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Bistable traveling waves for time periodic reaction-diffusion equations in strip with Dirichlet boundary condition

Jian Fang [a,*], Penglong Shao [a,b], Junping Shi [b]

[a] *School of Mathematics, Harbin Institute of Technology, Harbin, Heilongjiang, 150001, China*
[b] *Department of Mathematics, William & Mary, Williamsburg, VA, 23187-8795, USA*

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Abstract

We study the existence of bistable traveling wave solutions for time periodic reaction-diffusion equations in straight strips subject to the Dirichlet boundary condition. We first employ a monotone dynamical system framework to establish the existence of such a wave by assuming a bistability structure in terms of multiplicity and stability of periodic solutions in the section of the strip, which is then realized under a set of sufficient explicit conditions; in particular, the bistability structure appears if the reaction term is a time periodic smooth perturbation of the nonlinearity $\lambda u(1-u)(u-a)$ when $a \in (0, 1/2)$ and $\lambda > \lambda_*$ for some $\lambda_* > 0$.

MSC: 35C07; 35K57; 35B20

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1. Introduction

In this paper, we investigate the existence of periodic bistable traveling wave solutions of the following reaction-diffusion equation in a strip:

* Corresponding author.
E-mail address: jfang@hit.edu.cn (J. Fang).

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$$\begin{cases} u_t = u_{xx} + u_{yy} + g(t,u), & (x,y) \in \mathbb{R} \times (-1,1), \\ u(t,x,-1) = u(t,x,1) = 0, & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $g \in C^{1,2}(\mathbb{R} \times \mathbb{R})$ is ω-periodic in t for some $\omega > 0$ and $g(t,0) \equiv 0$ for $t \in \mathbb{R}$.

By an ω-periodic traveling wave solution $u(t,x,y)$ to (1.1), we mean that $u(t,x,y)$ is defined for all $t \in \mathbb{R}$ and it has the following properties: (i) $u(t,x,y) = U(t,\xi,y)$ with $\xi = x - ct$ for some function U and constant $c \in \mathbb{R}$, (ii) $U(t,x,y) = U(t+\omega,x,y)$, and (iii) the limits $U(t,\pm\infty,y)$ are two ordered and linearly stable ω-periodic solutions of the following problem:

$$\begin{cases} v_t = v_{yy} + g(t,u), & y \in (-1,1), \\ v(t,-1) = v(t,1) = 0. \end{cases} \tag{1.2}$$

The study of bistable traveling waves for reaction-diffusion equations may go back to the pioneer work by Fife and McLeod [13], where they revealed a type of propagation phenomenon for bistable reaction-diffusion equation:

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R}, \tag{1.3}$$

where $f \in C^1(\mathbb{R},\mathbb{R})$ satisfies

$$f(0) = f(1) = f(\alpha) = 0 \quad \text{for some } \alpha \in (0,1) \tag{1.4}$$

and

$$f'(0) < 0, \quad f'(1) < 0, \quad f'(\alpha) > 0, \quad f(u) < 0 \text{ for } u \in (0,\alpha), \quad f(u) > 0 \text{ for } u \in (\alpha,1). \tag{1.5}$$

A nonlinearity satisfying (1.4) and (1.5) is often called a bistable one. A typical example of bistable nonlinearity is $f(u) = u(1-u)(u-\alpha)$ with $\alpha \in (0,1)$. They found that there exists a unique $c \in \mathbb{R}$ such that (1.3) admits a traveling wave solution of the form $u(t,x) = U(x-ct)$ with $U(-\infty) = 1$ and $U(+\infty) = 0$. Further, such a traveling wave solution is unique up to a constant shift and it attracts other solutions as long as the initial value u_0 satisfies $\liminf_{x\to-\infty} u_0(x) > \alpha > \limsup_{x\to+\infty} u_0(x)$. Such a propagation phenomenon, in spirit, is then found to exist in various time-evolution systems, such as systems of reaction-diffusion [32], reaction-diffusion equations with time periodicity [1,2] or spatial periodicity [35,34,9,8], nonlocal dispersal equations [3], delayed reaction-diffusion equations [25,27,33], iterative maps [7] and a class of monotone dynamical systems [12].

The propagation phenomenon of bistable traveling waves was also found in reaction-diffusion equations in infinite cylinders. Consider

$$\begin{cases} u_t = \Delta u + f(u), & (x,y) \in \mathbb{R} \times \Omega, \\ \boldsymbol{B}u = 0, & (x,y) \in \mathbb{R} \times \partial\Omega, \end{cases} \tag{1.6}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n)$, $\Delta = \frac{\partial^2}{\partial x^2} + \Delta_y$ with $\Delta_y = \sum_{j=1}^n \frac{\partial^2}{\partial y_j^2}$, and boundary condition $\boldsymbol{B}u = 0$ is either Dirichlet type $u = 0$ or Neumann type

$\partial u/\partial \nu = 0$, where ν is the outward unit vector perpendicular to the wall of cylinder. Berestycki and Nirenberg in [6] developed an approximation idea, using a family of truncated problems in bounded domains to approximate the cylinder problem, to obtain the existence of traveling wave solutions when $\boldsymbol{B}u = \partial u/\partial \nu$. In particular, they showed that if $f \in C^1$ satisfies (1.4)-(1.5) and Ω is convex, then there exists a bistable traveling solution $U(x - ct, y)$ with some speed $c \in \mathbb{R}$ and

$$U(-\infty, y) \equiv 1, \quad U(+\infty, y) \equiv 0.$$

It is worthy mentioning that the convexity of Ω is crucial since it ensures that 0 and 1 are the only stable steady states for

$$\begin{cases} \Delta_y u + f(u) = 0, & y \in \Omega, \\ \partial u/\partial \nu = 0, & y \in \partial\Omega. \end{cases} \tag{1.7}$$

A parallel existence result of bistable waves to [6] for cooperative systems in cylinders with Neumann boundary condition was obtained in [12], where a class of monotone dynamical systems is studied. For Dirichlet boundary condition $\boldsymbol{B}u = u$, Gardner [14] utilized the Conley index theory to obtain the existence of bistable traveling wave $U(x - ct, y)$ in the case where $g(u) = u(u-1)(\alpha - u), \alpha \in (0, 1/2)$ and $\Omega = (0, L)$ provided that L is sufficiently large. Here $U(\pm\infty, y)$ are two stable steady states for

$$\begin{cases} \Delta_y u + f(u) = 0, & y \in \Omega, \\ u = 0, & y \in \partial\Omega. \end{cases} \tag{1.8}$$

Vega [31] proved a parallel result to [14] for high-dimensional Ω by assuming that (1.8) admits exactly two nonlinearly stable steady states, where a stable steady state is understood as a local minimum of an energy functional. Roquejoffre [24] proved the uniqueness and attractivity of such a wave for (1.6) with Neumann and Dirichlet boundary conditions, respectively. For more research works on the propagation dynamics of reaction-diffusion equations in cylinders, we refer [18,20,4,10,5,29,30] and references therein.

This work is devoted to the study of time periodic reaction-diffusion equations in an infinite strip (a cylinder with $\Omega = (-L, L)$) with Dirichlet boundary condition. The rest of this paper is organized as follows. In Section 2, we first verify that system (1.1) generates a solution semiflow that fits the dynamical system framework in [12], then we apply [12, Theorem 3.1], with a necessary modification due to the Dirichlet boundary condition, to the generated solution semiflow of (1.1) to obtain the existence of bistable traveling wave solutions under a bistability structure hypothesis (BSH), as stated in subsection 2.2. In Section 3, we give several sufficient conditions to realize the (BSH) so that the existence of bistable traveling wave solutions is achieved.

2. Existence of bistable traveling wave solution

In this section, we apply the monotone dynamical system result of [12] to establish the existence of traveling waves for (1.1) under a bistability structure hypothesis that will be explained later. For this purpose, we first show that the solution of (1.1) generates a monotone semiflow in appropriate phase spaces so that it fits the dynamical system setting of [12].

2.1. Solution semiflow generated by system (1.1)

Let X be the Banach space

$$X = \{u \in C[-1,1] : u(-1) = u(1) = 0\} \tag{2.1}$$

endowed with the norm

$$\|u\|_X = \max_{|y|\le 1} |u(y)|. \tag{2.2}$$

Define $X^+ = \{u \in X : u \ge 0\}$. Then X^+ is a positive cone of X but has an empty interior.[1] For u, v in X, we write $u \ge v$ if $u - v \in X^+$ and $u > v$ if $u \ge v$ but $u \ne v$. Then, the triple $(X, X^+, \|\cdot\|_X)$ is a Banach lattice [26].

Let $\mathcal{C} = C(\mathbb{R}, X)$ be the set of all continuous functions from $\mathbb{R}$ to X, and let $\mathcal{C}^+ = C(\mathbb{R}, X^+)$ be the set of all continuous functions from $\mathbb{R}$ to X^+. We equip $\mathcal{C}$ with the compact open topology, which can be induced by the following norm

$$\|\phi\|_{\mathcal{C}} = \sum_{k=1}^{\infty} \frac{\max_{|x|\le k} \|\phi(x,\cdot)\|_X}{2^k}, \quad \phi \in \mathcal{C}. \tag{2.3}$$

For $u, v \in \mathcal{C}$, we write $u \ge v$ if $u - v \in \mathcal{C}^+$ and $u > v$ if $u \ge v$ but $u \ne v$. For $u \le v$ in $\mathcal{C}$, we write $[u, v]_{\mathcal{C}}$ to denote the set $\{w \in \mathcal{C} : u \le w \le v\}$. Similarly, for $u \le v$ in X, $[u, v]_X$ denotes the set $\{w \in X : u \le w \le v\}$. For the sake of convenience, we also write $\mathcal{C}_v$ instead of $[0, v]_{\mathcal{C}}$ and X_v instead of $[0, v]_X$

To prove that system (1.1) generates an ω-periodic monotone semiflow, we begin with the following linear equation:

$$\begin{cases} U_t = U_{xx} + U_{yy}, & (x, y) \in \mathbb{R} \times (-1, 1),\ t > 0, \\ U(t, x, -1) = U(t, x, 1) = 0, & x \in \mathbb{R},\ t > 0, \\ U(0, x, y) = \phi(x, y), & (x, y) \in \mathbb{R} \times (-1, 1). \end{cases} \tag{2.4}$$

By the standard method one can solve (2.4) explicitly to obtain

$$U(t, x, y; \phi) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\eta)^2}{4t}} V(t, \eta, y; \phi)\, \mathrm{d}\eta, \tag{2.5}$$

where $V(t, \eta, y; \phi)$ solves

[1] In section 2.2, we introduce a subspace of X with a smaller cone that has nonempty interior, and then slightly modify the proof of [12, Theorem 3.1] so that it is applicable to the solution semiflow generated by the cylinder problem with Dirichlet boundary condition.

$$\begin{cases} V_t = V_{yy}, & t > 0, y \in (-1, 1), \\ V(t, \eta, \pm 1) = 0, & t > 0 \\ V(0, \eta, y) = \phi(\eta, y), & y \in (-1, 1) \end{cases} \tag{2.6}$$

for any $\eta \in \mathbb{R}$. For $t \geq 0$ define $S(t) : \mathcal{C} \to \mathcal{C}$ by

$$S(t)[\phi](x, y) = U(t, x, y; \phi). \tag{2.7}$$

Then it is easy to see that $S(0)$ is an identity map and $S(t+s) = S(t)S(s)$ for $t, s \geq 0$.

Lemma 2.1. *There exists $M > 0$ such that $\|S(t)\phi\|_{\mathcal{C}} \leq e^{Mt}\|\phi\|_{\mathcal{C}}$ for $t \geq 0$ and $\phi \in \mathcal{C}$.*

Proof. From (2.3) and (2.5) we obtain

$$\left\|S(t)[\phi]\right\|_{\mathcal{C}} \leq \frac{1}{\sqrt{4\pi t}} \sum_{k=1}^{\infty} 2^{-k} \max_{|x| \leq k} \int_{-\infty}^{\infty} e^{-\frac{(x-\eta)^2}{4t}} \|V(t, \eta, \cdot; \phi)\|_X \, \mathrm{d}\eta \tag{2.8}$$

Note that $\bar{V} \equiv \|\phi(\eta, \cdot; \phi)\|_X$ is a super solution of (2.6). It then follows from comparison principle that

$$\|V(t, \eta, \cdot; \phi)\|_X \leq \|\phi(\eta, \cdot)\|_X. \tag{2.9}$$

Hence,

$$\left\|S(t)[\phi]\right\|_{\mathcal{C}} \leq \frac{1}{\sqrt{4\pi t}} \sum_{k=1}^{\infty} 2^{-k} \max_{|x| \leq k} \int_{-\infty}^{\infty} e^{-\frac{(x-\eta)^2}{4t}} \|\phi(\eta, \cdot)\|_X \mathrm{d}\eta.$$

Furthermore, since

$$\begin{aligned} \max_{|x| \leq k} \int_{-\infty}^{\infty} e^{-\frac{(x-\eta)^2}{4t}} \|\phi(\eta, \cdot)\|_X \, \mathrm{d}\eta &= \max_{|x| \leq k} \sum_{l \geq 0} \int_{l \leq |\eta| \leq l+1} e^{-\frac{\eta^2}{4t}} \|\phi(x-\eta, \cdot)\|_X \, \mathrm{d}\eta, \\ &\leq \sum_{l \geq 0} \max_{|x| \leq k+l+1} \|\phi(x, \cdot)\|_X \int_{l \leq |\eta| \leq l+1} e^{-\frac{\eta^2}{4t}} \, \mathrm{d}\eta, \end{aligned} \tag{2.10}$$

we can obtain the following inequality by making the change of index $\tilde{l} = l + k + 1$ and dropping the tilde:

$$\left\|S(t)[\phi]\right\|_{\mathcal{C}} \leq \frac{1}{\sqrt{4\pi t}} \sum_{k=1}^{\infty} 2^{-k} \sum_{l \geq k+1} \max_{|x| \leq l} \|\phi(x, \cdot)\|_X \int_{l-k-1 \leq |\eta| \leq l-k} e^{-\frac{\eta^2}{4t}} \, \mathrm{d}\eta. \tag{2.11}$$

After swapping the order of sums, the above inequality becomes

$$\begin{aligned}\left\|S(t)[\phi]\right\|_{\mathcal{C}} &\le \sum_{l=2}^{\infty} 2^{-l} \max_{|x|\le l} \|\phi(x,\cdot)\|_X \left(\frac{1}{\sqrt{4\pi t}} \sum_{k=1}^{l-1} 2^{l-k} \int\limits_{l-k-1\le|\eta|\le l-k} e^{-\frac{\eta^2}{4t}}\, \mathrm{d}\eta \right) \\ &=: \sum_{l=2}^{\infty} 2^{-l} \max_{|x|\le l} \|\phi(x,\cdot)\|_X \cdot I \le \|\phi\|_{\mathcal{C}} \cdot I.\end{aligned} \tag{2.12}$$

Next we refine an argument of [11] to estimate I. Indeed, we make the change of index $n = l - k$ to obtain

$$I = \frac{1}{\sqrt{4\pi t}} \sum_{k=1}^{l-1} 2^{l-k} \int\limits_{l-k-1\le|\eta|\le l-k} e^{-\frac{\eta^2}{4dt}}\, \mathrm{d}\eta = \frac{1}{\sqrt{4\pi t}} \sum_{n=1}^{l-1} 2^{n} \int\limits_{n-1\le|\eta|\le n} e^{-\frac{\eta^2}{4t}}\, \mathrm{d}\eta.$$

Next we construct a super solution for the heat equation to estimate I. Indeed, let $\theta = 4 + \ln 2$ and set $\overline{w}(x,t) = e^{-\theta x + \theta^2 t}$. Then

$$\overline{w}_t = \overline{w}_{xx} \quad \text{with} \quad \overline{w}(x,0) = e^{-\theta x} \ge \mathbf{1}_{(-\infty,0]}(x),$$

which, in view of the comparison principle, implies that $\overline{w} \ge w$, where w satisfies

$$\begin{cases} w_t - w_{xx} = 0, \quad x \in \mathbb{R},\ t > 0, \\ w(x,0) = \mathbf{1}_{(-\infty,0]}(x) \end{cases} \tag{2.13}$$

Note that $w(t,x) = \frac{1}{\sqrt{4\pi t}} \int_x^{\infty} e^{-\frac{\eta^2}{4t}} d\eta$. Then, in particular, at $x = n - 1$ we obtain

$$e^{-\theta(n-1)} e^{\theta^2 t} = \bar{w}(t, n-1) \ge w(t, n-1) \ge \frac{1}{\sqrt{4\pi t}} \int\limits_{n-1}^{n} e^{-\frac{\eta^2}{4t}} d\eta. \tag{2.14}$$

Consequently,

$$\frac{1}{\sqrt{4\pi t}} \int\limits_{n-1\le|\eta|\le n} e^{-\frac{\eta^2}{4t}}\, \mathrm{d}\eta \le 2e^{-\theta(n-1)} e^{\theta^2 t}.$$

Further, we infer that

$$I \le 2\sum_{n=1}^{\infty} 2^n e^{-\theta(n-1)} e^{\theta^2 t} = 4e^{\theta^2 t} \sum_{n=1}^{\infty} e^{-(\theta - \ln 2)(n-1)} \le 4e^{\theta^2 t} \int\limits_0^{\infty} e^{-(\theta-\ln 2)x}\, \mathrm{d}x = \frac{4e^{\theta^2 t}}{\theta - \ln 2} = e^{\theta^2 t},$$

which, combining with (2.12), implies that

$$\left\|S(t)[\phi]\right\|_{\mathcal{C}} \le e^{\theta^2 t} \|\phi\|_{\mathcal{C}}. \tag{2.15}$$

This completes the proof. □

Lemma 2.2. *$S(t):\mathcal{C}^+\to\mathcal{C}^+$ is compact for $t>0$ and $\lim_{t\to 0^+} S(t)[\phi]=\phi$ for any bounded $\phi\in\mathcal{C}^+$.*

Proof. By the expression of $S(t)[\phi]$ as in (2.5)-(2.7) and the estimate in (2.9), we see that for any $\delta\in\mathbb{R}$,

$$\|S(t)[\phi](x_1,\cdot)-S(t)[\phi](x_2,\cdot)\|_X\le\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}}\left|e^{-\frac{(x+\delta-\eta)^2}{4t}}-e^{-\frac{(x-\eta)^2}{4t}}\right|\|\phi(\eta,\cdot)\|_X\mathrm{d}\eta,$$

which tends to 0 uniformly for x in any compact set as δ tends to 0. Then by [36, Theorem 9.7.5] we see that $S(t)$ is compact.

By the expression of $S(t)[\phi]$ we also have

$$\begin{aligned}\|S(t)[\phi]-\phi\|_{\mathcal{C}}&=\sum_{k=1}^{\infty}2^{-k}\max_{x\le k}\left\|\int_{\mathbb{R}}\frac{e^{-\frac{z^2}{4}}}{\sqrt{4\pi}}V(t,x-z\sqrt{t},\cdot;\phi)-\phi(x,\cdot)\right\|_X dz\\&\le\sum_{k=1}^{\infty}2^{-k}\max_{x\le k}\int_{\mathbb{R}}\frac{e^{-\frac{z^2}{4}}}{\sqrt{4\pi}}\|V(t,x-z\sqrt{t},\cdot;\phi)-\phi(x,\cdot)\|_X dz.\end{aligned}$$

Then the boundedness of V and ϕ yields that for any $\epsilon>0$ there exist z_0 and k_0 such that

$$\|S(t)[\phi]-\phi\|_{\mathcal{C}}\le\frac{\epsilon}{2}+\sum_{k\le k_0}2^{-k}\max_{x\le k_0}\int_{|z|\le z_0}\frac{e^{-\frac{z^2}{4}}}{\sqrt{4\pi}}\|V(t,x-z\sqrt{t},\cdot;\phi)-\phi(x,\cdot)\|_X dz. \tag{2.16}$$

Note that

$$\begin{aligned}\|V(t,x-z\sqrt{t},\cdot;\phi)-\phi(x,\cdot)\|_X&\le\|V(t,x-z\sqrt{t},\cdot;\phi)-\phi(x-z\sqrt{t},\cdot)\|_X\\&\quad+\|\phi(x-z\sqrt{t},\cdot)-\phi(x,\cdot)\|_X.\end{aligned}$$

Thus, by the continuity of ϕ we see that $\|V(t,x-z\sqrt{t},\cdot;\phi)-\phi(x,\cdot)\|_X\to 0$ locally uniformly for (x,z) as $t\to 0$. Therefore, for the above ϵ, z_0 and k_0, there exists t_0 such that

$$\|V(t,x-z\sqrt{t},\cdot;\phi)-\phi(x,\cdot)\|_X\le\epsilon/2,\quad t\le t_0,\ |x|\le k_0,\ |z|\le z_0,$$

which, combining with (2.16), implies that $\|S(t)[\phi]-\phi\|_{\mathcal{C}}\le\epsilon$ for $t\le t_0$, that is, $\lim_{t\to 0^+}S(t)[\phi]=\phi$. □

Using the linear semiflow $S(t)$ we may write the Cauchy problem (1.1) as the following integral form:

$$u(t;\phi)=S(t)[\phi]+\int_0^t S(t-s)[g(s,u(s;\phi))]\,\mathrm{d}s. \tag{2.17}$$

By [21, Corollary 5] we can infer that for each bounded $\phi \in \mathcal{C}^+$, (2.17) admits a solution $u(t;\phi)$ for $t \geq 0$ with $u(\cdot;\phi) \in \mathcal{C}^+$ and $u(t;\phi_1) \geq u(t;\phi_2)$ provided that $\phi_1 \geq \phi_2$. For $t \geq 0$ and any bounded subset $\mathcal{B} \subset \mathcal{C}^+$, define $Q_t : \mathcal{B} \to \mathcal{C}^+$ by

$$Q_t[\phi] = u(t;\phi), \quad \phi \in \mathcal{B}. \tag{2.18}$$

Now we are able to state the main result of this subsection.

Theorem 2.3. *For $t > 0$ and any bounded subset $\mathcal{B} \subset \mathcal{C}$, the map $Q_t : \mathcal{B} \to \mathcal{C}^+$ is monotone, compact and continuous.*

Proof. The monotonicity of Q_t follows directly from the comparison principle. The compactness of Q_t follows from the compactness of $S(t)$ and the boundedness of g. It then remains to show the continuity. Indeed, for $\phi_n \to \phi$ in $\mathcal{C}$ and any $t > 0$, by virtue of (2.15) and (2.17) we obtain

$$\begin{aligned}\left\| Q_t[\phi_n] - Q_t[\phi]\right\|_{\mathcal{C}} &\leq \|S(t)[\phi_n - \phi]\|_{\mathcal{C}} + \left\| \int_0^t S(t-s)\big(g(s, Q_t[\phi_n]) - g(s, Q_s[\phi])\big)\mathrm{d}s \right\|_{\mathcal{C}}, \\ &\leq e^{\theta^2 t}\|\phi_n - \phi\|_{\mathcal{C}} + \int_0^t L_g\, e^{\theta^2(t-s)} \left\| Q_s[\phi_n] - Q_s[\phi]\right\|_{\mathcal{C}} \mathrm{d}s,\end{aligned}$$

that is,

$$e^{-\theta^2 t}\left\| Q_t[\phi_n] - Q_t[\phi]\right\|_{\mathcal{C}} \leq \|\phi_n - \phi\|_{\mathcal{C}} + L_g \int_0^t e^{-\theta^2 s}\left\| Q_s[\phi_n] - Q_s[\phi]\right\|_{\mathcal{C}} \mathrm{d}s, \tag{2.19}$$

where L_g is a Lipschitz constant of g. In view of Gronwall's inequality, we infer that

$$\left\| Q_t[\phi_n] - Q_t[\phi]\right\|_{\mathcal{C}} \leq \|\phi_n - \phi\|_{\mathcal{C}} \left(e^{\theta^2 t} + L_g t e^{(L_g + \theta^2)t}\right). \tag{2.20}$$

Therefore, $\|Q_t[\phi_n] - Q_t[\phi]\|_{\mathcal{C}} \to 0$ as $n \to \infty$. □

2.2. *Existence of traveling waves under a bistability structure*

In this part, assuming the x-independent problem (1.2) has a bistability structure, we establish the existence of bistable traveling waves. For this purpose, we first recall the linear stability of ω-periodic solutions for (1.2). An ω-periodic solution $v(t,y)$ of (1.2) is said to be linearly stable/unstable if the following eigenvalue problem

$$\begin{cases} -\psi_t + \psi_{yy} + g_v(t,v)\psi = \mu\psi, & y \in (-1,1),\ t > 0, \\ \psi(t,-1) = \psi(t,1) = 0, & t > 0, \\ \psi(t+\omega, y) = \psi(t,y), & y \in (-1,1),\ t > 0 \end{cases} \tag{2.21}$$

has a negative/positive principal eigenvalue μ. We also define the bistability structure hypothesis as follows.

(BSH) Problem (1.2) has two ordered and linearly stable nonnegative ω-periodic solutions, say $v_0 = 0 < v_{max}$, between which any other ω-periodic solution is linearly unstable.

Since the interior of the cone X^+ is empty, the dynamical system theorem in [12] cannot be directly applied to the period map Q_ω of (1.1). However, for this particular cylinder problem, we may merge an idea of [19] into the proof of [12, Theorem 3.1] so that it is still applicable to (1.1). More precisely, we introduce a subspace Y with a smaller positive cone Y^+ and a stronger topology so that the interior $Int(Y^+)$ is not empty. Then, one can define a strong ordering $\gg_Y$. Going through every place where a strong ordering relation is involved in the proof of [12, Theorem 3.1], although a strong ordering in X is not applicable, the ordering $\gg_Y$ holds there provided that $(Y, \|\cdot\|_Y, Y^+)$ is appropriately chosen so that Q_ω maps X^+ into Y^+ and

$$\phi - \psi \in X^+ \setminus \{0\} \Longrightarrow Q_\omega[\phi] - Q_\omega[\psi] \in Int(Y^+). \tag{2.22}$$

For our problem, we choose

$$Y := \{u \in C^1[-1, 1] : u(-1) = u(1) = 0\}$$

endowed with C^1 norm. Let

$$Y^+ := \{u \in Y : u(y) \geq 0, y \in [-1, 1]\}, \quad \mathcal{D}^+ := C(\mathbb{R}, Y^+).$$

From the regularity of solutions (e.g., [16, section III.20]), we see that Q_ω maps $\mathcal{C}^+$ into $\mathcal{D}^+$. It then follows that all fixed points of Q_ω in X^+ are also in Y^+. Define

$$\beta := v_{\max}(0, \cdot). \tag{2.23}$$

Then $\beta \in Y^+$ and $Q_\omega[\beta] = \beta$. Further, by the strong maximum principle and the Hopf boundary lemma we can infer that

$$\beta \in Int(Y^+) \quad \text{and} \quad Q_\omega[\phi](x) \in Int(Y^+), \quad \forall \phi \in \mathcal{C}_\beta \setminus \{0\} \quad \text{and} \quad x \in \mathbb{R},$$

where

$$Int(Y^+) = \{u \in Y^+ : u'(1) < 0 < u'(-1)\}.$$

For u, v in Y we write $u \gg_Y v$ if $u - v \in Int(Y^+)$. Define $Y_\beta := \{u \in Y : \beta \geq u \geq 0\}$, where the ordering $\geq$ in Y has the same meaning as the ordering $\geq$ in X. Similarly, we may define $\mathcal{D}_\beta$. Then we can state the main result of this section.

Theorem 2.4. *Assume (BSH) holds. Then* (1.1) *admits a traveling wave solution with the form* $u(t, x, y) := U(t, x - ct, y)$ *such that* $U(t, \xi, y)$ *is* ω*-periodic in* t*, decreasing in* $\xi \in \mathbb{R}$ *and*

$$U(t, -\infty, y) = v_{max}(t, y), \quad U(t, +\infty, y) = 0,$$

where the limits hold uniformly in t and y.

Proof. We first show that $Q_t : \mathcal{C}_\beta \to \mathcal{C}_\beta$ satisfies several properties. Then according to the previous discussions, we have the following properties.

(A1) $T_x Q_\omega = Q_\omega T_x$ for all $x \in \mathbb{R}$, where $T_x[\phi] = \phi(\cdot - x)$.
(A2) $Q_t : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is continuous locally uniformly in $t \geq 0$.
(A3) $Q_\omega : \mathcal{C}_\beta \to \mathcal{D}_\beta$ is monotone in the sense that

$$\phi - \psi \in \mathcal{C}^+ \setminus \{0\} \Longrightarrow Q_\omega[\phi](x) - Q_\omega[\psi](x) \in Int(Y^+) \quad \text{for any } x \in \mathbb{R}, \tag{2.24}$$

which, combining with property (A1), implies that $Q_\omega : X_\beta \to Y_\beta$ is strongly monotone in the sense of (2.22).
(A4) $Q_\omega : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is compact with respect to the compact open topology.

By the hypothesis (BSH) we can infer that

(A5) 0 and β are two fixed points of Q_ω, and there exist $e_0, e_\beta \in Int(Y^+)$ and $\delta_0 > 0$ such that

$$\delta e_0 \gg_Y Q_\omega[\delta e_0], \quad Q[\beta - \delta e_\beta] \gg_Y \beta - \delta e_\beta, \quad \delta \in (0, \delta_0].$$

For any other fixed point α with $0 < \alpha < \beta$, there exists $e_\alpha \in Int(Y^+)$ and $\delta_\alpha > 0$ such that

$$\alpha - \delta e_\alpha \gg_Y Q_\omega[\alpha - \delta e_\alpha] \quad \text{and} \quad Q_\omega[\alpha + \delta e_\alpha] \gg_Y \alpha + \delta e_\alpha, \quad \delta \in (0, \delta_\alpha].$$

Indeed, the linear stability/instability of periodic solutions of (1.2) implies (A5). In particular, e_0 can be chosen to be the value at $t = 0$ of the principal time periodic eigenfunction $\psi(t, y)$ associated with the periodic solution 0, and e_α, e_β can be chosen in the same manner.

With the property (A5) we further obtain that all unstable fixed points of Q_ω between 0 and β are pairwise unordered [12, Proposition 2.1]. Then for each unstable α, we have a pair of monostable subsystems, one is $Q_\omega : [\alpha, \beta]_{\mathcal{C}} \to [\alpha, \beta]_{\mathcal{C}}$, the other is $Q_\omega : [0, \alpha]_{\mathcal{C}} \to [0, \alpha]_{\mathcal{C}}$. Let us first consider $Q_\omega : [\alpha, \beta]_{\mathcal{C}} \to [\alpha, \beta]_{\mathcal{C}}$. According to [19], it admits a rightward spreading speed $c_+^*(\alpha, \beta)$, which is bounded from below by $\frac{1}{\omega} \inf_{\rho > 0} \frac{\mu(\rho, \alpha)}{\rho}$. Here $\mu(\rho, \alpha)$ is the principal eigenvalue of the following problem

$$\begin{cases} -\psi_t + \psi_{yy} + [g_v(t, v(t, y; \alpha)) + d\rho^2]\psi = \mu\psi, & y \in (-1, 1),\ t > 0, \\ \psi(t, -1) = \psi(t, 1) = 0, & t > 0, \\ \psi(t + \omega, y) = \psi(t, y), & y \in (-1, 1),\ t > 0, \end{cases} \tag{2.25}$$

where $v(t, y; \alpha)$ is the periodic solution of (1.2) with the initial value α. Clearly, $\mu(\rho, \alpha) = \mu(0, \alpha) + d\rho^2$. Note that $\mu(0, \alpha) > 0$ due to the instability of v. Hence,

$$\frac{1}{\omega} c_+^*(\alpha, \beta) \geq \inf_{\rho > 0} \frac{\mu(0, \alpha) + \rho^2}{\rho} = 2\sqrt{d\mu(0, \alpha)} > 0.$$

Similarly, $Q_\omega : [\alpha, \beta]_{\mathcal{C}} \to [\alpha, \beta]_{\mathcal{C}}$ admits a leftward spreading speed $c^*_-(0, \alpha)$, which is also positive. Therefore, we obtain the following property:

(A6) $c^*_-(0, \alpha) + c^*_+(\alpha, \beta) > 0$ for every unstable fixed point $\alpha \in [0, \beta]_X$.

With the above established properties for Q_ω, we can proceed with the same steps as in the proof of [12, Theorem 3.1] to obtain a monotone function $\varphi \in \mathcal{C}_\beta$ and $c \in \mathbb{R}$ such that

$$T_{-c\omega} Q_\omega[\varphi] = \varphi \text{ with } \varphi(-\infty) = \beta \text{ and } \varphi(+\infty) = 0. \tag{2.26}$$

Define

$$U(t, \xi, \cdot) := T_{-ct} Q_t[\varphi](\xi), \quad t \geq 0, \xi \in \mathbb{R}. \tag{2.27}$$

Then it follows that U is periodic in $t \geq 0$ and nonincreasing in $\xi \in \mathbb{R}$. Let k_t be the integer part of t/ω. Hence, in view of (2.26), (2.27), (A1), (A2) and the fact $Q_\omega[\beta] = \beta$, we obtain

$$\begin{aligned} U(t, -\infty, \cdot) &= \lim_{\xi \to -\infty} T_{-ct} Q_t[\varphi](\xi) \\ &= \lim_{\xi \to -\infty} Q_{t-k_t\omega}(T_{-c\omega} Q_\omega)^{k_t}[\varphi](\xi + c(t - k_t\omega)) \\ &= Q_t[\beta] = v_{\max}(t, \cdot). \end{aligned}$$

Similarly, $U(t, +\infty, \cdot) = Q_t[0] = 0$. Therefore, a periodic extension of U to all $t \in \mathbb{R}$ is the required periodic traveling wave solution of (1.1).

Finally we show that $U(t, \xi)$ is strictly decreasing. Note that the time periodic function $W(t, \xi, y) := U_\xi(t, \xi, y)$ satisfies $W_t = W_{\xi\xi} + W_{yy} + cW_\xi + g_u(t, U)W$ and $W \geq 0$. It then follows from the parabolic strong maximum principle that $W > 0$ for $y \in (-1, 1)$. $\square$

3. Bistability structure

In this section, we aim to find several sets of sufficient conditions, under which the bistability structure hypothesis (BSH) holds. Hence, in these cases, the existence of periodic traveling wave solutions is guaranteed. In the remaining part of this paper, we always assume that

$$g(t, u) = \lambda f(u) + \varepsilon h(t, u), \tag{3.1}$$

where $f \in C^2(\mathbb{R})$ and $h \in C^{1,2}(\mathbb{R} \times \mathbb{R})$, and $\lambda > 0$ and $|\varepsilon| \ll 1$ are two parameters.

3.1. *The case of* $\varepsilon = 0$

When $\varepsilon = 0$, problem (1.2) becomes autonomous, for which we have the following observation.

Lemma 3.1. *If* $\varepsilon = 0$*, then any nontrivial time periodic solution of* (1.2) *is linearly unstable.*

Proof. Let $p(t, y)$ be a nontrivial ω-periodic solution. Then p_t satisfies

$$\begin{cases}(p_t)_t = (p_t)_{yy} + \lambda f'(p)p_t, & y \in (-1,1),\ t > 0,\\ p_t(t, \pm 1) = 0, & t > 0,\\ p_t(t, y) = p_t(t+\omega, y), & y \in (-1,1),\ t > 0,\end{cases}$$

which means that 0 is an eigenvalue with an associated eigenfunction p_t that changes sign in t. Then, by the Krein-Rutman theorem, one can infer that principal eigenvalue associated with the periodic solution p is positive. □

By Lemma 3.1, it then suffices to study the steady state problem:

$$\begin{cases}v_{yy} + \lambda f(v) = 0, & y \in (-1,1),\\ v(-1) = v(1) = 0.\end{cases} \tag{3.2}$$

More precisely, if (3.2) admits two ordered and linearly stable solutions, between which all other solutions are all linearly unstable, then the (BSH) holds.

The rest of this subsection deals with the exact multiplicity and linear stability of solutions for (3.2). It consists of two parts and both parts use λ as the parameter; One recalls a sharp result when f is of convex-concave, and the other gives an asymptotic result for a general f.

3.1.1. Exact multiplicity of steady states for (3.2) with convex-concave nonlinearity

There have been extensive research work on the existence and exact multiplicity of solutions set of (3.2) under different conditions. Here we recall a result from [23] that gives a sharp threshold result when f has a typical shape.

Definition 3.2. [23, Def. 2.1 and 2.2] *Let $f \in C^1[a, b]$. It is said to be superlinear (sublinear) in $[a, b]$ if $f(u)/u \le (\ge) f'(u)$ in $[a, b]$. It is said to be convex (concave) in $[a, b]$ if $f \in C^2[a, b]$ and $f''(u) \ge (\le) 0$ in $[a, b]$. It is said to be of sup-sub (sub-sup) in $[a, b]$ if there exists $c \in [a, b]$ such that $f(u)$ is superlinear (sublinear) in $[a, c]$, and sublinear (superlinear) in $[c, b]$. And it is said to be of convex-concave (concave-convex) if $f \in C^2[a, b]$ and there exists $c \in (a, b)$ such that f is convex (concave) in [a, c], and f is concave (convex) in $[c, b]$.*

The linear stability of a solution v for (3.2) is determined by the following eigenvalue problem

$$\begin{cases}\psi_{yy} + \lambda f'(v)\psi = \mu\psi, & y \in (-1,1),\\ \psi(-1) = \psi(1) = 0.\end{cases} \tag{3.3}$$

It is well known that (3.3) has a sequence of eigenvalues satisfying $\mu_1 > \mu_2 > \cdots > \mu_k > \cdots \to -\infty$. Recall that the Morse index $M(v)$ of the solution v stands for the number of positive eigenvalues μ's of (3.3). In particular, $M(v) = 0$ implies the principal eigenvalue $\mu_1 < 0$, while $M(v) = 1$ implies that $\mu_1 > 0 > \mu_2$.

Theorem 3.3. [23, Theorem 6.18] *Assume that $f \in C^2([0, \infty), \mathbb{R})$ is of convex-concave and sup-sub and satisfies the following properties:*

(f1) *f has exactly three zeros $0 < b < a$, $f'(0) < 0$, $f'(a) < 0$ and $\int_0^a f(s)\,ds > 0$.*

(f2) *$f''(u)$ changes sign only once in $(0,\infty)$, and there exists $\alpha \in (0,\infty)$ such that $f''(z) \geq 0$ in $(0,\alpha)$ and $f''(z) \leq 0$ in (α,∞).*

(f3) *Let θ be the unique positive number such that $f(\theta) > 0$, $F(\theta) = \int_0^\theta f(s)\,ds = 0$, $\rho = \alpha - (f(\alpha)/f'(\alpha))$, and $K(u) = uf'(u)/f(u)$. If $\rho > \theta$, then $K(z) \geq K(\theta)$ for $z \in [b,\theta]$, $K(z)$ is nonincreasing in $[\theta,\rho]$, and $K(z) \leq K(\rho)$ for $z \in [\rho,\alpha]$.*

Then, there exists $\lambda_* > 0$ such that (3.2) has no positive solution for $\lambda < \lambda_*$, exactly one positive solution for $\lambda = \lambda_*$, and exactly two positive solutions v_1, v_2 for $\lambda > \lambda_*$. Moreover, $v_2 > v_1 > 0$ with $M(v_1) = 1$ and $M(v_2) = 0$.

Remark 3.4. An estimate of λ_* can be found in [17]. The convexity condition (*f2*) and growth condition (*f3*) are essential for the exact two positive solutions for all $\lambda > \lambda_*$, as they are needed to establish the uniqueness of turning point on the bifurcation diagram. A typical example satisfying (*f1*) − (*f3*) is $f(u) = u(u-b)(a-u)$ with $0 < 2b < a$, see [22].

3.1.2. Exact multiplicity of steady states for (3.2) with general nonlinearity

In this subsection, we prove the exact multiplicity of positive solutions of (3.2) when λ is sufficiently large when f satisfies

(*f4*) f admits $m \geq 2$ positive zero points arranged in increasing order $0 < a_1 < a_2 < \cdots < a_{m-1} < a_m = a$ such that $f'(0) < 0$, $f'(a) < 0$ and

$$\int_0^a f(s)\,ds > 0, \quad \int_0^{a_i} f(s)\,ds < 0, \quad 1 \leq i \leq m-1. \tag{3.4}$$

Remark 3.5. If $m = 3$, then to meet requirements in (*f4*), one can set $f(u) = u(u-a_1)(a_2-u)(a_3-u)(a-u)$, for either $0 < a_1 < a_2 \leq 2a_1$, $a_2 < a_3 < a$ or $0 < 2a_1 < a_2 < \frac{5}{2}a_1$, $a_2 < a_3 \leq \frac{5a_1a_2-3a_2^2}{10a_1-5a_2}$ and $a_3 < a$, where a is chosen to validate

$$5a(a_1a_2 + a_1a_3 + a_2a_3) + 2a^3 - 3a^2(a_1+a_2+a_3) > 10a_1a_2a_3.$$

Our main result in this subsection is as follows.

Theorem 3.6. *Assume that f satisfies (f4). Then there exists $\lambda_{**} > 0$ such that (3.2) has exactly two positive solutions v_1, v_2 such that $v_2 > v_1$ for each $\lambda > \lambda_{**}$. Moreover, v_2 is linearly stable and v_1 is linearly unstable.*

We prove the results in Theorem 3.6 using the time mapping method in several steps. Through a change of variable $\lambda = L^2$, $z = yL$ and $u(z) = v(z/L)$, (3.2) is transformed into

$$\begin{cases} u_{zz} + f(u) = 0, & z \in (-L,L), \\ u(-L) = u(L) = 0. \end{cases} \tag{3.5}$$

In the sequel, let u' stand for u_z, u'' for u_{zz}. From results in [15] that every solution u of (3.5) satisfies that $u'(0)=0$, $u(0)=\max\limits_{[-L,L]} u$. Multiplying $u''+f(u)=0$ by u' and integrating from 0 to $z\in(0,L]$, we get

$$\frac{1}{2}(u'(z))^2+F(u(z))=F(u(0)), \tag{3.6}$$

where $F(u)=\int_0^u f(s)\,\mathrm{d}s$. Since $u'(z)<0$ for $z\in(0,L]$, from (3.6), the inverse function $z(u)$ of $u(z)|_{[0,L]}$ on $u\in[0,u(0)]$ satisfies

$$\frac{dz}{du}=\frac{1}{\sqrt{2}\sqrt{F(u(0))-F(u)}},\quad u\in(0,u(0)). \tag{3.7}$$

As in [28] and references therein, setting $\alpha=u(0)$ and integrating (3.7) with respect to u from α to 0, we obtain that

$$L=\frac{1}{\sqrt{2}}\int_0^{\alpha}\frac{1}{\sqrt{F(\alpha)-F(u)}}\,\mathrm{d}u \doteq T(\alpha). \tag{3.8}$$

The function $T(\alpha)$ is defined only if $F(\alpha)>F(u)$ for any $u\in[0,\alpha)$ (see [28]). By virtue of assumption (*f4*), there exists some $\alpha_*\in(a_{m-1},a)$ such that $F(\alpha_*)=0$, and the time mapping $T(\alpha)$ is defined only for $\alpha\in(\alpha_*,a)$. This establishes the following equivalence of positive solutions of (3.2) and the time mapping function.

Lemma 3.7. *Assume that f satisfies (f4). Then the equation (3.2) has a positive solution $(\lambda,v(y))$ if and only if there exists $\alpha\in(\alpha_*,a)$ such that (3.5) has a solution $u(z)$ satisfying $u(0)=\alpha$, $L=T(\alpha)$ and $\lambda=L^2$.*

Next we prove the following lemmas on the asymptotic behavior of $T(\alpha)$ and $T'(\alpha)$ near $\alpha=\alpha_*$ and $\alpha=a$, and the proofs of these technical estimates are postponed to the end of this subsection.

Lemma 3.8. *Suppose that f satisfies (f4), then*

$$\lim_{\alpha\to a^-}T(\alpha)=\infty,\qquad \lim_{\alpha\to\alpha_*^+}T(\alpha)=\infty.$$

Lemma 3.9. *Suppose that f satisfies (f4), then*

$$\lim_{\alpha\to a^-}T'(\alpha)=\infty,\qquad \lim_{\alpha\to\alpha_*^+}T'(\alpha)=-\infty.$$

Now we can complete the proof of Theorem 3.6.

Proof of Theorem 3.6. From Lemma 3.8, given $L>0$ sufficiently large, (3.5) has two solutions, say u_1 and u_2 with $u_1(0)=\alpha_1^L\in(\alpha_*,\alpha_*+\delta)$ and $u_2(0)=\alpha_2^L\in(a-\delta,a)$, respectively, for some $\delta>0$, and $\alpha_2^L>\alpha_1^L$. Moreover such $\alpha_1^L\in(\alpha_*,\alpha_*+\delta)$ and $\alpha_2^L\in(a-\delta,a)$ are unique from the monotonicity of $T(\alpha)$ in these intervals from Lemma 3.9. Since $T(\alpha)$ is continuous differentiable for $\alpha \in [\alpha_* + \delta, a - \delta]$, $T(\alpha)$ is bounded. Without loss of generality, we may assume that $T(\alpha) < L$ on $[\alpha_* + \delta, a - \delta]$. Therefore for $L > 0$ (or $\lambda > 0$) sufficiently large (say $\lambda > \lambda_{**}$), (3.5) has exactly two solutions $u_1(z)$ and $u_2(z)$.

Let $v_1(y) = u_1(Ly)$ and $v_2(y) = u_1(Ly)$. Then v_1 and v_2 are the exactly two positive solutions of problem (3.2) with $\lambda = L^2$. To prove that $v_2(y) > v_1(y)$ for $y \in (-1, 1)$, we recall that for fixed $\lambda > 0$, (3.2) has a maximal solution v_M which is the iteration limit of the upper solution $v = a$ in the upper-lower solution method. Since v_1 and v_2 are the only positive solutions of (3.2), and $v_2(0) = \alpha_2^L > \alpha_1^L = v_1(0)$, we must have $v_M(y) \equiv v_2(y)$. Then the maximizing property of v_M and strong maximum principle imply that $v_2(y) = v_M(y) > v_1(y)$ for $y \in (-1, 1)$.

Finally we prove the stability of $v_i(y)$ for $i = 0, 1, 2$. It is clear that $\mu_1^0 = -\pi^2/4 + \lambda f'(0) < 0$, where $\eta_1 = \pi^2/4 > 0$ is the principal eigenvalue of $\psi'' + \eta\psi = 0$ on $(-1, 1)$ with $\psi(\pm 1) = 0$. For v_1 and v_2, we differentiate $\lambda(\alpha) = T^2(\alpha)$ to obtain that $\lambda'(v_1(0)) = 2T(v_1(0))T'(v_1(0)) < 0$ and $\lambda'(v_2(0)) = 2T(v_2(0))T'(v_2(0)) > 0$. From the arguments in Page 110 of [23], the linearized operator $L\phi = \phi'' + \lambda f'(v_i(y))\phi$ is always disconjugate on the interval $(0, 1)$ as $v_i'(y) < 0$ on $(0, 1)$. Then from [23, Proposition 6.1], the Morse index of v_2 is 0 and it is linearly stable, and the Morse index of v_1 is 1 and it is unstable. □

Finally we give the proofs of Lemmas 3.8 and 3.9.

Proof of Lemma 3.8. Through the change of variable $u = \alpha v$, we can rewrite the time mapping as

$$T(\alpha) = \frac{1}{\sqrt{2}} \int_0^1 \frac{1}{\sqrt{G(\alpha, v)}} \, \mathrm{d}v, \tag{3.9}$$

where

$$G(\alpha, v) = \frac{1}{\alpha^2}[F(\alpha) - F(\alpha v)], \quad \alpha \in (\alpha_*, a). \tag{3.10}$$

We first prove that $\lim\limits_{\alpha \to a^-} T(\alpha) = \infty$. From the Taylor expansion of $G(\alpha, v)$ at $v = 1$, it follows that

$$G(\alpha, v) = \frac{f(\alpha)(1 - v)}{\alpha} - \frac{f'(\alpha)(v - 1)^2}{2} + o((v - 1)^2). \tag{3.11}$$

Since $f'(a) < 0$, then $f'(\alpha) < 0$ for α close to a. For any $0 < \varepsilon < -f'(\alpha)/4$, there exists $\delta > 0$ such that for $v \in (1 - \delta, 1)$,

$$G(\alpha, v) < \frac{f(\alpha)(1 - v)}{\alpha} - \frac{f'(\alpha)(v - 1)^2}{2} + \varepsilon(v - 1)^2. \tag{3.12}$$

Then we have

$$\int_0^1 \frac{1}{\sqrt{G(\alpha,v)}}\,\mathrm{d}v > \int_{1-\delta}^1 \frac{1}{\sqrt{G(\alpha,v)}}\,\mathrm{d}v > \int_{1-\delta}^1 \frac{1}{\sqrt{\frac{f(\alpha)(1-v)}{\alpha} - \frac{f'(\alpha)(v-1)^2}{2} + \varepsilon(v-1)^2}}\,\mathrm{d}v,$$

$$= \int_0^\delta \frac{1}{\sqrt{\frac{f(\alpha)}{\alpha}x + \left(-\frac{f'(\alpha)}{2} + \varepsilon\right)x^2}}\,\mathrm{d}x = 2\sqrt{\frac{1}{B(\alpha)}}\sinh^{-1}\left(\frac{B(\alpha)\delta}{A(\alpha)}\right),$$

where

$$A(\alpha) = \frac{f(\alpha)}{\alpha}, \qquad B(\alpha) = -\frac{f'(\alpha)}{2} + \varepsilon.$$

Since $f(a) = 0$, it follows that $\lim_{\alpha\to a^-} T(\alpha) = \infty$.

Next we prove that $\lim_{\alpha\to\alpha_*^+} T(\alpha) = \infty$. Here we have the Taylor expansion of $G(\alpha, v)$ at $v = 0$:

$$G(\alpha,v) = \frac{F(\alpha)}{\alpha^2} - \frac{f'(0)}{2}v^2 + o(v^2),$$

and for any $0 < \varepsilon < -f'(0)/2$, there exists $\delta > 0$ such that for $0 < v < \delta$,

$$G(\alpha,v) < \frac{F(\alpha)}{\alpha^2} - \frac{f'(0)}{2}v^2 + \varepsilon v^2.$$

Then we have

$$\int_0^1 \frac{1}{\sqrt{G(\alpha,v)}}\,\mathrm{d}v > \int_0^\delta \frac{1}{\sqrt{G(\alpha,v)}}\,\mathrm{d}v > \int_0^\delta \frac{1}{\sqrt{\frac{F(\alpha)}{\alpha^2} + (\frac{-f'(0)}{2} + \varepsilon)v^2}}\,\mathrm{d}v,$$

$$= \frac{1}{\sqrt{\frac{-f'(0)}{2} + \varepsilon}} \ln\left| v + \sqrt{v^2 + \frac{F(\alpha)}{\alpha^2(\frac{-f'(0)}{2} + \varepsilon)}}\right| \Bigg|_0^\delta.$$

Therefore, $\lim_{\alpha\to\alpha_*^+} T(\alpha) = \infty$ as $F(\alpha) \to 0$ when α_*^+. □

Proof of Lemma 3.9. Differentiating $T(\alpha)$ with respect to α, we have

$$T'(\alpha) = \frac{1}{\sqrt{2}}\int_0^1 \frac{H(\alpha,v) - \frac{1}{2}\alpha H_\alpha(\alpha,v)}{(H(\alpha,v))^{3/2}}\,\mathrm{d}v,$$

where $H(\alpha, v) = F(\alpha) - F(\alpha v)$, $H_\alpha(\alpha, v) = \dfrac{\partial[F(\alpha) - F(\alpha v)]}{\partial\alpha} = f(\alpha) - vf(\alpha v)$.

First we study the behavior of $T'(\alpha)$ as $\alpha \to a^-$. The Taylor expansions of $H(\alpha, v)$ and $H_\alpha(\alpha, v)$ at $v = 1$ are

$$H(\alpha, v) = -\alpha f(\alpha)(v-1) - \frac{1}{2}\alpha^2 f'(\alpha)(v-1)^2 + o((v-1)^2), \tag{3.13}$$

and

$$H_\alpha(\alpha, v) = (-f(\alpha) - \alpha f'(\alpha))(v-1) + (-\alpha f'(\alpha) - \frac{1}{2}\alpha^2 f''(\alpha))(v-1)^2 + o((v-1)^2). \tag{3.14}$$

For any $\delta > 0$, we have

$$\sqrt{2}T'(\alpha) = \int_0^{1-\delta} \frac{H(\alpha, v) - \frac{1}{2}\alpha H_\alpha(\alpha, v)}{(H(\alpha, v))^{3/2}}\, \mathrm{d}v + \int_{1-\delta}^{1} \frac{H(\alpha, v) - \frac{1}{2}\alpha H_\alpha(\alpha, v)}{(H(\alpha, v))^{3/2}}\, \mathrm{d}v. \tag{3.15}$$

Since the first term in (3.15) is bounded, we just need to show the second term, denoted by $J_1(\alpha)$, tends to ∞ when $\alpha \to a^-$. After substituting (3.13) and (3.14) into $J_1(\alpha)$, for some certain $\varepsilon > 0$, we have

$$\begin{aligned} J_1(\alpha) &\geq \int_{1-\delta}^{1} \frac{[-\frac{1}{2}\alpha f(\alpha) + \frac{1}{2}\alpha^2 f'(\alpha)](v-1) + \left(\frac{1}{4}\alpha^3 f''(\alpha) - \varepsilon\right)(v-1)^2}{(-\alpha f(\alpha)(v-1) + \left(-\frac{1}{2}\alpha^2 f'(\alpha) + \varepsilon\right)(v-1)^2)^{3/2}}\, \mathrm{d}v \\ &= \int_0^{\delta} \frac{[\frac{1}{2}\alpha f(\alpha) - \frac{1}{2}\alpha^2 f'(\alpha)]x + \left(\frac{1}{4}\alpha^3 f''(\alpha) - \varepsilon\right)x^2}{\left(\alpha f(\alpha)x + \left(-\frac{1}{2}\alpha^2 f'(\alpha) + \varepsilon\right)x^2\right)^{3/2}}\, \mathrm{d}x. \end{aligned} \tag{3.16}$$

Note that $f(\alpha) > 0$, and $f'(\alpha) < 0$ for α close to a. By choosing δ smaller, we may assume that for $x \in [0, \delta]$,

$$\frac{1}{2}\left(\alpha f(\alpha) - \alpha^2 f'(\alpha)\right)x + \left(\frac{1}{4}\alpha^3 f''(\alpha) - \varepsilon\right)x^2 > \frac{1}{4}\left(\alpha f(\alpha) - \alpha^2 f'(\alpha)\right)x. \tag{3.17}$$

Then from (3.17) and (3.16), we obtain

$$\begin{aligned} J_1(\alpha) &\geq \int_0^{\delta} \frac{\frac{1}{4}[\alpha f(\alpha) - \alpha^2 f'(\alpha)]x}{\left(\alpha f(\alpha)x + \left(-\frac{1}{2}\alpha^2 f'(\alpha) + \varepsilon\right)x^2\right)^{3/2}}\, \mathrm{d}x \\ &= \frac{[\alpha f(\alpha) - \frac{1}{2}\alpha^2 f'(\alpha)]\delta^{1/2}}{2\alpha f(\alpha)\left(\alpha f(\alpha) + \left(-\frac{1}{2}\alpha^2 f'(\alpha) + \varepsilon\right)\delta\right)^{1/2}}. \end{aligned} \tag{3.18}$$

Since $f(\alpha) \to f(a) = 0$, and $f'(\alpha) \to f'(a) < 0$ as $\alpha \to a^-$, and δ is fixed, (3.18) implies that $J_1(\alpha) \to \infty$, and consequently $T'(\alpha) \to \infty$, as $\alpha \to a^-$.

Finally we consider the asymptotic behavior of $T'(\alpha)$ as $\alpha \to \alpha_*^+$. Similar as above, we can break $T'(\alpha)$ into the sum of three integrals on $(0, \delta)$ and $(\delta, 1)$, for some $\delta > 0$. Since the integral on $(\delta, 1)$ is bounded, we only need to show the integral on $(0, \delta)$ (denoted by $J_2(\alpha)$) tends to $-\infty$ as α tends to α_*^+. The Taylor expansions of $H(\alpha, v)$ and $H_\alpha(\alpha, v)$ around $v = 0$ are, respectively,

$$H(\alpha, v) = F(\alpha) - \frac{1}{2}\alpha^2 f'(0)v^2 + o(v^2), \tag{3.19}$$

and

$$H_\alpha(\alpha, v) = f(\alpha) - \alpha^2 f'(0)v^2 + o(v^2). \tag{3.20}$$

Substituting (3.19) and (3.20) into the corresponding integral on $(0, \delta)$, we have

$$J_2(\alpha) \le \int_0^\delta \frac{\left(F(\alpha) - \frac{1}{2}\alpha f(\alpha)\right) + \left(-\alpha^2 f'(0) + \frac{1}{2}\alpha^3 f'(0) + \varepsilon\right) v^2}{\left(F(\alpha) + \left(-\frac{1}{2}\alpha^2 f'(0) - \varepsilon\right) v^2\right)^{3/2}} \mathrm{d}v. \tag{3.21}$$

By choosing δ smaller, we may assume that for $v \in [0, \delta]$,

$$\left(F(\alpha) - \frac{1}{2}\alpha f(\alpha)\right) + \left(-\alpha^2 f'(0) + \frac{1}{2}\alpha^3 f'(0) + \varepsilon\right) v^2 < \frac{1}{2}\left(F(\alpha) - \frac{1}{2}\alpha f(\alpha)\right). \tag{3.22}$$

Then from (3.17) and (3.21), we have

$$\begin{aligned} J_2(\alpha) &\le \int_0^\delta \frac{\left(F(\alpha) - \frac{1}{2}\alpha f(\alpha)\right)}{2\left(F(\alpha) + \left(-\frac{1}{2}\alpha^2 f'(0) - \varepsilon\right) v^2\right)^{3/2}} \mathrm{d}v, \\ &= \frac{\left(F(\alpha) - \frac{1}{2}\alpha f(\alpha)\right)\delta}{2F(\alpha)(\left(F(\alpha) + \left(-\frac{1}{2}\alpha^2 f'(0) - \varepsilon\right)\delta^2\right)^{3/2})}. \end{aligned} \tag{3.23}$$

When $\alpha \to \alpha_*$, we have $F(\alpha) \to F(\alpha_*) = 0$ and $F(\alpha) - \frac{1}{2}\alpha f(\alpha) \to -\frac{1}{2}\alpha_* f(\alpha_*) < 0$, thus $J_2(\alpha) \to -\infty$, which implies that $T'(\alpha) \to -\infty$ as $\alpha \to \alpha_*^+$. □

3.2. *The case of* $|\varepsilon| \ll 1$

In this part, we assume that (BSH) holds for $\varepsilon = 0$. Namely, 0 and v_{max} are two linearly stable solution of (3.2), and any other solution of (3.2) between 0 and v_{max} is linearly unstable. Then, by employing a perturbation argument we aim to show that the (BSH) condition persists when ε varies near 0.

Lemma 3.10. *There exists* $\varepsilon_0 > 0$ *such that* (1.1) *possesses a linearly stable, positive* ω*-periodic solution* p^ε_{max} *with* $p^\varepsilon_{max}(0, \cdot) \to v_{max}$ *in* $C_0^{2+\delta}$ *as* $\varepsilon \to 0$.

Proof. Let $\{\hat{Q}^\varepsilon_t\}_{t \ge 0}$ be the solution semiflow of (1.1). For $\varphi \in C_0^{2+\delta}[-1, 1]$, define

$$Z(t, \cdot; \phi) \doteq \left(D_\varphi \widehat{Q}^\varepsilon_t[\varphi]\Big|_{\varepsilon=0 \varphi=v_{max}} \right) \phi$$

and then $Z(t, y; \phi)$ satisfies

$$\begin{cases} Z_t = Z_{yy} + \lambda f_u(v_{max})Z, & y \in (-1,1),\ t>0,\\ Z(t,\pm 1) = 0, & t>0,\\ Z(0,y) = \phi(y), & y \in (-1,1). \end{cases}$$

Let us also define $F : C_0^{2+\delta}[-1,1] \times \mathbb{R} \to C_0^{2+\delta}[-1,1]$ by

$$F(\varphi, \varepsilon) \doteq \varphi - \widehat{Q}^{\varepsilon}_{\omega}[\varphi]. \tag{3.24}$$

We claim that $D_\varphi F(v_{max}, 0)$ is one-to-one. Note that $D_\varphi F(v_{max}, 0)\phi = 0$ implies $Z(\omega, \cdot; \phi) = \phi$. From that it follows that $Z(t, \cdot; \phi)$ is ω-periodic in t, then we further infer that 0 is an eigenvalue related to v_{max} if and only if $\phi \not\equiv 0$. Recall that v_{max} is a linearly stable trivial periodic solution for (1.1) with $\varepsilon = 0$. Therefore, 0 can not be an eigenvalue. Thus, $\phi = 0$, which implies $D_\varphi F(v_{max}, 0)$ is one-to-one. According to the results from [16, section II.12.3], $Z(\omega, \cdot; \phi)$ is compact. By virtue of the Fredholm theory for compact operators, $D_\varphi F(v_{max}, 0)$ is bijective. Note that $F(v_{max}, 0) = 0$, then the implicit function theorem implies that there exists some sufficiently small $\varepsilon_0 > 0$ such that $F(\varphi(\varepsilon), \varepsilon) = 0$, that is, $\widehat{Q}^{\varepsilon}_{\omega}[\varphi(\varepsilon)] = \varphi(\varepsilon)$, for $|\varepsilon| < \varepsilon_0$. In addition, we denote the periodic solutions around v_{max} by $p^{\varepsilon}_{max}(t, y)$.

To prove that p^{ε}_{max} is linearly stable, let us assume, for the sake of contradiction, that there exists a sequence $\{\varepsilon_n\}_{n=1,2,\ldots}$ such that $\varepsilon_0 > \cdots > \varepsilon_n > \varepsilon_{n+1} > \cdots > 0$, $\varepsilon_n \to 0$, as $n \to \infty$, and $\mu_1^{\varepsilon_n} \geq 0$, here $\mu_1^{\varepsilon_n}$ and ψ^{ε_n} denote the principal eigenvalue and eigenfunction of the corresponding periodic parabolic eigenvalue problem related to $p^{\varepsilon_n}_{max}$, then, we have

$$\begin{cases} (\psi^{\varepsilon_n})_t + \mu_1^{\varepsilon_n}\psi^{\varepsilon_n} = (\psi^{\varepsilon_n})_{yy} + \left[\lambda f'(p^{\varepsilon_n}_{max}) + \varepsilon_n h_u(t, p^{\varepsilon_n}_{max})\right]\psi^{\varepsilon_n}, & y \in (-1,1),\ t>0,\\ \psi^{\varepsilon_n}(t, \pm 1) = 0, & t>0,\\ \psi^{\varepsilon_n}(t,y) = \psi^{\varepsilon_n}(t+\omega, y), & y \in (-1,1),\ t>0,\\ \|\psi^{\varepsilon_n}(0,\cdot)\|_\infty = 1. \end{cases} \tag{3.25}$$

On the other hand, by the implicit function theorem, we have $\varepsilon_n \to 0$ and $p^{\varepsilon_n}_{max} \to v_{\max}$ as $n \to \infty$, we infer that $p^{\varepsilon_n}_{max}$ is bounded from above by $v_{max} + 1$, then by $a + 1$ for all large n's. Recall that $h_u(t, u)$ is uniformly bounded in $t \geq 0$ and $u \geq 0$, we deduce, for all large n's, that

$$|\mu_1^{\varepsilon_n}| \leq \sup_{t\in[0,\omega] s\in[0,a+1]} \left[\lambda |f'(s)| + \varepsilon_n |h_u(t,s)|\right] + \frac{\pi^2}{4} < \infty,$$

through [16, Section II, Lemma 15.3 and 15.5], where $\pi^2/4$ is the principal eigenvalue of $\psi'' + \eta\psi = 0$ on $(-1, 1)$ with $\psi(\pm 1) = 0$. Thus we can obtain that, by standard regularity theory,

$$\left(\psi^{\varepsilon_n}, p^{\varepsilon_n}_{max}, \mu_1^{\varepsilon_n}\right) \to (\psi, v_{max}, \mu_1) \text{ as } n \to \infty.$$

Then $\mu_1 \geq 0$ and ψ solves the following eigenvalue problem

$$\begin{cases} \mu_1\psi = \psi_{yy} + \lambda f'(v_{max})\psi, & y \in (-1,1),\ t>0,\\ \psi(\pm 1) = 0, \end{cases}$$

which implies that v_{max} is not a linearly stable solution to (3.2), a contradiction to the assumption at the very beginning of this subsection. Therefore, p^{ε}_{max} is linearly stable for all small ε. □

Now we are ready to present the main result of this subsection.

Theorem 3.11. *Assume that* (3.2) *admits two linearly stable steady states, say* 0 *and* v_{max}*, between which all other steady states are linearly unstable. Then there exists* $\varepsilon_0 > 0$ *such that* (1.1) *admits the bistability structure for* $|\varepsilon| < \varepsilon_0$*, that is, (BSH) holds for* (1.1) *and* $|\varepsilon| < \varepsilon_0$.

Proof. With the help of Lemma 3.10, we see that (1.1) possesses a positive linearly stable ω-periodic solution p^{ε}_{max}. Since $f(0) = h(t, 0) = 0$ for all $t \geq 0$, $f'(0) < 0$, and $h_u(t, 0)$ is locally uniformly bounded, we deduce that, for all sufficiently small ε, 0 is linearly stable according to [16, Page 35, Section II, Lemma 14.2] and [16, Page 73, Section III, Theorem 23.2]. Therefore, it suffices to show that any other time periodic solution, say p^{ε}, is linearly unstable for the same ε as above. To that end, like in Lemma 3.10, let us assume for the sake of contradiction that there exists $\varepsilon_n \to 0$ as $n \to \infty$, such that $0 < p^{\varepsilon_n} < p^{\varepsilon_n}_{max}$ with $\mu^{\varepsilon_n}_1 \leq 0$, and denote the corresponding eigenfunction by ψ^{ε_n}, then

$$\begin{cases} \psi^{\varepsilon_n}_t + \mu^{\varepsilon_n}_1 \psi^{\varepsilon_n} = \psi^{\varepsilon_n}_{yy} + \left[\lambda f'(p^{\varepsilon_n}) + \varepsilon_n h_u(t, p^{\varepsilon_n})\right] \psi^{\varepsilon_n}, & y \in (-1, 1),\ t > 0, \\ \psi^{\varepsilon_n}(t, \pm 1) = 0, & t > 0, \\ \psi^{\varepsilon_n}(t, y) = \psi^{\varepsilon_n}(t + \omega, y), & y \in (-1, 1),\ t > 0, \\ \|\psi^{\varepsilon_n}(0, \cdot)\|_\infty = 1. \end{cases}$$

Moreover, by Lemma 3.10, we infer from the fact that $\sup\limits_{t\in[0,\omega]} |p^{\varepsilon_n}_{max}(t, \cdot) - v_{max}(\cdot)|_\infty \to 0$ as $\varepsilon_n \to 0$ that p^{ε_n} is bounded from above by $v_{max} + 1$ for all large n's. Recall that $\mu^{\varepsilon_n}_1$ is uniformly bounded in large n's as in Lemma 3.10, thus we can obtain that, by standard regularity theory,

$$\left(\psi^{\varepsilon_n}, p^{\varepsilon_n}, \mu^{\varepsilon_n}_1\right) \to (\psi^\infty, p^\infty, \mu^\infty_1) \text{ as } n \to \infty,$$

where ψ^∞ and p^∞ satisfy

$$\begin{cases} \psi^\infty_t + \mu^\infty_1 \psi^\infty = \psi^\infty_{yy} + \lambda f'(p^\infty)\psi^\infty, & y \in (-1, 1),\ t > 0, \\ \psi^\infty(t, \pm 1) = 0, & t > 0, \\ \psi^\infty(t, y) = \psi^\infty(t + \omega, y), & y \in (-1, 1),\ t > 0, \\ \|\psi^\infty(0, \cdot)\|_\infty = 1 \end{cases}$$

and

$$\begin{cases} p^\infty_t = p^\infty_{yy} + \lambda f(p^\infty), & y \in (-1, 1), t > 0, \\ p^\infty(t, \pm 1) = 0, & t > 0, \\ p^\infty(t, y) = p^\infty(t + \omega, y), & y \in (-1, 1), t > 0. \end{cases}$$

Note that $\mu^\infty_1 \leq 0$. Thus by virtue of Lemma 3.1, p^∞ is independent of t and it is a solution of (3.2). As already assumed that for (3.2) any solution between two linearly stable ones is linearly unstable, we obtain $p^\infty \in \{0, v_{max}\}$. Finally, to reach a contradiction, we show that p^∞ is neither 0 nor v_{max}. Indeed, let $\hat{Q}^{\varepsilon_n}_\omega$ be the related period map for (1.2) with $\varepsilon = \varepsilon_n$. Let $p^{\varepsilon_n}_{max}$ be the linearly stable solution established in Lemma 3.10. Then there exists η, independent of n, such that

$$(\widehat{Q}^{\varepsilon_n}_{\omega})^k[\phi] \to p^{\varepsilon_n}_{max}(0,\cdot), \quad \forall \phi \in B_n := [p^{\varepsilon_n}_{max}(0,\cdot) - \eta\psi^{\varepsilon_n}(0,\cdot), p^{\varepsilon_n}_{max}(0,\cdot) + \eta\psi^{\varepsilon_n}(0,\cdot)]_X.$$

Thus, $p^{\varepsilon_n} \notin B_n$. Passing n to infinity, we obtain that p^∞ is outside of a neighborhood of v_{max}. Consequently, $p^\infty \neq v_{\max}$. For a similar reason, $p^\infty \neq 0$. Therefore, the contradiction is reached. □

Combining with Theorems 3.3 and 3.11, we obtain the following corollary.

Corollary 3.12. *If $g(t,u) = \lambda u(1-u)(u-\alpha) + \varepsilon h(t,u)$ with $\alpha \in (0, 1/2)$ and $h \in C^{1,2}(\mathbb{R}\times\mathbb{R})$ is ω-periodic in t. Then there exists $\lambda_* > 0$ such that for each $\lambda > \lambda_*$ there exists $\varepsilon_0 > 0$ such that problem (1.1) admits an ω-periodic bistable traveling wave as long as $|\varepsilon| \leq \epsilon_0$.*

Data availability

No data was used for the research described in the article.

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