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GLOBAL STABILITY OF SPATIALLY NONHOMOGENEOUS STEADY STATE SOLUTION IN A DIFFUSIVE HOLLING-TANNER PREDATOR-PREY MODEL

WENJIE NI, JUNPING SHI, AND MINGXIN WANG

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Abstract. The global stability of the nonhomogeneous positive steady state solution to a diffusive Holling-Tanner predator-prey model in a heterogeneous environment is proved by using a newly constructed Lyapunov function and estimates of nonconstant steady state solutions. The techniques developed here can be adapted for other spatially heterogeneous consumer-resource models.

1. Introduction

In this paper we study the global dynamics of the following diffusive Holling-Tanner predator-prey model in a heterogeneous environment:

$$\begin{cases} u_t = d_1(x)\Delta u + u\left(a(x) - u - \frac{bv}{1+ru}\right), & x \in \Omega,\ t > 0, \\ v_t = d_2(x)\Delta v + \mu v\left(1 - \frac{v}{u}\right), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here $u(x,t)$ and $v(x,t)$ are the density functions of prey and predator respectively, and Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$; a no-flux boundary condition is imposed on $\partial\Omega$ so that the ecosystem is closed to exterior environment. $d_1(x)$ and $d_2(x)$ are the spatially dependent diffusion coefficient functions of prey and predator respectively; $a(x)$ is the spatially heterogeneous resource function, and other parameters b, r and μ are assumed to be constants. The non-spatial version of (1.1) was introduced in [12, 24] as one of prototypical mathematical models describing predator-prey interactions.

For the non-spatial ODE model corresponding to (1.1), it is known that for certain parameter range the unique positive steady state is globally asymptotically stable, while in other parameter range a unique limit cycle exists [8, 9]. The spatial model (1.1) in a homogeneous environment (assuming d_i, a are constants) was first studied in [19]. The global stability of the positive constant steady state solution for the homogeneous case was proved in [2, 20, 23] under different conditions on parameters, and spatiotemporal pattern formation for the homogeneous system (1.1) was considered in [11]. When $r = 0$ in (1.1), the system becomes to the

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Leslie-Gower predator-prey model. The global and local stability for that case including delay effect was investigated in [3,5,21,22] when a and d_i are constants, see also [4,13,15,16] for related work.

We define the nonlinearities in (1.1) to be

$$f(y,u,v) := u\left(y - u - \frac{bv}{1+ru}\right), \quad g(u,v) := \mu v\left(1 - \frac{v}{u}\right), \tag{1.2}$$

and we also denote

$$\bar{a} := \max_{\overline{\Omega}} a(x), \quad \underline{a} := \min_{\overline{\Omega}} a(x). \tag{1.3}$$

Our results on the global dynamics of (1.1) are as follows:

Theorem 1.1. *Let $\mu > 0$ and $r \geq 0$ be constants. Suppose that a, $d_i \in C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$, and $a(x) > 0$, $d_i(x) > 0$ on $\overline{\Omega}$; and the initial functions $u_0 \in C(\overline{\Omega})$ and $v_0 \in C(\overline{\Omega})$ satisfy $u_0 \geq, \not\equiv 0$ and $v_0 \geq, \not\equiv 0$ on $\overline{\Omega}$. Assume that*

$$0 < b < \underline{a}/\bar{a} := \frac{\min_{\overline{\Omega}} a(x)}{\max_{\overline{\Omega}} a(x)}. \tag{1.4}$$

(i) *There exists a unique positive solution $(\underline{u}_\infty, \bar{u}_\infty, \underline{v}_\infty, \bar{v}_\infty)$ to the system of equations:*

$$f(\bar{a}, \bar{u}_\infty, \underline{v}_\infty) = 0, \quad f(\underline{a}, \underline{u}_\infty, \bar{v}_\infty) = 0, \quad g(\bar{u}_\infty, \bar{v}_\infty) = 0, \quad g(\underline{u}_\infty, \underline{v}_\infty) = 0.$$

(ii) *Let $(u(x,t), v(x,t))$ be the positive solution of problem (1.1). Then*

$$\begin{cases} \underline{u}_\infty \leq \liminf_{t\to\infty} u(x,t) \leq \limsup_{t\to\infty} u(x,t) \leq \bar{u}_\infty, \\ \underline{v}_\infty \leq \liminf_{t\to\infty} v(x,t) \leq \limsup_{t\to\infty} v(x,t) \leq \bar{v}_\infty. \end{cases} \tag{1.5}$$

(iii) *If in addition,*

$$b < (1 + 2r\underline{u}_\infty - r\underline{a})\left[\frac{\min d_1(x) \min d_2(x)}{\max d_1(x) \max d_2(x)}\right]^{1/2} \left(\frac{\underline{u}_\infty}{\bar{u}_\infty}\right)^{5/2}, \tag{1.6}$$

then the problem (1.1) has a unique positive steady state solution (u_*, v_*), and $\lim_{t\to\infty} u(x,t) = u_*(x)$ and $\lim_{t\to\infty} v(x,t) = v_*(x)$ in $C^2(\overline{\Omega})$.

Remark 1.2.

(1) If the reaction function $g(u,v)$ in the equation of predator is changed to $\tilde{g}(u,v) := \mu(1 - v/(k+u))$ for some constant $k \geq 0$, and $\bar{a} > \underline{a}$, one can similarly show the conclusions in Theorem 1.1 with the condition $b < \underline{a}/\bar{a}$ replaced by $b < \underline{a}/(\bar{a}+k)$ and the condition(1.6) replaced by

$$b < \left(1+2r\underline{u}_\infty - r\underline{a} + \frac{rbk}{1+r\underline{u}_\infty}\right)\left[\frac{\min d_1(x) \min d_2(x)}{\max d_1(x) \max d_2(x)}\right]^{1/2} \left(\frac{\underline{u}_\infty + k}{\bar{u}_\infty + k}\right)^2 \left(\frac{\underline{u}_\infty}{\bar{u}_\infty}\right)^{1/2},$$

respectively.

(2) The global stability of the positive steady state solution of (1.1) in the heterogeneous environment in Theorem 1.1 also holds for $r = 0$ which is the Leslie-Gower predator-prey model, and in that case, the condition (1.6) is simplified to

$$b < \left[\frac{\min d_1(x) \min d_2(x)}{\max d_1(x) \max d_2(x)}\right]^{1/2} \left(\frac{\min a(x) - b \max a(x)}{\max a(x) - b \min a(x)}\right)^{5/2}.$$

If d_1, d_2 and a are all constants, then the problem (1.1) admits a unique positive constant steady state which can be solved as

(1.7)

$$u_\infty = \underline{u}_\infty = \bar{u}_\infty = \underline{v}_\infty = \bar{v}_\infty = \frac{-(b+1-ar)+\sqrt{(b+1-ar)^2+4ar}}{2r}, \quad \text{when } r>0,$$

$$u_\infty = \underline{u}_\infty = \bar{u}_\infty = \underline{v}_\infty = \bar{v}_\infty = \frac{a}{1+b}, \quad \text{when } r=0.$$

Then the global stability of the constant steady state (u_∞, u_∞) in Theorem 1.1 holds under the assumption $b<1$, which is the earlier result of [2]. In this case (1.6) is not needed as part (ii) already implies the global stability. Part (iii) of Theorem 1.1 shows the global stability in the heterogeneous environment, and the condition (1.6) depends on the level of heterogeneity of $a(x)$ and $d_i(x)$. Indeed (1.6) can be satisfied when a is nearly constant or when b is sufficiently small (see Remark 2.7). We also remark that the global stability of the positive constant steady state of (1.1) is proved in [22, 23] with weaker condition on b but constant a and $d_1 = d_2$.

The proof of global stability combines the upper-lower solution method used in [2, 23] and a newly developed Lyapunov functional method. For spatially homogeneous case, the upper-lower solution method alone can prove the global stability of the constant steady state of (1.1), but in the spatially heterogeneous case, it only proves that the solutions are attracted into a rectangle defined as in (1.5). The Lyapunov function we use inside the attraction zone takes the form

$$\int_\Omega \int_{u_*(x)}^{u(x,t)} \frac{u_*(x)}{d_1(x)} \frac{s-u_*(x)}{s} \mathrm{d}s\mathrm{d}x + \int_\Omega \int_{v_*(x)}^{v(x,t)} \frac{v_*(x)}{d_2(x)} \frac{s-v_*(x)}{s} \mathrm{d}s\mathrm{d}x$$

where $(u_*(x), v_*(x))$ is a positive steady state solution of (1.1). The form of the Lyapunov function when d_i and a are constants is well-known, and here we use a spatially heterogenous form with weigh functions $u_*(x)/d_1(x)$ and $v_*(x)/d_2(x)$ which is first used in [14] for proving the global stability of positive steady state of diffusive Lotka-Volterra competition system in the heterogeneous environment. It turns out that the weight functions encode the spatial heterogeneity of the environment so a non-constant steady state is achieved asymptotically. The new Lyapunov function developed here may be a useful tool to explore more general diffusive predator-prey models in the nonhomogeneous environment [4, 6].

2. Proof of main results

2.1. Existence of positive solutions. In the subsection, we show the existence and uniqueness of the solution to (1.1), and the existence of positive solution to the corresponding steady state problem:

$$\begin{cases} -d_1(x)\Delta u = u\left(a(x) - u - \frac{bv}{1+ru}\right), & x \in \Omega, \\ -d_2(x)\Delta v = \mu v\left(1-\frac{v}{u}\right), & x\in\Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

We recall that the system (1.1) is called to be uniformly persistent (see, e.g., [7, Page 390]) if all positive solutions satisfy $\liminf_{t\to\infty} u(x,t) > 0$ and $\liminf_{t\to\infty} v(x,t) > 0$ for all $x \in \bar{\Omega}$, and it is permanent (see, e.g., [1, 10]) if they also satisfy

$\limsup_{t\to\infty} u(x,t) \le M$ and $\limsup_{t\to\infty} v(x,t) \le M$ for some $M > 0$. The following result shows the basic dynamics of (1.1).

Proposition 2.1. *Let $b > 0$, $\mu > 0$ and $r \ge 0$ be constants. Suppose that a, $d_i \in C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$, and $a(x) > 0$, $d_i(x) > 0$ on $\overline{\Omega}$. The initial functions $u_0 \in C(\overline{\Omega})$ and $v_0 \in C(\overline{\Omega})$ satisfy $u_0 \ge, \not\equiv 0$ and $v_0 \ge, \not\equiv 0$ on $\overline{\Omega}$.*

(1) *The problem (1.1) has a unique globally-defined solution $(u(x,t), v(x,t))$ satisfying $u(x,t) > 0$, $v(x,t) > 0$ for $(x,t) \in \overline{\Omega} \times (0,\infty)$.*

(2) *For any given small $\epsilon_1 > 0$, there exist a constant $T_1 > 0$ determined by ϵ_1 and a constant $\epsilon_2 \in (0, \epsilon_1]$ depending on initial functions such that*

$$\epsilon_2 \le u(x,t), v(x,t) \le \bar{a} + \epsilon_1, \quad \forall\, x \in \overline{\Omega},\ t \ge T_1, \tag{2.2}$$

which implies that the problem (1.1) is permanent. Moreover the problem (1.1) has a positive steady state solution $(u_*(x), v_*(x))$ lying in $[\epsilon_2, \bar{a}+\epsilon_1] \times [\epsilon_2, \bar{a}+\epsilon_1]$.

(3) *There exits a constant $C = C(\epsilon_2) > 0$ such that*

$$\max_{t\ge T_1} \|u(\cdot,t)\|_{C^{2+\alpha}(\overline{\Omega})},\ \max_{t\ge T_1} \|v(\cdot,t)\|_{C^{2+\alpha}(\overline{\Omega})} \le C. \tag{2.3}$$

Proof. (1) We will use the upper and lower solutions method to prove the existence and uniqueness of positive solution of problem (1.1). Clearly, the problem (1.1) is a mixed quasi-monotone system in the domain $\{u > 0,\ v \ge 0\}$. Denote

$$M = \max\left\{\bar{a},\ \max_{x\in\overline{\Omega}} u_0(x),\ \max_{x\in\overline{\Omega}} v_0(x)\right\}.$$

Let $\underline{v}(x,t) = 0$, $\bar{u}(x,t) = \bar{v}(x,t) \equiv M$, and let $\underline{u}(x,t)$ be the unique positive solution of

$$\begin{cases} u_t = d_1(x)\Delta u + u\left(a(x) - u - \frac{bM}{1+ru}\right), & x \in \Omega,\ t > 0,\\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0,\\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Then $(\bar{u}(x,t), \bar{v}(x,t))$ and $(\underline{u}(x,t), \underline{v}(x,t))$ are a pair of coupled ordered upper and lower solutions of the problem (1.1). Hence (1.1) has a unique global solution $(u(x,t), v(x,t))$ satisfying

$$0 < \underline{u}(x,t) \le u(x,t) \le M, \quad 0 \le v(x,t) \le M, \quad \forall\, x \in \overline{\Omega},\ t \ge 0. \tag{2.4}$$

Moreover, by the strong maximum principle we also have $v(x,t) > 0$ for $x \in \overline{\Omega}$ and $t > 0$.

(2) From the first equation of (1.1), $u(x,t)$ satisfies

$$\begin{cases} u_t \le d_1(x)\Delta u + u\left(\max_{x\in\bar{\Omega}} a(x) - u\right), & x \in \Omega,\ t > 0,\\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t \ge 0,\\ u(x,0) = u_0(x) > 0, & x \in \Omega. \end{cases}$$

It is deduced by the comparison principle of parabolic equations that

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} u(x,t) \le \max_{x\in\bar{\Omega}} a(x) = \bar{a}.$$

Thus, for any given $\varepsilon > 0$, there is a $T > 0$ such that $u(x,t) < \bar{a}+\varepsilon$ for $x \in \overline{\Omega}$, $t \ge T$. From the second equation of (1.1), $v(x,t)$ satisfies

$$v_t \le d_2\Delta v + \mu v\big(1 - v/(\bar{a}+\varepsilon)\big), \quad x > \Omega,\ t > T.$$

Thanks to the boundary condition $\frac{\partial v}{\partial \nu} = 0$, we could use the comparison principle of parabolic equations to conclude that $\limsup_{t\to\infty} \max_{\overline{\Omega}} v(x,t) \le \bar{a} + \varepsilon$. The arbitrariness of ε implies

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} v(x,t) \le \bar{a}.$$

Hence, for a small fixed $\epsilon_1 > 0$ satisfying

$$b < (\underline{a} - \epsilon_1)/(\bar{a} - \epsilon_1), \tag{2.5}$$

there exists a $T_1 > 0$ such that the following estimates hold

$$u(x,t), v(x,t) \le \bar{a} + \epsilon, \quad x \in \overline{\Omega},\ t \ge T_1.$$

Since $u(x,T_1), v(x,T_1) > 0$ on $\overline{\Omega}$, we can choose a small constant $\epsilon_2 > 0$ belonging to $(0, \epsilon_1]$ such that $u(x,T), v(x,T) > \epsilon_2$ on $\overline{\Omega}$. Denote

$$\bar{u}_1 := \bar{a} + \epsilon_1, \quad \underline{u}_1 := \epsilon_2, \quad \bar{v}_1 := \bar{a} + \epsilon_1, \quad \underline{v}_1 := \epsilon_2.$$

Then from $\epsilon_2 \le \epsilon_1$ and (2.5), we obtain that

$$\begin{cases} a(x) - \bar{u}_1 - \frac{b\underline{v}_1}{1 + r\bar{u}_1} \le a(x) - (\bar{a} + \epsilon_1) < 0, \\ a(x) - \underline{u}_1 - \frac{b\bar{v}_1}{1 + r\bar{u}_1} \ge a(x) - \epsilon_1 - b(\bar{a} + \epsilon_1) > 0, \\ 1 - \underline{v}_1/\underline{u}_1 = 1 - \bar{v}_1/\bar{u}_1 = 0, \end{cases} \tag{2.6}$$

which indicates that $(\bar{u}_1, \underline{u}_1, \bar{v}_1, \underline{v}_1)$ is a pair of coupled ordered upper and lower solutions of the problem (1.1) with initial density $(u(x,T_1), v(x,T_1))$. Hence (2.2) holds. A simple calculation shows that $(\bar{u}_1, \underline{u}_1, \bar{v}_1, \underline{v}_1)$ is also the coupled ordered upper and lower solutions of the problem (2.1). Thus the problem (2.1) has a positive solution (u_*, v_*) in the region $[\underline{u}_1, \bar{u}_1] \times [\underline{u}_1, \bar{u}_1]$. For (3), recalling that $u(x,t), v(x,t) > \epsilon_2$ for $x \in \bar{\Omega}$, $t \ge T_1$, we could show (2.3) by the similar arguments as [25, Theorem 2.1]. The proof is completed. □

2.2. **Estimates for positive solutions and steady state solutions.** From Proposition 2.1, under the assumption (1.4) every positive solution of (1.1) has a positive lower bound which may depend on its initial value. In this subsection, a uniform lower bound for positive solutions of (1.1) is obtained. Moreover, by an iterating process using the idea of [18], we obtain more accurate estimates for positive solutions and steady solutions of problem (1.1).

Denote

$$\bar{u}_1 = \bar{v}_1 := \bar{a} + \epsilon_1, \quad \underline{u}_1 = \underline{v}_1 := \epsilon_2, \quad Q := [\underline{u}_1, \bar{u}_1] \times [\underline{v}_1, \bar{v}_1]. \tag{2.7}$$

where ϵ_1 and ϵ_2 are given by Proposition 2.1. It is clear that

$$\begin{cases} f_y(y,u,v) \ge 0,\ f_v(y,u,v) \le 0,\ g_u(y,u,v) \ge 0, & y,u,v > 0, \\ |f(y,u_1,v_1) - f(y,u_2,v_2)| \le K(|u_1 - u_2| + |v_1 - v_2|), & y \ge 0, (u,v) \in Q, \\ |g(u_1,v_1) - g(u_2,v_2)| \le K(|u_1 - u_2| + |v_1 - v_2|), & (u,v) \in Q, \end{cases} \tag{2.8}$$

for some $K > 0$. With $\underline{u}_1$, $\bar{u}_1$, $\underline{v}_1$ and $\bar{v}_1$ given by (2.7), we define the following iterative sequences:

$$\begin{cases} \bar{u}_{i+1} = \bar{u}_i + \frac{1}{K} f(\bar{a}, \bar{u}_i, \underline{v}_i), \quad \underline{u}_{i+1} = \underline{u}_i + \frac{1}{K} f(\underline{a}, \underline{u}_i, \bar{v}_i), & i = 1, 2, \cdots, \\ \bar{v}_{i+1} = \bar{v}_i + \frac{1}{K} g(\bar{u}_i, \bar{v}_i), \quad \underline{v}_{i+1} = \underline{v}_i + \frac{1}{K} g(\underline{u}_i, \underline{v}_i), & i = 1, 2, \cdots, \end{cases}$$

The iterative sequence defined above satisfy the following monotonicity and convergence properties.

Lemma 2.2. *Suppose that* (1.4) *holds.*

(i) *The sequences of constants* $\{\bar{u}_i\}_i^\infty$, $\{\underline{u}_i\}_{i=1}^\infty$, $\{\bar{v}_i\}_{i=1}^\infty$ *and* $\{\underline{u}_i\}_{i=1}^\infty$ *satisfy*

$$\begin{cases} 0<\underline{u}_1\le\cdots\le\underline{u}_i\le\underline{u}_{i+1}\le\cdots\le\bar{u}_{i+1}\le\bar{u}_i\cdots\le\bar{u}_1, \\ 0<\underline{v}_1\le\cdots\le\underline{v}_i\le\underline{v}_{i+1}\le\cdots\le\bar{v}_{i+1}\le\bar{v}_i\cdots\le\bar{v}_1. \end{cases}$$

(ii) *Denote* $\bar{u}_\infty:=\lim_{i\to\infty}\bar{u}_i$, $\underline{u}_\infty:=\lim_{i\to\infty}\underline{u}_i$, $\bar{v}_\infty:=\lim_{i\to\infty}\bar{v}_i$ *and* $\underline{v}_\infty:=\lim_{i\to\infty}\underline{v}_i$. *Then*

$$f(\bar{a},\bar{u}_\infty,\underline{v}_\infty)=0,\ f(\underline{a},\underline{u}_\infty,\bar{v}_\infty)=0,\quad g(\bar{u}_\infty,\bar{v}_\infty)=0,\ g(\underline{u}_\infty,\underline{v}_\infty)=0. \tag{2.9}$$

Proof. (i) We prove the monotonicity of $\underline{u}_i$, $\bar{u}_i$, $\underline{v}_i$ and $\bar{v}_i$ with respect to i inductively. First, we show that

$$\underline{u}_1\le\underline{u}_2\le\bar{u}_2\le\bar{u}_1,\quad \underline{v}_1\le\underline{v}_2\le\bar{v}_2\le\bar{v}_1. \tag{2.10}$$

From (2.6),

$$\begin{cases} \underline{u}_1\le\bar{u}_1,\ \underline{v}_1\le\bar{v}_1, \\ f(\bar{a},\bar{u}_1,\underline{v}_1)\le0,\ f(\underline{a},\underline{u}_1,\bar{v}_1)\ge0,\ g(\bar{u}_1,\bar{v}_1)\le0,\ g(\underline{u}_1,\underline{v}_1)\ge0. \end{cases} \tag{2.11}$$

Then the definitions of $\underline{u}_2$, $\bar{u}_2$, $\underline{v}_2$ and $\bar{v}_2$ give $\underline{u}_1\le\underline{u}_2$, $\bar{u}_2\le\bar{u}_1$, $\underline{v}_1\le\underline{v}_2$ and $\bar{v}_2\le\bar{v}_1$. From $\bar{a}\ge\underline{a}$ and (2.8), we derive

$$\begin{aligned}\bar{u}_2-\underline{u}_2&=\bar{u}_1+\frac{1}{K}f(\bar{a},\bar{u}_1,\underline{v}_1)-\underline{u}_1-\frac{1}{K}f(\underline{a},\underline{u}_1,\bar{v}_1)\\ &\ge\bar{u}_1-\underline{u}_1+\frac{1}{K}f(\underline{a},\bar{u}_1,\bar{v}_1)-\frac{1}{K}f(\underline{a},\underline{u}_1,\bar{v}_1)\ge0.\end{aligned}$$

Similarly, we obtain $\bar{v}_2\ge\underline{v}_2$. Therefore, (2.10) holds. Suppose that for $i\in\mathbb{N}$, we have

$$\underline{u}_1\le\underline{u}_2\le\cdots\le\underline{u}_i\le\bar{u}_i\cdots\le\bar{u}_2\le\bar{u}_1,\quad \underline{v}_1\le\underline{v}_2\le\cdots\le\underline{v}_i\le\bar{v}_i\cdots\le\bar{v}_2\le\bar{v}_1.$$

From (2.8), it follows

$$\begin{aligned}\bar{u}_{i+1}-\underline{u}_{i+1}&=\bar{u}_i+\frac{1}{K}f(\bar{a},\bar{u}_i,\underline{v}_i)-\underline{u}_i-\frac{1}{K}f(\underline{a},\underline{u}_i,\bar{v}_i),\\ &\ge\bar{u}_i-\underline{u}_i+\frac{1}{K}f(\underline{a},\bar{u}_i,\bar{v}_i)-\frac{1}{K}f(\underline{a},\underline{u}_i,\bar{v}_i)\ge0,\\ \bar{u}_{i+1}-\bar{u}_i&=\bar{u}_i+\frac{1}{K}f(\bar{a},\bar{u}_i,\underline{v}_i)-\bar{u}_{i-1}-\frac{1}{K}f(\bar{a},\bar{u}_{i-1},\underline{v}_{i-1})\\ &\le\bar{u}_i-\bar{u}_{i-1}+\frac{1}{K}f(\bar{a},\bar{u}_i,\underline{v}_{i-1})-\frac{1}{K}f(\bar{a},\bar{u}_{i-1},\underline{v}_{i-1})\le0.\end{aligned}$$

Similarly, we can show that $\underline{u}_{i+1}\ge\underline{u}_i,\underline{v}_{i+1}\ge\underline{v}_i,\bar{v}_{i+1}\ge\underline{v}_{i+1}$ and $\bar{v}_{i+1}\le\bar{v}_i$. Therefore the conclusion in (i) holds.

(ii) The formulas in (i) imply that the sequences $\{\bar{u}_i\}_i^\infty$, $\{\underline{u}_i\}_{i=1}^\infty$, $\{\bar{v}_i\}_{i=1}^\infty$ and $\{\underline{u}_i\}_{i=1}^\infty$ converge to some constants, respectively. Then (2.9) follows from the definitions of $\bar{u}_i,\bar{v}_i,\underline{u}_i$ and $\underline{v}_i$. The proof is completed. □

The above lemma states that the system of equations (2.9) admits a positive solution. We next show that the positive solution of (2.9) is unique, which in fact is the conclusion of Theorem 1.1 (i).

Proof of Theorem 1.1 *(i).* From the definition of f and g, every possible positive solution $(\underline{u},\bar{u},\underline{v},\bar{v})$ of (2.9) satisfies $\bar{v}=\bar{u}$, $\underline{v}=\underline{u}$ and

$$(\bar{a}-\bar{u})(1+r\bar{u})=b\underline{u},\qquad(\underline{a}-\underline{u})(1+r\underline{u})=b\bar{u}, \tag{2.12}$$

which is equivalent to

$$h(\underline{u}) = 0 \quad \text{for} \quad \underline{u} \in (0, \underline{a}) \quad \text{and} \quad \bar{u} = (\underline{a} - \underline{u})(1 + r\underline{u})/b, \tag{2.13}$$

where

$$\begin{aligned} h(\tau) :=& [b\bar{a} - (\underline{a} - \tau)(1 + r\tau)][b + r(\underline{a} - \tau)(1 + r\tau)] - b^3\tau \\ =& b^2\bar{a} + (b\bar{a}r - b)(\underline{a} - \tau)(1 + r\tau) - r(\underline{a} - \tau)^2(1 + r\tau)^2 - b^3\tau. \end{aligned}$$

Case 1. $r > 0$.

Making use of $b\bar{a} < \underline{a}$ (see (1.4)) and $\underline{a} \leq \bar{a}$, we get

$$h(0) = (b\bar{a} - \underline{a})(b + \underline{a}r) < 0,$$
$$h(\underline{a}) = b^2\bar{a} - b^3\underline{a} = b^2(\bar{a} - b\underline{a}) > 0.$$

Note $\lim_{|\tau| \to \infty} h(\tau) = -\infty$. We see that the equation $h(\tau) = 0$ in $\tau \in [0, \underline{a}]$ either admits a unique zero or has at least two zeros. In the later case, h satisfies

$$h'(\tau) \geq 0 \quad \text{for all } \tau \leq 0, \tag{2.14}$$

which will be excluded in the following discussion.

Direct calculation yields

$$\begin{aligned} h'(\tau) =& (b\bar{a}r - b)(\underline{a}r - 1 - 2r\tau) - 2r(\underline{a} - \tau)(1 + r\tau)(\underline{a}r - 1 - 2r\tau) - b^3 \\ =& [(b\bar{a}r - b) - 2r(\underline{a} - \tau)(1 + r\tau)](\underline{a}r - 1 - 2r\tau) - b^3. \end{aligned}$$

If $\underline{a}r \geq 1$, then from $b\bar{a} < \underline{a}$,

$$h'(0) = (b\bar{a}r - b - 2r\underline{a})(\underline{a}r - 1) - b^3 < (-b - \underline{a}r)(\underline{a}r - 1) \leq 0.$$

On the other hand, if $\underline{a}r < 1$, then $(\underline{a}r - 1)/(2r) < 0$ and

$$h\left(\frac{\underline{a}r - 1}{2r}\right) = -b^3 < 0.$$

Thus, (2.14) is impossible for any $r > 0$. Consequently, $h(\tau) = 0$ has only one zero in $\tau \in [0, \underline{a}]$.

Case 2. $r = 0$.

Clearly, $h(\tau) = [b\bar{a} - (\underline{a} - \tau)]b - b^3\tau = b[b\bar{a} - \underline{a} + (1 - b^2)\tau]$, and from (2.13),

$$\underline{u}_\infty = \underline{u}_\infty = \frac{\underline{a} - b\bar{a}}{1 - b^2}, \; \bar{u}_\infty = \bar{u}_\infty = \frac{\bar{a} - b\underline{a}}{1 - b^2}. \tag{2.15}$$

The proof is completed. □

Remark 2.3. For any given $k \geq 0$, if $b < \underline{a}/(\bar{a} + k)$ and $\bar{a} > \underline{a}$, then the equations

$$(\bar{a} - \bar{u})(1 + r\bar{u}) = b\underline{u} + bk, \qquad (\underline{a} - \underline{u})(1 + r\underline{u}) = b\bar{u} + bk \tag{2.16}$$

with $\bar{u} > \underline{u}$ still admit a unique positive solution. In fact, the existence of solutions of (2.16) can be obtained by the similar arguments as Lemma 2.2. For the case of $r = 0$, the proof of uniqueness is quite straightforward by just using (2.16). For $r > 0$, by defining

$$\tilde{h}(\tau) := [b\bar{a} + bk - (\underline{a} - \tau)(1 + r\tau)][b - bkr + r(\underline{a} - \tau)(1 + r\tau)] - b^3\tau - b^3k,$$

one can easily verify that $\tilde{h}(0) < 0$ and $\tilde{h}(\tau_*) > 0$ with τ_* a positive constant satisfying $(\underline{a} - \tau_*)(1 + r\tau_*) = b\tau_* + bk$, and also $\tilde{h}'(c) < 0$ for some $c \leq 0$, which allows us to similarly obtain the existence and uniqueness of solutions of $\tilde{h}$ in $[0, \tau_*]$. We claim that $\underline{u} < \tau_*$, otherwise $(\underline{a} - \underline{u})(1 + r\underline{u}) - bk = b\bar{u} \leq b\underline{u}$ which contradicts with $\bar{u} > \underline{u}$. Hence, $\bar{u}$ and $\underline{u}$ satisfy $\tilde{h}(\underline{u}) = 0$, $\underline{u} < \tau_*$ and $\bar{u} = [(\underline{a} - \underline{u})(1 + r\underline{u}) - bk]/b$. This leads to the uniqueness of solution of (2.16).

We call that $(\underline{u}_s(x), \bar{u}_s(x), \underline{v}_s(x), \bar{v}_s(x))$ is a pair of quasi-solution of problem (2.1) if $(\underline{u}_s(x), \bar{u}_s(x), \underline{v}_s(x), \bar{v}_s(x))$ satisfies $\underline{u}_s(x) \le \bar{u}_s(x)$, $\underline{v}_s(x) \le \bar{v}_s(x)$ and

$$\begin{cases} -d_1(x)\Delta \bar{u}_s = f(x, \bar{u}_s(x), \underline{v}_s(x)), & x \in \Omega, \\ -d_1(x)\Delta \underline{u}_s = f(x, \underline{u}_s(x), \bar{v}_s(x)), & x \in \Omega, \\ -d_2(x)\Delta \bar{v}_s = g(\bar{u}_s, \bar{u}_s), & x \in \Omega, \\ -d_2(x)\Delta \underline{v}_s = g(\underline{u}_s, \underline{u}_s), & x \in \Omega, \\ \frac{\partial \bar{u}_s}{\partial \nu} = \frac{\partial \underline{u}_s}{\partial \nu} = \frac{\partial \bar{v}_s}{\partial \nu} = \frac{\partial \underline{v}_s}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Proposition 2.4. *Suppose* (1.4) *holds.*

(i) *Let* $(\underline{u}_s(x), \bar{u}_s(x), \underline{v}_s(x), \bar{v}_s(x))$ *with* $\underline{u}_s(x)$, $\bar{u}_s(x) \in [\underline{u}_1, \bar{v}_1]$, *and* $\underline{v}_s(x)$, $\bar{v}_s(x) \in [\underline{v}_1, \bar{u}_1]$ *be a positive quasi-solution of problem* (2.1). *Then*

$$\underline{u}_\infty \le \underline{u}_s(x) \le \bar{u}_s(x) \le \bar{u}_\infty, \quad \underline{v}_\infty \le \underline{v}_s(x) \le \bar{v}_s(x) \le \bar{v}_\infty, \tag{2.17}$$

where $\underline{u}_\infty, \underline{v}_\infty$, $\bar{u}_\infty, \bar{v}_\infty$ *are defined by Lemma* 2.2.

(ii) *Let* $(u(x,t), v(x,t))$ *be the positive solution of problem* (1.1), *and let* $(u_*(x), v_*(x))$ *be a positive steady state solution of* (1.1). *Then the following estimates hold*

(2.18)
$$\begin{cases} \underline{u}_\infty \le \liminf_{t\to\infty} u(x,t) \le \limsup_{t\to\infty} u(x,t) \le \bar{u}_\infty, \quad \underline{u}_\infty \le u_*(x) \le \bar{u}_\infty, \\ \underline{v}_\infty \le \liminf_{t\to\infty} v(x,t) \le \limsup_{t\to\infty} v(x,t) \le \bar{v}_\infty, \quad \underline{v}_\infty \le v_*(x) \le \bar{v}_\infty. \end{cases}$$

Proof. (i) By Lemma 2.2, in order to prove (2.17), it is sufficient to show that

$$\underline{u}_i \le \underline{u}_s(x) \le \bar{u}_s(x) \le \bar{u}_i, \quad \underline{v}_i \le \underline{v}_s(x) \le \bar{v}_s(x) \le \bar{v}_i, \quad \forall\, i = 1, 2, \cdots \tag{2.19}$$

The proof is by induction on i. Since $\bar{u}_1 = \bar{v}_1 = \bar{a} + \epsilon_1$ and $\underline{u}_1 = \underline{v}_1 = \epsilon_2$, the inequalities in (2.19) hold for $i = 1$. Assuming the inequalities in (2.19) hold for $i \le j_0$ where $j_0 \ge 2$ is an integer, we will prove it for $i = j_0 + 1$. Making use of (2.8), we deduce that

$$\begin{aligned} & - d_1(x)\Delta \bar{u}_{j_0+1} + K\bar{u}_{j_0+1} - Ku_s - f(x, \bar{u}_s, \underline{v}_s) \\ =& K\bar{u}_{j_0+1} - Ku_s - f(x, \bar{u}_s, \underline{v}_s) = K\bar{u}_{j_0} + f(\bar{a}, \bar{u}_{j_0}, \underline{v}_{j_0}) - Ku_s - f(x, \bar{u}_s, \underline{v}_s) \\ \ge& K\bar{u}_{j_0} - Ku_s + f(x, \bar{u}_{j_0}, \underline{v}_s) - f(x, \bar{u}_s, \underline{v}_s) \ge 0. \end{aligned}$$

Denote $w(x) = \bar{u}_{j_0+1} - \bar{u}_s(x)$. Then w satisfies $-d_1(x)\Delta w + Kw \ge 0$ in Ω with $\partial_\nu w = 0$ on $\partial\Omega$. It is derived by the maximum principle of elliptic equations that $w \ge 0$. Ans so $\bar{u}_{j_0+1} \ge \bar{u}_s(x)$ on $\bar{\Omega}$. Similarly, we can prove that $\underline{u}_{j_0+1} \le \underline{u}_s(x)$, $\underline{v}_{j_0+1} \le \underline{v}_s(x)$ and $\bar{v}_s(x) \le \bar{v}_{j_0+1}$. Thus the inequalities in (2.19) hold.

(ii) If the initial densities u_0, v_0 lie in the region $[\epsilon_2, \bar{a}+\epsilon_1]$, by (2.17) and Theorem 3.2 in [17], the solution $(u(x,t), v(x,t))$ satisfies the estimates in (2.18). Recalling (2.2), we obtain that any positive solution $(u(x,t), v(x,t))$ of (1.1) satisfies the estimates in (2.18).

For any positive steady state solution $(u_*(x), v_*(x))$ of (1.1), $(u_*(x), u_*(x), v_*(x), v_*(x))$ is a pair of positive quasi-solution of problem (2.1). Combining this fact with (2.2) and (2.17), we obtain the estimates for $(u_*(x), v_*(x))$ in (2.18). The proof is completed. □

We remark that the conclusions of Proposition 2.4 hold for some general functions f and g satisfying (2.8) and (2.11).

Proof of Theorem 1.1 *(ii).* It is clear that Theorem 1.1 (ii) follows directly from Proposition 2.4. □

2.3. **Global stability of positive steady state solution.** To prove the global stability of positive steady state solution of (1.1), we need the following two lemmas.

Lemma 2.5 ([26, Theorem 1.1] or [14, Lemma 2.2])**.** *Let $\delta > 0$ be a constant, and let the two functions $\psi, h \in C([\delta, \infty))$ satisfy $\psi(t) \geq 0$ and $\int_\delta^\infty h(t)dt < \infty$, respectively. Assume that $\varphi \in C^1([\delta, \infty))$ is bounded from below and satisfies*

$$\varphi'(t) \leq -\psi(t) + h(t) \quad \text{in } [\delta, \infty).$$

If one of the following conditions holds:

(i) *ψ is uniformly continuous in $[\delta, \infty)$,*
(ii) *$\psi \in C^1([\delta, \infty))$ and $\psi'(t) \leq K$ in $[\delta, \infty)$ for some constant $K > 0$,*
(iii) *$\psi \in C^\beta([\delta, \infty))$ with $0 < \beta < 1$, and for $\tau > 0$ there exists $K > 0$ just depending on τ such that $\|\psi\|_{C^\beta([x,x+\tau])} \leq K$ for all $x \geq \delta$,*

then $\lim_{t\to\infty} \psi(t) = 0$.

Lemma 2.6 ([14, Lemma 2.3])**.** *Let w, $w_* \in C^2(\overline{\Omega})$ be two positive functions. If $\frac{\partial w}{\partial \nu} = 0$ and $\frac{\partial w_*}{\partial \nu} = 0$ on $\partial\Omega$, then*

$$\int_\Omega \frac{w_*[w - w_*]}{w}\left(\Delta w - \frac{w}{w_*}\Delta w_*\right)\mathrm{d}x \leq -\int_\Omega w^2\left|\nabla \frac{w_*}{w}\right|^2 \mathrm{d}x \leq 0. \tag{2.20}$$

With the help of the above results, we now show the global stability of positive steady state solution of (1.1) using Lyapunov functional method.

Proof of Theorem 1.1 *(iii).* Let $(u(x,t), v(x,t))$ be the solution of (1.1). Define a function $G : [0, \infty) \to \mathbb{R}$ by

$$G(t) := \int_\Omega \int_{u_*(x)}^{u(x,t)} \frac{u_*(x)}{d_1(x)} \frac{s - u_*(x)}{s} \mathrm{d}s\mathrm{d}x + \eta \int_\Omega \int_{v_*(x)}^{v(x,t)} \frac{v_*(x)}{d_2(x)} \frac{s - v_*(x)}{s} \mathrm{d}s\mathrm{d}x$$

with $\eta > 0$ to be determined later. Then $G(t) \geq 0$ for $t \geq 0$. Making use of (2.20), we deduce

$$\begin{aligned}
\frac{dG(t)}{dt} &= \int_\Omega \frac{u_*(u-u_*)}{d_1 u} u_t \mathrm{d}x + \eta \int_\Omega \frac{v_*(v-v_*)}{d_2 v} v_t \mathrm{d}x \\
&= \int_\Omega \left(\frac{u_*(u-u_*)}{d_1 u} \left[d_1 \Delta u + f(x,u,v) \right] + \eta \frac{v_*(v-v_*)}{d_2 v} \left[d_2 \Delta v + g(u,v) \right] \right) \mathrm{d}x \\
&= \int_\Omega \frac{u_*(u-u_*)}{d_1 u} \left(d_1 \Delta u + f(x,u,v) - \frac{u}{u_*} d_1 \Delta u_* - \frac{u}{u_*} f(x,u_*,v_*) \right) \mathrm{d}x \\
&\quad + \eta \int_\Omega \frac{v_*(v-v_*)}{d_2 v} \left(d_2 \Delta v + g(u,v) - \frac{v}{v_*} d_2 \Delta v_* - \frac{v}{v_*} g(u_*,v_*) \right) \mathrm{d}x \\
&= \int_\Omega \left[\frac{u_*(u-u_*)}{u} \left(\Delta u - \frac{u}{u_*} \Delta u_* \right) + \frac{u_*(u-u_*)}{d_1} \left(\frac{f(x,u,v)}{u} - \frac{f(x,u_*,v_*)}{u_*} \right) \right] \mathrm{d}x \\
&\quad + \eta \int_\Omega \left[\frac{v_*(v-v_*)}{v} \left(\Delta v - \frac{v}{v_*} \Delta v_* \right) + \frac{v_*(v-v_*)}{d_2} \left(\frac{g(u,v)}{v} - \frac{g(u_*,v_*)}{v_*} \right) \right] \mathrm{d}x \\
&\leq \int_\Omega \frac{u_*(u-u_*)}{d_1} \left(\frac{f(x,u,v)}{u} - \frac{f(x,u_*,v_*)}{u_*} \right) \mathrm{d}x \\
&\quad + \eta \int_\Omega \frac{v_*(v-v_*)}{d_2} \left(\frac{g(u,v)}{v} - \frac{g(u_*,v_*)}{v_*} \right) \mathrm{d}x.
\end{aligned}$$

By the definition of f and g in (2.7), we derive

$$\begin{aligned}
\frac{dG(t)}{dt} &\leq \int_\Omega \frac{u_*(u-u_*)}{d_1} \left(-u - \frac{bv}{1+ru} + u_* + \frac{bv_*}{1+ru_*} \right) \mathrm{d}x \\
&\quad + \int_\Omega \eta \frac{v_*(v-v_*)}{d_2} \mu \left(-\frac{v}{u} + \frac{v_*}{u_*} \right) \mathrm{d}x \\
&= \int_\Omega \frac{-u_*[(1+ru)(1+ru_*)-brv_*](u-u_*)^2 - bu_*(1+ru_*)(u-u_*)(v-v_*)}{d_1(1+ru)(1+ru_*)} \mathrm{d}x \\
&\quad + \int_\Omega \frac{\eta\mu v_*^2(u-u_*)(v-v_*) - \eta\mu u_* v_*(v-v_*)^2}{d_2 u u_*} \mathrm{d}x \\
&= \int_\Omega \frac{E}{d_1 d_2 u u_* (1+ru)(1+ru_*)} \mathrm{d}x,
\end{aligned}$$

with

$$\begin{aligned}
E :=& - d_2 u u_*^2[(1+ru)(1+ru_*)-brv_*](u-u_*)^2 - bd_2 u u_*^2(1+ru_*)(u-u_*)(v-v_*) \\
&+ d_1(1+ru)(1+ru_*)[\eta\mu v_*^2(u-u_*)(v-v_*) - \eta\mu u_* v_*(v-v_*)^2] \\
=& A(u-u_*)^2 + B(u-u_*)(v-v_*) + C(v-v_*)^2,
\end{aligned}$$

where

$$\begin{aligned}
&A := - d_2 u u_*^2[(1+ru)(1+ru_*) - brv_*], \ C := -d_1 \eta\mu u_* v_*(1+ru)(1+ru_*), \\
&B := - bd_2 u u_*^2(1+ru_*) + d_1 \eta\mu v_*^2(1+ru)(1+ru_*).
\end{aligned}$$

Next we choose a suitable $\eta > 0$ such that $2\sqrt{AC} > |B|$, which then yields

$$\frac{dG(t)}{dt} \le -\int_\Omega \frac{\delta(u-u_*)^2 + \delta(v-v_*)^2}{d_1 d_2 u u_*(1+ru)(1+ru_*)} \mathrm{d}x =: \psi(t) \le 0, \tag{2.21}$$

for some $0 < \delta \ll 1$. Denote $\bar{d}_i = \max_{x\in\overline{\Omega}} d_i(x)$, $\underline{d}_i = \min_{x\in\overline{\Omega}} d_i(x)$ and

$$\eta = \sqrt{\frac{\underline{d}_2 \bar{d}_2 (\underline{u}_\infty - \epsilon)(\bar{u}_\infty - \epsilon)\underline{u}_\infty^3 \bar{u}_\infty}{\underline{d}_1 \bar{d}_1 \underline{v}_\infty \bar{v}_\infty^3}} \frac{b}{\mu[1 + r(\underline{u}_\infty - \epsilon)]}$$

for some small $\epsilon > 0$. From (2.18), there exists $T > 1$ such that $u(x,t) \ge \underline{u}_\infty - \epsilon$ and $v(x,t) \ge \underline{v}_\infty - \epsilon$ for all $t \ge T$. A simple calculation gives

$$\begin{aligned}
&2\sqrt{AC} - |B| \ge 2\sqrt{AC} - [bd_2 u u_*^2(1+ru_*) + d_1\eta\mu v_*^2(1+ru)(1+ru_*)]\\
=&2\sqrt{d_1 d_2 \eta\mu u u_*^3 v_*(1+ru)(1+ru_*)[(1+ru)(1+ru_*) - brv_*]}\\
&- [bd_2 u u_*^2(1+ru_*) + d_1\eta\mu v_*^2(1+ru)(1+ru_*)]\\
=&(1+ru)(1+ru_*)\sqrt{d_1 d_2 u u_*^3 v_*}\left[2\sqrt{\eta\mu - \frac{b\eta\mu r v_*}{(1+ru)(1+ru_*)}}\right.\\
&\left. - \left(b\sqrt{\frac{d_2 u u_*}{d_1 v_*}}\frac{1}{1+ru} + \eta\mu\sqrt{\frac{d_1 v_*^3}{d_2 u u_*^3}}\right)\right]\\
=:&E_1\left[2\sqrt{\eta\mu - \frac{b\eta\mu r v_*}{(1+ru)(1+ru_*)}} - \left(b\sqrt{\frac{d_2 u u_*}{d_1 v_*}}\frac{1}{1+ru} + \eta\mu\sqrt{\frac{d_1 v_*^3}{d_2 u u_*^3}}\right)\right].
\end{aligned}$$

Taking advantages of (2.18), $\underline{v}_\infty = \underline{u}_\infty$, $\bar{v}_\infty = \bar{u}_\infty$ and the definition of η, we derive that for $t \ge T$,

$$\begin{aligned}
&2\sqrt{AC} - |B| \ge E_1\left[2\sqrt{\eta\mu - \frac{b\eta\mu r\bar{v}_\infty}{[1+r(\underline{u}_\infty - \epsilon)](1+r\underline{u}_\infty)}}\right.\\
&\left. - \left(b\sqrt{\frac{\bar{d}_2(\bar{u}_\infty - \epsilon)\bar{u}_\infty}{\underline{d}_1 \underline{v}_\infty}}\frac{1}{1+r(\underline{u}_\infty - \epsilon)} + \eta\mu\sqrt{\frac{\bar{d}_1 \bar{v}_\infty^3}{\underline{d}_2(\underline{u}_\infty - \epsilon)\underline{u}_\infty^3}}\right)\right]\\
=&E_1\left(2\sqrt{\eta\mu - \frac{b\eta\mu r\bar{v}_\infty}{[1+r(\underline{u}_\infty-\epsilon)](1+r\underline{u}_\infty)}} - 2\sqrt{\frac{b\eta\mu}{1+r(\underline{u}_\infty-\epsilon)}\sqrt{\frac{\bar{d}_1\bar{d}_2(\bar{u}_\infty-\epsilon)\bar{u}_\infty\bar{v}_\infty^3}{\underline{d}_1\underline{d}_2(\underline{u}_\infty-\epsilon)\underline{u}_\infty^3\underline{v}_\infty}}}\right)\\
=&E_1\left(2\sqrt{\eta\mu - \frac{b\eta\mu r\bar{u}_\infty}{[1+r(\underline{u}_\infty-\epsilon)](1+r\underline{u}_\infty)}} - 2\sqrt{\frac{b\eta\mu}{1+r(\underline{u}_\infty-\epsilon)}\sqrt{\frac{\bar{d}_1\bar{d}_2(\bar{u}_\infty-\epsilon)\bar{u}_\infty^4}{\underline{d}_1\underline{d}_2(\underline{u}_\infty-\epsilon)\underline{u}_\infty^4}}}\right).
\end{aligned}$$

Then $2\sqrt{AC} - B > 0$ follows from (1.6) (with $\epsilon \to 0$) and $(\underline{a} - \underline{u})(1 + r\underline{u}) = b\bar{u}$ (See (2.12)). Thus (2.21) holds for $t \ge T$.

Next we show the global stability of the positive steady state solution (u_*, v_*). By (2.3) and the definition of $\psi(t)$, we see that $|\psi'(t)| < C_1$ in $t \in [T, \infty)$ for some $C_1 > 0$. Then it follows from Lemma 2.5 that

$$\lim_{t\to\infty}\psi(t) = -\int_\Omega \frac{\delta(u-u_*)^2 + \delta(v-v_*)^2}{d_1 d_2 u u_*}\mathrm{d}x = 0.$$

Recalling that $u(t,x) \geq \epsilon_2 > 0$ for $t \geq T_1$ by (2.2), we have

$$\lim_{t\to\infty} u(x,t) = u_*(x), \quad \lim_{t\to\infty} v(x,t) = v_*(x) \ \text{ in } \ L^2(\overline{\Omega}). \tag{2.22}$$

The estimate (2.3) also implies that the set $\{u(\cdot,t) : t \geq 1\}$ is relatively compact in $C^2(\overline{\Omega})$. Therefore, we may assume that

$$\|u(x,t_k) - \tilde{u}(x)\|_{C^2(\overline{\Omega})}, \|v(x,t_k) - \tilde{v}(x)\|_{C^2(\overline{\Omega})} \to 0 \quad \text{as} \quad t_k \to \infty$$

for some functions $\tilde{u}, \tilde{v} \in C^2(\overline{\Omega})$. Combining this with (2.22), we could conclude that $\tilde{u}(x) \equiv u_*(x)$ and $\tilde{v}(x) \equiv v_*(x)$ for $x \in \overline{\Omega}$. Thus $\lim_{t\to\infty} u(x,t) = u_*(x)$ and $\lim_{t\to\infty} v(x,t) = v_*(x)$ in $C^2(\overline{\Omega})$. The proof is finished. □

Remark 2.7. The condition (1.6) for the global stability of positive steady state is an implicit one as the quasi-steady state $(\bar{u}_\infty, \underline{u}_\infty, \bar{v}_\infty, \underline{v}_\infty)$ cannot be solved explicitly except when $r = 0$ (see (1.7)). We observe that (1.6) holds in the following cases:

(1) $b > 0$ is sufficiently small. In fact, from (2.12) we see $\bar{u}_\infty \approx \bar{a} - \frac{\underline{u}_\infty}{(1+r\bar{u}_\infty)} b \to \bar{a}$ and $\underline{u}_\infty \approx \underline{a} - \frac{\bar{u}_\infty}{(1+r\underline{u}_\infty)} b \to \underline{a}$ as $b \to 0$ since $\bar{u}_\infty,\, \underline{u}_\infty \in [0,\bar{a}]$. Hence,

$$(1 + 2r\underline{u}_\infty - r\underline{a}) \left(\frac{\underline{u}_\infty}{\bar{u}_\infty}\right)^{5/2} \to (1 + r\underline{a}) \left(\underline{a}/\bar{a}\right)^{5/2} \quad \text{as } b \to 0,$$

which immediately implies that (1.6) is satisfied for small $b > 0$.

(2) $a_A = \bar{a} - \underline{a}$ is sufficiently small. For any $M > 1$, there exists $\tilde{a} > 0$ such that when $0 < a_A < \tilde{a}$, we have $\bar{u}_\infty/\underline{u}_\infty < M$. Then (1.6) holds if b satisfies

$$b < (1 + 2r\underline{u}_\infty - r\underline{a}) \left[\frac{\min d_1(x) \min d_2(x)}{\max d_1(x) \max d_2(x)}\right]^{1/2} M^{-5/2}. \tag{2.23}$$

Remark 2.8. A condition for the global stability of positive steady state solution of (1.1) weaker than (1.6) is possible. In the proof of Theorem 1.1 (iii), we use $|B| \geq bd_2uu_*^2(1+ru_*) + d_1\eta\mu v_*^2(1+ru)(1+ru_*)$. By applying $|B| = |-bd_2uu_*^2(1+ru_*) + d_1\eta\mu v_*^2(1+ru)(1+ru_*)|$, one may derive a weaker restriction on b. In fact, a similar calculation as above yields,

$$\begin{aligned} 2\sqrt{AC} - |B| &= E_1 \left[2\sqrt{\eta\mu - \frac{b\eta\mu r v_*}{(1+ru)(1+ru_*)}} - \left|-b\sqrt{\frac{d_2uu_*}{d_1v_*}}\frac{1}{1+ru} + \eta\mu\sqrt{\frac{d_1v_*^3}{d_2uu_*^3}}\right|\right] \\ &= E_1\sqrt{\eta}\left[2\sqrt{\mu - \frac{b\mu r v_*}{(1+ru)(1+ru_*)}} - \mu\sqrt{\frac{d_1v_*^3}{d_2uu_*^3}}\left|-\frac{bd_2uu_*^2}{\mu d_1 v_*^2(1+ru)}\frac{1}{\sqrt{\eta}} + \sqrt{\eta}\right|\right] > 0, \end{aligned}$$

for large $t > 0$, if

$$b < 8\frac{D_1 + D_2}{(D_2 - D_1)^2}\frac{\min d_2(x)}{\max d_1(x)}\frac{(1 + 2r\underline{u}_\infty - r\underline{a})\underline{u}_\infty^4}{(1 + r\underline{u}_\infty)\bar{u}_\infty^3}, \tag{2.24}$$

with

$$D_1 := \frac{\underline{u}_\infty^3 \min d_2(x)}{\bar{u}_\infty^2(1 + r\bar{u}_\infty)\max d_1(x)},\ D_2 := \frac{\bar{u}_\infty^3 \max d_2(x)}{\underline{u}_\infty^2(1 + r\underline{u}_\infty)\min d_1(x)},$$

where we have used the fact that $(\underline{a} - \underline{u})(1 + r\underline{u}) = b\bar{u}$ (see (2.12)) and for any $\mu > 0$,

$$\inf_{k>0}\max_{D\in[D_1,D_2]}\left|k - \frac{bD/\mu}{k}\right| = \frac{bD_2/\mu - bD_1/\mu}{\sqrt{2(bD_1/\mu + bD_2/\mu)}} = \sqrt{b/\mu}\frac{D_2 - D_1}{\sqrt{2(D_1 + D_2)}}.$$

Clearly, $D_1 - D_2 = 0$ if d_i and m are constants, and hence (2.24) holds if $1+2r\underline{u}_\infty - r\underline{a} > 0$ which is weaker than (1.6). Thus (2.24) is a good alternative for (1.6) when $\bar{a} - \underline{a}$, $\max d_i - \min d_i$ and r are small.

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School of Science and Technology, University of New England, Armidale, New South Wales 2351, Australia
Email address: wni2@une.edu.au

Department of Mathematics, William & Mary, Williamsburg, Virginia 23187-8795
Email address: jxshix@wm.edu

School of Mathematics, Harbin Institute of Technology, Harbin 150001, People's Republic of China
Email address: mxwang@hit.edu.cn