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Standing Waves of Coupled Schrödinger Equations with Quadratic Interactions from Raman Amplification in a Plasma

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Abstract. The standing wave solutions of a coupled nonlinear Schrödinger equations with quadratic nonlinearities from Raman amplification of laser beam in a plasma are considered. For both the original three-wave system and a reduced two-wave system, the existence/nonexistence, continuous dependence and asymptotic behavior of positive ground state solutions are established. In particular, multiple positive standing wave solutions are found via a combination of variational and bifurcation methods for the attractive interaction case, which has not been found for the conventional nonlinear Schrödinger systems with cubic nonlinearities.

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Contents

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1. Introduction and Main Results

1.1. Physical Models

Raman amplification in a plasma is an instability phenomenon taking place when an incident laser field propagates into a plasma. A Schrödinger type model has been established to describe such phenomenon [19,22,59]:

$$\begin{cases} (i\partial_t + iv_C\partial_y + \alpha_1\partial_y^2 + \alpha_2\Delta_\perp)\mathcal{A}_C = \dfrac{b^2}{2}n\mathcal{A}_C - \chi(\nabla\cdot E)\mathcal{A}_R e^{-i\theta}, \\ (i\partial_t + iv_R\partial_y + \gamma_1\partial_y^2 + \gamma_2\Delta_\perp)\mathcal{A}_R = \dfrac{bc}{2}n\mathcal{A}_R - \chi(\nabla\cdot E^*)\mathcal{A}_C e^{i\theta}, \\ (i\partial_t + \delta_1\Delta)E = \dfrac{b}{2}nE + \chi\nabla(\mathcal{A}_R^*\mathcal{A}_C e^{i\theta}), \\ (\partial_t^2 - v_s^2\triangle)n = a\Delta(|E|^2 + b|\mathcal{A}_C|^2 + c|\mathcal{A}_R|^2), \end{cases} \tag{1.1}$$

where the vector fields $\mathcal{A}_R, \mathcal{A}_C, E : \mathbb{R}^3 \to \mathbb{C}^3$ are the incident laser field, the backscattered Raman field and the electronic plasma-wave, respectively, and n is the variation of the density of the ions; the parameters v_C and v_R are group velocities, $\alpha_1, \alpha_2, \gamma_1, \gamma_2, \chi$ and δ_1 are dispersion coefficients, a, b, c are the coefficients of the nonlinearities, $\theta = k_1 y - w_1 t$, $w_1 = \delta_1 k_1^2$, w_1 is the main frequency of the Raman component, k_1 is the wave number and $\Delta_\perp = \partial_x^2 + \partial_z^2$.

The model (1.1) was derived to describe nonlinear interaction between a laser beam and a plasma. From a physical point of view, when an incident laser field enters a plasma, it is backscattered by a Raman type process. These two waves interact to create an electronic plasma wave; then the three waves combine to create a variation of the density of the ions which has an influence on the three proceedings waves. The system (1.1) describing this phenomenon is composed by three Schrödinger equations coupled to a wave equation and reads in a suitable dimensionless form. For a complete description of this model as well as a precise description of the physical coefficients, we refer to [19,20, 22].

Following simplifications in [22], a subsystem of (1.1) with nonlinear effects can be simplified and converted into a system of three-wave nonlinear Schrödinger equations:

$$\begin{cases} i\partial_t v_1 = -\Delta v_1 - \mu_1|v_1|^{p-2}v_1 - \chi\bar{v_2}v_3, \\ i\partial_t v_2 = -\Delta v_2 - \mu_2|v_2|^{p-2}v_2 - \chi\bar{v_1}v_3, \\ i\partial_t v_3 = -\Delta v_3 - \mu_3|v_3|^{p-2}v_1 - \chi v_1 v_2, \end{cases} \tag{1.2}$$

where v_i $(i = 1, 2, 3)$ are complex valued functions of $(t, x) \in \mathbb{R}\times\mathbb{R}^N$, $p > 2$, $N \le 3$, $\mu_i > 0$ $(i = 1, 2, 3)$ and $\chi > 0$. Orbital stability of standing wave solutions in form of $(e^{iwt}\psi_w, 0, 0)$, $(0, e^{iwt}\psi_w, 0)$ and $(0, 0, e^{iwt}\psi_w)$ were considered in [21,22], see also [52] for the case of dimension $N = 4, 5$. In the present paper we look for standing waves of (1.2) of the form

$$(v_1(t,x), v_2(t,x), v_3(t,x)) = (u(x)e^{i\lambda_1 t}, v(x)e^{i\lambda_2 t}, w(x)e^{i\lambda_3 t}), \tag{1.3}$$

where u, v and w are real-valued functions of $x \in \mathbb{R}^N$. One can see that if $\lambda_3 = \lambda_1 + \lambda_2$, substituting (1.3) into (1.2), then we find that (u, v, w) is a solution of the system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 |u|^{p-2} u + \chi v w, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 |v|^{p-2} v + \chi u w, & \text{in } \mathbb{R}^N, \\ -\Delta w + \lambda_3 w = \mu_3 |w|^{p-2} w + \chi u v, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.4}$$

Here p satisfies $2 < p < 2^*$, $2^* = 2N/(N-2)$ if $N \geq 3$, and $2^* = \infty$ if $N = 1, 2$. When $3 < p < 2^*$ and $\mu_1 = \mu_2 = \mu_3 = 1$, the existence of a positive least energy solution of (1.4) for $\chi > 0$ sufficiently large was proved in [56]. The existence and nonexistence of solutions to (1.4) for the case of $2 < p < 3$ have been shown in [69]. In the present paper, we consider the existence, nonexistence and multiplicity of solutions to system (1.4) with $p = 3$, i.e.,

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 |u| u + \beta v w, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 |v| v + \beta u w, & \text{in } \mathbb{R}^N, \\ -\Delta w + \lambda_3 w = \mu_3 |w| w + \beta u v, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.5}$$

Here the parameters $\lambda_i, \mu_i > 0$ for $i = 1, 2, 3$ and $\beta \in \mathbb{R}$.

A special case of (1.5) with $u = w$, $\lambda_1 = \lambda_3$ and $\mu_1 = \mu_3$ reduces to a two-wave equation

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 |u| u + \beta v u, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 |v| v + \dfrac{\beta}{2} u^2, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.6}$$

Indeed, under the assumptions we know that $\tilde{u} = \tilde{w} = \sqrt{2}u = \sqrt{2}w$. Then the first and third equations of (1.5) becomes $-\Delta \tilde{u} + \lambda_1 \tilde{u} = \tilde{\mu}_1 |\tilde{u}| \tilde{u} + \beta v \tilde{u}$, and the second equation (1.5) become $-\Delta v + \lambda_2 v = \mu_2 |v| v + \frac{\beta}{2} \tilde{u}^2$, where $\tilde{\mu}_1 = \frac{\mu_1}{2}$. Hence, we can get the system (1.6) by re-notation the parameters. The existence and multiplicity of positive solutions of (1.6) when $\lambda_1 = \lambda_2 > 0$ and $\mu_2 = 1$ have been investigated in [23], and the orbital stability of standing waves for the corresponding evolution equation was also considered. When $\mu_i = 0$ for $1 \leq i \leq 3$, the systems (1.5) and (1.6) are in the same form as the Schrödinger system of Second Harmonic Generation (SHG) [73–75], and the existence of ground state and multi-pulse solutions for $2 \leq N \leq 5$ has been investigated in [75] (see also [68]). In the papers [24,25,35], the authors show the existence of bound state and ground state of Schrödinger-KdV system with cubic nonlinearity in dimension one, where the terms $\mu_1 |u| u$ and $\mu_2 |v| v$ are replaced by u^3 and $\frac{1}{2} v^2$ in system (1.6).

Our main results of the present paper are for three cases of (1.5) and (1.6):

1. Under the assumption of $\lambda_1 \neq \lambda_2$ and $\lambda_i > 0$, the existence/nonexistence, uniqueness/multiplicity, asymptotical behavior and bifurcation of positive solutions or nontrivial solutions of the two-wave system (1.6) (see Sect. 1.2);
2. Liouville's type results of the two-wave system (1.6) when $\lambda_1 = \lambda_2 = 0$ (see Sect. 1.3);
3. The existence/nonexistence, multiplicity, asymptotical behavior and bifurcation of positive solutions or nontrivial solutions of the three-wave system (1.5) (see Sect. 1.4).

The two-wave system (1.6) and three-wave system (1.5) considered here are both systems of nonlinear Schrödinger type equations with quadratic interaction terms. More often nonlinear Schrödinger equations arising from physical applications have cubic interactions. For example, the Kerr effect in nonlinear optics, Gross–Pitaevskii equation of Bose–Einstein condensate [8,9,14,65,66]. Models of multiple wave interactions also often inherit such cubic nonlinear effect, see for example [1,2,4–6,11,16,17,31,34,40–42,48,50,51,53,60,62–64,67] and the references therein. But in recent years there have been increasing interests in nonlinear Schrödinger type equations with quadratic nonlinearities, which arises from nonlinear optical effects such as Second Harmonic Generation (SHG) [13,36,45,55,74,75], and Lotka-Volterra competition models in ecology [26,29,30].

One of our main findings in this paper is that the two-wave system (1.6) and three-wave system (1.5) have multiple positive solutions for certain parameter values (by combining the variational and bifurcation methods). This is quite different from other Schrödinger type two-wave or three-wave systems. For the cubic two-wave system studied in [2,4,34,42,49], the positive solution appears to be unique for all $\beta > 0$ when it exists (though not proved except the special case considered in [71]), and the uniqueness is numerically verified in [38] for some parameters. On the other hand, it is known when $\beta < 0$ the cubic Schrödinger system has multiple positive solutions [4]. When $\mu_1 = \mu_2 = 0$ in (1.7), it becomes the Second-Harmonic Generation type I model considered in [75], and in that case, the uniqueness of the positive solution is known for $\lambda = 0$ and $\lambda = 1$ and conjectured for all $\lambda > 0$ and $\beta > 0$ (see [18,75]). Hence, the nonuniqueness for (1.6) obtained here provides an example of possible multiple standing waves for Schrödinger type systems with attractive interaction.

The remaining part of the paper is organized as follows: in Sects. 1.2, 1.3 and 1.4, we state our main results on two-wave system (1.6) and three-wave system (1.5), respectively. Some preliminaries for two-wave system are reviewed in Sect. 2. We prove the existence/nonexistence results for two-wave system in Sect. 3, and bifurcation and asymptotic behavior of solutions of two-wave system are considered in Sect. 4. Some Liouville type results for two-wave system are proved in Sect. 5. Finally, results for three-wave system are proved in Sect. 6.

Throughout the paper, we use the following notation for the function spaces: $X = H^1(\mathbb{R}^N)$, $X_r = H^1_r(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : u$ is radially sym metric}, $Y = L^2(\mathbb{R}^N)$ and $Y_r = L_r^2(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : u$ is radially symmetric}. Also for any linear space Z, Z^k is the k-fold Cartesian product of Z. In this paper we will consider Z^2 and Z^3 for $Z = X, X_r, Y, Y_r$.

1.2. General Two-Wave System

Let

$$\tilde{u}(x) = u\left(\frac{x}{\sqrt{\lambda_1}}\right),\ \tilde{v}(x) = v\left(\frac{x}{\sqrt{\lambda_1}}\right),\ \tilde{\beta} = \frac{\beta}{\lambda_1},\ \tilde{\mu}_1 = \frac{\mu_1}{\lambda_1},\ \tilde{\mu}_2 = \frac{\mu_2}{\lambda_1},\ \tilde{\lambda}_2 = \frac{\lambda_2}{\lambda_1}.$$

Then $(\tilde{u}(x), \tilde{v}(x))$ satisfies (1.6) when the parameters $\lambda_1, \lambda_2, \mu_1, \mu_1, \beta$ are replaced by $1, \tilde{\lambda}_2, \tilde{\mu}_1, \tilde{\mu}_1, \tilde{\beta}$, respectively. In the following we shall only consider

$$\begin{cases} -\Delta u + u = \mu_1 |u| u + \beta v u, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda v = \mu_2 |v| v + \dfrac{\beta}{2} u^2, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.7}$$

as the solutions of (1.6) can be easily converted from the ones of (1.7) via above scaling. The energy functional associated with (1.7) is defined by

$$\begin{aligned} \mathscr{J}(u,v) = & \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) + \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + \lambda v^2) - \frac{\beta}{2}\int_{\mathbb{R}^N} u^2 v \\ & - \frac{1}{3}\int_{\mathbb{R}^N} (\mu_1 |u|^3 + \mu_2 |v|^3) \quad \text{for } (u,v) \in X^2, \end{aligned} \tag{1.8}$$

where $N \le 6$. We say that (u, v) is a weak solution of (1.7) if $(u, v) \in X^2$, and

$$\begin{aligned} \mathscr{J}'(u,v)(\phi,\varphi) = & \int_{\mathbb{R}^N} (\nabla u \nabla \phi + u\phi) + \int_{\mathbb{R}^N} (\nabla v \nabla \varphi + \lambda v \varphi) - \beta \int_{\mathbb{R}^N} uv\phi \\ & - \frac{\beta}{2}\int_{\mathbb{R}^N} u^2 \varphi - \int_{\mathbb{R}^N} (\mu_1 |u| u\phi + \mu_2 |v| v\varphi) = 0, \end{aligned}$$

for each $(\phi, \varphi) \in X^2$.

A solution (u, v) of (1.7) is nontrivial if $u \ne 0$ and $v \ne 0$. A solution (u, v) with $u > 0$ and $v > 0$ is a positive solution. A solution is called a ground state solution (or positive ground state solution) if its energy is minimal among all the nontrivial solutions (or all the nontrivial positive solutions) of (1.7). The system (1.7) also possesses semitrivial solutions of type $(0, v)$.

In the variational setting we consider a weak solution (u, v) of (1.7) in the space X^2, and we will also consider the problem in the subspace of radially symmetric functions X_r^2. We say that $(u, v) \in X_r^2$ is a radial solution, and it is called a radial ground state solution (or positive radial ground state solution) if its energy is minimal among all the nontrivial radial solutions (or all the positive radial solutions) of (1.7).

A more general notation of stability of a solution of (1.7) can be defined through the Morse index. Let (u, v) be a nonnegative solution of (1.7). Then for each $(\phi,\varphi)\in X^2$, one has that

$$\begin{aligned} L(\phi,\varphi) &\equiv \mathscr{J}''(u,v)[(\phi,\varphi),(\phi,\varphi)] \\ &= \int_{\mathbb{R}^N}(|\nabla\phi|^2+\phi^2+|\nabla\varphi|^2+\lambda\varphi^2)-\beta\int_{\mathbb{R}^N}(v\phi^2+2u\varphi\phi) \\ &\quad -2\int_{\mathbb{R}^N}(\mu_1 u\phi^2+\mu_2 v\varphi^2). \end{aligned}$$

Let S^- be the negative subspace of X^2 such that $L|_{S^-}$ is negatively definite. Then the Morse index of a nonnegative solution (u,v) is defined as $M(u,v)=\dim S^-$. Similarly we define S_r^- to be the subspace of X_r^2 such that $L|_{S_r^-}$ is negatively definite, and the radial Morse index to be $M_r(u,v)=\dim S_r^-$.

To state our first existence result for (1.7), we set

$$\begin{aligned} \bar\beta_0 &= \frac{2\mu_2\left[\left(\lambda^{\frac{N-6}{6}}+\lambda^{-\frac{N}{3}}\right)^{\frac{3}{2}}-\left(\frac{\mu_1}{\mu_2}+\lambda^{-\frac{N}{2}}\right)\right]}{3\min\left\{1,\lambda^{-\frac{N}{2}}\right\}}, \\ \bar\beta_1 &= \frac{2}{3}\left[\mu_2\lambda^{\frac{N-6}{4}}\left(2+(\lambda-1)\sigma_0\right)^{\frac{3}{2}}-(\mu_2+\mu_1)\right], \\ \bar\beta_2 &= \frac{2}{3}\left[\mu_2\left(2+\left(\frac{1}{\lambda}-1\right)\sigma_0\right)^{\frac{3}{2}}-(\mu_1+\mu_2)\right], \end{aligned} \tag{1.9}$$

and

$$\hat\beta_0=\min\{\bar\beta_0,\bar\beta_1,\bar\beta_2\}, \tag{1.10}$$

where $\sigma_0=|w_0|^2_{L^2(\mathbb{R}^N)}/|w_0|^3_{L^3(\mathbb{R}^N)}$, and w_0 is the unique positive solution of

$$-\Delta u+u=u^2,\ \ u\in X_r\equiv H_r^1(\mathbb{R}^N). \tag{1.11}$$

The following is a set of basic existence results for the nontrivial solutions of (1.7).

Theorem 1.1. *Suppose that $\mu_1,\mu_2,\lambda>0$ and $\beta\in\mathbb{R}$.*

(i) *For $1\le N\le 5$, (1.7) possesses a nontrivial radial ground state solution $z\in X_r^2$ if one of the following conditions holds:*

(A_1) $\mu_1,\mu_2>0$, $\lambda>\left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}$ *and* $\beta\in\mathbb{R}$; *or*

(A_2) $\mu_1,\mu_2>0$, $0<\lambda\le\left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}$ *and* $\beta>\hat\beta_0$.

In addition, if $\beta>0$, then z is also a positive ground state solution of (1.7).

(ii) *If $N=6$ and $\lambda>0$, or $N>6$, $\lambda,\mu_1,\mu_2>0$ and $\beta>0$, then (1.7) has no positive solution.*

(iii) *For $N=1$, (1.7) has a nontrivial radial ground state solution $z\in X_r^2$ if*
(A_3) $\mu_1,\lambda,\beta>0$, *and* $-\infty<\mu_2\le 0$.

(iv) *For any $\beta_*>0$, there exists a $K_{\beta_*}>0$ such that for $(u,v)\in S_{\beta_*}$,*

$$|u|_\infty+|v|_\infty\le K_{\beta_*},$$

where $S_{\beta_*} = \{(u, v) : (u, v)$ *is nontrivial radial positive solution of* (1.7) *with* $\beta \in [0, \beta_*]\}$.

Remark 1.2.

1. The range of β in the existence results of (i) of Theorem 1.1 is not the best one which we can obtain. We give the present form to avoid heavy notations at this stage, and the more precise range for β can be found in section 3.1 (see (3.10)).
2. If $\lambda = 1$, then for the constants defined in (1.9), $\hat{\beta}_0 = \bar{\beta}_0 = \bar{\beta}_1 = \bar{\beta}_2 = 2(\mu_2 - \mu_1)/3 \geq 0$ as the condition (A_2) is satisfied. For general $\lambda > 0$, one can also show that $\hat{\beta}_0 > 0$ under the condition (A_2).
3. If $\mu_2 = 0$ in (1.7), then the conclusions (i) and (iii) still hold for each $\lambda, \mu_1 > 0$.

Remark 1.3. Comparing to the case of $2 < p < 3$ in [69], although the conclusions (i)–(ii) and (iv) in Theorem 1.1 are similar to the case $2 < p < 3$, we have the following differences. First of all, due to the sub-quadratic term $|u|^{p-2}u(2 < p < 3)$ in (1.4), the energy functional is neither bounded from below on Nehari manifolds nor satisfies the Mountain-Pass condition in X^2. The author combined the Mountain-Pass theorem in convex set and Nehari manifolds methods to overcome the difficulty and prove the existence positive solutions. This is the main contribution of the previous paper [69]. On the other hand, as we have already pointed out that one of our main findings in this paper is that the two-wave system (1.6) and three-wave system (1.5) have multiple positive solutions for certain parameter values (by combining the variational and local bifurcation methods). We also prove the Liouville's type results of the systems (1.6) and (1.5). This is quite different from other cubic Schrödinger type two-wave or three-wave systems.

In the next result we show the existence of small amplitude positive solutions which bifurcate from the known semi-trivial solutions of (1.7), and in some cases, the uniqueness of positive solution can also be proved. We denote S_1 to be the best constant of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^3(\mathbb{R}^N)$:

$$S_1 = \inf_{u \in H^1(\mathbb{R}^N)\backslash\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)}{\left(\displaystyle\int_{\mathbb{R}^N} u^3\right)^{\frac{2}{3}}}.$$

More generally the equation

$$-\Delta u + \lambda u = \mu u^2, \quad u \in X_r \equiv H_r^1(\mathbb{R}^N), \tag{1.12}$$

has a unique positive solution (see Lemma 2.1)

$$w_{\lambda,\mu}(x) = \frac{\lambda}{\mu} w_0(\sqrt{\lambda} x). \tag{1.13}$$

By the regularity results one knows that $w_{\lambda,\mu}(x) \in X_p^r{:=}W^{2,p}(\mathbb{R}^N)$ for $p \in \left(\frac{N}{2}, \infty\right) \cap (1, \infty)$. Now we have the following main results.

Theorem 1.4. *Suppose that $\mu_1, \mu_2, \lambda > 0$. Then the following results hold:*

(i) *Let $\beta_1 > 0$ be the principal eigenvalue of*

$$-\Delta\phi + \phi = \beta w_{\lambda,\mu_2}\phi, \quad \phi \in X_p^r = W^{2,p}(\mathbb{R}^N), \tag{1.14}$$

and let $\phi_{1,\beta_1} > 0$ be the corresponding positive eigenfunction. Then there exists $\tau_0 > 0$ such that when $\beta \in (\beta_1 - \tau_0, \beta_1)$, (1.7) has a positive solution $(u_{1\beta}, v_{1\beta})$ in the form of

$$u_{1\beta} = \frac{\beta_1 - \beta}{\mu_1} \frac{\displaystyle\int_{\mathbb{R}^N} \phi_{1,\beta_1}^2 w_{\lambda,\mu_2}}{\displaystyle\int_{\mathbb{R}^N} \phi_{1,\beta_1}^3} \phi_{1,\beta_1} + o(\beta - \beta_1), \; v_{1\beta} = w_{\lambda,\mu_2} + o(\beta - \beta_1).$$

Moreover, if either (A_1) or (A_2) holds, then $(u_{1\beta}, v_{1\beta})$ is not a ground state solution, and $M(u_{1\beta}, v_{1\beta}) = 2$ for $\beta \in (\beta_1 - \tau_0, \beta_1)$. Additionally, we have that $M(0, w_{\lambda,\mu_2}) = 2$ for $\beta_1 < \beta < \beta_1 + \tau_0$ and $M(0, w_{\lambda,\mu_2}) = 1$ for $0 < \beta < \beta_1$.

(ii) *There exists $\tau_1 > 0$ such that when $\beta \in (-\tau_1, \tau_1)$, (1.7) has exactly two nontrivial solutions $(u_{2\beta}, v_{2\beta})$ and $(u_{3\beta}, v_{3\beta})$ in the form of*

$$\begin{aligned} u_{2\beta} &= w_{1,\mu_1} + \beta(-\Delta + 1 - 2\mu_1 w_{1,\mu_1})^{-1}(w_{1,\mu_1} w_{\lambda,\mu_2}) + o(\beta), \\ v_{2\beta} &= w_{\lambda,\mu_2} + \frac{\beta}{2}(-\Delta + \lambda - 2\mu_2 w_{\lambda,\mu_2})^{-1}(w_{1,\mu_1}^2) + o(\beta), \\ u_{3\beta} &= w_{1,\mu_1} + o(\beta), \quad v_{3\beta} = \frac{\beta}{2}(-\Delta + \lambda)^{-1}(w_{1,\mu_1}^2) + o(\beta); \end{aligned} \tag{1.15}$$

when $\beta \in (0, \tau_1)$, both of $(u_{2\beta}, v_{2\beta})$ and $(u_{3\beta}, v_{3\beta})$ are positive, and when $\beta \in (-\tau_1, 0)$, $(u_{2\beta}, v_{2\beta})$ is positive but $(u_{3\beta}, v_{3\beta})$ satisfies $u_{3\beta} > 0$, $v_{3\beta} < 0$. Moreover, $(u_{2\beta}, v_{2\beta})$ is not a ground state solution with $M(u_{2\beta}, v_{2\beta}) = 2$, and $(u_{3\beta}, v_{3\beta})$ is a ground state solution with $M(u_{3\beta}, v_{3\beta}) = 1$.

(iii) *Suppose that $\lambda = 1$, $\beta = \mu_2 > 0$ and $1 \le N \le 5$, and let (u_0, v_0) be a positive solution of (1.7). Then $u_0 = \frac{2\mu_1}{\mu_2} v_0$, and v_0 satisfies*

$$-\Delta v + v = \left(\mu_2 + \frac{2\mu_1^2}{\mu_2}\right) v^2, \; v \in H^1(\mathbb{R}^N). \tag{1.16}$$

The existence of the principal eigenvalue β_1 of (1.14) for fixed $\lambda, \mu_2 > 0$ will be proved in Lemma 4.1. Part (i) shows the local bifurcation of positive solutions from the branch of the semi-trivial solutions $\{(\beta, 0, w_{\lambda,\mu_2}) : \beta > 0\}$, and part (ii) shows the continuation of the decoupled solution $(w_{1,\mu_1}, w_{\lambda,\mu_2})$ to small $\beta > 0$. These two results can be illustrated by Fig. 1. Moreover combining the existence of positive ground state in Theorem 1.1 and the bifurcation of a non-ground state positive solution in Theorem 1.4, we obtain the following result of existence of multiple positive solutions of (1.7).

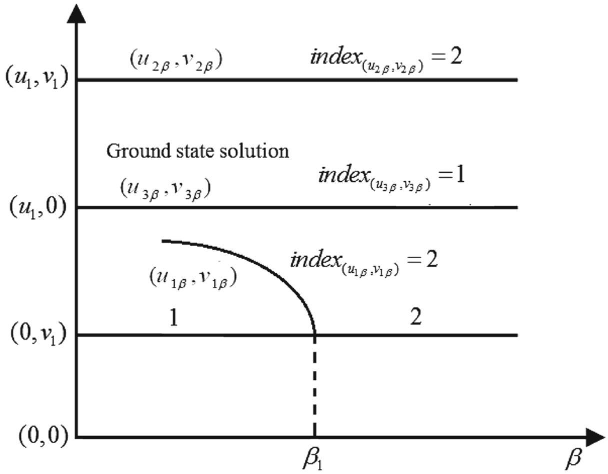

FIGURE 1. Bifurcation diagram for the two-wave system (1.7)

Corollary 1.5. *Suppose that* $\mu_1, \mu_2 > 0$.

1. *If* $\lambda > \left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}$ *and* β_1 *is the principal eigenvalue of* (1.14) *for fixed* $\lambda, \mu_2 > 0$, *then* (1.7) *has at least two positive solutions when* $\beta \in (\beta_1 - \tau_0, \beta_1)$ *for some* $\tau_0 > 0$.
2. *If* $0 < \lambda \leq \left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}$ *and the principal eigenvalue* β_1 *of* (1.14) *for fixed* $\lambda, \mu_2 > 0$ *satisfies* $\beta_1 > \hat{\beta}_0$ *(defined in* (1.9) *and* (1.10)*), then* (1.7) *has at least two positive solutions when* $\beta \in (\beta_1 - \tau_0, \beta_1)$ *for some* $\tau_0 > 0$.
3. *For any* $\lambda > 0$, *then* (1.7) *has exactly two positive solutions when* $\beta \in (0, \tau_1)$ *for some* $\tau_1 > 0$.

Remark 1.6.

1. Corollary 1.5 identifies two intervals of β in which the two-wave equation (1.7) has multiple positive solutions.
2. Note that when $\beta \in (0, \tau_1)$, (1.7) has exactly two positive solutions since when $\beta = 0$, there are exactly four nonnegative solutions of (1.7): $(0, 0)$, $(0, w_{1,\mu_2})$, $(w_{\lambda,\mu_1}, 0)$ and $(w_{1,\mu_1}, w_{\lambda,\mu_2})$. The first two remains as nonnegative solutions of (1.7) for $\beta \in (0, \tau_1)$, and the latter two perturb to positive solutions $(u_{2\beta}, v_{2\beta})$ and $(u_{3\beta}, v_{3\beta})$ for $\beta \in (0, \tau_1)$. There are no other positive solutions from Theorem 1.1 part (iv).
3. The solution $(u_{3\beta}, v_{3\beta})$ is a ground state as it achieves the minimal energy among all non-trivial solutions of (1.7), but the semi-trivial solution $(0, w_{1,\mu_2})$ may have smaller energy.

The next result is concerned with the asymptotical behavior of the positive ground state solutions of (1.7).

Theorem 1.7. *Assume that* $1 \le N \le 5$.

(i) *Let* $(u_{\beta_n}, v_{\beta_n})$ *be any positive ground state solution of* (1.7) *with* $\beta = \beta_n > 0$. *Then, passing to a subsequence, as* $\beta_n \to \beta_* \ge 0$, *one has that* $(u_{\beta_n}, v_{\beta_n}) \to (u_0, v_0)$ *strongly in* X_r^2 *as* $n \to \infty$. *Then the following results hold.*

  (1) *If* $\beta_* = 0$ *and* $\lambda > \left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}$, *then* $\lim_{n\to\infty} C^{\beta_n} = \frac{S_1^3}{6\mu_1^2}$ *and* $(u_0, v_0) = (\mu_1^{-1} w_0, 0)$, *where* C^{β_n} *denotes the least energy level of* (1.7) *with* $\beta = \beta_n$.

  (2) *If* $\beta_* > 0$ *and* $\lambda > 0$, *then* (u_0, v_0) *is a positive ground state solution of* (1.7) *with* $\beta = \beta_*$.

(ii) *Assume that* (A_1) *and* $\beta > 0$ *hold. Let* $(u_{\lambda_n}, v_{\lambda_n})$ *be any positive ground state solution of* (1.7) *with* $\lambda = \lambda_n$, *where* $\lambda_n \to \infty$ *as* $n \to \infty$. *Then, passing to a subsequence, as* $\lambda_n \to \infty$, *we have that*

$$(u_{\lambda_n}, v_{\lambda_n}) \to (\mu_1^{-1} w_0, 0) \text{ in } X_r^2, \quad \sqrt{\lambda_n} v_n \to 0 \text{ in } Y_r \equiv L_r^2(\mathbb{R}^N),$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \lambda_n v_{\lambda_n} \varphi = \lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{\beta}{2} u_{\lambda_n}^2 \varphi = \int_{\mathbb{R}^N} \frac{\beta}{2\mu_1^2} w_0^2 \varphi, \ \forall \varphi \in C_0^\infty(\mathbb{R}^N). \tag{1.17}$$

(iii) *Assume that* $\mu_2 = 0$, (A_1) *and* $\beta > 0$ *hold. Let* $(u_{\lambda_n}, v_{\lambda_n})$ *be any positive ground state solution of* (1.7) *with* $\lambda = \lambda_n > 0$ *and* $\mu_2 = 0$. *Then, passing to a subsequence, as* $\lambda_n \to \lambda_* \ge 0$, *we have that* $(u_{\lambda_n}, v_{\lambda_n}) \to (u_0, v_0)$ *in* $H_r^1(\mathbb{R}^N) \times D_r^{1,2}(\mathbb{R}^N)$, *and* (u_0, v_0) *is a positive ground state solution of* (1.7) *with* $\mu_2 = 0$ *and* $\lambda = \lambda_*$.

Remark 1.8. If $\mu_1 = \mu_2 = 0$, the system (1.7) reduces to the well-known Second-Harmonic Generation type I model ($\chi^{(2)}$-Model), which was studied in [75] recently. Let (u_λ, v_λ) be a solution of (1.7) with $\mu_1 = \mu_2 = 0$. It was known via formal argument [3,13] and later proved in [75] that if λ is large enough, $v_\lambda \approx u_\lambda^2/(2\lambda)$ and u_λ is a solution of (1.11) with $\mu_1 = \beta$. Here we prove this conclusion in an integral sense (see (1.17)) for more general situation.

1.3. The Limiting System

In this part we focus on the case when $\lambda_1 = \lambda_2 = 0$ in (1.6). The existence and multiplicity of nontrivial solutions of (1.6) for $\lambda_1 = \lambda_2 > 0$ were shown in [23]. Here we shall establish some Liouville's type results of system (1.6) for the case $\lambda_1 = \lambda_2 = 0$, which is

$$\begin{cases} -\Delta u = \mu_1 |u| u + \beta v u, & \text{in } \mathbb{R}^N, \\ -\Delta v = \mu_2 |v| v + \frac{\beta}{2} u^2, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.18}$$

We have the following Liouville's type results for (1.18).

Theorem 1.9. *Assume that* $\mu_1, \mu_2 > 0$, *and* $\beta > -\left(2\mu_1^2 \mu_2\right)^{\frac{1}{3}}$. *Then the following results hold.*

(i) *If* $1 \leq N \leq 4$, *and* (u, v) *is a nonnegative classical solution of* (1.18), *then* $(u, v) \equiv (0, 0)$.

(ii) *If* $1 \leq N \leq 4$, *and* (u, v) *is a nonnegative classical solution of*

$$\begin{cases} -\Delta u = \mu_1 u^2 + \beta v u, & \text{in } \mathbb{R}^N_+, \\ -\Delta v = \mu_2 v^2 + \dfrac{\beta}{2} u^2, & \text{in } \mathbb{R}^N_+, \\ u = v = 0, & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{1.19}$$

then $(u, v) \equiv (0, 0)$, *where* $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$.

(iii) *Let* $\Omega \subset \mathbb{R}^N$ *be a smooth bounded domain in* $\mathbb{R}^N$ *with* $1 \leq N \leq 4$, *and let* (u, v) *be any nonnegative solution of*

$$\begin{cases} -\Delta u = \mu_1 u^2 + \beta v u, & \text{in } \Omega, \\ -\Delta v = \mu_2 v^2 + \dfrac{\beta}{2} u^2, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.20}$$

then we have $\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)} \leq C$, *where* $C = C(\beta, \mu_1, \mu_2, \Omega)$.

Results in Theorem 1.9 are closely related to the ones in Theorems 2.1 and 2.2 of [32] (also see [33,58]), in which Louville's type results for cubic Schrödinger system were proved. We notice that the results in [32] hold for $1 \leq N \leq 3$, while our results hold for $1 \leq N \leq 4$ as the nonlinearity here is quadratic. It is an open question for the case $N = 5$. In order to prove these results, we shall apply the general theorems given in [33,58].

1.4. Three-Wave Systems

In this subsection, we describe our results for the three-wave system (1.5). Note that the system (1.5) possesses semi-trivial solutions of type $(u, 0, 0)$, $(0, v, 0)$ and $(0, 0, w)$. So, similar to the two-wave system, we give the following definitions of solutions of (1.5). A solution (u, v, w) of (1.5) is nontrivial if $u \neq 0$, $v \neq 0$ and $w \neq 0$. A solution (u, v, w) with $u > 0$, $v > 0$ and $w > 0$ is called a positive solution. A solution is called a ground state solution (or positive ground state solution) if its energy is minimal among all the nontrivial solutions (or all the nontrivial positive solutions) of (1.5). Here the energy functional corresponded to (1.5) is given by

$$\begin{aligned} \tilde{\mathscr{J}}(u, v, w) = &\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda_1 u^2 + |\nabla v|^2 + \lambda_2 v^2 + |\nabla w|^2 + \lambda_3 w^2) \\ &- \frac{1}{3} \int_{\mathbb{R}^N} (\mu_1 |u|^3 + \mu_2 |v|^3 + \mu_3 |w|^3) - \beta \int_{\mathbb{R}^N} uvw, \end{aligned} \tag{1.21}$$

where $(u, v, w) \in X^3$. Similar to the two-wave system (1.7), we can also consider the problem in the subspace of radially symmetric functions X_r^3, and define the radial ground state solution (or the positive radial ground state solution).

In the present paper we consider the case when λ_1, λ_2 and λ_3 are pairwisely distinct. The case $\lambda_1 = \lambda_2 = \lambda_3$ will be considered in a forthcoming works. Without loss of generality we assume that $\lambda_1 = 1$, $\lambda_2 \neq \lambda_3$, $\lambda_2 \neq 1$ and $\lambda_3 \neq 1$ in (1.5). Set

$$\begin{aligned}
\hat{\beta}_1 &= \frac{\mu_1\left[\left(1+\lambda_2^{\frac{2-N}{2}}+\lambda_3^{\frac{2-N}{2}}\right)^{\frac{3}{2}}-\left(1+\frac{\mu_2}{\mu_1}\lambda_2^{-\frac{N}{2}}+\frac{\mu_3}{\mu_1}\lambda_3^{-\frac{N}{2}}\right)\right]}{3\min\left\{1,\lambda_2^{-\frac{N}{2}},\lambda_3^{-\frac{N}{2}}\right\}},\\
\hat{\beta}_2 &= \frac{1}{3}\left[\left(3+(\frac{1+\lambda_2}{\lambda_3}-2)\sigma_0\right)^{\frac{3}{2}}\mu_3-(\mu_1+\mu_2+\mu_3)\right],\\
\hat{\beta}_3 &= \frac{1}{3}\left[\left(3+(\frac{1+\lambda_3}{\lambda_2}-2)\sigma_0\right)^{\frac{3}{2}}\mu_2-(\mu_1+\mu_2+\mu_3)\right],
\end{aligned} \tag{1.22}$$

where $\sigma_0 = |w_0|^2_{L^2(\mathbb{R}^N)}/|w_0|^3_{L^3(\mathbb{R}^N)}$, and w_0 is the unique positive solution of (1.11). We make the following assumptions.

To state our results, we define the following conditions.

(B_1) $\mu_1,\mu_2,\mu_3>0$, $\lambda_2 \geq \left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}$, $\lambda_3 \geq \left(\frac{\mu_3}{\mu_1}\right)^{\frac{4}{6-N}}$ and $\beta > \hat{\beta}_1 > 0$.

(B_2) $\mu_1,\mu_2,\mu_3>0$, $\lambda_2 \geq \left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}$, $0<\lambda_3 \leq \left(\frac{\mu_3}{\mu_1}\right)^{\frac{4}{6-N}}$ and $\beta > \hat{\beta}_2$.

(B_3) $\mu_1,\mu_2,\mu_3>0$, $0<\lambda_2 \leq \left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}$, $\lambda_3 \geq \left(\frac{\mu_3}{\mu_1}\right)^{\frac{4}{6-N}}$ and $\beta > \hat{\beta}_3$.

(B_4) $\mu_1,\mu_2,\mu_3>0$, $0<\lambda_2 < \left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}$, $0<\lambda_3 < \left(\frac{\mu_3}{\mu_1}\right)^{\frac{4}{6-N}}$ and $\beta > \max\left\{\hat{\beta}_2,\hat{\beta}_3\right\}$.

(B_5) $\mu_1,\mu_2,\mu_3,\lambda_2,\lambda_3>0$, and $\beta > \max\left\{\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3\right\}$.

Then we have the following existence and nonexistence results for the three-wave system.

Theorem 1.10. *Suppose that* $\mu_1,\mu_2,\mu_3,\lambda_2,\lambda_3>0$, $\lambda_1=1$, $\lambda_2\neq\lambda_3$, $\lambda_2\neq 1$, $\lambda_3 \neq 1$ *and* $\beta\in\mathbb{R}$.

(i) *If* $1\leq N\leq 5$ *and one of* (B_1)-(B_5) *holds, then* (1.5) *possesses a nontrivial radial solution* $z\in X_r^3$. *Moreover, if* $\beta>0$, *then* z *is a positive ground state solution of* (1.5).

(ii) *For any* $\tilde{\beta}_*>0$, *there exists a* $\tilde{K}_{\tilde{\beta}_*}>0$ *such that for* $(u,v,w)\in\tilde{S}_{\tilde{\beta}_*}$

$$|u|_\infty+|v|_\infty+|w|_\infty \leq \tilde{K}_{\tilde{\beta}_*},$$

where $\tilde{S}_{\tilde{\beta}_*}=\{(u,v,w):(u,v,w)$ *is nontrivial radial positive solution of* (1.7) *with* $\beta\in[0,\tilde{\beta}_*]\}$.

(iii) *If* $N=6$, *or* $N>6$ *and* $\beta>0$, *then* (1.5) *has no positive solution.*

Note that, similar to the two-wave system case, the ranges of β in Theorem 1.10 can be improved. We can also give other conditions to guarantee the conclusions of Theorem 1.10 hold (see Sect. 6). Next we have the following bifurcation results for the system (1.5).

Theorem 1.11. *Suppose that $\mu_1, \mu_2, \mu_3, \lambda_2, \lambda_3 > 0$ and $\lambda_1 = 1$. Then the following results hold.*

(i) *There exists $\tau_2 > 0$ such that when $\beta \in (\beta_2 - \tau_2, \beta_2)$, (1.5) has a positive solution $(u_{1\beta}, v_{1\beta}, w_{1\beta})$ in the form of*

$$\begin{aligned} u_{1\beta}(x) &= w_{1,\mu_1}(x) + o(\beta - \beta_2), \\ v_{1\beta}(x) &= \frac{2(\beta_2 - \beta)\cos\theta_2 \displaystyle\int_{\mathbb{R}^N} \phi_{2,\beta_2}^2 w_{1,\mu_1}}{\displaystyle\int_{\mathbb{R}^N} (\mu_2 \cos^3\theta_2 + \mu_3 \sin^3\theta_2)\phi_{2,\beta_2}^3} \phi_{2,\beta_2}(x) + o(\beta - \beta_2), \\ w_{1\beta}(x) &= \frac{2(\beta_2 - \beta)\sin\theta_2 \displaystyle\int_{\mathbb{R}^N} \phi_{2,\beta_2}^2 w_{1,\mu_1}}{\displaystyle\int_{\mathbb{R}^N} (\mu_2 \cos^3\theta_2 + \mu_3 \sin^3\theta_2)\phi_{2,\beta_2}^3} \phi_{2,\beta_2}(x) + o(\beta - \beta_2), \end{aligned} \tag{1.23}$$

where $\beta_2 > 0$ is the principal eigenvalue of

$$-\Delta\phi + \frac{\lambda_2 + \lambda_3}{2}\phi - \frac{\sqrt{(\lambda_2 - \lambda_3)^2 + 4\beta^2 w_{1,\mu_1}^2}}{2}\phi = 0, \quad \phi \in X_p^r, \tag{1.24}$$

$\phi_{2,\beta_2} > 0$ is the corresponding eigenfunction, and $\theta_2 : \mathbb{R}^N \to (0, \pi/2)$ is a continuous function depending on $\lambda_2, \lambda_3, \beta_2, w_{1,\mu_1}$. Similarly (1.5) has a positive solution $(u_{2\beta}, v_{2\beta}, w_{2\beta})$ for $\beta \in (\beta_3 - \tau_3, \beta_3)$ in a similar form as (1.23) near $(0, w_{\lambda_2,\mu_2}, 0)$, where β_3 is the principal eigenvalue of an eigenvalue problem similar to (1.24) and $\tau_3 > 0$; and (1.5) has a positive solution $(u_{3\beta}, v_{3\beta}, w_{3\beta})$ for $\beta \in (\beta_4 - \tau_4, \beta_4)$ in a similar form as (1.23) near $(0, 0, w_{\lambda_3,\mu_3})$, where β_4 is the principal eigenvalue of an eigenvalue problem similar to (1.24) and $\tau_4 > 0$. Moreover, each of the bifurcating positive solution $(u_{i\beta}, v_{i\beta}, w_{i\beta})$ ($i = 1, 2, 3$) has Morse index $M(u_{i\beta}, v_{i\beta}, w_{i\beta}) = 2$, and if one of (B_i) ($1 \le i \le 5$) holds for λ_i, μ_i and β, then $(u_{i\beta}, v_{i\beta}, w_{i\beta})$ ($i = 1, 2, 3$) is not a ground state solution.

(ii) *There exists $\tau_5 > 0$ such that when $\beta \in (-\tau_5, \tau_5)$, (1.5) has exact four nontrivial solutions $(u_{i\beta}, v_{i\beta}, w_{i\beta})$ ($i = 4, 5, 6, 7$) of the form*

$$\begin{aligned}
&u_{4\beta} = w_{1,\mu_1} + (-\Delta + 1 - 2\mu_1 w_{1,\mu_1})^{-1} w_{\lambda_2,\mu_2} w_{\lambda_3,\mu_3} \beta + o(\beta),\\
&v_{4\beta} = w_{\lambda_2,\mu_2} + (-\Delta + \lambda_2 - 2\mu_2 w_{\lambda_2,\mu_2})^{-1} w_{1,\mu_1} w_{\lambda_3,\mu_3} \beta + o(\beta),\\
&w_{4\beta} = w_{\lambda_3,\mu_3} + (-\Delta + \lambda_3 - 2\mu_3 w_{\lambda_3,\mu_3})^{-1} w_{1,\mu_1} w_{\lambda_2,\mu_2} \beta + o(\beta),\\
&u_{5\beta} = w_{1,\mu_1} + o(\beta),\ \ v_{5\beta} = w_{\lambda_2,\mu_2} + o(\beta),\\
&w_{5\beta} = (-\Delta + \lambda_3)^{-1} w_{1,\mu_1} w_{\lambda_2,\mu_2} \beta + o(\beta),\\
&u_{6\beta} = w_{1,\mu_1} + o(\beta),\ \ v_{6\beta} = (-\Delta + \lambda_2)^{-1} w_{1,\mu_1} w_{\lambda_3,\mu_3} \beta + o(\beta),\\
&w_{6\beta} = w_{\lambda_3,\mu_3} + o(\beta),\\
&u_{7\beta} = (-\Delta + 1)^{-1} w_{\lambda_2,\mu_2} w_{\lambda_3,\mu_3} \beta + o(\beta),\\
&v_{7\beta} = w_{\lambda_2,\mu_2} + o(\beta),\ \ w_{7\beta} = w_{\lambda_3,\mu_3} + o(\beta);
\end{aligned} \tag{1.25}$$

when $\beta \in (0, \tau_5)$, all four solutions above are all positive, and when $\beta \in (-\tau_5, 0)$, the four solutions have sign patterns $(u_{4\beta}, v_{4\beta}, w_{4\beta}) = (+,+,+)$, $(u_{5\beta}, v_{5\beta}, w_{5\beta}) = (+,+,-)$, $(u_{6\beta}, v_{6\beta}, w_{6\beta}) = (+,-,+)$ and $(u_{7\beta}, v_{7\beta}, w_{7\beta}) = (-,+,+)$; these solutions have Morse index $M(u_{4\beta}, v_{4\beta}, w_{4\beta}) = 3$, and $M(u_{i\beta}, v_{i\beta}, w_{i\beta}) = 2$ for $i = 5, 6, 7$. Moreover, if

$$\lambda_2 < \left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}, \ \ \textit{and} \ \ \lambda_3 \geq \left(\frac{\mu_3}{\mu_1}\right)^{\frac{4}{6-N}},$$

$$\textit{or} \ \ \lambda_2 \geq \left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}, \lambda_3 \geq \left(\frac{\mu_3}{\mu_1}\right)^{\frac{4}{6-N}}, \ \ \textit{and} \ \ \lambda_2 \leq \lambda_3,$$

then $(u_{5\beta}, v_{5\beta}, w_{5\beta})$ is a ground state solution; if

$$\lambda_2 \geq \left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}, \ \ \textit{and} \ \ \lambda_3 < \left(\frac{\mu_3}{\mu_1}\right)^{\frac{4}{6-N}},$$

$$\textit{or} \ \ \lambda_2 \geq \left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}, \lambda_3 \geq \left(\frac{\mu_3}{\mu_1}\right)^{\frac{4}{6-N}}, \ \ \textit{and} \ \ \lambda_2 > \lambda_3,$$

then $(u_{6\beta}, v_{6\beta}, w_{6\beta})$ is a ground state solution; and

$$\lambda_2 < \left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}, \ \ \textit{and} \ \ \lambda_3 < \left(\frac{\mu_3}{\mu_1}\right)^{\frac{4}{6-N}},$$

then $(u_{7\beta}, v_{7\beta}, w_{7\beta})$ is a ground state solution.

Remark 1.12.

1. The principal eigenvalue of (1.24) can be defined through a variational way, see Sect. 6 for details.
2. The bifurcation result in Theorem 1.11 holds when some or even all of λ_1, λ_2 and λ_3 are equal. For example, if $\lambda_2 = \lambda_3 \neq 1$, then $v_{1\beta} = w_{1\beta}$ and $\theta_2 \equiv \pi/4$.
3. An illustration of the bifurcation of positive solutions of (1.5) is shown in Fig. 2, in which we assume that $\beta_2 < \beta_3 < \beta_4$.

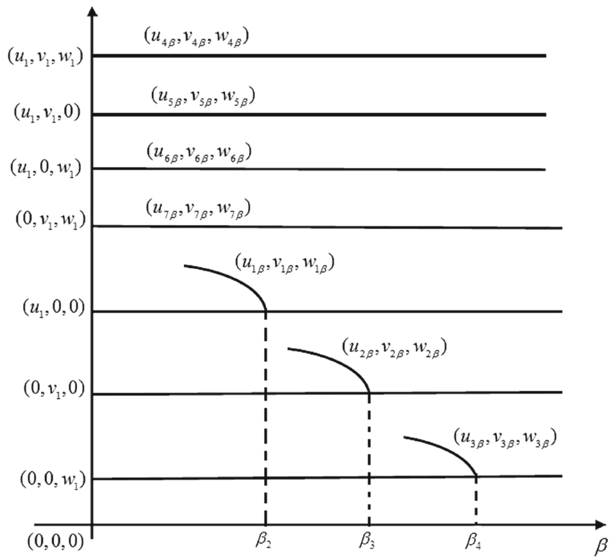

FIGURE 2. Bifurcation diagram for the three-wave system (1.5)

4. Part (ii) of Theorem 1.11 show that (1.5) has exactly four positive solutions when $\beta \in (0, \tau_5)$ for fixed $\mu_i, \lambda_i > 0$. Note that $(u_{4\beta}, v_{4\beta}, w_{4\beta})$ has the highest Morse index and also the highest energy. Here one of $(u_{i\beta}, v_{i\beta}, w_{i\beta})$ with $i = 5, 6, 7$ is the ground state solution as it achieves the minimal energy among the non-trivial solutions of (1.5), but indeed one of semi-trivial solutions has lower energy than all of $(u_{i\beta}, v_{i\beta}, w_{i\beta})$ with $i = 5, 6, 7$. Also the Morse index of each semi-trivial solution is 1.
5. In Theorem 1.4 (iii), we prove the uniqueness of positive solution for the two-wave system (1.7) in a special case. The uniqueness of positive solution to the three-wave system (1.5) is not true in general as it may possess multiple synchronous positive solutions, see our forthcoming paper [39, Theorem 1.1].

2. Preliminaries

In the present paper we use the following notations:

- $\|\cdot\|$ is the norm of $X = H^1(\mathbb{R}^N)$ defined by $\|u\|^2 = \displaystyle\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)$;
- $\|\cdot\|_M$ is an equivalent norm of $X = H^1(\mathbb{R}^N)$ defined by $\|u\|_M^2 = \displaystyle\int_{\mathbb{R}^N} (|\nabla u|^2 + M|u|^2)$, for a positive function or constant M.
- For $z = (u, v) \in X^2$, $\|z\|_E^2 = \|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2$.
- $X_p = W^{2,p}(\mathbb{R}^N)$, where $p \in \left(\frac{N}{2}, \infty\right) \cap (1, \infty)$, and X_p^r denotes the subspace of X_p consisting of radially symmetric functions.

- $|\cdot|_p$ is the norm of $L^p(\mathbb{R}^N)$ defined by $|u|_p = \left(\int_{\mathbb{R}^N} |u|^p\right)^{1/p}$ for $0 < p < \infty$.
- $2^* = \frac{2N}{N-2}$ if $N \geq 3$, and $2^* = \infty$ if $N = 1, 2$.
- c or $C_i (i = 1, \cdots)$ denotes different positive constants.

Define

$$S_{\lambda,\mu} = \inf_{u\in X\setminus\{0\}} \frac{\|u\|_\lambda^2}{\left(\int_{\mathbb{R}^N} \mu|u|^3\right)^{\frac{2}{3}}} \quad \text{and} \quad T_{\lambda,\mu} = \inf_{u\in\mathcal{M}_0} \left\{\frac{1}{2}\|u\|_\lambda^2 - \frac{1}{3}\int_{\mathbb{R}^N} \mu|u|^3\right\}, \tag{2.1}$$

where $\mathcal{M}_0 = \{u \in X : u \neq 0, \|u\|_\lambda^2 = \mu \int_{\mathbb{R}^N} |u|^3\}$. A direct computation shows that the following results hold.

Lemma 2.1. *Assume that $\lambda, \mu > 0$, then $T_{\lambda,\mu}$ is attained by the unique positive solution $w_{\lambda,\mu}(x)$ (defined in (1.13)) of (1.12). Moreover, we have*

$$T_{\lambda,\mu} = \frac{1}{6}S_{\lambda,\mu}^3 \quad \text{and} \quad S_{\lambda,\mu} = \frac{\lambda^{1-\frac{N}{6}}}{\mu^{\frac{2}{3}}} S_{1,1} := \frac{\lambda^{1-\frac{N}{6}}}{\mu^{\frac{2}{3}}} S_1, \tag{2.2}$$

where $S_{1,1} = S_1 = \left(\int_{\mathbb{R}^N} w_0^3\right)^{\frac{1}{3}}$ and w_0 is the unique positive solution of (1.11).

In order to find nontrivial critical points for $\mathscr{J}$, we consider the following Nehari manifold for (1.7).

$$\mathscr{N} = \left\{z = (u, v) \in X^2 \setminus \{(0, 0)\} : \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) + \int_{\mathbb{R}^N} (|\nabla v|^2 + \lambda v^2) = \int_{\mathbb{R}^N} (\mu_1|u|^3 + \mu_2|v|^3) + \frac{3\beta}{2}\int_{\mathbb{R}^N} u^2 v\right\}. \tag{2.3}$$

Apparently all nontrivial solutions of (1.7) are contained in $\mathscr{N}$. The definition of $\mathscr{N}$ implies that for $(u, v) \in \mathscr{N}$,

$$\mathscr{J}|_{\mathscr{N}}(u, v) = \frac{1}{6}(\|u\|^2 + \|v\|_\lambda^2) = \frac{1}{6}\int_{\mathbb{R}^N} \left(\mu_1|u|^3 + \mu_2|v|^3 + \frac{3\beta}{2}u^2 v\right). \tag{2.4}$$

Moreover, for each $(u, v) \in \mathscr{N}$, it follows from Hölder and Young's inequalities that

$$\begin{aligned} \|u\|^2 + \|v\|_\lambda^2 &= \frac{3\beta}{2}\int_{\mathbb{R}^N} u^2 v + \int_{\mathbb{R}^N} (\mu_1|u|^3 + \mu_2|v|^3) \\ &\leq c\left(\|u\|^3 + \|v\|_\lambda^3 + \|u\|^2\|v\|_\lambda\right) \leq c\left(\|u\|^3 + \|v\|_\lambda^3\right), \end{aligned} \tag{2.5}$$

for some $c > 0$. Thus, one deduces from (2.4) and (2.5) that $\mathscr{J}$ is bounded uniformly away from zero on $\mathscr{N}$.

Set the ground state energy and the radial ground state energy to be

$$C = \inf_{z\in\mathscr{N}} \mathscr{J}(z), \quad C_r = \inf_{z\in\mathscr{N}\cap X_r^2} \mathscr{J}(z). \tag{2.6}$$

The following lemma shows the role of C_r and C.

Lemma 2.2. *Suppose that* $\lambda, \mu_1, \mu_2 > 0$, $\beta \in \mathbb{R}$, *and* C *or* C_r *is attained by some* $z_0 \in \mathscr{N}$, *then* z_0 *is a solution of* (1.7).

Proof. Assume that $z_0 = (u_0, v_0) \in \mathscr{N}$ is such that $\mathscr{J}(u_0, v_0) = C$. According to Theorem 4.1.1 of [15], there exists a Lagrangian multiplier $\ell \in \mathbb{R}$ such that

$$\mathscr{J}'(u_0, v_0) = \ell g'(u_0, v_0), \tag{2.7}$$

where $g(u_0, v_0) = \mathscr{J}'(u_0, v_0)(u_0, v_0)$. We infer from $z_0 = (u_0, v_0) \in \mathscr{N}$ that

$$\begin{aligned} & g'(u_0, v_0)(u_0, v_0) \\ & \quad = 2\int_{\mathbb{R}^N} (|\nabla u_0|^2 + u_0^2 + |\nabla v_0|^2 + \lambda v_0^2) \\ & \qquad - \frac{9\beta}{2}\int_{\mathbb{R}^N} u_0^2 v_0 - 3\int_{\mathbb{R}^N} (\mu_1 |u_0|^3 + \mu_2 |v_0|^3) \\ & \quad = -\int_{\mathbb{R}^N} (|\nabla u_0|^2 + u_0^2 + |\nabla v_0|^2 + \lambda v_0^2) < 0. \end{aligned} \tag{2.8}$$

Multiplying the equation (2.7) by (u_0, v_0), it follows from (2.8) that $\ell = 0$. Thus, we have $\mathscr{J}'(u_0, v_0) = 0$. That is, z_0 is a critical point of $\mathscr{J}$ and a solution of (1.7). If C_r is attained, one can similarly prove the conclusion. □

Now we prove a basic existence result. That is, both C and C_r are attained by some (possibly semitrivial) $z \in \mathscr{N}$.

Lemma 2.3. *Suppose that* $\lambda, \mu_1, \mu_2 > 0$ *and* $\beta \in \mathbb{R}$, *and* $1 \le N \le 5$. *Then* C *(or* C_r*) defined in* (2.6) *is attained by some* $z(\neq (0,0)) \in \mathscr{N}$ *(or* $\mathscr{N} \cap X_r^2$*).*

Proof. From (2.4) and (2.5), we know that there exists $\delta > 0$ such that $C_r \ge C \ge \delta > 0$. Since $(w_{1,\mu_1}, 0) \in \mathscr{N}$, $\mathscr{N} \cap X_r^2 \neq \emptyset$. We first prove C can be attained by a nontrivial z. Let $\{(u_n, v_n)\} \subset \mathscr{N}$ be a minimizing sequence. By using the Ekeland's variational principle type arguments (see [70, Lemma 3.10] or [72]), there exists a sequence (still denoted by $\{(u_n, v_n)\}$) on $\mathscr{N}$ such that

$$\mathscr{J}(u_n, v_n) \to C, \quad \mathscr{J}'(u_n, v_n) \to 0, \text{ as } n \to \infty, \tag{2.9}$$

which also implies the boundedness of $\{(u_n, v_n)\}$. Without loss of generality we assume that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in X^2, $(u_n, v_n) \to (u_0, v_0)$ in $[L^p_{loc}(\mathbb{R}^N)]^2$ for $p \in (2, 2^*)$.

We claim that $\{(u_n, v_n)\}$ is nonvanishing. That is, there exist $y_n \in \mathbb{R}^N$ and $R > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} (u_n^2 + v_n^2) \ge \sigma > 0, \tag{2.10}$$

where $B_R(y_n) = \{y \in \mathbb{R}^N : |y - y_n| \le R\}$. If (2.10) is not satisfied, then we have $\{(u_n, v_n)\}$ is vanishing, *i.e.*, for any $R > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} (u_n^2 + v_n^2) = 0. \tag{2.11}$$

According to Lions' concentration compactness lemma (see for example [72, Lemma 1.21]) we have that $u_n \to 0$ and $v_n \to 0$ in $L^p(\mathbb{R}^N)$ for $\forall p \in (2, 2^*)$. So, we infer from $\mathscr{J}'(u_n, v_n)(u_n, v_n) = 0$ that

$$\|u_n\|^2 + \|v_n\|_\lambda^2 = \int_{\mathbb{R}^N} (\mu_1 |u_n|^3 + \mu_2 |v_n|^3 + \frac{3\beta}{2} u_n^2 v_n) \to 0,\ n \to \infty. \tag{2.12}$$

Hence, one sees that

$$0 < \delta \leq C = \frac{1}{6}(\|u_n\|^2 + \|v_n\|_\lambda^2) + o(1) \to 0,\ n \to \infty. \tag{2.13}$$

This is a contradiction. Thus, (2.10) holds.

Set $\tilde{u}_n = u_n(x + y_n)$ and $\tilde{v}_n = v_n(x + y_n)$. Due to the invariance by translations, we can assume that $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}_0, \tilde{v}_0)$ in X^2, $(\tilde{u}_n, \tilde{v}_n) \to (\tilde{u}_0, \tilde{v}_0)$ in $[L^p_{loc}(\mathbb{R}^N)]^2$ for $p \in (2, 2^*)$ as $n \to \infty$. Moreover, it follows from (2.10) that

$$\liminf_{n\to\infty} \int_{B_R(0)} (\tilde{u}_n^2 + \tilde{v}_n^2) \geq \sigma > 0. \tag{2.14}$$

So we have $\tilde{u}_0 \neq 0$ or $\tilde{v}_0 \neq 0$. Hence, we infer from (2.9) that $\mathscr{J}'(\tilde{u}_0, \tilde{v}_0) = 0$ and $\tilde{z}_0 = (\tilde{u}_0, \tilde{v}_0) \in \mathscr{N}$. Furthermore, one deduces from the weak lower semicontinuity of the norms that

$$\begin{aligned} C &\leq \mathscr{J}(\tilde{u}_0, \tilde{v}_0) = \frac{1}{6}(\|\tilde{u}_0\|^2 + \|\tilde{v}_0\|_\lambda^2) \leq \liminf_{n\to\infty} \frac{1}{6}(\|\tilde{u}_n\|^2 + \|\tilde{v}_n\|_\lambda^2) \\ &= \liminf_{n\to\infty} \mathscr{J}(\tilde{u}_n, \tilde{v}_n) = \liminf_{n\to\infty} \mathscr{J}(u_n, v_n) = C. \end{aligned} \tag{2.15}$$

Therefore, $\tilde{z}_0 = (\tilde{u}_0, \tilde{v}_0) \neq (0, 0)$ is a ground state solution of (1.7). The proof for C_r is similar. □

3. Existence Results for the Two-Wave System

In Sect. 2, we have found a ground state solution $z = (u, v) \neq (0, 0)$ of (1.7). In this section we show that under some additional conditions, (1.7) has a nontrivial ground state solution, i.e., $u \not\equiv 0$ and $v \not\equiv 0$. We consider the following two cases separately (i) Existence for $1 \leq N \leq 5$ and positive μ_2; (ii) Existence for $N = 1$ and possibly negative μ_2.

3.1. Existence Results for $1 \leq N \leq 5$ and Positive μ_2

From Lemmas 2.2 and 2.3, we have found a ground state solution $z = (u, v) \neq (0, 0)$ of (1.7). So, we only need to exclude the case that $z = (u, v) = (0, v)$. To accomplish this, we show that for the ground state z and under (A_1) or (A_2), we have

$$C = \mathscr{J}(z) < \mathscr{J}(0, w_{\lambda,\mu_2}) = \frac{1}{6\mu_2^2} \lambda^{3-\frac{N}{2}} S_1^3, \tag{3.1}$$

where $w_{\lambda,\mu_2} = \dfrac{\lambda}{\mu_2} w_0(\sqrt{\lambda} x)$ is given in Lemma 2.1. We first prove that when (A_1) is satisfied, C is attained by a nontrivial solution.

Lemma 3.1. *If* (A_1) *and* $2 \leq N \leq 5$ *are satisfied, then the infimum* $C > 0$ $(C_r > 0)$ *is attained by a nontrivial (radial) solution of* (1.7)*. Moreover, if* $\beta > 0$*, then* $C = C_r > 0$ *is attained by a nontrivial positive radial solution of* (1.7)*.*

Proof. We only prove that $C > 0$ is attained by a nontrivial solution of (1.7). The conclusion that $C > 0$ has been proved in (2.5), and from Lemma 2.3, C is attained by some $(u_0, v_0) \in \mathscr{N}$. To prove that $u_0 \neq 0$ and $v_0 \neq 0$, it suffices to check that (3.1) holds. That is, we only need to show that

$$C = \mathscr{J}(u_0, u_0) < \mathscr{J}(0, w_{\lambda,\mu_2}) = \frac{\mu_2}{6}\int_{\mathbb{R}^N} w_{\lambda,\mu_2}^3 = \frac{\lambda^{3-\frac{N}{2}}}{6\mu_2^2} S_1^3. \tag{3.2}$$

Since $(w_{1,\mu_1}, 0) \in \mathscr{N} \cap E_r$, it follows from $\lambda > \left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}$ that

$$C = \mathscr{J}(u_0, v_0) \leq C_r \leq \mathscr{J}(w_{1,\mu_1}, 0) = \frac{\mu_1}{6}\int_{\mathbb{R}^N} w_{1,\mu_1}^3 = \frac{1}{6\mu_1^2} S_1^3 < \frac{\lambda^{3-\frac{N}{2}}}{6\mu_2^2} S_1^3. \tag{3.3}$$

Thus, (3.1) holds and $C > 0$ is attained by a nontrivial radial solution of (1.7).

Now we assume that $\beta > 0$. Let (u_0, v_0) be the nontrivial ground state solution of (1.7) obtained above such that $u_0 \neq 0$ and $v_0 \neq 0$. It is easy to check that there exists a unique $t_0 > 0$ such that $(t_0|u_0|, t_0|v_0|) \in \mathscr{N}$. That is,

$$\begin{aligned}\int_{\mathbb{R}^N}(|\nabla u_0|^2 + |u_0|^2) + \int_{\mathbb{R}^N}(|\nabla v_0|^2 + \lambda|v_0|^2) = t_0\int_{\mathbb{R}^N}(\mu_1|u_0|^3 + \mu_2|v_0|^3)\\ + t_0\frac{3\beta}{2}\int_{\mathbb{R}^N} u_0^2|v_0|.\end{aligned} \tag{3.4}$$

We deduce from (3.4) and $(u_0, v_0) \in \mathscr{N}$ that

$$\begin{aligned} t_0 &= \frac{\int_{\mathbb{R}^N}(|\nabla u_0|^2 + u_0^2) + \int_{\mathbb{R}^N}(|\nabla v_0|^2 + \lambda v_0^2)}{\frac{3\beta}{2}\int_{\mathbb{R}^N} u_0^2|v_0| + \int_{\mathbb{R}^N}(\mu_1|u_0|^3 + \mu_2|v_0|^3)}\\ &\leq \frac{\int_{\mathbb{R}^N}(|\nabla u_0|^2 + u_0^2) + \int_{\mathbb{R}^N}(|\nabla v_0|^2 + \lambda v_0^2)}{\frac{3\beta}{2}\int_{\mathbb{R}^N} u_0^2 v_0 + \int_{\mathbb{R}^N}(\mu_1|u_0|^3 + \mu_2|v_0|^3)} = 1. \end{aligned} \tag{3.5}$$

So one sees from (2.4) and (3.5) that

$$\begin{aligned} C &\leq \mathscr{J}(t_0|u_0|, t_0|v_0|) = \frac{t_0^2}{6}\int_{\mathbb{R}^N}(|\nabla u_0|^2 + u_0^2 + |\nabla v_0|^2 + \lambda v_0^2)\\ &\leq \frac{1}{6}\int_{\mathbb{R}^N}(|\nabla u_0|^2 + u_0^2 + |\nabla v_0|^2 + \lambda v_0^2) = \mathscr{J}(u_0, v_0) = C. \end{aligned} \tag{3.6}$$

Thus, $(u_1, v_1) = (t_0|u_0|, t_0|v_0|) \in \mathscr{N}$ is a nontrivial nonnegative radial ground state solution of (1.7). Moreover, since $u_1 \neq 0$, $v_1 \neq 0$ and (u_1, v_1) is a solution of (1.7), then the strong maximum principle yields that $u_1 > 0$ and $v_1 > 0$. Hence, when $\beta > 0$, C is attained by a positive ground state (u_1, v_1) of (1.7).

Similarly, one can prove that C_r is also attained by a positive ground state of (1.7)

Now we prove that when $\beta > 0$, C can also be attained by a positive radial ground state of (1.7). Let $\{(u_n, v_n)\} \subset \mathcal{N}$ be a minimizing sequence such that $\mathcal{J}(u_n, v_n) \to C$ as $n \to \infty$. Since $\beta > 0$, as in the last paragraph we can assume that $u_n \geq 0$ and $v_n \geq 0$. Let u_n^* and v_n^* be the radial functions obtained by Schwarz symmetrization from u_n and v_n. Then from [46, Theorem 3.7] we infer that

$$\begin{aligned} &\|u_n^*\|_1^2 \leq \|u_n\|_1^2,\ \|v_n^*\|_\lambda^2 \leq \|v_n\|_\lambda^2,\ |u_n^*|_3^3 = |u_n|_3^3, \\ &|v_n^*|_3^3 = |v_n|_3^3,\ \int_{\mathbb{R}^N} (u_n^*)^2 v_n^* \geq \int_{\mathbb{R}^N} u_n^2 v_n. \end{aligned} \tag{3.7}$$

Let $t_n > 0$ be such that $(t_n u_n^*, t_n v_n^*) \in \mathcal{N} \cap X_r^2$. As in (3.5), one can check that $t_n \leq 1$. So we infer from (3.7) that $\mathcal{J}(t_n u_n^*, t_n v_n^*) \leq \mathcal{J}(u_n^*, v_n^*) \leq \mathcal{J}(u_n, v_n)$. Now we can proceed as in the radial case to show that $(u_n, v_n) \to (u_0, v_0)$ in X_r^2 as $n \to \infty$, and (u_0, v_0) is a positive radial ground state solution of (1.7) such that $\mathcal{J}(u_0, v_0) = C$. Since u_0 and v_0 are radial symmetry, it follows that $C = C_r$ in this case. □

Next we prove the existence result under the condition (A_2).

Lemma 3.2. *If (A_2) is satisfied, then $C = C_r > 0$ is attained by a nontrivial positive radial solution of (1.7).*

Proof. We use a test function $z_0(x) = (w_0, w_1) \equiv (w_0(x), w_0(\sqrt{\lambda}x))$ to estimate the ground state energy level. It is straightforward to verify that there exists a unique $t_0 > 0$ such that $t_0 z_0 \in \mathcal{N}$, and

$$t_0 = \frac{\|w_0\|_1^2 + \|w_1\|_\lambda^2}{\int_{\mathbb{R}^N} \left(\mu_1 w_0^3 + \mu_2 w_1^3 + \frac{3\beta}{2} w_1 w_0^2\right)} = \frac{1 + \lambda^{1-N/2}}{\mu_1 + \lambda^{-\frac{N}{2}}\mu_2 + \frac{3\beta}{2S_1^3}\int_{\mathbb{R}^N} w_0^2(x) w_0(\sqrt{\lambda}x)}. \tag{3.8}$$

From Lemma 2.3 we know that there exists $z \in \mathcal{N}$ such that $\mathcal{J}(z) = C$. Since $t_0 z_0 \in \mathcal{N}$, it follows that

$$C = \mathcal{J}(z) \leq \mathcal{J}(t_0 z_0) = \frac{t_0^3 S_1^3}{6}\left(\mu_1 + \lambda^{-\frac{N}{2}}\mu_2 + \frac{3\beta}{2S_1^3}\int_{\mathbb{R}^N} w_0^2(x) w_0(\sqrt{\lambda}x)\right)$$

So to guarantee that (3.1) holds to exclude the semi-trivial solution as the global minimizer, it suffices to have

$$\frac{t_0^3 S_1^3}{6}\left(\mu_1 + \lambda^{\frac{-N}{2}}\mu_2 + \frac{3\beta}{2S_1^3}\int_{\mathbb{R}^N} w_0^2(x) w_0(\sqrt{\lambda}x)\right) < \frac{\lambda^{3-\frac{N}{2}} S_1^3}{6\mu_2^2}. \tag{3.9}$$

A simple computation shows that if

$$\beta > \tilde{\beta}_0 = \frac{2S_1^3 \mu_2 \left[\left(\lambda^{\frac{N-6}{6}} + \lambda^{-\frac{N}{3}}\right)^{\frac{3}{2}} - \left(\frac{\mu_1}{\mu_2} + \lambda^{-\frac{N}{2}}\right)\right]}{3\int_{\mathbb{R}^N} w_0^2(x) w_0(\sqrt{\lambda}x)}, \tag{3.10}$$

then (3.9) holds. Moreover, one can check that $\tilde{\beta}_0 > 0$ if $\lambda \leq \left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}$. To obtain the value $\bar{\beta}_0$ in (1.9), we infer from $w_0(r)$ is strictly decreasing in $r > 0$ that

$$\min\left\{1, \lambda^{-\frac{N}{2}}\right\} S_1^3 \leq \int_{\mathbb{R}^N} w_0^2(x) w_0(\sqrt{\lambda}x) \leq \max\left\{1, \lambda^{-\frac{N}{2}}\right\} S_1^3. \tag{3.11}$$

Hence,

$$\begin{aligned} &\frac{2\mu_2\left[\left(\lambda^{\frac{N-6}{6}} + \lambda^{-\frac{N}{3}}\right)^{\frac{3}{2}} - \left(\frac{\mu_1}{\mu_2} + \lambda^{-\frac{N}{2}}\right)\right]}{3\max\left\{1, \lambda^{-\frac{N}{2}}\right\}} \leq \tilde{\beta}_0 \\ &\leq \frac{2\mu_2\left[\left(\lambda^{\frac{N-6}{6}} + \lambda^{-\frac{N}{3}}\right)^{\frac{3}{2}} - \left(\frac{\mu_1}{\mu_2} + \lambda^{-\frac{N}{2}}\right)\right]}{3\min\left\{1, \lambda^{\frac{-N}{2}}\right\}} \equiv \hat{\beta}_0. \end{aligned} \tag{3.12}$$

Hence, when $\beta > \bar{\beta}_0$, (3.9) still holds. We can use test functions $z_1(x) = (w_0(x), w_0(x))$ or $z_2(x) = (w_0(\sqrt{\lambda}x), w_0(\sqrt{\lambda}x))$ to replace z_0 in the above arguments to obtain $\bar{\beta}_1$ and $\bar{\beta}_2$ respectively. Hence when $\beta > \min\{\bar{\beta}_0, \bar{\beta}_1, \bar{\beta}_2\} \equiv \hat{\beta}_0$, the energy minimizer is non-trivial. By using the same argument as in Lemma 3.1, one can show that the minimizer (u, v) is positive and radially symmetric. Since z_0, z_1 and z_2 are all radial, the proof above is also valid for C_r. In particular we have $C = C_r$. □

Remark 3.3. In (1.9), $\hat{\beta}_0$ can be replaced by $\beta_0 = \min\{\tilde{\beta}_0, \bar{\beta}_1, \bar{\beta}_2\}$ and $\hat{\beta}_0 \geq \beta_0$. But $\hat{\beta}_0$ is more explicit.

Finally, we prove the nonexistence of positive solutions of (1.7) when $N \geq 6$. If (u, v) is a positive solution of (1.7), we can use a standard method to deduce the following Pohozaev identity (see [57, Theorem 1]). That is, for each $a \in \mathbb{R}$,

$$\begin{aligned} &\left[\frac{N}{2} - (a+1)\right] \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) + \left(\frac{N}{2} - a\right) \int_{\mathbb{R}^N} (u^2 + \lambda v^2) \\ &\quad + \left(a - \frac{N}{3}\right) \int_{\mathbb{R}^N} (\mu_1 |u|^3 + \mu_2 |v|^3) + \frac{\beta}{2}(3a - N) \int_{\mathbb{R}^N} u^2 v = 0. \end{aligned} \tag{3.13}$$

In particular, we let $a = N/2 - 1$. Then (3.14) reduces to

$$\int_{\mathbb{R}^N} (u^2 + \lambda v^2) = \frac{6-N}{6} \int_{\mathbb{R}^N} (\mu_1 |u|^3 + \mu_2 |v|^3) + \frac{6-N}{4} \beta \int_{\mathbb{R}^N} u^2 v. \tag{3.14}$$

Thus, $u = v = 0$ if $N = 6$ and $\lambda > 0$, or $N > 6$, $\lambda, \mu_1, \mu_2 > 0$ and $\beta > 0$.

Proof of Theorem 1.1 (i)–(ii) *and* (iv). The existence of a radial ground state solution follows from Lemmas 3.1 (A_1) and 3.2 (A_2), and the nonexistence of positive solutions follows from (3.14). This proves (i) and (ii) of Theorem 1.1. Finally, in order to prove the conclusion (iv), one can follow the line of the proof of [42, Proposition 2.2], and we omit the details here. □

3.2. Existence for $N = 1$ and Possibly Negative μ_2

In this subsection, we prove the existence of a positive ground state solution when $N = 1$ and $\mu_2 \leq 0$. We first have the following estimate for the minimizing sequences.

Lemma 3.4. *If* (A_3) *holds, and* $\{(u_n, v_n)\} \subset \mathscr{N}$ *is a minimizing sequence such that* $\mathscr{J}(u_n, v_n) \to C$ *as* $n \to \infty$*, then there exists* $\sigma > 0$ *such that* $\max_{t\in\mathbb{R}} |u_n(t)| \geq \sigma > 0$.

Proof. By the Schwarz symmetrization principle (see Lemma 3.1), we may assume that (u_n, v_n) is nonnegative and radial, and also from Lemma 3.1, we know that $\{(u_n, v_n)\}$ is bounded in X_r^2. Without loss of generality we assume that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in X_r^2, and $(u_n, v_n) \to (u_0, v_0)$ in $[L^\infty_{loc}(\mathbb{R})]^2$. Also it follows from $\mathscr{J}'(u_n, v_n)(u_n, v_n) = 0$ that

$$\|u_n\|_1^2 + \|v_n\|_\lambda^2 = \int_{\mathbb{R}^N} (\mu_1|u_n|^3 + \mu_2|v_n|^3) + \frac{3\beta}{2}\int_{\mathbb{R}} u_n^2 v_n. \tag{3.15}$$

Since $\mu_2 \leq 0$, we infer from the boundedness of $\{(u_n, v_n)\}$, $\mathscr{J}(u_n, v_n) \to C$ and (3.15) that

$$\begin{aligned} 6C + o(1) = \|u_n\|_1^2 + \|v_n\|_\lambda^2 &= \int_{\mathbb{R}^N} (\mu_1|u_n|^3 + \mu_2|v_n|^3) + \frac{3\beta}{2}\int_{\mathbb{R}} u_n^2 v_n \\ &\leq \int_{\mathbb{R}} \mu_1|u_n|^3 + \frac{3\beta}{2}\int_{\mathbb{R}} |u_n|^2|v_n| \\ &\leq \max_{t\in\mathbb{R}} |u_n(t)| \left(\int_{\mathbb{R}} \mu_1 u_n^2 + \frac{3\beta}{2}\int_{\mathbb{R}} |u_n||v_n|\right) \\ &\leq c \max_{t\in\mathbb{R}} |u_n(t)|. \end{aligned} \tag{3.16}$$

This gives the conclusion for n large enough. □

Proof of Theorem 1.1 (iii). Let $\{(u_n, v_n)\} \subset \mathscr{N}$ be a minimizing sequence as in lemma 3.4, then from Lemma 3.4, for each n we can choose t_n such that

$$|u_n(t_n)| \geq \sigma > 0. \tag{3.17}$$

We define

$$(\tilde{u}_n(t), \tilde{v}_n(t)) = (u_n(t + t_n), v_n(t + t_n)). \tag{3.18}$$

Since $\mathscr{J}(\tilde{u}_n, \tilde{v}_n) = \mathscr{J}(u_n, v_n)$ and $\|(\tilde{u}_n, \tilde{v}_n)\|_E = \|(u_n, v_n)\|_E$, it follows that $\{(\tilde{u}_n, \tilde{v}_n)\} \subset \mathscr{N}$ and $\{(\tilde{u}_n, \tilde{v}_n)\}$ is bounded. Without loss of generality we assume that $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}, \tilde{v})$ in X^2, and $(\tilde{u}_n, \tilde{v}_n) \to (\tilde{u}, \tilde{v})$ in $[L^\infty_{loc}(\mathbb{R})]^2$. Then (3.17) implies that

$$|\tilde{u}(0)| \geq \sigma > 0. \tag{3.19}$$

That is, $\tilde{u} \not\equiv 0$. By the Ekeland's variational principle (Lemma 2.3), we can assume that $\mathscr{J}'(\tilde{u}_n, \tilde{v}_n) \to 0$. Hence the weak convergence of $(\tilde{u}_n, \tilde{v}_n)$ implies that $(\tilde{u}, \tilde{v})$ is a weak solution of (1.7). Now we can follow the same argument as in Lemma 3.1 to obtain desired result. □

4. Properties of Solutions of Two-Wave System

4.1. Bifurcation and Continuation of Positive Solutions

In this subsection we prove the existence of nontrivial solutions of (1.7) by using bifurcation theory. As in [43], we consider our problem in X_p^2 and L_r^p. First, in order to show the existence of a principal eigenvalue of (1.14), we need consider the following eigenvalue problem (for a fixed $\beta \geq 0$)

$$-\Delta\phi + \phi - \beta w_{\lambda,\mu_2}\phi = \theta(\beta)\phi, \quad \phi \in X_p^r, \tag{4.1}$$

where $\beta, \lambda, \mu_2 > 0$, and w_{λ,μ_2} is the unique positive solution of (1.13). We define the Rayleigh quotient associated with (4.1):

$$\chi_1(\beta) = \inf_{\phi \in X_p^r \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} \left(|\nabla\phi|^2 + \phi^2 - \beta w_{\lambda,\mu_2}\phi^2\right)}{\displaystyle\int_{\mathbb{R}^N} \phi^2}. \tag{4.2}$$

Now we show the following result on the eigenvalue problems (4.1) and (1.14).

Lemma 4.1. *Suppose that $\lambda, \mu_2 > 0$, then for each $\beta > 0$, (4.1) has a unique principal eigenvalue $\chi_1(\beta)$ (which is defined by (4.2)) with a positive eigenfunction $\phi_{1,\beta}$. Moreover,*

$$\lim_{\beta \to 0^+} \chi_1(\beta) = 1, \quad \lim_{\beta \to \infty} \chi_1(\beta) = -\infty, \quad \chi_1'(\beta) < 0 \ \textit{ for } \ \beta > 0. \tag{4.3}$$

In particular there exists $\beta_1 > 0$ such that $\chi_1(\beta_1) = 0$, hence β_1 is the principal eigenvalue of (1.14).

Proof. First from [10, Section 3, Theorem 3.4], the principal eigenvalue $\chi_1(\beta)$ of (4.1) exists and it is defined by (4.2). It is easy to see that $(\chi_1(\beta), \phi_{1,\beta})$ is differentiable with respect to β. Differentiating (4.1) in β, we obtain that

$$-\Delta\phi' + \phi' - \beta w_{\lambda,\mu_2}\phi' - w_{\lambda,\mu_2}\phi = \chi_1'(\beta)\phi + \chi_1(\beta)\phi', \quad \phi \in X_p^r, \tag{4.4}$$

where $\phi' = \dfrac{\partial\phi}{\partial\beta}$. Multiplying (4.1) by ϕ', multiplying (4.4) by ϕ, subtracting and integrating, we obtain that

$$\chi_1'(\beta)\int_{\mathbb{R}^N} \phi^2 = -\int_{\mathbb{R}^N} w_{\lambda,\mu_2}\phi^2, \tag{4.5}$$

which implies that $\chi_1'(\beta) < 0$. Since $w_{\lambda,\mu_2}(x)$ is bounded for $x \in \mathbb{R}^N$, then from (4.2), we have

$$\int_{\mathbb{R}^N} \left(|\nabla\phi|^2 + \phi^2 - \beta w_{\lambda,\mu_2}\phi^2\right) \geq (1 - \beta||w_{\lambda,\mu_2}||_\infty)\int_{\mathbb{R}^N} \phi^2,$$

for $\phi \in X_p^r$. Therefore, $\chi_1(\beta) \geq 1 - \beta||w_{\lambda,\mu_2}||_\infty$, which implies that $\lim\limits_{\beta \to 0^+} \chi_1(\beta) \geq 1$. On the other hand, for each $R > 0$, let (λ_R, ϕ_R) be the principal eigen-pair of the following eigenvalue problem

$$\begin{cases} -\Delta\phi_R(y) = \lambda_R\phi_R(y), & \text{in } B_R(0), \\ \phi_R(0) = 0, & \text{on } \partial B_R(0), \end{cases} \tag{4.6}$$

satisfying $\displaystyle\int_{B_R(0)} \phi_R^2 = 1$. Then we know that $\lambda_R = \lambda_1 R^{-2}$. We extend ϕ_R to be zero outside of $B_R(0)$, and use it as a test function for (4.2), then we obtain that

$$\chi_1(\beta) \le 1 + R^{-2} - \beta \int_{B_R(0)} w_{\lambda,\mu_2}\phi_R^2, \tag{4.7}$$

for all $\beta > 0$ and $R > 0$. Hence, $\chi_1(\beta) \le 1$ for all $\beta > 0$ and in particular $\lim_{\beta\to 0^+} \chi_1(\beta) = 1$. On the other hand, fix an $R > 0$, then $w_{\lambda,\mu_2}(x) \ge \delta > 0$ for $|x| \le R$. Hence from (4.7), we have $\chi_1(\beta) \le 1 + R^{-2} - \beta\delta$ for $\beta > 0$, which implies that $\lim_{\beta\to\infty} \chi_1(\beta) = -\infty$. The existence of a unique β_1 such that $\theta(\beta_1) = 0$ follows immediately from (4.3). □

Now we are ready to give the proof of the conclusion (i) of Theorem 1.4.

Proof of Theorem 1.4 (i). Set $\mathcal{S}_* = \{(\beta, u, v) = (\beta, 0, w_{\lambda,\mu_2}) : \beta > 0\}$, where $w_{\lambda,\mu_2}(x) = \lambda\mu_2^{-1} w_0(\sqrt{\lambda}x)$ is the unique positive solution of (1.13), and β_1 is given in Lemma 4.1. We shall consider the bifurcation of nontrivial solutions of (1.7) from the semitrivial branch $\mathcal{S}_*$ near $(\beta_1, 0, w_{\lambda,\mu_2})$. To accomplish this we apply the bifurcation results of Crandall and Rabinowitz [27]. First, we define $F : \mathbb{R} \times (X_p^r)^2 \to (L_r^p)^2$ by

$$F(\beta, u, v) = \begin{pmatrix} \Delta u - u + \mu_1 u^2 + \beta uv \\ \Delta v - \lambda v + \mu_2 v^2 + \frac{\beta}{2} u^2 \end{pmatrix}. \tag{4.8}$$

Clearly, for $(\phi, \psi), (\phi_1, \psi_1), (\phi_2, \psi_2) \in (X_p^r)^2$, one sees that

$$\begin{aligned} &F_{(u,v)}(\beta, u, v)[(\phi, \psi)] = \begin{pmatrix} \Delta\phi - \phi + 2\mu_1 u\phi + \beta u\psi + \beta v\phi \\ \Delta\psi - \lambda\psi + 2\mu_2 v\psi + \beta u\phi \end{pmatrix}, \\ &F_{(u,v)(u,v)}(\beta, u, v)[(\phi_1, \psi_1)(\phi_2, \psi_2)] = \begin{pmatrix} 2\mu_1\phi_1\phi_2 + \beta\phi_2\psi_1 + \beta\phi_1\psi_2 \\ 2\mu_2\psi_1\psi_2 + \beta\phi_1\phi_2 \end{pmatrix}, \\ &F_\beta(\beta, u, v) = \begin{pmatrix} uv \\ \frac{1}{2}u^2 \end{pmatrix}, \quad \text{and} \quad F_{\beta(u,v)}(\beta, u, v)[(\phi, \psi)] = \begin{pmatrix} u\psi + v\phi \\ u\phi \end{pmatrix}. \end{aligned} \tag{4.9}$$

We define

$$\mathcal{L}_0(\phi, \psi) = F_{(u,v)}(\beta_1, 0, w_{\lambda,\mu_2})(\phi, \psi) = \begin{pmatrix} \Delta\phi - \phi + \beta_1 w_{\lambda,\mu_2}\phi \\ \Delta\psi - \lambda\psi + 2\mu_2 w_{\lambda,\mu_2}\psi \end{pmatrix} \equiv \begin{pmatrix} \mathcal{L}_1(\phi) \\ \mathcal{L}_2(\psi) \end{pmatrix}. \tag{4.10}$$

From Lemma 4.1, the null space $N(\mathcal{L}_1) = span\{\phi_{1,\beta_1}\}$. From [47, Lemma 2.1] (also see [44,54]), the solution space of $\mathcal{L}_2(\psi) = 0$ in X is $N_1 \equiv span\{\partial w_{\lambda,\mu_2}/\partial x_j : 1 \le j \le N\}$. Hence null space $N(\mathcal{L}_2) = N_1 \cap X_p^r = \{0\}$. So the null space $N(\mathcal{L}_0) = span\{(\phi_{1,\beta_1}, 0)\}$, and ϕ_{1,β_1} is the principal eigenfunction of (4.1). The range space of $\mathcal{L}_0$ is defined by

$$R(\mathcal{L}_0) = \left\{ (f, g) \in Y_r^2 : \int_{\mathbb{R}^N} f\phi_{1,\beta_1} = 0 \right\}. \tag{4.11}$$

Thus, $\dim N(\mathcal{L}_0)=\operatorname{codim} R(\mathcal{L}_0)=1$. Since $\int_{\mathbb{R}^N} w_{\lambda,\mu_2}\phi_{1,\beta_1}^2 > 0$, it follows from (4.11) that

$$F_{\beta(u,v)}(\beta_1, 0, w_{\lambda,\mu_2})(\phi_{1,\beta_1}, 0) = \begin{pmatrix} \phi_{1,\beta_1} w_{\lambda,\mu_2} \\ 0 \end{pmatrix} \notin R(\mathcal{L}_0). \tag{4.12}$$

Thus, we can apply the result of [27] to conclude that the set of positive solutions to (1.7) near $(\beta_1, 0, w_{\lambda,\mu_2})$ is a smooth curve

$$\Gamma = \{(\beta(s), u_{1\beta}(s), v_{1\beta}(s)) : s \in (0, \tilde{\tau}_0)\}, \tag{4.13}$$

such that $\beta(s)=\beta_1+\beta'(0)s+o(s)$, $u_{1\beta}(s) = s\phi_{1,\beta_1}+o(s)$ and $v_{1\beta}(s) = w_{\lambda,\mu_2}+o(s)$, where $\tilde{\tau}_0 > 0$ is a small constant. Moreover, $\beta'(0)$ can be calculated as (see for example [37,61])

$$\begin{aligned} \beta'(0) &= -\frac{\langle F_{(u,v)(u,v)}(\beta_1, 0, w_{\lambda,\mu_2})[(\phi_{1,\beta_1}, 0), (\phi_{1,\beta_1}, 0)], \ell\rangle}{2\langle F_{\beta(u,v)}(\beta_1, 0, w_{\lambda,\mu_2})[(\phi_{1,\beta_1}, 0)], \ell\rangle} \\ &= -\frac{\mu_1 \int_{\mathbb{R}^N} \phi_{1,\beta_1}^3}{\int_{\mathbb{R}^N} w_{\lambda,\mu_2}\phi_{1,\beta_1}^2} < 0, \end{aligned} \tag{4.14}$$

where ℓ is a linear functional on $(L_r^p)^2$ defined as $\langle (f,g), \ell\rangle = \int_{\mathbb{R}^N} f\phi_{1,\beta_1}$. Hence, we infer from (4.13)–(4.14) that for $\beta_1 - \tau_0 < \beta < \beta_1$,

$$\begin{aligned} u_{1\beta} &= \frac{\beta-\beta_1}{\beta'(0)}\phi_{1,\beta_1} + o(\beta-\beta_1) = \frac{\beta_1-\beta}{\mu_1}\frac{\int_{\mathbb{R}^N} \phi_{1,\beta_1}^2 w_{\lambda,\mu_2}}{\int_{\mathbb{R}^N} \phi_{1,\beta_1}^3}\phi_{1,\beta_1} + o(\beta-\beta_1), \\ v_{1\beta} &= w_{\lambda,\mu_2} + o(\beta-\beta_1). \end{aligned} \tag{4.15}$$

Furthermore, by using the same argument as in [43, Theorem 5.1], one can deduce that $(u_{1\beta}, v_{1\beta})$ is positive solution. In the next we can also show that $(u_{1\beta}, v_{1\beta})$ is not a ground state solution. In fact, under the condition (A_1) or (A_2), we have shown (see (3.1)) that the ground state energy satisfies (3.1). On the other hand, when $\tau_0 > 0$ is sufficiently small, we infer from (4.15) that for $\beta \in (\beta_1 - \tau_0, \beta_1)$,

$$\begin{aligned} \mathscr{J}(u_{1\beta}, v_{1\beta}) &= \frac{1}{6\mu_2^2}\lambda^{3-\frac{N}{2}}S_1^3 + o(s) + \frac{1}{2}\|u_{1\beta}\|^2 - \frac{1}{3}\int_{\mathbb{R}^N} u_{1\beta}^3 - \frac{\beta}{2}\int_{\mathbb{R}^N} u_{1\beta}^2 v_{1\beta} \\ &= \frac{1}{6\mu_2^2}\lambda^{3-\frac{N}{2}}S_1^3 - \frac{(\beta-\beta_1)^3}{6\mu_1^2}\frac{\left(\int_{\mathbb{R}^N} \phi_{1,\beta_1} w_{\lambda,\mu_2}\right)^3}{(\int_{\mathbb{R}^N} \phi_{1,\beta_1}^3)^2} + o(s) \\ &= \frac{1}{6\mu_2^2}\lambda^{3-\frac{N}{2}}S_1^3 + o(s). \end{aligned} \tag{4.16}$$

We infer from (3.1) that there exists $\epsilon_0 > 0$ small such that $C < \frac{1}{6\mu_2^2}\lambda^{3-\frac{N}{2}}S_1^3 - \epsilon_0 = \mathscr{J}(0, w_{\lambda,\mu_2}) - \epsilon_0$. Hence, for $\tau_0 > 0$ small and $\beta \in (\beta_1 - \tau_0, \beta_1)$,

$\mathscr{J}(u_{1\beta}, v_{1\beta}) > \mathscr{J}(0, w_{\lambda,\mu_2}) - \epsilon_0 > C$. So when the condition (A_1) or (A_2) is satisfied, $(u_{1\beta}, v_{1\beta})$ is not a ground state solution.

Finally we use the "principle of exchange of stability" [28, Corollary 1.13 and Theorem 1.16] to calculate the Morse index for the nonnegative solutions $(u_{1\beta}, v_{1\beta})$ and $(0, w_{\lambda,\mu_2})$. For that purpose we consider the eigenvalue problem

$$\mathcal{L}_\beta \begin{pmatrix} \phi \\ \psi \end{pmatrix} = F_{(u,v)}(\beta, 0, w_{\lambda,\mu_2}) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \Delta\phi - \phi + \beta w_{\lambda,\mu_2}\phi \\ \Delta\psi - \lambda\psi + 2\mu_2 w_{\lambda,\mu_2}\psi \end{pmatrix} = \gamma(\beta) \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{4.17}$$

where $\gamma(\beta) : (\beta_1 - \tau_0, \beta_1 + \tau_0) \to \mathbb{R}$ is the simple eigenvalue of $\mathcal{L}_\beta$ satisfying $\gamma(\beta_1) = 0$. Notice that the eigenvalues of the problem (4.17) are given by $\sigma_p(\mathcal{L}_\beta) = \sigma_p(\mathcal{L}_1) \cup \sigma_p(\mathcal{L}_2)$, where $\sigma_p(L)$ denotes the eigenvalues of a linear operator L, and $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined as in (4.10) with β_1 replaced by β. Apparently $\gamma(\beta)$ is determined by $\mathcal{L}_1$, hence $\gamma(\beta) = -\chi_1(\beta)$ defined in Lemma 4.1. Thus $\gamma'(\beta) > 0$ from (4.5).

Now consider the eigenvalue problem at the bifurcating solution $(u_{1\beta}, v_{1\beta})$:

$$\begin{aligned} &F_{(u,v)}(\beta, u_{1\beta}(s), v_{1\beta}(s)) \begin{pmatrix} \phi \\ \psi \end{pmatrix} \\ &\quad= \begin{pmatrix} \Delta\phi - \phi + 2\mu_1 u_{1\beta}(s)\phi + \beta(s)u_{1\beta}(s)\psi + \beta v_{1\beta}(s)\phi \\ \Delta\psi - \lambda\psi + 2\mu_2 v_{1\beta}(s)\psi + \beta u_{1\beta}(s)\phi \end{pmatrix} = \xi(s) \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \end{aligned} \tag{4.18}$$

Then from [28, Theorem 1.16], we have

$$\lim_{s\to 0,\ \xi(s)\neq 0} \frac{-s\beta'(s)\gamma'(\beta_1)}{\xi(s)} = 1. \tag{4.19}$$

From (4.14) we have $\beta'(0) < 0$, hence we infer from (4.19) that $\xi(s) > 0$ for $s \in (0, \tau_1)$.

According to [7, Theorem 4.4], $\mathcal{L}_2$ has exactly one positive eigenvalue for $\beta > 0$ (note that $\mathcal{L}_2$ is independent of β). Therefore $\mathcal{L}_\beta$ has exactly one positive eigenvalue when $0 < \beta < \beta_1$, and has exactly two positive eigenvalues when $\beta_1 < \beta < \beta_1 + \tau_0$ for small $\tau_0 > 0$. From $\xi(s) > 0$ and the continuity of eigenvalues, we know that the eigenvalue problem (4.18) has two positive eigenvalues when $s \in (0, \tilde{\tau}_0)$. From the definition of Morse index, we know that $M(u_{1\beta}, v_{1\beta}) = 2$ for $\beta \in (\beta_1 - \tau_0, \beta_1)$, and $M(0, w_{\lambda,\mu_2}) = 1$ for $0 < \beta < \beta_1$ and $M(0, w_{\lambda,\mu_2}) = 2$ for $\beta_1 < \beta < \beta_1 + \tau_0$. □

Next we are ready to prove the conclusion (ii) of Theorem 1.4.

Proof of Theorem 1.4 (ii). Notice that

$$z_0(x) = (w_{1,\mu_1}(x), w_{\lambda,\mu_2}(x)) \equiv (\mu_1^{-1} w_0(x), \lambda\mu_2^{-1} w_0(\sqrt{\lambda}x)) \tag{4.20}$$

is the unique positive solution of (1.7) with $\beta = 0$. Recall the mapping defined in (4.8), and we have

$$F_{(u,v)}(0, w_{1,\mu_1}, w_{\lambda,\mu_2})[(\phi, \psi)] = \begin{pmatrix} \Delta\phi - \phi + 2\mu_1 w_{1,\mu_1}\phi \\ \Delta\psi - \lambda\psi + 2\mu_2 w_{\lambda,\mu_2}\psi \end{pmatrix}. \tag{4.21}$$

It is well-known that $\mathcal{L}_3 = \Delta - 1 + 2\mu_1 w_{1,\mu_1}$ and $\mathcal{L}_4 = \Delta - \lambda + 2\mu_2 w_{\lambda,\mu_2}$ are both invertible in X_p^r ([7, Theorem 4.4]), hence z_0 is non-degenerate in X_p^r, *i.e.*, $\left[F_{(u,v)}(0,z_0)\right]^{-1}$ exists. By the implicit function theorem, there exist $\widetilde{\beta}_0 > 0, R_0 > 0$ and $z_2(\beta) : (-\widetilde{\beta}_0, \widetilde{\beta}_0) \to B_{R_0}(z_0)$ such that for any $\beta \in (-\widetilde{\beta}_0, \widetilde{\beta}_0)$, $F(\beta, z_2(\beta)) = F(\beta, u_2(\beta), v_2(\beta)) = 0$.

Moreover, we can solve (ϕ, ψ) from

$$F_{(u,v)}(0, w_{1,\mu_1}, w_{\lambda,\mu_2}) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \Delta\phi - \phi + 2\mu_1 w_{1,\mu_1}\phi \\ \Delta\psi - \lambda\psi + 2\mu_2 w_{\lambda,\mu_2}\psi \end{pmatrix} = -\begin{pmatrix} w_{1,\mu_1} w_{\lambda,\mu_2} \\ \frac{1}{2} w_{1,\mu_1}^2 \end{pmatrix}, \tag{4.22}$$

to obtain that

$$\phi = (-\Delta + 1 - 2\mu_1 w_{1,\mu_1})^{-1}(w_{1,\mu_1} w_{\lambda,\mu_2}), \quad \psi = \frac{1}{2}(-\Delta + \lambda - 2\mu_2 w_{\lambda,\mu_2})^{-1}(w_{1,\mu_1}^2). \tag{4.23}$$

This gives the expression of $(u_{2\beta}, v_{2\beta})$ in (1.15). A similar argument using implicit function theorem at the semitrivial solution $\tilde{z}_0 = (w_{1,\mu_1}, 0)$ at $\beta = 0$ as above, one can obtain the existence of another positive solution $(u_{3\beta}, v_{3\beta})$ for small $\beta > 0$ as in (1.15).

Since $(u_{2\beta}, v_{2\beta})$ and $(u_{3\beta}, v_{3\beta})$ are both obtained from the implicit function theorem, then their stability are same as the ones of unperturbed solutions $(w_{1,\mu_1}, w_{\lambda,\mu_2})$ and $(w_{1,\mu_1}, 0)$ respectively. Again from [7, Theorem 4.4], each of $\mathcal{L}_3$ and $\mathcal{L}_4$ has exactly one positive eigenvalue, thus $M(u_{2\beta}, v_{2\beta}) = 2$ for $\beta \in (0, \tau_1)$, where $\tau_1 > 0$ small. Similarly $M(u_{3\beta}, v_{3\beta}) = 1$ for $\beta \in (0, \tau_1)$. From Theorem 1.1 part (iv) and the implicit function theorem, $(u_{2\beta}, v_{2\beta})$ and $(u_{3\beta}, v_{3\beta})$ are the only positive solutions of (1.7). From Lemma 3.1, the ground state solution can always be chosen as positive when $\beta > 0$, hence $(u_{3\beta}, v_{3\beta})$ must be a ground state solution, and $(u_{2\beta}, v_{2\beta})$ is not a ground state. If $\beta \in (-\tau_1, 0)$, $(u_{3\beta}, v_{3\beta})$ is a opposite sign ground state solution, i.e., $u_{3\beta} > 0$ and $v_{3\beta} < 0$, as if there is a nontrivial solution other than $(u_{2\beta}, v_{2\beta})$ and $(u_{3\beta}, v_{3\beta})$, its energy necessarily goes to infinity as $\beta \to 0^-$. □

4.2. Uniqueness of Positive Solutions

In this subsection we prove the uniqueness of positive solution of (1.7) stated in the conclusion (iii) of Theorem 1.4.

Proof of Theorem 1.4 (iii). If $\lambda = 1$ and $\beta = \mu_2$, we first look for positive synchronized solutions of (1.7) of the form (xw_0, yw_0), where $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$. It is easy to calculate that if $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$ satisfies

$$1 = \mu_1 x + \mu_2 y, \quad 2y = 2\mu_2 y^2 + \mu_2 x^2, \tag{4.24}$$

then (xw_0, yw_0) is solution of (1.7). A simple computation shows that (4.24) has a unique positive solution $\widehat{z}_0 = (u_0, v_0) \equiv (x_0 w_0, y_0 w_0)$ if and only if

$$(x_0, y_0) = \left(\frac{2\mu_1}{2\mu_1^2 + \mu_2^2}, \frac{\mu_2}{2\mu_1^2 + \mu_2^2} \right). \tag{4.25}$$

Next we shall prove that $\widehat{z}_0$ is a unique positive solution of (1.7). Let $k_0 = \frac{2\mu_1}{\beta}$ and $\hat{u}_0 = k_0^{-1} u_0$. Then $(v_0, \hat{u}_0)$ satisfies

$$\begin{cases} -\Delta v_0 + v_0 = \mu_2 v_0^2 + \frac{\beta}{2} k_0^2 \hat{u}_0^2, & \text{in } \mathbb{R}^N \\ -\Delta \hat{u}_0 + \hat{u}_0 = k_0 \mu_1 \hat{u}_0^2 + \beta \hat{u}_0 v_0, & \text{in } \mathbb{R}^N. \end{cases} \tag{4.26}$$

If $N = 1$, by using the same arguments as in proof of [71, Theorem 1.1], we know that $v_0 = \hat{u}_0$ holds. For the case $2 \leq N \leq 5$, we shall use the idea of the proof of [71, Theorem 4.2] (also see Appendix II of [12]) to prove the result. Let $\Omega_1 = \{x \in \mathbb{R}^N : v_0(x) > \hat{u}_0(x)\}$. Thus, Ω_1 is a piecewise C^1 smooth domain. Multiplying the first equation in (4.26) by $\hat{u}_0$ and the second equation in (4.26) by v_0 and then integrating by parts on Ω_1 and subtracting together, we obtain that for $\beta = \mu_2$,

$$\begin{aligned} &\int_{\partial\Omega_1} (\hat{u}_0 \frac{\partial v_0}{\partial n} - v_0 \frac{\partial \hat{u}_0}{\partial n}) + (\mu_2 - \beta) \int_{\Omega_1} \hat{u}_0^2 v_0^2 + \frac{2\mu_1^2}{\beta} \int_{\Omega_1} \hat{u}_0^2 (\hat{u}_0 - v_0) \\ &\quad = \int_{\partial\Omega_1} (\hat{u}_0 \frac{\partial v_0}{\partial n} - v_0 \frac{\partial \hat{u}_0}{\partial n}) + \frac{2\mu_1^2}{\beta} \int_{\Omega_1} \hat{u}_0^2 (\hat{u}_0 - v_0) = 0, \end{aligned} \tag{4.27}$$

where n denotes the unit outward normal to $\partial\Omega_1$. Since $v_0(x) - \hat{u}_0(x) > 0$ in Ω_1 and $v_0(x) - \hat{u}_0(x) = 0$ in $\partial\Omega_1$, it follows that

$$\begin{aligned} \int_{\partial\Omega_1} \left(\hat{u}_0 \frac{\partial v_0}{\partial n} - v_0 \frac{\partial \hat{u}_0}{\partial n} \right) &= \int_{\partial\Omega_1} (\hat{u}_0 - v_0) \frac{\partial v_0}{\partial n} + \int_{\partial\Omega_1} v_0 \frac{\partial (v_0 - \hat{u}_0)}{\partial n} \\ &= \int_{\partial\Omega_1} v_0 \frac{\partial (v_0 - \hat{u}_0)}{\partial n} \leq 0. \end{aligned} \tag{4.28}$$

Moreover, one sees that

$$\frac{2\mu_1^2}{\beta} \int_{\Omega_1} \hat{u}_0^2 (\hat{u}_0 - v_0) \leq 0. \tag{4.29}$$

So (4.27)–(4.29) imply that $\Omega_1 = \emptyset$. Similarly, we set $\Omega_2 = \{x \in \mathbb{R}^N : v_0(x) < \hat{u}_0(x)\}$, and one can check that $\Omega_2 = \emptyset$. So we have $v_0(x) = \hat{u}_0(x)$ in $\mathbb{R}^N$ which shows the uniqueness of positive solution of (1.7). □

4.3. Asymptotical Behavior of Positive Solutions

In this subsection we study the asymptotical behavior of positive ground state solutions of (1.7). For the convenience of notations, we use C^β, C_r^β, $\mathscr{J}_\beta$ and $\mathscr{N}_\beta$ (or C^λ, C_r^λ, $\mathscr{J}_\lambda$ and $\mathscr{N}_\lambda$) instead of C, C_r, $\mathscr{J}$ and $\mathscr{N}$ (see (2.3)–(2.6)) to emphasis the dependence on β (or λ).

To prove Theorem 1.7, we first prove some properties of C^β and C^λ.

Lemma 4.2. *For $\lambda, \beta, \mu_1, \mu_2 > 0$, the following results hold.*

(1) *C^β is non-increasing in $\beta > 0$, and $\lim_{\beta \to \beta_*} C^\beta = C^{\beta_*}$, where $\beta_* \geq 0$.*

(2) *C^λ is non-decreasing in $\lambda > 0$, and $\lim_{\lambda \to (\lambda_*)^+} C^\lambda = C^{\lambda_*}$, where $\lambda_* > 0$.*

Proof. Since the proof of (1) and (2) are similar, we only give the proof of (1) here. We first prove that C^β is a non-increasing function on β. Indeed, for $\beta_1 \geq \beta_2 > 0$, we let (u_1, v_1) and (u_2, v_2) be the positive ground state solutions corresponding to $\beta = \beta_1$ and $\beta = \beta_2$, respectively. Then there exists a unique $t_1 > 0$ such that $t_1(u_2, v_2) \in \mathscr{N}^{\beta_1}$. So, one sees that

$$t_1 = \frac{\|u_2\|^2 + \|v_2\|_\lambda^2}{\int_{\mathbb{R}^N}(\mu_1|u_2|^3 + \mu_2|v_2|^3 + \frac{3\beta_1}{2}u_2^2 v_2)} \leq \frac{\|u_2\|^2 + \|v_2\|_\lambda^2}{\int_{\mathbb{R}^N}(\mu_1|u_2|^3 + \mu_2|v_2|^3 + \frac{3\beta_2}{2}u_2^2 v_2)} = 1.$$

Hence we obtain that

$$\begin{aligned}\mathscr{I}_{\beta_1}(u_1, v_1) \leq \mathscr{I}_{\beta_1}(t_1 u_2, t_1 v_2) &= \frac{t_1^2}{6}(\|u_2\|^2 + \|v_2\|_\lambda^2) \leq \frac{1}{6}(\|u_2\|^2 + \|v_2\|_\lambda^2)\\ &= \mathscr{I}_{\beta_2}(u_2, v_2).\end{aligned}$$

Thus, C^β is a non-increasing function on $\beta > 0$.

Let $\{\beta_n\}$ be a sequence satisfying $\beta_n > 0$ and $\beta_n \to \beta_* \geq 0$ as $n \to \infty$, and let $(u_{\beta_n}, v_{\beta_n})$ be a positive radial ground state solution of (1.7) with $\beta = \beta_n$. We use $(u_n, v_n) = (u_{\beta_n}, v_{\beta_n})$ for simplicity of notation. One infers from (3.2) that

$$\mathscr{I}_{\beta_n}(u_n, v_n) = \frac{1}{6}(\|u_n\|_1^2 + \|v_n\|_\lambda^2) = C^{\beta_n} \leq C_r^{\beta_n} \leq \frac{1}{6\mu_1^2} S_1^2. \tag{4.30}$$

Thus, $\{(u_n, v_n)\}$ is bounded in X_r^2. Without loss of generality we assume that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in X_r^2, and $(u_n, v_n) \to (u_0, v_0)$ in $[L_{loc}^p(\mathbb{R}^N)]^2$ for $p \in (2, 2^*)$. Moreover, $u_0, v_0 \geq 0$ in $\mathbb{R}^N$, and (u_0, v_0) is a solution of (1.7) with $\beta = \beta_*$. Similar to (2.5), one can prove that $\|u_n\|_1^2 + \|v_n\|_\lambda^2 \geq \delta > 0$. On the other hand, by using the same arguments as in Lemma 2.3, we can prove that $\{(u_n, v_n)\}$ is nonvanishing, i.e., (2.10) holds. Since the system (1.7) is invariant under the translation $u_n(\cdot) \mapsto u_n(\cdot + y_n)$, we can assume that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in X_r^2, $(u_0, v_0) \neq (0, 0)$ and $(u_0, v_0) \in \mathscr{N}_{\beta_*}$. It follows from Fatou's lemma that

$$\begin{aligned}C^{\beta_*} &\leq \mathscr{I}_{\beta_*}(u_0, v_0) = \frac{1}{6}\int_{\mathbb{R}^N}(\mu_1|u_0|^3 + \mu_2|v_0|^3 + \frac{3\beta_*}{2}u_0^2 v_0)\\ &\leq \frac{1}{6}\liminf_{n\to\infty}\int_{\mathbb{R}^N}(\mu_1|u_n|^3 + \mu_2|v_n|^3 + \frac{3\beta_n}{2}u_n^2 v_n)\\ &= \lim_{n\to\infty}\mathscr{I}_{\beta_n}(u_n, v_n) = \lim_{n\to\infty} C^{\beta_n}.\end{aligned} \tag{4.31}$$

Moreover, by the definition of C^{β_n}, we know that

$$\begin{aligned}C^{\beta_n} &\leq \max_{t\geq 0}\mathscr{I}_{\beta_n}(tu_0, tv_0)\\ &= \max_{t\geq 0}\left\{\frac{t^2}{2}(\|u_0\|^2 + \|v_0\|_\lambda^2) - \frac{t^3}{6}\int_{\mathbb{R}^N}(\mu_1|u_0|^3 + \mu_2|v_0|^3 + \frac{3\beta_n}{2}u_0^2 v_0)\right\}\\ &= \frac{2}{3}\frac{\left(\|u_0\|^2 + \|v_0\|_\lambda^2\right)^3}{\left(\int_{\mathbb{R}^N}(\mu_1|u_0|^3 + \mu_2|v_0|^3 + \frac{3\beta_n}{2}u_0^2 v_0)\right)^2}\end{aligned}$$

$$= \frac{2}{3} \frac{\left(\|u_0\|^2 + \|v_0\|_\lambda^2\right)^3}{\left(\int_{\mathbb{R}^N} (\mu_1 |u_0|^3 + \mu_2 |v_0|^3 + \frac{3\beta_*}{2} u_0^2 v_0)\right)^2} + o(1)$$

$$= \mathscr{I}_{\beta_*}(u_0, v_0) + o(1) = C^{\beta_*} + o(1). \tag{4.32}$$

That is,

$$\limsup_{n\to\infty} C^{\beta_n} \leq C^{\beta_*}. \tag{4.33}$$

Combining (4.31) and (4.33) we infer that $\lim_{\beta\to\beta_*} C^\beta = C^{\beta_*}$. □

Now we are ready to give the proof of Theorem 1.7.

Proof of Theorem 1.7. (i) We first consider the case of $\beta_* = 0$. We take $\beta_n > \beta_*$ as in Lemma 4.2. Thus, we know that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in X_r^2, $(u_0, v_0) \in \mathscr{N}_{\beta_*} \setminus \{(0,0)\}$ is a nonnegative solution of (1.7), where $(u_n, v_n) = (u_{\beta_n}, v_{\beta_n})$.

First we consider the case $N \geq 2$. We show that $u_0 \neq 0$. Assume, on the contrary, that $u_0 = 0$ and $v_0 \neq 0$. Then from Lemma 2.1 we know that $v_0 = w_{\lambda,\mu_2}$ is the unique positive solution of (1.12) with $\mu = \mu_2$. So we conclude that

$$\begin{aligned}
\frac{\lambda^{3-\frac{N}{2}}}{6\mu_2^2} S_1^3 &= \mathscr{I}_0(0, w_{\lambda,\mu_2}) = \mathscr{I}_0(u_0, v_0) = \frac{1}{6} \int_{\mathbb{R}^N} \left(\mu_1 |u_0|^3 + \mu_2 |v_0|^3\right) \\
&\leq \frac{1}{6} \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left(\mu_1 |u_n|^3 + \mu_2 |v_n|^3 + \frac{3\beta_n}{2} u_n^2 v_n\right) \\
&= \lim_{n\to\infty} \mathscr{I}_{\beta_n}(u_n, v_n) = \lim_{n\to\infty} C^{\beta_n} \leq \frac{1}{6\mu_1^2} S_1^3,
\end{aligned} \tag{4.34}$$

which contradicts $\lambda > \left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}$. Hence $u_0 \neq 0$. Furthermore, if $v_0 \neq 0$, by similar arguments as in (4.34) we can obtain a contradiction. Thus, we have that $u_0 \neq 0$ and $v_0 = 0$. On the other hand, since

$$\|u_n\|_1^2 = \mu_1 \int_{\mathbb{R}^N} |u_n|^3 + \beta_n \int_{\mathbb{R}^N} u_n^2 v_n, \tag{4.35}$$

it follows that

$$\|u_0\|_1^2 \leq \liminf_{n\to\infty} \|u_n\|_1^2 = \liminf_{n\to\infty} \left(\mu_1 \int_{\mathbb{R}^N} |u_n|^3 + \beta_n \int_{\mathbb{R}^N} u_n^2 v_n\right) = \mu_1 \int_{\mathbb{R}^N} |u_0|^3. \tag{4.36}$$

We infer from u_0 is a solution of the first equation of (1.7) that $\|u_0\|_1^2 = \mu_1 \int_{\mathbb{R}^N} |u_0|^3$ and $\lim_{n\to\infty} \|u_n\|_1^2 = \|u_0\|_1^2$. Similarly, we can prove $\lim_{n\to\infty} \|v_n\|_\lambda^2 = 0$. Thus, from Brezis-Lieb Lemma (see [72]) we infer that $(u_n, v_n) \to (u_0, 0)$ in E as $n \to \infty$.

Next we study the case of $N = 1$ and $\beta_* = 0$. As in the case $N \geq 2$, one can prove that $u_0 = w_{1,\mu_1}$ and $v_0 = 0$. We only need to check that

$(u_n, v_n) \to (u_0, 0)$ in X_r^2. By using the same arguments as in (4.34), one deduces that

$$\begin{aligned}\frac{1}{6\mu_1^2} S_1^3 &= \mathscr{I}_0(w_{1,\mu_1}) = \mathscr{I}_0(u_0, v_0) = \frac{\mu_1}{6}\int_{\mathbb{R}} |u_0|^3 \\ &\le \frac{1}{6}\liminf_{n\to\infty}\int_{\mathbb{R}}\left(\mu_1|u_n|^3 + \mu_2|v_n|^3 + \frac{3\beta_n}{2}u_n^2 v_n\right) \\ &= \lim_{n\to\infty}\mathscr{I}_{\beta_n}(u_n, v_n) = \lim_{n\to\infty} C^{\beta_n} \le \frac{1}{6\mu_1^2} S_1^3.\end{aligned} \tag{4.37}$$

Thus, it follows that $(u_n, v_n) \to (u_0, 0)$ in $L^3(\mathbb{R}) \times L^3(\mathbb{R})$. So, by using the same arguments as in (4.35) and (4.36), one deduces that $(u_n, v_n) \to (u_0, 0)$ in X_r^2 as $n \to \infty$.

At last we prove the case of $\beta^* > 0$ which is part (2) of (i). We take β_n the same as in Lemma 4.2. Hence, we know that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in X_r^2, $(u_0, v_0) \in \mathscr{N}_{\beta_*} \setminus \{(0, 0)\}$ is a nonnegative solution of (1.6), where $(u_n, v_n) = (u_{\beta_n}, v_{\beta_n})$. We claim that $u_0 \ne 0$ and $v_0 \ne 0$. Since (u_0, v_0) is a nonnegative solution of (1.6), it follows that $(u_0, v_0) = (0, v_0)$ or $u_0 \ne 0$ and $v_0 \ne 0$. Assume, on the contrary, that $u_0 = 0$ and $v_0 \ne 0$. It is easy to see that $v_0 = w_{\lambda,\mu_2}$, where w_{λ,μ_2} is the unique positive solution of (1.12) with $(\lambda, \mu) = (\lambda, \mu_2)$. By using similar arguments as in (4.34), one obtains the contradiction. Hence, we know that $u_0 \ne 0$ and $v_0 \ne 0$. One infers from Lemma 4.2 that

$$\begin{aligned}C^{\hat\beta_0} &\le \mathscr{I}_{\hat\beta_0}(u_0, v_0) = \frac{1}{6}(\|u_0\|_1^2 + \|v_0\|_\lambda^2) \le \lim_{n\to\infty}\mathscr{I}_{\beta_n}(u_n, v_n) \\ &= \frac{1}{6}\lim_{n\to\infty}(\|u_n\|_1^2 + \|v_n\|_\lambda^2) = \lim_{n\to\infty} C^{\beta_n} \le C^{\hat\beta_0}.\end{aligned}$$

So $(u_n, v_n) \to (u_0, v_0)$ in X_r^2 as $n \to \infty$, and (u_0, v_0) is a positive ground state solution of (1.7) with $\beta = \hat\beta_0$.

(ii) Let $(u_n, v_n) = (u_{\lambda_n}, v_{\lambda_n})$ be any radial positive ground state solution of (1.7) with $\lambda = \lambda_n$, and $\lambda_n \to \infty$, as $n \to \infty$. We first consider the case of $N \ge 2$. As in (4.30) we know that

$$\mathscr{I}_{\lambda_n}(u_n, v_n) = \frac{1}{6}(\|u_n\|_1^2 + \|v_n\|_{\lambda_n}^2) = C^{\lambda_n} \le \frac{1}{6\mu_1^2} S_1^3. \tag{4.38}$$

Set $\tilde v_n = \sqrt{\lambda_n} v_n$. It follows from $\lambda_n \to \infty$ that $\|u_n\|, \|v_n\|$ and $|\tilde v_n|_2$ are bounded. Without loss of generality we assume that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in X_r^2, $(u_n, v_n) \to (u_0, v_0)$ in $L_r^p(\mathbb{R}^N) \times L_r^p(\mathbb{R}^N)$ $(\forall p \in (2, 2^*))$, and $\tilde v_n \rightharpoonup \tilde v_0$ in $L^2(\mathbb{R}^N)$. We first claim $v_0 \equiv 0$. Indeed, we infer from (u_n, v_n) satisfies the second equation of (1.7) that

$$\int_{\mathbb{R}^N} v_0^2 = \lim_{n\to\infty}\int_{\mathbb{R}^N} v_n^2 = \lim_{n\to\infty}\lambda_n^{-1}\left[\int_{\mathbb{R}^N}\left(\mu_2|v_n|^3 + \frac{\beta}{2}u_n^2 v_n\right) - \int_{\mathbb{R}^N}|\nabla v_n|^2\right] = 0. \tag{4.39}$$

Thus $v_0 \equiv 0$. Moreover, since (u_n, v_n) satisfies the first equation of (1.7), we have that for each $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left[\int_{\mathbb{R}^N} (\nabla u_n \nabla \varphi + u_n \varphi) - \int_{\mathbb{R}^N} \left(\mu_1 u_n^2 \varphi + \beta v_n u_n \varphi \right) \right] \\ &= \int_{\mathbb{R}^N} (\nabla u_0 \nabla \varphi + u_0 \varphi - \mu_1 u_0^2 \varphi). \end{aligned} \tag{4.40}$$

These, together with Lemma 2.1 we know that $u_0 \neq 0$ is a solution of (1.12) with $(\lambda, \mu) = (1, \mu_1)$.

Next we shall prove that $\|u_n - u_0\|, \|v_n\|, |\tilde{v}_n|_2 \to 0$ as $n \to \infty$. We only prove $|\tilde{v}_n|_2 \to 0$ as $n \to \infty$. As in (4.37) we know that

$$\begin{aligned} \frac{1}{6}\left(\|u_0\|^2 + |\tilde{v}_0|_2^2\right) &\le \liminf_{n\to\infty} \frac{1}{6}\left(\|u_n\|^2 + |\tilde{v}_n|_2^2)\right) \le \liminf_{n\to\infty} \frac{1}{6}\left(\|u_n\|^2 + \|v_n\|_{\lambda_n}^2\right) \\ &= C_r^{\lambda_n} \le \frac{1}{6\mu_1^2} S_1^3 \le \frac{1}{6} \frac{(\|u_0\|_1^2)^3}{(\int_{\mathbb{R}^N} \mu_1 u_0^3)^2} \le \frac{1}{6}\|u_0\|^2. \end{aligned} \tag{4.41}$$

Thus, we obtain that $\tilde{v}_0 = 0$, and $|\tilde{v}_n|_2 \to 0$, as $n \to \infty$. Finally, we infer from $(u_n, v_n) \to (u_0, v_0)$ in $L_r^p(\mathbb{R}^N) \times L_r^p(\mathbb{R}^N)$ $(\forall p \in (2, 6))$ that for each $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \lambda_n v_n \varphi = \lim_{n\to\infty} \left[\int_{\mathbb{R}^N} (\mu_2 v_n^2 \varphi + \frac{\beta}{2} u_n^2 \varphi) - \int_{\mathbb{R}^N} (\nabla v_n \nabla \varphi) \right] = \frac{\beta}{2} \int_{\mathbb{R}^N} u_0^2 \varphi. \tag{4.42}$$

Next we consider the case of $N = 1$. Comparing to the proof of the case of $N \ge 2$, the only difference is that the embedding $H_r^1(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$ is not compact. But since the embedding $H_r^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ is continuous, and we can use this to establish (4.40). In fact, since $\|v_n\|$ is bounded, it follows that there exists a subsequence $\{v_n\}$ such that $v_n \rightharpoonup v_0$ in $H_r^1(\mathbb{R})$. Moreover, we infer from the continuous embedding that v_n^2 is also bounded in $H_r^1(\mathbb{R})$. So, we can assume that $v_n^2 \rightharpoonup v$ in $H_r^1(\mathbb{R})$. Thus by local compactness of Sobolev imbedding we have $v = v_0^2$. This fact also holds for the sequence $\{u_n\}$. So one sees that for each $\varphi \in C_0^\infty(\mathbb{R})$, (4.40) also holds. The remaining part of the proof is the same as that for the case of $N \ge 2$.

(iii) We first consider the case $\lambda_* = 0$. Let $(u_n, v_n) = (u_{\lambda_n}, v_{\lambda_n})$ be any radial positive ground state solution of (1.7), where $\lambda = \lambda_n$, $\mu_2 = 0$ and $\lambda_n \to 0$ as $n \to \infty$. As in (4.30), we know that

$$\frac{1}{6}\left(\|u_n\|_1^2 + \int_{\mathbb{R}^N} |\nabla v_n|^2 \right) \le \mathscr{I}_{\lambda_n}(u_n, v_n) = \frac{1}{6}\left(\|u_n\|_1^2 + \|v_n\|_{\lambda_n}^2\right) \le \frac{1}{6\mu_1^2} S_1^2. \tag{4.43}$$

So, (u_n, v_n) is bounded in $H_r^1(\mathbb{R}^N) \times D_r^{1,2}(\mathbb{R}^N)$. Without loss of generality we assume that $u_n \rightharpoonup u_0$ in $H_r^1(\mathbb{R}^N)$, $v_n \rightharpoonup v_0$ in $D_r^{1,2}(\mathbb{R}^N)$. Moreover, it follows from (4.43) and Hölder inequality that for each $\psi \in C_0^\infty(\mathbb{R}^N)$,

$$\lambda_n \int_{\mathbb{R}^N} v_n^2 \le \frac{1}{6\mu_1^2} S_1^2, \; \int_{\mathbb{R}^N} \lambda_n v_n \psi \le \sqrt{\lambda_n} \left(\int_{\mathbb{R}^N} \lambda_n v_n^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \psi^2 \right)^{\frac{1}{2}}. \tag{4.44}$$

Thus, we obtain that $\int_{\mathbb{R}^N} \lambda_n v_n \psi \to 0$ for each $\psi \in C_0^\infty(\mathbb{R}^N)$ as $n \to \infty$. This implies that for each $(\varphi, \psi) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$,

$$\lim_{n\to\infty} \left[\int_{\mathbb{R}^N} (\nabla u_n \nabla \varphi + u_n \varphi + \nabla v_n \nabla \psi + \lambda_n v_n \psi) - (\mu_1 u_n^2 \varphi + \frac{\beta}{2} u_n^2 \psi + \beta v_n u_n \varphi) \right]$$
$$= \int_{\mathbb{R}^N} (\nabla u_0 \nabla \varphi + u_0 \varphi + \nabla v_0 \nabla \psi) - (\mu_1 u_0^2 \varphi + \frac{\beta}{2} u_0^2 \psi + \beta v_0 u_0 \varphi) = 0. \qquad (4.45)$$

That is, (u_0, v_0) is a solution of (1.7) with $\lambda = 0$ and $\mu_2 = 0$. By using the same argument as the one in Lemma 2.3, we can prove that $\{(u_n, v_n)\}$ is nonvanishing, i.e., (2.10) holds. Since the system (1.7) is invariant under the translation $u_n(\cdot) \mapsto u_n(\cdot + y_n)$, we can assume that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in X_r^2, $(u_0, v_0) \neq (0, 0)$ and $(u_0, v_0) \in \mathcal{N}_0$, where $\mathcal{N}_\lambda|_{\lambda=0}$. Thus $u_0 \neq 0$ and $v_0 \neq 0$ is a solution of (1.7) with $(\lambda, \mu_2) = (0, 0)$.

Finally, by using the arguments of (4.35)–(4.36) we have that $(u_n, v_n) \to (u_0, v_0)$ in $H_r^1(\mathbb{R}^N) \times D_r^{1,2}(\mathbb{R}^N)$. The proof of the conclusion for $\lambda_n \to \lambda_* > 0$ is almost the same as the one for $\lambda_* = 0$, we omit the details. □

5. Liouville Type Results for the Two-Wave System

In this part we mainly focus on the proof of Theorem 1.9. To accomplish this we shall apply the general results of [33, 58]. Precisely, we shall apply the results [58, Theorems 4 and 6] to prove the conclusions in Theorem 1.9 (i). Base on this, we can use [33, Theorems 1.2 and 1.3] to get the results in Theorem 1.9 (ii) and (iii).

We first present an elementary algebraic result which will be used in verifying the conditions of [58, Theorems 4 and 6].

Lemma 5.1. *Suppose that $\mu_1, \mu_2 > 0$. Let $\beta_\sharp = \left(2\mu_1^2 \mu_2\right)^{\frac{1}{3}}$. Then when $\beta > -\beta_\sharp$,*

(1) *for any $u, v \geq 0$, $h(u, v) = \mu_1 u^3 + \mu_2 v^3 + \frac{3\beta}{2} u^2 v \geq 0$, and $h(u, v) = 0$ if and only if $u = v = 0$.*
(2) *there exist positive α_0, α_1 and α_2 such that $\alpha_1(\mu_1 u^2 + \beta uv) + \alpha_2(\mu_2 v^2 + \frac{\beta}{2} u^2) \geq \alpha_0(\alpha_1 u + \alpha_2 v)^2$ for any $u, v \geq 0$.*
(3) *There exists some constant $\sigma > 0$ such that*

$$2N \left(\frac{\mu_1}{3} u^3 + \frac{\mu_2}{3} v^3 + \frac{\beta}{2} u^2 v \right) - (N-2) \left(\mu_1 u^3 + \mu_2 v^3 + 2\beta u^2 v \right) \geq \sigma(u^3 + v^3)$$

for $1 \leq N \leq 4$.

Proof. (1) It is clear that $h(u, 0) = \mu_1 u^3 \geq 0$. We prove $h(u, v) \geq 0$ for $v > 0$. Set $t = u/v \geq 0$ and define the function

$$k(t) = \mu_1 t^3 + \frac{3\beta}{2} t^2 + \mu_2, \quad t \geq 0.$$

It suffices to show that $\min_{t\geq 0} k(t) \geq 0$. By using some elementary calculations, we know that $\min_{t \geq 0} k(t) = k\left(-\frac{\beta}{\mu_1}\right) = \frac{\beta^3}{2\mu_1^2} + \mu_2$. Thus when

$\beta > -\beta_\sharp$, we have $\min_{t\geq 0} k(t) > 0$. The second part of the conclusion (1) is easy to verify.

(2) We set $\alpha_1 = 1$ and $\alpha_2 = \left(\frac{\mu_1}{2\mu_2}\right)^{1/3}$. It is clear true when $u = 0$. Similar to (1), we set $t = v/u$ and consider the function

$$f(t) = \frac{(\mu_1 + \beta t) + \alpha_2(\mu_2 t^2 + \frac{\beta}{2})}{(1 + \alpha_2 t)^2}, \quad t \geq 0. \tag{5.1}$$

Then $f(0) = \mu_1 + \frac{\beta\alpha_2}{2} > 0$ as $\beta > -\beta_\sharp$ and $\lim_{t\to\infty} f(t) = \frac{\mu_2}{\alpha_2} > 0$. Moreover, for any $t > 0$, using $\beta > -\beta_\sharp$, we have

$$\begin{aligned}(1 + \alpha_2 t)^2 f(t) > g(t) &:= \mu_1 - \beta_\sharp t + \alpha_2 \left(\mu_2 t^2 + \frac{-\beta_\sharp}{2}\right) \\ &\geq \frac{\mu_1}{2\alpha_2^2}(t - \alpha_2)^2 = \beta_\sharp (t - \alpha_2)^2 \geq 0.\end{aligned} \tag{5.2}$$

Hence $\alpha_0 = \min_{t\geq 0} f(t) > 0$ and the conclusion of part (2) of this lemma holds.

(3) A direct computation shows that

$$\begin{aligned}&2N\left(\frac{\mu_1}{3}u^3 + \frac{\mu_2}{3}v^3 + \frac{\beta}{2}u^2 v\right) - (N-2)\left(\mu_1 u^3 + \mu_2 v^3 + 2\beta u^2 v\right) \\ &= \frac{6-N}{3}\left[\mu_1 u^3 + \mu_2 v^3 + \frac{3}{2}\beta u^2 v\right]\end{aligned}$$

Since $\beta > \beta_\sharp = \left(2\mu_1^2\mu_2\right)^{\frac{1}{3}}$, one deduces that there $\sigma > 0$ such that the conclusion (3) holds. □

Now we are ready to give the proof of (i)–(iii) of Theorem 1.9.

Proof of Theorem 1.9. From Lemma 5.1, we verify that the conditions of [58, Theorems 4 and 6] are satisfied. Thus, we infer from [58, Theorems 4 and 6] that the results in part (i) hold. On the other hand, from this we deduce from [33, Theorems 1.2 and 1.3] that the results of Theorem (ii) and (iii) hold. □

6. Existence Results for the Three-Wave System

In this section we prove the results for the three-wave system (1.5). In the following we always assume that $\lambda_1 = 1$, and $\lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3 > 0$. First we point out that (1.5) has three semi-trivial solutions of the form $(u, 0, 0)$, $(0, v, 0)$ and $(0, 0, w)$, where $u, v, w \neq 0$. These are the only possible semitrivial solutions. Here we are interested in the non-trivial positive solution (u, v, w), where $u, v, w > 0$.

We define the following Nehari manifold

$$\begin{aligned}\tilde{\mathscr{N}} = \Big\{ z = (u,v,w) \in X^3 \setminus \{(0,0,0)\} :\ & \|u\|^2 + \|v\|^2_{\lambda_2} + \|w\|^2_{\lambda_3} \\ &= 3\beta \int_{\mathbb{R}^N} uvw + \int_{\mathbb{R}^N} (\mu_1|u|^3 + \mu_2|v|^3 + \mu_3|w|^3) \Big\},\end{aligned} \tag{6.1}$$

and we also define

$$\tilde{C} = \inf_{(u,v,w)\in\tilde{\mathscr{N}}} \tilde{\mathscr{J}}(u,v,w), \quad \tilde{C}_r = \inf_{(u,v,w)\in\tilde{\mathscr{N}}\cap X_r^3} \tilde{\mathscr{J}}(u,v,w). \tag{6.2}$$

where $\tilde{\mathscr{J}}$ is defined in (1.21). From the definition of $\tilde{\mathscr{N}}$, we know that for $(u,v,w) \in \tilde{\mathscr{N}}$,

$$\begin{aligned}\tilde{\mathscr{J}}|_{\tilde{\mathscr{N}}}(u,v,w) &= \frac{1}{6}(\|u\|^2 + \|v\|^2_{\lambda_2} + \|w\|^2_{\lambda_3}) \\ &= \frac{1}{6}\int_{\mathbb{R}^N} (\mu_1|u|^3 + \mu_2|v|^3 + \mu_3|w|^3 + 3\beta uvw).\end{aligned} \tag{6.3}$$

As in (2.4) and (2.5), one can check that $\tilde{\mathscr{J}}|_{\tilde{\mathscr{N}}}$ is bounded from below away from zero on $\tilde{\mathscr{N}}$.

Similar to the two-wave system, we can prove the following basic results corresponding to Lemmas 2.2 and 2.3.

Lemma 6.1. *Let $\tilde{C}$ and $\tilde{C}_r$ be defined as in* (6.2).

1. *If $\tilde{C}$ or $\tilde{C}_r$ is attained by some $z \in \tilde{\mathscr{N}}$, then z is a solution of* (1.5).
2. *Assume that $1 \le N \le 5$, $\lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3 > 0$ and $\beta \in \mathbb{R}$. Then $\tilde{C} > 0$ (or $\tilde{C}_r > 0$) is attained by some $z \in \tilde{\mathscr{N}}$ (or $\tilde{\mathscr{N}} \cap X_r^3$).*

Next, to prove the existence of nontrivial solutions for (1.5), we exclude the possibility of $\tilde{C}$ or $\tilde{C}_r$ is achieved by one of the semi-trivial solutions: $(u,0,0)$, $(0,v,0)$ and $(0,0,w)$. First, if the condition (B_1) holds, one has the following result.

Lemma 6.2. *Suppose that $1 \le N \le 5$, and (B_1) holds. Then the infimum $\tilde{C}_r$ and $\tilde{C} > 0$ are attained by a nontrivial solution of* (1.5). *Furthermore, we have $\tilde{C} = \tilde{C}_r$.*

Proof. We first prove that $\tilde{C} > 0$ is attained by a nontrivial solution of (1.5). From Lemma 6.1 we know that $\tilde{C} > 0$ is attained by some $z = (\tilde{u}_0, \tilde{v}_0, \tilde{w}_0) \in \tilde{\mathscr{N}}$. So, to exclude the possibility of $z = (\tilde{u}_0, 0, 0)$, $(0, \tilde{v}_0, 0)$ or $(0, 0, \tilde{w}_0)$, we only need to show that

$$\tilde{C} = \tilde{\mathscr{J}}(\tilde{u}_0, \tilde{v}_0, \tilde{w}_0) \le \tilde{C}_r < \min\Big\{ \tilde{\mathscr{J}}(w_{1,\mu_1}, 0, 0), \tilde{\mathscr{J}}(0, w_{\lambda_2,\mu_2}, 0), \tilde{\mathscr{J}}(0, 0, w_{\lambda_3,\mu_3}) \Big\}, \tag{6.4}$$

where w_{λ_i,μ_i} is defined in (1.13). We infer from $\lambda_2 \geq \left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}$ and $\lambda_3 \geq \left(\frac{\mu_3}{\mu_1}\right)^{\frac{4}{6-N}}$ that

$$\begin{aligned}\tilde{\mathscr{J}}(w_{1,\mu_1},0,0) = \frac{S_1^3}{6\mu_1^2} &\leq \min\left\{\frac{S_1^3\lambda_2^{3-\frac{N}{2}}}{6\mu_2^2}, \frac{S_1^3\lambda_3^{3-\frac{N}{2}}}{6\mu_3^2}\right\} \\ &= \min\left\{\tilde{\mathscr{J}}(0,w_{\lambda_2,\mu_2},0), \tilde{\mathscr{J}}(0,0,w_{\lambda_3,\mu_3})\right\}.\end{aligned} \tag{6.5}$$

So it suffices to show that

$$\tilde{C} = \tilde{\mathscr{J}}(\tilde{u}_0,\tilde{v}_0,\tilde{w}_0) \leq \tilde{C}_r < \frac{S_1^3}{6\mu_1^2}. \tag{6.6}$$

For $\tilde{z}_0 = (w_0, w_1, w_2) = (w_0(x), w_0(\sqrt{\lambda_2}x), w_0(\sqrt{\lambda_3}x))$, there exists a unique $\tilde{t}_0 > 0$ such that $\tilde{t}_0\tilde{z}_0 \in \tilde{\mathscr{N}}$. Moreover, we know that

$$\begin{aligned}\tilde{t}_0 &= \frac{\|w_0\|_1^2 + \|w_1\|_{\lambda_2}^2 + \|w_2\|_{\lambda_3}^2}{\int_{\mathbb{R}^N}(\mu_1 w_0^3 + \mu_2 w_1^3 + \mu_3 w_2^3 + 3\beta w_0 w_1 w_2)} \\ &= \frac{S_1^3\left(1 + \lambda_2^{1-\frac{N}{2}} + \lambda_3^{1-\frac{N}{2}}\right)}{S_1^3\left(\mu_1 + \mu_2\lambda_2^{-\frac{N}{2}} + \mu_3\lambda_2^{-\frac{N}{2}} + \frac{3}{S_1^3}\beta\int_{\mathbb{R}^N} w_0 w_1 w_2\right)}.\end{aligned} \tag{6.7}$$

So it follows from $\tilde{t}_0\tilde{z}_0 \in \tilde{\mathscr{N}}$ that

$$\begin{aligned}\tilde{C} &= \tilde{\mathscr{J}}(u,v,w) \leq \tilde{C}_r \\ &\leq \tilde{\mathscr{J}}(\tilde{t}_0\tilde{z}_0) = \frac{\tilde{t}_0^3}{6}S_1^3\left(\mu_1 + \mu_2\lambda_2^{-\frac{N}{2}} + \mu_3\lambda_2^{-\frac{N}{2}} + \frac{3}{S_1^3}\beta\int_{\mathbb{R}^N} w_0 w_1 w_2\right).\end{aligned} \tag{6.8}$$

If

$$\frac{\tilde{t}_0^3}{6}S_1^3\left(\mu_1 + \mu_2\lambda_2^{\frac{-N}{2}} + \mu_3\lambda_2^{\frac{-N}{2}} + \frac{3}{S_1^3}\beta\int_{\mathbb{R}^N} w_0 w_1 w_2\right) < \frac{S_1^3}{6\mu_1^2}, \tag{6.9}$$

we know that (6.4) holds. Hence, a direct computation shows that if

$$\beta > \tilde{\beta}_1 = \frac{S_1^3\mu_1\left[\left(1 + \lambda_2^{\frac{2-N}{2}} + \lambda_3^{\frac{2-N}{2}}\right)^{\frac{3}{2}} - \left(1 + \frac{\mu_2}{\mu_1}\lambda_2^{-\frac{N}{2}} + \frac{\mu_3}{\mu_1}\lambda_3^{-\frac{N}{2}}\right)\right]}{3\int_{\mathbb{R}^N} w_0 w_0(\sqrt{\lambda_2}x) w_0(\sqrt{\lambda_3}x)}, \tag{6.10}$$

then $\tilde{C}$ is attained by a nontrivial solution of (1.5). We claim that $\hat{\beta}_1 \geq \tilde{\beta}_1$. Indeed, since $w_0(r)$ is strictly decreasing in r, it follows that

$$\begin{aligned}\min\left\{1, \lambda_2^{-\frac{N}{2}}, \lambda_3^{-\frac{N}{2}}\right\} S_1^3 &\leq \int_{\mathbb{R}^N} w_0 w_0(\sqrt{\lambda_2}x) w_0(\sqrt{\lambda_3}x) \\ &\leq \max\left\{1, \lambda_2^{-\frac{N}{2}}, \lambda_3^{-\frac{N}{2}}\right\} S_1^3.\end{aligned}$$

Substituting this into (6.10), we obtain that

$$\frac{\mu_1\left[\left(1+\lambda_2^{\frac{2-N}{2}}+\lambda_3^{\frac{2-N}{2}}\right)^{\frac{3}{2}}-\left(1+\frac{\mu_2}{\mu_1}\lambda_2^{\frac{-N}{2}}+\frac{\mu_3}{\mu_1}\lambda_3^{\frac{-N}{2}}\right)\right]}{3\max\left\{1,\lambda_2^{\frac{-N}{2}},\lambda_3^{\frac{-N}{2}}\right\}}=\tilde{\beta}_2\le\tilde{\beta}_1$$

$$\le\frac{\mu_1\left[\left(1+\lambda_2^{\frac{2-N}{2}}+\lambda_3^{\frac{2-N}{2}}\right)^{\frac{3}{2}}-\left(1+\frac{\mu_2}{\mu_1}\lambda_2^{-\frac{N}{2}}+\frac{\mu_3}{\mu_1}\lambda_3^{-\frac{N}{2}}\right)\right]}{3\min\left\{1,\lambda_2^{-\frac{N}{2}},\lambda_3^{-\frac{N}{2}}\right\}}:=\hat{\beta}_1.$$

Hence, if $\beta>\hat{\beta}_1$, the inequality (6.4) holds, and $\tilde{C}$ is attained by a nontrivial solution of (1.5). Moreover, it follows from $\lambda_2\ge\left(\frac{\mu_2}{\mu_1}\right)^{\frac{4}{6-N}}$ and $\lambda_3\ge\left(\frac{\mu_3}{\mu_1}\right)^{\frac{4}{6-N}}$ that

$$\begin{aligned}&\left(1+\lambda_2^{\frac{2-N}{2}}+\lambda_3^{\frac{2-N}{2}}\right)^{\frac{3}{2}}-\left(1+\frac{\mu_2}{\mu_1}\lambda_2^{-\frac{N}{2}}+\frac{\mu_3}{\mu_1}\lambda_3^{-\frac{N}{2}}\right)\\&\quad\ge\left(1+\lambda_2^{\frac{2-N}{2}}+\lambda_3^{\frac{2-N}{2}}\right)^{\frac{3}{2}}-\left(1+\lambda_2^{\frac{6-3N}{4}}+\lambda_3^{\frac{6-3N}{4}}\right)>0.\end{aligned}$$

Thus $\hat{\beta}_1\ge\tilde{\beta}_1>0$.

Next we show the existence of a positive radial ground state solution of (1.5) for $\beta>\hat{\beta}_1$. Since $\beta>\hat{\beta}_1>0$, by using the same argument as in Lemma 3.1, one can prove that $z=(\tilde{u}_0,\tilde{v}_0,\tilde{w}_0)$ is radial. Moreover, we can prove that $\tilde{C}$ is attained by some positive z_0. Indeed, it is easily to check that there exist a unique $t_0>0$ such that $(t_0|\tilde{u}_0|,t_0|\tilde{v}_0|,t_0|\tilde{w}_0|)\in\tilde{\mathscr{N}}\cap\tilde{E}_r$. It follows that

$$\begin{aligned}&\|(|\tilde{u}_0|)\|_1^2+\|(|\tilde{v}_0|)\|_{\lambda_2}^2+\|(|\tilde{w}_0|)\|_{\lambda_3}^2=\|\tilde{u}_0\|_1^2+\|\tilde{v}_0\|_{\lambda_2}^2+\|\tilde{w}_0\|_{\lambda_3}^2\\&\quad=t_0\int_{\mathbb{R}^N}(\mu_1|\tilde{u}_0|^3+\mu_2|\tilde{v}_0|^3+\mu_3|\tilde{w}_0|^3)+t_0 3\beta\int_{\mathbb{R}^N}|\tilde{u}_0||\tilde{v}_0||\tilde{w}_0|.\end{aligned}\tag{6.11}$$

We deduce from (6.11) and $(\tilde{u}_0,\tilde{v}_0,\tilde{w}_0)\in\tilde{\mathscr{N}}$ that

$$\begin{aligned}t_0&=\frac{\|\tilde{u}_0\|_1^2+\|\tilde{v}_0\|_{\lambda_2}^2+\|\tilde{w}_0\|_{\lambda_3}^2}{\int_{\mathbb{R}^N}(\mu_1|\tilde{u}_0|^3+\mu_2|\tilde{v}_0|^3+\mu_3|\tilde{w}_0|^3)+3\beta\int_{\mathbb{R}^N}|\tilde{u}_0||\tilde{v}_0||\tilde{w}_0|}\\&\le\frac{\|\tilde{u}_0\|_1^2+\|\tilde{v}_0\|_{\lambda_2}^2+\|\tilde{w}_0\|_{\lambda_3}^2}{\int_{\mathbb{R}^N}(\mu_1|\tilde{u}_0|^3+\mu_2|\tilde{v}_0|^3+\mu_3|\tilde{w}_0|^3)+3\beta\int_{\mathbb{R}^N}\tilde{u}_0\tilde{v}_0\tilde{w}_0}=1.\end{aligned}\tag{6.12}$$

So one sees from (6.4) and (6.12) that

$$\begin{aligned}\tilde{C}&\le\mathscr{J}(t_0|\tilde{u}_0|,t_0|\tilde{v}_0|,t_0|\tilde{w}_0|)=\frac{t_0^2}{6}(\|\tilde{u}_0\|_1^2+\|\tilde{v}_0\|_{\lambda_2}^2+\|\tilde{w}_0\|_{\lambda_3}^2)\\&\le\frac{1}{6}(\|\tilde{u}_0\|_1^2+\|\tilde{v}_0\|_{\lambda_2}^2+\|\tilde{w}_0\|_{\lambda_3}^2)=\mathscr{J}(\tilde{u}_0,\tilde{v}_0,\tilde{w}_0)\le\tilde{C}.\end{aligned}\tag{6.13}$$

Thus, $z_0=(z_0^1,z_0^2,z_0^3)=(t_0|\tilde{u}_0|,t_0|\tilde{v}_0|,t_0|\tilde{w}_0|)$ is a nonnegative radial ground state solution of (1.5). From the condition (B_1), one deduces that $\tilde{u}_0\ne0$, $\tilde{v}_0\ne0$ and $\tilde{w}_0\ne0$. So applying the maximum principle to each equation of (1.5) yields that $z_0^i > 0$ for $i = 1, 2, 3$. Similarly, we can prove that $\tilde{C}_r$ is attained by a positive solution of (1.5) and clearly, we have $C = \tilde{C}_r$. □

Remark 6.3.

(1) In the Lemma 6.2, $\hat{\beta}_1$ can be replaced by $\tilde{\beta}_1$ and $\hat{\beta}_1 \geq \tilde{\beta}_1$. But $\hat{\beta}_1$ is more explicit.

(2) It is also possible to give other conditions under which (6.4) holds. For instance, we replace $\tilde{z}_0$ by $\tilde{z}_3 = (w_0, w_0, w_0)$ or $\tilde{z}_1 = (w_0(\sqrt{\lambda_2}x), w_0(\sqrt{\lambda_2}x), w_0(\sqrt{\lambda_2}x))$ or $\tilde{z}_2 = (w_0(\sqrt{\lambda_3}x), w_0(\sqrt{\lambda_3}x),\ w_0(\sqrt{\lambda_3}x))$ in Lemma 6.2. Then a direct computation shows that if

$$\beta > \tilde{\beta}_3 = \frac{1}{3}\left[(3 + (\lambda_2 + \lambda_3 - 2)\sigma_0)^{\frac{3}{2}} \mu_1 - (\mu_1 + \mu_2 + \mu_3)\right] \quad \text{or}$$
$$\beta > \tilde{\beta}_4 = \frac{1}{3}\left[\lambda_2^{\frac{6-N}{4}} \left(3 + (\frac{1+\lambda_3}{\lambda_2} - 2)\sigma_0\right)^{\frac{3}{2}} \mu_1 - (\mu_1 + \mu_2 + \mu_3)\right] \quad \text{or}$$
$$\beta > \tilde{\beta}_5 = \frac{1}{3}\left[\lambda_3^{\frac{6-N}{4}} \left(3 + (\frac{1+\lambda_2}{\lambda_3} - 2)\sigma_0\right)^{\frac{3}{2}} \mu_1 - (\mu_1 + \mu_2 + \mu_3)\right], \tag{6.14}$$

where $\sigma_0 = |w_0|^2_{L^2(\mathbb{R}^N)}/|w_0|^3_{L^3(\mathbb{R}^N)}$, then the conclusion of Lemma 6.2 still holds.

Next we study the existence of a ground state solution for the case that (B_2) or (B_3) holds.

Lemma 6.4. *Assume that $1 \leq N \leq 5$, and either (B_2) or (B_3) holds, then the conclusion of Lemma 6.2 remains true.*

Proof. From Lemma 6.1 we know that $\tilde{C} > 0$ is attained by some $z = (\tilde{u}_0, \tilde{v}_0, \tilde{w}_0) \in \tilde{\mathscr{N}}$. If (B_2) holds, we know that

$$\tilde{\mathscr{J}}(0, 0, w_{\lambda_3,\mu_3}) \leq \tilde{\mathscr{J}}(w_{1,\mu_1}, 0, 0) \leq \tilde{\mathscr{J}}(0, w_{\lambda_2,\mu_2}, 0). \tag{6.15}$$

So, to guarantee (6.4) hold, we only need to show that

$$\tilde{C} = \tilde{\mathscr{J}}(\tilde{u}_0, \tilde{v}_0, \tilde{w}_0) < \tilde{\mathscr{J}}(0, 0, w_{\lambda_3,\mu_3}) = \frac{S_1^3 \lambda_3^{3-\frac{N}{2}}}{6\mu_3^2}. \tag{6.16}$$

As in Lemma 6.2, we use $\tilde{z}_1 = (w_0(\sqrt{\lambda_3}x), w_0(\sqrt{\lambda_3}x), w_0(\sqrt{\lambda_3}x))$ as a test function. Then there exists $\tilde{t}_1 > 0$ such that $\tilde{t}_1\tilde{z}_1 \in \tilde{\mathscr{N}}$ and

$$\tilde{t}_1 = \frac{3\lambda_3 + (1 + \lambda_2 - 2\lambda_3)\sigma_0}{\mu_1 + \mu_2 + \mu_3 + 3\beta}, \tag{6.17}$$

where $\sigma_0 = |w_0|^2_{L^2(\mathbb{R}^N)}/|w_0|^3_{L^3(\mathbb{R}^N)}$. So it suffices to show that

$$\tilde{C} = \tilde{\mathscr{J}}(\tilde{u}_0, \tilde{v}_0, \tilde{w}_0) \leq \tilde{\mathscr{J}}(\tilde{t}_1\tilde{z}_1) < \frac{S_1^3 \lambda_3^{3-\frac{N}{2}}}{6\mu_3^2}. \tag{6.18}$$

A direct computation shows that if $\beta > \hat{\beta}_2$ (defined in (1.22)), then (6.18) holds. The remaining part of the proof is the same as that in Lemma 6.2. If

(B_3) holds, by similar arguments as above one can show that if $\beta > \hat{\beta}_3$ defined in (1.22), then the conclusion of the Lemma 6.2 remains true. □

Finally, we consider the case that (B_4) or (B_5) holds.

Lemma 6.5. *Assume that* $1 \leq N \leq 5$, *and either* (B_4) *or* (B_5) *holds. Then the conclusion of Lemma* 6.2 *remains true.*

Proof. If (B_4) holds, one can check that

$$\tilde{\mathscr{J}}(0, w_{\lambda_2,\mu_2}, 0) < \tilde{\mathscr{J}}(w_{1,\mu_1}, 0, 0) \quad \text{and} \quad \tilde{\mathscr{J}}(0, 0, w_{\lambda_3,\mu_3}) < \tilde{\mathscr{J}}(w_{1,\mu_1}, 0, 0). \tag{6.19}$$

So it suffices to prove that

$$\tilde{C} = \tilde{\mathscr{J}}(\tilde{u}_0, \tilde{v}_0, \tilde{w}_0) < \min\left\{\tilde{\mathscr{J}}(0, w_{\lambda_2,\mu_2}, 0), \tilde{\mathscr{J}}(0, 0, w_{\lambda_3,\mu_3})\right\}. \tag{6.20}$$

By the proof of Lemma 6.2 we know that if $\beta > \max\{\hat{\beta}_2, \hat{\beta}_3\}$, then (6.20) holds. Hence, the conclusion of Lemma 6.2 remains true if (B_4) holds. Finally, if (B_5) holds, as in the proof of Lemmas 6.2 and 6.4, we know that (6.4) holds. Then the conclusion of Lemma 6.2 remains true. □

Remark 6.6. Similar to Remark 6.3, it is possible to find other conditions to guarantee (6.4) holds in Lemmas 6.4 and 6.5. Here we omit details and leave it to the interested readers.

Now we are ready to prove Theorem 1.10.

Proof of Theorem 1.10. The conclusion (i) follows from Lemmas 6.2, 6.4 and 6.5. The conclusion (ii) follows from the arguments of the proof of [42, Proposition 2.2]. For (iii), assume by contradiction, if (u, v, w) is a positive solution of (1.5), then the following Pohozaev identity holds (see [57, Theorem 1]): for any $a \in \mathbb{R}$,

$$\begin{aligned}\left[\frac{N}{2} - (a+1)\right] \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2) + \left(\frac{N}{2} - a\right) \int_{\mathbb{R}^N} (u^2 + \lambda_2 v^2 + \lambda_3 w^2) \\ + \left(a - \frac{N}{3}\right) \int_{\mathbb{R}^N} (\mu_1 |u|^3 + \mu_2 |v|^3 + \mu_3 |w|^3) + \beta(3a - N) \int_{\mathbb{R}^N} uvw = 0.\end{aligned} \tag{6.21}$$

Let $a = \dfrac{N}{2} - 1$ in (6.21). Then one has

$$\begin{aligned}\int_{\mathbb{R}^N} (u^2 + \lambda_2 v^2 + \lambda_3 w^2) + \frac{N-6}{6} \int_{\mathbb{R}^N} (\mu_1 |u|^3 + \mu_2 |v|^3 + \mu_3 |w|^3) \\ + \frac{N-6}{4} \beta \int_{\mathbb{R}^N} uvw = 0.\end{aligned} \tag{6.22}$$

Thus, $u = v = w = 0$ if $N = 6$ and $\lambda_2, \lambda_3 > 0$, or $N > 6$, $\lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3 > 0$ and $\beta > 0$. □

Next we prove the bifurcation result Theorem 1.11 for the three-wave system.

Proof of Theorem 1.11 (i). In the following we only give the proof of the result at the bifurcation point $(\beta, u, v, w) = (\beta_2, w_{1,\mu_1}, 0, 0)$. We denote $u_0 = w_{1,\mu_1}$. First, we consider an eigenvalue problem

$$-\Delta\phi + \frac{\lambda_2 + \lambda_3}{2}\phi - \frac{\sqrt{(\lambda_2 - \lambda_3)^2 + 4\beta^2 u_0^2}}{2}\phi = \theta(\beta)\phi, \quad \phi \in X_p^r. \tag{6.23}$$

Set

$$\chi_2(\beta) = \frac{\lambda_2 + \lambda_3}{2} + \inf_{\phi \in X_p^r \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} 2|\nabla\phi|^2 - \int_{\mathbb{R}^N} \sqrt{(\lambda_2 - \lambda_3)^2 + 4\beta^2 u_0^2}\,\phi^2}{\displaystyle 2\int_{\mathbb{R}^N} \phi^2}. \tag{6.24}$$

As in the proof of Lemma 4.1, one can verify that

$$\lim_{\beta \to 0^+} \chi_2(\beta) = \frac{\lambda_2 + \lambda_3 - |\lambda_2 - \lambda_3|}{2} > 0, \quad \lim_{\beta \to \infty} \chi_2(\beta) = -\infty, \tag{6.25}$$

and $\chi_2'(\beta) < 0$ for $\beta > 0$. Hence we know that there exists $\beta_2 > 0$ such that $\chi_2(\beta_2) = 0$ is the principal eigenvalue of the problem (6.23), and the corresponding positive eigenvalue function is denoted by ϕ_{2,β_2}. Thus as in Lemma 4.1, (1.24) has a principal eigenvalue $\beta_2 > 0$, and the corresponding positive eigenfunction is ϕ_{2,β_2}.

Set $\mathcal{S} = \{(\beta, w_{1,\mu_1}, 0, 0) : \beta > 0\}$. We shall consider the bifurcation of nontrivial solutions of (1.5) from the semi-trivial branch $\mathcal{S}$ near $(\beta_2, w_{1,\mu_1}, 0, 0)$. To accomplish this we also apply the bifurcation result in [27]. We define $G : \mathbb{R} \times \left(X_p^r\right)^3 \to \left(L_p^r\right)^3$ by

$$G(\beta, u, v, w) = \begin{pmatrix} \Delta u - u + \mu_1 u^2 + \beta v w \\ \Delta v - \lambda_2 v + \mu_2 v^2 + \beta u w \\ \Delta w - \lambda_3 w + \mu_3 w^2 + \beta u v \end{pmatrix}. \tag{6.26}$$

For $(\phi_1, \phi_2, \phi_3), (\psi_1, \psi_2, \psi_3) \in \left(X_p^r\right)^3$, one sees that

$$\begin{aligned}
&G_{(u,v,w)}(\beta, u, v, w)[(\phi_1, \phi_2, \phi_3)] \\
&\quad = \begin{pmatrix} \Delta\phi_1 - \phi_1 + 2\mu_1 u\phi_1 + \beta v\phi_3 + \beta w\phi_2 \\ \Delta\phi_2 - \lambda_2\phi_2 + 2\mu_2 v\phi_2 + \beta w\phi_1 + \beta u\phi_3 \\ \Delta\phi_3 - \lambda_3\phi_3 + 2\mu_3 w\phi_3 + \beta v\phi_1 + \beta u\phi_2 \end{pmatrix}, \\
&G_{(u,v,w)(u,v,w)}(\beta, u, v, w)[(\phi_1, \phi_2, \phi_3), (\psi_1, \psi_2, \psi_3)] \\
&\quad = \begin{pmatrix} 2\mu_1\psi_1\phi_1 + \beta\phi_2\psi_3 + \beta\phi_3\psi_2 \\ 2\mu_2\phi_2\psi_2 + \beta\phi_1\psi_3 + \beta\phi_3\psi_1 \\ 2\mu_3\phi_3\psi_3 + \beta\phi_1\psi_2 + \beta\phi_2\psi_1 \end{pmatrix}, \\
&G_\beta(\beta, u, v, w) \\
&\quad = \begin{pmatrix} vw \\ uw \\ uv \end{pmatrix}, \text{ and } G_{\beta(u,v,w)}(\beta, u, v, w)[(\phi_1, \phi_2, \phi_3)] = \begin{pmatrix} v\phi_3 + w\phi_2 \\ u\phi_3 + w\phi_1 \\ u\phi_2 + v\phi_1 \end{pmatrix}.
\end{aligned} \tag{6.27}$$

We define

$$\mathcal{L}_5[(\phi_1,\phi_2,\phi_3)] = G_{(u,v,w)}(\beta_2, w_{1,\mu_1}, 0, 0)[(\phi_1,\phi_2,\phi_3)] = \begin{pmatrix} \Delta\phi_1 - \phi_1 + 2\mu_1 w_{1,\mu_1}\phi_1 \\ \Delta\phi_2 - \lambda_2\phi_2 + \beta_2 w_{1,\mu_1}\phi_3 \\ \Delta\phi_3 - \lambda_3\phi_3 + \beta_2 w_{1,\mu_1}\phi_2 \end{pmatrix}. \tag{6.28}$$

Next we characterize $N(\mathcal{L}_5)$ and $R(\mathcal{L}_5)$. Again from [47, Lemma 2.1], the only solution of $\Delta\phi_1 - \phi_1 + 2\mu_1 w_{1,\mu_1}\phi_1 = 0$ in X_p^r is 0, hence $N(\mathcal{L}_5) = \{(0,\phi_2,\phi_3) : \mathcal{L}_6[(\phi_2,\phi_3)] = (0,0)\}$ where $\mathcal{L}_6$ is defined by

$$\mathcal{L}_6[(\phi_2,\phi_3)] = \begin{pmatrix} \Delta\phi_2 - \lambda_2\phi_2 + \beta_2 u_0\phi_3 \\ \Delta\phi_3 - \lambda_3\phi_3 + \beta_2 u_0\phi_2 \end{pmatrix}, \tag{6.29}$$

where $u_0 = w_{1,\mu_1}$. To solve $\mathcal{L}_6[(\phi_2,\phi_3)] = (0,0)$, we set

$$a_1 = \lambda_3 - \lambda_2 + \sqrt{(\lambda_3-\lambda_2)^2 + 4\beta_2^2 u_0^2},\ a_2 = \lambda_3 - \lambda_2 - \sqrt{(\lambda_3-\lambda_2)^2 + 4\beta_2^2 u_0^2},$$
$$a_3 = a_1^2 + 4\beta_2^2 u_0^2, \quad \text{and} \quad a_4 = a_2^2 + 4\beta_2^2 u_0^2. \tag{6.30}$$

By using an orthonormal transformation,

$$\begin{pmatrix} \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \frac{a_1}{\sqrt{a_3}} & \frac{a_2}{\sqrt{a_4}} \\ \frac{2\beta_2 u_0}{\sqrt{a_3}} & \frac{2\beta_2 u_0}{\sqrt{a_4}} \end{pmatrix} \begin{pmatrix} \Psi_2 \\ \Psi_3 \end{pmatrix} = \begin{pmatrix} \frac{a_1}{\sqrt{a_3}}\Psi_2 + \frac{a_2}{\sqrt{a_4}}\Psi_3 \\ \frac{2\beta_2 u_0}{\sqrt{a_3}}\Psi_2 + \frac{2\beta_2 u_0}{\sqrt{a_4}}\Psi_3 \end{pmatrix}, \tag{6.31}$$

$\mathcal{L}_6[(\phi_2,\phi_3)] = (0,0)$ is transformed into two decoupled equations:

$$\begin{cases} \Delta\Psi_2 - \dfrac{\lambda_2+\lambda_3}{2}\Psi_2 + \dfrac{\sqrt{(\lambda_2-\lambda_3)^2 + 4\beta_2^2 u_0^2}}{2}\Psi_2 = 0, \\ \Delta\Psi_3 - \dfrac{\lambda_2+\lambda_3}{2}\Psi_3 - \dfrac{\sqrt{(\lambda_2-\lambda_3)^2 + 4\beta_2^2 u_0^2}}{2}\Psi_3 = 0. \end{cases} \tag{6.32}$$

Thus, we know that $\Psi_2 = \phi_{2,\beta_2}$ and $\Psi_3 = 0$ from the fact that $\chi_2(\beta_2) = 0$ is the principal eigenvalue of the problem (6.23). Hence the only solution to $\mathcal{L}_6[(\phi_2,\phi_3)] = (0,0)$ is

$$(\phi_2,\phi_3) = \left(\frac{a_1}{\sqrt{a_1^2 + 4\beta_2^2 u_0^2}}\phi_{2,\beta_2}, \frac{2\beta_2 u_0}{\sqrt{a_1^2 + 4\beta_2^2 u_0^2}}\phi_{2,\beta_2} \right). \tag{6.33}$$

Define $\theta_2 : \mathbb{R}^N \to (0,\pi/2)$ by

$$\theta_2(x) = \tan^{-1}\left(\frac{2\beta_2 u_0}{a_1}\right) = \tan^{-1}\left(\frac{2\beta_2 w_{1,\mu_1}(x)}{\lambda_3 - \lambda_2 + \sqrt{(\lambda_3-\lambda_2)^2 + 4\beta_2^2 w_{1,\mu_1}^2(x)}} \right). \tag{6.34}$$

Then the subspace $N(\mathcal{L}_5) = \mathit{span}\,\{(0, \cos\theta_2\phi_{2,\beta_2}, \sin\theta_2\phi_{2,\beta_2})\}$. Noticing that the linear operator $\mathcal{L}_6$ is indeed self-adjoint, then we obtain that the range space of $\mathcal{L}_5$ is defined by

$$R(\mathcal{L}) = \left\{ (f_1,f_2,f_3) \in (L_r^p)^3 : \int_{\mathbb{R}^N} (f_2\cos\theta_2 + f_3\sin\theta_2)\phi_{2,\beta_2} = 0 \right\}. \tag{6.35}$$

This shows that dim$N(\mathcal{L}_5)$=codim$R(\mathcal{L}_5)$=1. Finally it is also straightforward to use (6.35) to check that

$$G_{\beta(u,v,w)}(\beta_2, u_0, 0, 0)\,[(0, \phi_2, \phi_3)] = \begin{pmatrix} 0 \\ u_0 \cos\theta_2 \phi_{2,\beta_2} \\ u_0 \sin\theta_2 \phi_{2,\beta_2} \end{pmatrix} \notin R(\mathcal{L}_5), \tag{6.36}$$

where (ϕ_2, ϕ_3) is defined in (6.33). Thus we can apply the result of [27] to conclude that the set of positive solutions to (1.7) near $(\beta_2, u_0, 0, 0)$ is a smooth curve

$$\Gamma = \{(\beta(s), u_{1\beta}(s), v_{1\beta}(s), w_{1\beta}(s)) : s \in (0, \bar{\tau}_2)\}, \tag{6.37}$$

such that $\beta(s) = \beta_2 + \beta'(0)s + o(s)$, $u_{1\beta}(s) = u_0 + o(s)$ and $v_{1\beta}(s) = \cos\theta_2\phi_{2,\beta_2} s + o(s)$ and $w_{1\beta}(s) = \sin\theta_2\phi_{2,\beta_2} s + o(s)$. Moreover, $\beta'(0)$ can be calculated (see for example [37,61])

$$\begin{aligned} \beta'(0) &= -\frac{\langle G_{(u,v,w)(u,v,w)}(\beta_2, u_0, 0, 0)[(0, \phi_2, \phi_3)(0, \phi_2, \phi_3)], \ell\rangle}{2\langle G_{\beta(u,v,w)}(\beta_2, u_0, 0, 0)[(0, \phi_2, \phi_3)], \ell\rangle} \\ &= -\frac{\displaystyle\int_{\mathbb{R}^N} (\mu_2 \cos^3\theta_2 + \mu_3 \sin^3\theta_2)\phi_{2,\beta_2}^3}{\displaystyle 2\int_{\mathbb{R}^N} u_0 \sin\theta_2 \cos\theta_2 \phi_{2,\beta_2}^2} < 0, \end{aligned} \tag{6.38}$$

where ℓ is a linear functional on $(L^p_r)^3$ defined as $\langle[(f_1, f_2, f_3]), \ell\rangle = \int_{\mathbb{R}^N} (f_2 \cos\theta_2 + f_3 \sin\theta_2)\phi_{2,\beta_2}$. Hence we infer from (6.37)–(6.38) that (1.5) has a nontrivial solution $(u_{1\beta}, v_{1\beta}, w_{1\beta})$ in the form of (1.23) for $\beta_2 - \tau_2 < \beta < \beta_2$. Moreover, by using the arguments of [43, Theorem 5.1], one deduces that $(u_{1\beta}, v_{1\beta}, w_{1\beta})$ is positive solution.

The Morse index of $(u_{1\beta}, v_{1\beta}, w_{1\beta})$ can be calculated similar to the two-wave system case, which is omitted. Under the condition (B_i) $(1 \le i \le 5)$, we have shown that the ground state energy satisfies (6.4). On the other hand, when $\tau_2 > 0$ is sufficiently small, we infer from $\mathcal{L}_6[(\cos\theta_2\phi_{2,\beta_2}, \sin\theta_2\phi_{2,\beta_2})] = 0$ and (1.23) that for $\beta \in (\beta_2 - \tau_2, \beta_2)$,

$$\begin{aligned} \tilde{\mathscr{J}}(u_{1\beta}, v_{1\beta}, w_{1\beta}) &= \frac{S_1^3}{6\mu_1^2} + o(s) + \frac{1}{2}\left(\|v_{1\beta}\|_{\lambda_2}^2 + \|w_{1\beta}\|_{\lambda_3}^2\right) \\ &\quad - \frac{1}{3}\int_{\mathbb{R}^N} (\mu_2 v_{1\beta}^3 + \mu_3 w_{1\beta}^3) - \beta \int_{\mathbb{R}^N} u_{1\beta} v_{1\beta} w_{1\beta} \\ &= \frac{S_1^3}{6\mu_1^2} - \frac{2(\beta-\beta_2)^3}{3} \frac{\left(\int_{\mathbb{R}^N} \phi_{2,\beta_2} w_{1,\mu_1} \sin\theta_2 \cos\theta_2\right)^3}{\left(\int_{\mathbb{R}^N} (\mu_2 \cos^3\theta_2 + \mu_3 \sin^3\theta_2)\phi_{2,\beta_2}^3\right)^2} + o(s) \\ &= \frac{S_1^3}{6\mu_1^2} + o(s). \end{aligned} \tag{6.39}$$

One deduces from (6.4) that $\tilde{C}_r < \tilde{\mathscr{J}}(u_0, 0, 0) = \dfrac{S_1^3}{6\mu_1^2}$. Hence, for $\tau_2 > 0$ sufficiently small and $\beta \in (\beta_2 - \tau_2, \beta_2)$, $\tilde{\mathscr{J}}(u_{1\beta}, v_{1\beta}, w_{1\beta}) > \tilde{C}_r$. So when one of the condition (B_i) $(1 \le i \le 5)$ is satisfied, $(u_{1\beta}, v_{1\beta}, w_{1\beta})$ is not a ground state solution. □

Next we prove the conclusion (ii) of Theorem 1.11.

Proof of Theorem 1.11 (ii). Set

$$u_1(x) = \frac{1}{\mu_1} w_0(x),\ v_1(x) = \frac{\lambda_2}{\mu_2} w_0(\sqrt{\lambda_2} x),\ w_1(x) = \frac{\lambda_3}{\mu_3} w_0(\sqrt{\lambda_3} x). \tag{6.40}$$

Then we know that when $\beta = 0$, (1.5) has the following non-trivial non-negative solutions

$$\begin{aligned} &z_1 = (u_1, v_1, w_1),\ z_2 = (u_1, v_1, 0),\ z_3 = (u_1, 0, w_1),\ z_4 = (0, v_1, w_1), \\ &z_5 = (u_1, 0, 0),\ z_6 = (0, v_1, 0),\ \ z_7 = (0, 0, w_1). \end{aligned} \tag{6.41}$$

We shall apply the implicit function theorem at z_i with parameter $\beta = 0$ for $i = 1, 2, 3, 4$. Since the proof is essentially the same for each of $i = 1, 2, 3, 4$, we only present the proof for z_1. Since z_1 is nondegenerate in X_p^r, *i.e.*, $\left[G_{(u,v,w)}(0, z_1)\right]^{-1} = \left[G_{(u,v,w)}(0, u_1, v_1, w_1)\right]^{-1}$ exists. By the implicit function theorem, there exist $\widetilde{\beta}_0 > 0, R_0 > 0$ and $\tilde{z}_1(\beta) : (-\widetilde{\beta}_0, \widetilde{\beta}_0) \to B_{R_0}(z_1)$ such that for any $\beta \in (-\widetilde{\beta}_0, \widetilde{\beta}_0)$, $G(\beta, \tilde{z}_1(\beta)) = G(\beta, \tilde{u}_1(\beta), \tilde{v}_1(\beta), \tilde{w}_1(\beta)) = 0$.

For each $(\phi_1, \phi_2, \phi_3) \in \left(X_p^R\right)^3$, one infers from (6.27) that

$$G_{(u,v,w)}(0, u_1, v_1, w_1)[(\phi_1, \phi_2, \phi_3)] = \begin{pmatrix} \Delta\phi_1 - \phi_1 + 2\mu_1 u_1 \phi_1 \\ \Delta\phi_2 - \lambda_2\phi_2 + 2\mu_2 v_1 \phi_2 \\ \Delta\phi_3 - \lambda_3\phi_3 + 2\mu_3 w_1 \phi_3 \end{pmatrix} = -\begin{pmatrix} w_1 v_1 \\ u_1 w_1 \\ u_1 v_1 \end{pmatrix}. \tag{6.42}$$

Hence one has that

$$\begin{aligned} &\phi_1 = (-\Delta + 1 - 2\mu_1 u_1)^{-1} w_1 v_1, \quad \phi_2 = (-\Delta + \lambda_2 - 2\mu_2 v_1)^{-1} u_1 w_1 \\ &\phi_3 = (-\Delta + \lambda_3 - 2\mu_3 w_1)^{-1} u_1 v_1, \end{aligned} \tag{6.43}$$

which implies the form of $(u_{4\beta}, v_{4\beta}, w_{4\beta})$ in (1.25). Similarly by using the implicit function theorem at z_i for $i = 2, 3, 4$, we can obtain the positive solutions $(u_{i\beta}, v_{i\beta}, w_{i\beta})$ with $i = 5, 6, 7$ as in (1.25). Note that the implicit function theorem can also be applied at z_i for $i = 5, 6, 7$ but will only yield semi-trivial solutions.

Hence there exists $\tau_5 > 0$ such that when $\beta \in (0, \tau_5)$, (1.5) has exactly four positive solutions, and the sign information of $(u_{i\beta}, v_{i\beta}, w_{i\beta})$ with $i = 4, 5, 6, 7$ and $\beta \in (-\tau_5, 0)$ can also be easily obtained by using the form in (1.25). The Morse indices of all solutions can be obtained similarly as in the proof of Theorem 1.4, by using the stability information of each solution when $\beta = 0$. Finally the energy of the four positive solutions can be compared with the ones of three semi-trivial ones, and we can conclude that one of $(u_{i\beta}, v_{i\beta}, w_{i\beta})$ with $i = 5, 6, 7$ is the ground state solution under proper conditions on λ_2 and λ_3. □

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