3) *$f(u)=u^p+u^q$, for $0<q<1<p\leqslant n/(n-2)$.*

7. SOLUTION IN LIMIT CASES

In this section, we give the proof of the uniqueness of the ground state solution or the solution of overdetermined problem.

First we assume that $f\in C^1(\mathbf{R}^+)$, $f(0)=0$, $f'(u)\leqslant 0$ in $(0,\delta)$, $f(u)<0$ in $(0,b)$ for $b>0$, $f(b)=0$, $f'(b)>0$, (6.7) holds, and either $f(u)>0$ in (b,∞), or there exists $c>\theta$ such that $f(u)>0$ in (b,c) and $f(u)<0$ in (c,∞). For any $g\in G$, $u(r,g)$ is a ground state solution. In the case of $f(u)>0$ in (b,∞), if in addition f satisfies (A.1), then it was proved by Berestycki *et al.* [7] that $G\neq\varnothing$. The uniqueness of ground state was studied in [38, 34, 42, 43] and many others.

PROPOSITION 7.1. (1) *Let f be as in Theorem 6.11 or Theorem 6.13, and $f(0)=0$. Then there exists $g\geqslant\theta$ such that $N=(g,\infty)$, $G=\{g\}$, and $P=[\theta,g)$;*

(2) *Let f be as in Theorem 6.18 and $f(0)=0$. Then there exists $g\geqslant\theta$ such that $T=N=(g,\infty)$ (resp. (g,c)), $G=\{g\}$, and $P=[\theta,g)$.*

Our proof is based on the proofs of Theorem 1 of [34] and Lemma 1 of [19]. The former one is the same result here for Theorem (6.11), and the latter one essentially proves this result for f in Theorem 6.18 and $f<0$ for large u. So our proof here is sketchy and refer some details to [34, 19].

Proof. First we notice that in Theorem 6.11, we assume that $f'(u)\leqslant 0$ near 0 and $f'(b)>0$. These are also true for Theorems 6.13, 6.18 since the conditions on f implies them. Let $w(r,d)$ be the solution of (5.10). The proof in [34] consists of the following steps:

(1) For $g \in G$, $w(r, g)$ must change sign in $(0, \infty)$;

(2) For $g \in G$, $w(r, g)$ changes sign exactly once;

(3) For $g \in G$, $\limsup_{r \to \infty} w(r, g) = K < 0$;

(4) If $g \in G$, then $(g, g+\varepsilon) \subset N$.

Note that Step (4) implies $G = \{g\}$, since in all these theorems we have proved the solution curve is unique (which do not depend on Proposition 7.1), therefore G has only one element otherwise N is not connected. For f in Theorem 6.11, Kwong and Zhang [34] have proved the conclusions in Proposition 7.1. So we only consider f in Theorem 6.13 or Theorem 6.18. In the following, we use $u(r)$ and $w(r)$ to denote $u(r, g)$ and $w(r, g)$.

For Step (1), suppose we have $w(r) > 0$ in $\mathbf{R}^+$. Following [19], we consider test function $u_r(r)$ and we have

$$W_1'(r) \equiv [r^{n-1}(w(r)\, u_{rr}(r) - w_r(r)\, u_r(r))]' = (n-1)\, r^{n-3} u_r(r)\, w(r) < 0,$$

if $n \geq 2$. And $\lim_{r \to 0^+} W_1(r) = 0$, thus $W_1(r) < 0$ for $r > 0$. In particular, $-u_r(r)/w(r)$ is strictly increasing. Since $-u_r(0)/w(0) = 0$, then $-u_r(r)/w(r) \geq C > 0$ for some $C > 0$ for r large. Hence $w(r) < -C^{-1} u_r(r)$ for r large. Suppose that $w(r) > 0$ in $\mathbf{R}^+$, then $w(r) \to 0$ as $r \to \infty$. By (5.10), we have $(r^{n-1} w')' = -r^{n-1} f'(u(r))\, w(r) \geq 0$ and $r^{n-1} w'(r) < 0$ for r large. It follows that $r^{n-1} w'(r)$ is bounded as $r \to \infty$ and $\lim_{r \to \infty} r^{n-1} w'(r)\, u_r(r) = 0$ since $u_r(r) \to 0$ as $r \to \infty$. On the other hand, we have $(r^{n-1} u')' = -r^{n-1} f(u(r)) \geq 0$ and $r^{n-1} u'(r) < 0$ for r large, then $0 \geq -k = \lim_{r \to \infty} r^{n-1} u'(r)$ and $\lim_{r \to \infty} (r^{n-1} u')' = \lim_{r \to \infty} [r^{n-1} u''(r) + (n-1)\, r^{n-2} u'(r)] = 0$. But $\lim_{r \to \infty} r^{n-2} u'(r) = 0$ since $\lim_{r \to \infty} r \cdot r^{n-2} u'(r) = -k$, hence $\lim_{r \to \infty} r^{n-1} u''(r) = 0$ and $\lim_{r \to \infty} r^{n-1} w(r)\, u_{rr}(r) = 0$. So we obtain $\lim_{r \to \infty} W_1(r) = 0$, which contradicts $W_1(r) < 0$ and $W_1'(r) < 0$ for all $r > 0$. Therefore w has at least one zero in $(0, \infty)$ if $n \geq 2$. The case for $n = 1$ is similar, see [19, p. 156].

Step (2) can be proved in a similar way as proving the solution of (2.2) positive in finite ball, which we have done for f in Theorem 6.11 or Theorem 6.18 in [41]. In fact, let $0 \leq r_1 < r_2$ such that $u(r_2) = \theta$ and $u(r_1) = \rho$, then by the results in [41], w does not change sign in $[0, r_1]$ and w changes sign at most once in $[r_1, r_2]$. We can prove w does not change sign in (r_2, ∞) by applying Lemmas 15 and 16 of [34] instead of Lemma 4.13. For Step (3), we have proved that the unique zero r_0 of w must belong to $[r_1, r_2]$. Similar to the proof of (C) in Proposition 6.12, we use a test function $v(r) = r u_r(r) + \mu u(r)$ for some $\mu > 0$, then

$$\begin{aligned} W_2'(r) &\equiv [r^{n-1}(w(r)\, v_r(r) - w_r(r)\, v(r))]' \\ &= r^{n-1} w(r)[\mu f'(u) u - (\mu+2) f(u)]. \end{aligned}$$

Let μ satisfy $\gamma = K_f(u(r_0)) = (\mu+2)/\mu$, then

$$\gamma f(u) - uf'(u) \leqslant 0 \quad \text{in } [0, u(r_0)],$$

$$\gamma f(u) - uf'(u) \geqslant 0 \quad \text{in } [u_0, \min(c, \infty)).$$

Thus $W_2'(r) < 0$ for $r > 0$ and $W_2(0) = 0$. On the other hand, similar to the proof of Step (1), we can prove $\liminf_{r \to \infty} W_2(r) \geqslant 0$ (see [19, p. 155]), which is a contradiction.

Finally, Step (4) for f in Theorem 6.13 or Theorem 6.18 is same as that of Theorem 6.11, since in the proof of Lemma 10 of [34], only the property of f near 0, namely $f'(u) \leqslant 0$ near 0 (which is true since f is superlinear near 0), is used. This finishes the proof of Proposition 7.1. ∎

Next we assume that $f \in C^1(\mathbf{R}^+)$, $f(0) < 0$, $f(u) < 0$ in $(0, b)$ for $b > 0$, $f(b) = 0$, $f'(b) > 0$, (6.7) holds, and either $f(u) > 0$ in (b, ∞) or there exists $c > \theta$ such that $f(u) > 0$ in (b, c) and $f(u) < 0$ in (c, ∞). By Lemma 5.1, we have $G = \varnothing$.

PROPOSITION 7.2. (1) *Let f be as in Theorem* 6.11 *or Theorem* 6.13 *and $f(0) < 0$. Then there exists $g > \theta$ such that $N = (g, \infty)$, $B = \{g\}$, and $P = [\theta, g)$;*

(2) *Let f be as in Theorem* 6.16, *or Theorem* 6.18 *and $f(0) < 0$. Then there exists $g > \theta$ such that $N = (g, \infty)$ (resp. (g, c)), $B = \{g\}$, and $P = [\theta, g)$.*

Proof. Our proof follows the proof of Proposition 7.1 closely. Similar to the four steps which we use to prove Proposition 7.1, we prove in the following four steps:

(1) For $g \in B$, $w(r, g)$ must change sign in $(0, R(d))$;

(2) For $g \in B$, $w(r, g)$ changes sign exactly once in $(0, R(d))$;

(3) For $g \in B$, $w(R(g), g) < 0$;

(4) If $g \in B$, then $(g, g+\varepsilon) \subset N$ and $(g, g-\varepsilon) \subset P$ for some $\varepsilon > 0$.

For Step (1), we can directly apply Lemma 5.3. Step (2) can be proved in the same way as proving the solution of (2.4) positive, which we have done for these f's. Step (3) is straightforward. Suppose it is not true, then $w(R(g), g) = 0$, by Step (2), $w(r, g)$ is a solution to (2.4) and changes sign once. But we have proved for any solution w to (2.4), w is of one sign, that is a contradiction. Finally, we prove Step (4). By Step (3) and the continuity of $w(r, d)$, there exists $\varepsilon > 0$ such that $w(R(g), d) < 0$ for $d \in (g-\varepsilon, g+\varepsilon)$. We also observe that Step (3) implies that if $d \in (g, g+\varepsilon)$ for $\varepsilon > 0$ small, then there exists $r_1 > 0$ such that $u(r, d) > u(r, g)$ for

$r \in (0, r_1)$ and $u(r, d) < u(r, g)$ for $r \in (r_1, min(R(d), R(g)))$. In particular, $R(d) \leqslant R(g)$. Suppose $R(d) = R(g)$, we have

$$u(R(g), d) - u(R(g), g) = \int_g^d w(R(g), s)\, ds < 0, \tag{7.1}$$

that is a contradiction since the left-hand side is 0. Therefore $R(d) < R(g)$. Suppose $d \in B$, then $u(R(d), d) = u_r(R(d), d) = 0$ and $u_{rr}(R(d), d) = -f(0) > 0$, then $u(R(g), d) > 0$ if ε is small enough. Suppose $d \in P$, we also have $u(R(g), d) > 0$. But in either case, it contradicts with (7.1). Therefore $d \in N$ for $d \in (g, g + \varepsilon)$. Similarly, we can prove that $(g - \varepsilon, g) \subset P$.

Now we can easily finish the proof of Proposition 7.2. Since $B \neq \varnothing$, we can define $g = \inf B$, then $(\theta, g) \subset P$, $g \in B$ and $(g, g + \varepsilon) \subset N$. Let $g_1 = \sup \{d > g: (g, d) \subset N\}$. First we assume that $f(u) > 0$ for all $u > b$. Suppose $g_1 < \infty$, then $g_1 \in B$, but $(g_1 - \varepsilon, g_1) \subset N$, which contradicts with Step (4). So $g_1 = \infty$. Similarly, we can prove that if $f(c) = 0$, $f(u) < 0$ for $u > c$, then $g_1 = c$. ∎

8. CONCLUDING REMARKS

(1) (Lions' conjecture). For all f in theorems in Section 6, we have showed the bifurcation diagram of solutions to (1.1) is the same as the graph of $u/f(u)$. In all diagrams, if $f(u)/u$ is monotone, then the bifurcation diagram is also monotone; if $u/f(u)$ has a local minimum or local maximum, then the bifurcation diagram also has a turning point. For example, let $f(u) = u(u - b)(c - u)$ with $0 < 2b < c$, the graph of $u/f(u)$ is as Fig. 18a, and the bifurcation diagram of positive solution of (1.1) is shown in Fig. 18b. (See Theorem 6.18.) We notice that the graph of $u/f(u)$ between b and c has the same shape of the bifurcation curve. The graph of $u/f(u)$ between 0 and b corresponds to the positive solution of (1.1) with negative λ.

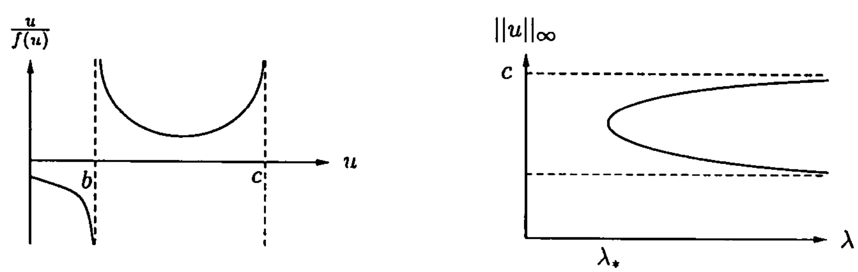


FIG. 18. An illustration of Lions' conjecture.

Similar to Theorem 6.2, we can prove the solution curve for negative λ is monotone. So Lions' conjecture is true for all $\lambda \in \mathbf{R}$ for this case.

But for $f(u)=e^u$, Lions' conjecture is only true for $n=1, 2$ [26]. We can accredit the failure to (i) f is supercritical; (ii) there exists higher ($\geqslant 2$) Morse indices solutions. We believe if the growth rate of f is below some critical exponent, then Lions' conjecture should be true for such $f(u)$. We also conjecture that, for given f, if the Morse index of any solution is either 0 or 1, then Lions's conjecture should be true. To conclude this remark, we notice that the following partial converse of Lions' conjecture is even true for general domain:

PROPOSITION 8.1. *Let $f \in C^1[0, \infty)$. For $\lambda > 0$, if there is no $u \geqslant 0$ such that $\lambda_1 u = \lambda f(u)$, then* (1.1) *has no solution with this λ.*

Proof. Since f is C^1, then the range of $f(u)/u$ is connected. So $\lambda_1 u - \lambda f(u)$ is either positive or negative for all $u \geqslant 0$. We multiply (1.1) by ϕ_1, the eigenfunction corresponding to λ_1, and integrate over Ω. Then we obtain that

$$\int_\Omega [\lambda_1 u - \lambda f(u)] \phi_1 \, dx = 0,$$

but ϕ_1 and $\lambda_1 u - \lambda f(u)$ are both of one sign, that is a contradiction. ∎

This proposition is useful for estimating the value λ_* at the unique turning point for theorems in Section 6. For example, for $f(u) = u(u-b)(c-u)$ with $0 < 2b < c$, λ_* in Theorem 6.18 satisfies $\lambda_* \geqslant 4\lambda_1 (c-b)^{-2}$ by Proposition 8.1.

(2) (Some "counterexamples"). First we construct an example to show that the growth condition on $K_f(u)$ in Theorems 6.18 and 6.19 cannot be easily dropped. Our example is based on an example by Lin and Ni [35]. Their example is $f(u) = u^{(q+1)/2} + u^q$, where $1 < (q+1)/2 < (n+2)/(n-2) < q$. For this nonlinearity, they showed that there exists an interval $(0, \beta) \subseteq T$ but $\beta \notin T$, where $T = \{d > 0 \mid (2.2)$ has a solution with $u(0) = d\}$. The boundedness of $(0, \beta)$ implies there is a turning point in $(0, \beta)$, and they also showed at all turning points in $(0, \beta)$, w cannot be of same sign (in fact, Theorem 3.10 also implies this.) Our example is constructed as follows: let $f_1(u) = u^{(q+1)/2} + u^q$ in $[0, \beta]$, there exists $c > \beta$ such that $f_1(c) = 0$, $f_1(u) > 0$ for $u \in (0, c)$, and there exists $\alpha \in (\beta, c)$ such that $f''(u) > 0$ in $(0, \alpha)$ and $f''(u) < 0$ in (α, c). Since there is a stable solution branch bifurcating from constant $u = c$, and there is another solution branch in $(0, \beta)$, so there are at least two disjoint solution branches for (2.2), and there are at least 4 positive solutions of (2.2) when λ is large.

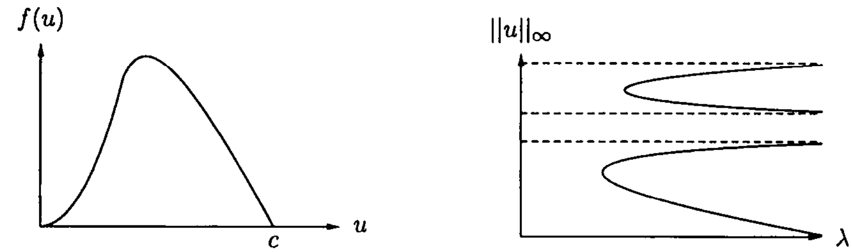


FIG. 19. An Counterexample.

That is quite different from Theorem 6.19, even though f_1 is also convex-concave. So the growth condition in Theorems 6.18 and 6.19 may be not optimal, but some growth conditions are always needed (see Fig. 19).

Next we remark that when f'' changes sign only once, but $(f(u)/u)'$ changes sign twice (this could happen when $f(0) \neq 0$), then it is possible that there are two turning points on the same solution curve. In fact, it is well known that, for $f(u) = exp(au/(a+u))$, the solution curve bifurcating from $(0, 0)$ has at least two turning points if $a > 0$ is large enough. Note that $f(0) > 0$, f is concave-convex and $(f(u)/u)'$ changes sign twice. So Theorem 3.13 is in some sense optimal for the nonlinearity f that f'' changes sign exactly once. This again indicates Lions' conjecture is true for f growing under some critical exponent, even for f satisfying $(f(u)/u)'$ changes sign more than once.

APPENDIX

A. *Collection of Existence Theorems*

In this section, we review some existence results for the positive solutions of (1.1). Most of them were obtained by using the methods other than bifurcation method and the domain can be any bounded smooth domain.

We first consider the case that f is asymptotic sublinear.

THEOREM A.1. *Let $f \in C^1(\mathbf{R}^+)$ and f be asymptotic sublinear. Assume either there exists $c > 0$ such that $f(c) = 0$, $f'(c) < 0$, and $f(u) < 0$ for $u > c$, or $f(u) > 0$ for $u > 0$ large and $f'(\infty) = 0$. If $f(0) \leqslant 0$, we also assume that (6.7) holds. Then there exists $\Lambda > 0$, such that (1.1) has a solution u_λ for all $\lambda > \Lambda$. Moreover, u_λ is stable for λ large enough.*

The result in Theorem A.1 is well known in the case of $f(0) \geqslant 0$ (see [1]), and the case of $f(0) < 0$ was proved by Castro *et al.* [10].

The existence of positive solutions for asymptotic superlinear f seems to be the more complicated. Usually we must add some growth conditions on

f to guarantee the existence. The following theorem is a special case of Theorems 1.1, 1.2, and 2.1 in Lions [36]:

THEOREM A.2. *Let $f \in C^1(\mathbf{R})$ and f be asymptotic superlinear.*

(1) *Suppose $f(0)=0$, f satisfies*

$$\lim_{u \to \infty} \frac{f(u)}{u^l} = 0, \qquad \text{with} \quad l = \frac{n+2}{n-2} \text{ if } n \geqslant 3,\ l < \infty \text{ if } n = 1, 2. \quad \text{(A.1)}$$

Then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, (1.1) has a positive solution. Moreover, if $f'(0) > 0$, $f(u) > 0$ for all $u > 0$, and f satisfies

$$\overline{\lim_{u \to \infty}} \frac{uf(u) - \theta F(u)}{u^2 f(u)^{2/n}} \leqslant 0, \qquad \text{for some} \quad 0 < \theta < \frac{2n}{n-2} \ (\text{if } n \geqslant 3), \quad \text{(A.2)}$$

then there exists $\lambda_* \geqslant \lambda_0$, such that for $0 < \lambda \leqslant \lambda_*$, there exists a minimum solution $\underline{u}_\lambda$ to (1.1); if $\lambda_* > \lambda_0$, then for $\lambda_0 < \lambda < \lambda_*$, there exists at least one solution u_λ to (1.1) distinct from $\underline{u}_\lambda$, i.e., satisfying $u_\lambda > \underline{u}_\lambda$.

(2) *Suppose $f(0) > 0$, $f(u) > 0$ for all $u > 0$. In addition, f satisfies (A.1) and (A.2), then there exists $\lambda_* > 0$, such that for $0 < \lambda \leqslant \lambda_*$, there exists a minimum solution $\underline{u}_\lambda$ to (1.1); for $\lambda > \lambda_*$, there exists no solution to (1.1); for $0 < \lambda < \lambda_*$, there exists at least one solution u_λ to (1.1) distinct from $\underline{u}_\lambda$, i.e. satisfying $u_\lambda > \underline{u}_\lambda$.*

For $f(0) < 0$, we have the following two results: Castro and Shivaji [11] proved for the domain being a ball, if $f(0) < 0$, asymptotic superlinear, $f' \geqslant 0$ and for some $k \in (0, 1)$,

$$\lim_{u \to \infty} \left[\frac{u}{f(u)}\right]^{n/2} \left[F(ku) - \frac{n-2}{2n} uf(u)\right] = \infty, \quad \text{(A.3)}$$

then (1.1) has a solution for $\lambda \in (0, \bar{\lambda})$; Ambrosetti *et al.* [3] proved for the general domains, if $f(0) < 0$, asymptotic superlinear, and

$$\lim_{u \to \infty} \frac{f(u)}{u^l} = b > 0, \qquad \text{with} \quad l < \frac{n+2}{n-2} \text{ if } n \geqslant 3,\ l < \infty \text{ if } n = 1, 2, \quad \text{(A.4)}$$

then (1.1) has a solution for $\lambda \in (0, \bar{\lambda})$. We summarize the results for asymptotic superlinear f as following theorem:

THEOREM A.3. *If $f \in C^1(\mathbf{R}^+)$, f is asymptotic superlinear, $f(0) < 0$, and f satisfies either $f' \geqslant 0$ and (A.3), or (A.4), then there exists $\bar{\lambda} > 0$, such that for $0 < \lambda < \bar{\lambda}$, there exists a solution to (1.1).*

B. *Bifurcation Approach for Hölder Continuous Nonlinearities*

In this section, we sketch a bifurcation theory for equation (1.1) and f satisfying (3.16). Let $X=\{u\in C^{2,\beta}(\overline{B^n}): u|_{\partial B^n}=0\}$, and $Y=C^{\beta}(\overline{B^n})$. Define $F: \mathbf{R}^+\times X\to Y$ by

$$F(\lambda, u)=\Delta u+\lambda f(u), \tag{B.1}$$

then if $f\in C^1(\mathbf{R})$, then $F\in C^1(\mathbf{R}^+\times X, Y)$ and all bifurcation theorems based on the Implicit Function Theorem can be applied; if $f\in C^2(\mathbf{R})$, then $F\in C^2(\mathbf{R}^+\times X, Y)$ and formula (3.13) is true. However, if f only satisfies (3.16), we need the following method to overcome the nonsmoothness at $u=0$. In the following we assume that $0<\beta<q$, where q is the power in (3.16.)

First we extend f to $\{u\leqslant 0\}$ oddly, then $f\in C^i(\mathbf{R}\backslash\{0\})\cap C^0(\mathbf{R})$. We notice that for $m\in\mathbf{N}$ and $m>2$, $g(u)\equiv u^m f(u)\in C^i(\mathbf{R})$ by the limit condition in (3.16). Motivated by this fact, we define a new operator,

$$F^*(\lambda, u)=u^m\,\Delta u+\lambda g(u), \tag{B.2}$$

where $m\in\mathbf{N}$ and $m>2$. Then it is standard to verify that $F^*\in C^i(\mathbf{R}^+\times X, Y)$ and

$$F_u^*(\lambda, u)w=mu^{m-1}\,\Delta u\cdot w+u^m\,\Delta w+\lambda g'(u)w \tag{B.3}$$

where

$$g'(u)=\begin{cases} mu^{m-1}f(u)+u^m f'(u), & u\neq 0,\\ 0, & u=0.\end{cases} \tag{B.4}$$

Since $\beta\in(0,q)$, then by Schauder's estimate, for f satisfying (3.16), a positive solution u of (1.1) belongs to $C^{2,\beta}(\overline{B^n})$. Let u_λ be a positive solution of (1.1), then $F^*(\lambda, u_\lambda)=0$, and hence

$$mu^{m-1}\,\Delta u\cdot w+\lambda mu^{m-1}f(u)w=0. \tag{B.5}$$

By (B.3) and (B.5), we have

$$F_u^*(\lambda, u_\lambda)w=u_\lambda^m\,\Delta w+\lambda u_\lambda^m f'(u_\lambda)w, \tag{B.6}$$

where we should understand that $u^m f'(u)|_{u=0}=0$. Since $u>0$ in B^n, then $\Delta w+\lambda f'(u_\lambda)w=0$ for any $x\in B^n$. So w is a solution of (2.3). (By Remark 3.1 in [4], for f satisfying (3.16), the spectrum theory can still be carried over, so the solutions of (2.3) in this case still make sense.) Therefore F_u^* is invertible at a positive solution u_λ if and only if (2.3) has no nontrivial solution. Then the classical bifurcation theory, especially the Implicit Function

Theorem and Crandall–Rabinowitz Theorem ((1) and (4) in Theorem 3.1), are still true if $i=1$ in (3.16). The bifurcation from the trivial solutions or from the infinity can also be obtained by using a proper scaling (see, for example, [3, 5]), we omit the detail here. In [4], the authors also prove that if f satisfies (3.16), (A.1) and asymptotic superlinear, then there exist two solutions to (1.1) for $\lambda \in (0, \lambda_*)$. So in the proof of Theorem 6.21, we can use the result in [4] instead of Theorem A.1 in such case.

Next we assume that $i=2$ in (3.16). (3.13) cannot be obtained from our new equation. But we can still use a similar proof to show that at a turning point (λ_*, u_*),

$$\lambda''(0) = \frac{-\lambda_* \int_{B^n} u_*^m f''(u_*) w^3 \, dx}{\int_{B^n} u_*^m f(u_*) w \, dx}, \tag{B.7}$$

where w is the solution of (2.3) and $u^m f''(u)|_{u=0} = 0$. For the term $\int_{B^n} u_*^m f(u_*) w \, dx$, we do not know if a simple identity like (3.14) exists, but if $f(u) \geqslant 0$ for $u > 0$ and $w > 0$ (which is true for all our applications), then we have $\int_{B^n} u_*^m f(u_*) w \, dx > 0$; for the other term $\int_{B^n} u_*^m f''(u_*) w^3 \, dx$, we can use the same proof of Theorems 3.10, 3.12, and 3.13 to determine its sign. In particular, the conclusions of Theorems 3.10, 3.12 and 3.13 are still true if we replace $f \in C^2[0, \infty)$ by (3.16) with $i=2$.

REFERENCES

1. H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach space, *SIAM Rev.* **18** (1976), 620–709.
2. H. Amann, Nonlinear eigenvalue problems having precisely two solutions, *Math. Z.* **150** (1976), 27–37.
3. A. Ambrosetti, D. Arcoya, and B. Buffoni, Positive solutions for some semi-positone problems via bifurcation theory, *Differential Integral Equations* **7** (1994), 655–663.
4. A. Ambrosetti, H. Brézis, and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.* **122** (1994), 519–543.
5. A. Ambrosetti and P. Hess, Positive solutions of asymptotically linear elliptic eigenvalue problems, *J. Math. Anal. Appl.* **73** (1980), 411–422.
6. H. Berestycki and P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, *Arch. Rational Mech. Anal.* **82** (1983), 313–345.
7. H. Berestycki, P.-L. Lions, and L. A. Peletier, An ODE approach to the existence of positive solutions for semilinear problems in $\mathbf{R}^N$, *Indiana Univ. Math. J.* **30** (1981), 141–157.
8. A. Castro and S. Gadam, Uniqueness of stable and unstable positive solutions for semipositone problems, *Nonlinear Anal.* **22** (1994), 425–429.
9. A. Castro, S. Gadam, and R. Shivaji, Branches of radial solutions for semipositone problems, *J. Differential Equations* **120** (1995), 30–45.
10. A. Castro, J. B. Garner, and R. Shivaji, Existence results for classes of sublinear semipositone problems, *Results Math.* **23** (1993), 214–220.

11. A. Castro and R. Shivaji, Nonnegative solutions for a class of radially symmetric nonpositone problems, *Proc. Amer. Math. Soc.* **106** (1989), 735–740.
12. C. C. Chen and C.-S. Lin, Uniqueness of the ground state solutions of $\Delta u + f(u) = 0$ in $\mathbf{R}^n$, $n \geqslant 3$, *Comm. Partial Differential Equations* **16** (1991), 1549–1572.
13. X.-Y. Chen and P. Peter, Asymptotic periodicity of positive solutions of reaction diffusion equations on a ball, *J. Reine Angew. Math.* **472** (1996), 17–51.
14. C. B. Clemons and C. K. R. T. Jones, A geometric proof of the Kwong–McLeod uniqueness result, *SIAM J. Math. Anal.* **24** (1993), 436–443.
15. C. V. Coffman, Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions, *Arch. Rational Mech. Anal.* **46** (1972), 81–95.
16. M. G. Crandall and P. H. Rabinowitz, Bifurcation from simple eigenvalues, *J. Funct. Anal.* **8** (1971), 321–340.
17. M. G. Crandall and P. H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability, *Arch. Rational Mech. Anal.* **52** (1973), 161–180.
18. M. G. Crandall and P. H. Rabinowitz, Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems, *Arch. Rational Mech. Anal.* **58** (1975), 207–218.
19. E. N. Dancer, On positive solutions of some singularly perturbed problems where the nonlinearity changes sign, *Topol. Methods Nonlinear Anal.* **5** (1995), 141–175.
20. E. N. Dancer, A note on asymptotic uniqueness for some nonlinearities which change sign, preprint, 1995.
21. E. N. Dancer and K. Schmitt, On positive solutions of semilinear elliptic equations, *Proc. Amer. Math. Soc.* **101** (1987), 445–452.
22. K. Deimling, "Nonlinear Functional Analysis," Springer-Verlag, Berlin/New York, 1985.
23. R. Gardner and L. A. Peletier, The set of positive solutions of semilinear equations in large balls, *Proc. Roy. Soc. Edinburgh Sect. A* **104** (1986), 53–72.
24. B. Gidas, W. M. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209–243.
25. P. Hartman, "Ordinary Differential Equations," Birkhäuser, Boston, 1982.
26. D. D. Joseph and T. S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, *Arch. Rational Mech. Anal.* **49** (1972–1973), 241–269.
27. I. I. Kolodner, Heavy rotating string—A nonlinear eigenvalue problem, *Comm. Pure Appl. Math.* **8** (1955), 395–408.
28. P. Korman, Solution curves for semilinear equations on a ball, *Proc. Amer. Math. Soc.* **125** (1997), 1997–2005.
29. P. Korman, Y. Li, and T. Ouyang, Exact multiplicity results for boundary value problems with nonlinearities generalising cubic, *Proc. Roy. Soc. Edinburgh Sect. A* **126** (1996), 599–616.
30. P. Korman, Y. Li, and T. Ouyang, An exact multiplicity result for a class of semilinear equations, *Comm. Partial Differential Equations* **22** (1997), 661–684.
31. M. K. Kwong, On the Kolodner–Coffman method for the uniqueness problem of Emden–Fowler BVP, *Z. Angew. Math. Phys.* **41** (1990), 79–104.
32. M. K. Kwong and Y. Li, Uniqueness of radial solutions of semilinear elliptic equations, *Trans. Amer. Math. Soc.* **333** (1992), 339–363.
33. M. K. Kwong, J. B. McLeod, L. A. Peletier, and W. C. Troy, On ground state solutions of $-\Delta u = u^p - u^q$, *J. Differential Equations* **95** (1992), 218–239.
34. M. K. Kwong and L. Q. Zhang, Uniqueness of the positive solution of $\Delta u + f(u) = 0$ in an annulus, *Differential Integral Equations* **4** (1991), 583–599.
35. C.-S. Lin and W.-M. Ni, A counterexample to the nodal domain conjecture and a related semilinear equation, *Proc. Amer. Math. Soc.* **102** (1988), 271–277.

36. P.-L. Lions, On the existence of positive solutions of semilinear elliptic equations, *SIAM Rev.* **24** (1982), 441–467.
37. K. McLeod, Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbf{R}^n$, II, *Trans. Amer. Math. Soc.* **339** (1993), 495–505.
38. K. McLeod and J. Serrin, Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbf{R}^n$, *Arch. Rational Mech. Anal.* **99** (1987), 115–145.
39. W.-M. Ni, Uniqueness of solutions of nonlinear Dirichlet problems, *J. Differential Equations* **50** (1983), 289–304.
40. W.-M. Ni and R. D. Nussbaum, Uniqueness and nonuniqueness for positive radial solutions of $\Delta u + f(u, r) = 0$, *Comm. Pure Appl. Math.* **38** (1985), 67–108.
41. T. Ouyang and J. Shi, Exact multiplicity of positive solutions for a class of semilinear problems, *J. Differential Equations* **146** (1998), 121–156.
42. L. A. Peletier and J. Serrin, Uniqueness of positive solutions of semilinear equations in $\mathbf{R}^n$, *Arch. Rational Mech. Anal.* **81** (1983), 181–197.
43. L. A. Peletier and J. Serrin, Uniqueness of nonnegative solutions of semilinear equations in $\mathbf{R}^n$, *J. Differential Equations* **61** (1986), 380–397.
44. P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.* **7** (1971), 487–513.
45. P. H. Rabinowitz, On bifurcation from infinity, *J. Differential Equations* **14** (1973), 462–475.
46. R. Schaaf, Global solution branches of two-point boundary value problems, *in* "Lecture Notes in Mathematics," Vol. 1458, Springer-Verlag, Berlin, 1990.
47. J. Shi, Ph.D. dissertation, Brigham Young University, 1998.
48. J. Shi, Exact multiplicity of solutions to superlinear and sublinear problems, submitted for publication.
49. J. Shi and J. Wang, Morse indices and exact multiplicity of solutions to semilinear elliptic problems, *Proc. Amer. Math. Soc.*, in press.
50. J. Shi and M. Yao, On a singular nonlinear semilinear elliptic problem, *Proc. Roy. Soc. Edinburgh Sect. A* **128** (1998), 1389–1401.
51. J. Smoller and A. Wasserman, Global bifurcation of steady-state solutions, *J. Differential Equations* **39** (1981), 269–290.
52. J. A. Smoller and A. G. Wasserman, Existence, uniqueness, and nondegeneracy of positive solutions of semilinear elliptic equations, *Comm. Math. Phys.* **95** (1984), 129–159.