Calc. Var. (2020) 59:132
https://doi.org/10.1007/s00526-020-01794-6

Calculus of Variations

Global stability of nonhomogeneous equilibrium solution for the diffusive Lotka–Volterra competition model

Wenjie Ni[1,3] · Junping Shi[2] · Mingxin Wang[1]

Received: 22 September 2019 / Accepted: 31 May 2020

Abstract

A diffusive Lotka–Volterra competition model is considered and the combined effect of spatial dispersal and spatial variations of resource on the population persistence and exclusion is studied. A new Lyapunov functional method and a new integral inequality are developed to prove the global stability of non-constant equilibrium solutions in heterogeneous environment. The general result is applied to show that in a two-species system in which the diffusion coefficients, resource functions and competition rates are all spatially heterogeneous, the positive equilibrium solution is globally asymptotically stable when it exists, and it can also be applied to the system with arbitrary number of species under the assumption of spatially heterogeneous resource distribution, for which the monotone dynamical system theory is not applicable.

Mathematics Subject Classification 35K51 · 35B09 · 35B35 · 92D25

1 Introduction

The uneven distribution of resources due to the effect of geological and environmental characteristics greatly enriches the diversity of ecosystems. In the past few decades, the phenomenon

---

Communicated by P. Rabinowitz.

Partially supported by a grant from China Scholarship Council, NSF Grant DMS-1715651, and NSFC Grant 11771110.

✉ Junping Shi
jxshix@wm.edu

Wenjie Ni
nwj1033159832@163.com

Mingxin Wang
mxwang@hit.edu.cn

1 School of Mathematics, Harbin Institute of Technology, Harbin 150001, China

2 Department of Mathematics, William & Mary, Williamsburg, VA 23187-8795, USA

3 Present Address: School of Science and Technology, University of New England, Armidale, NSW 2351, Australia

of spatial heterogeneity of resources has attracted the attention of many researchers from both biology and mathematics, see [4,23,25,32], for example. The dynamical properties of mathematical models with spatial heterogeneity are more complicated, especially the global stability of non-constant equilibrium solutions.

In this paper we consider the global dynamics of the diffusive Lotka–Volterra competition model of multiple species in a non-homogeneous environment:

$$\begin{cases} \partial_t u_i = d_i(x)\Delta u_i + u_i\left(m_i(x) - \sum_{j=1}^{k} a_{ij}(x)u_j\right), & x \in \Omega, \ \ t > 0, \ 1 \le i \le k, \\ \partial_\nu u_i = 0, & x \in \partial\Omega, \ t > 0, \ 1 \le i \le k, \\ u_i(x,0) = \varphi_i(x) \ge, \not\equiv 0, & x \in \overline{\Omega}, \ 1 \le i \le k, \end{cases} \tag{1.1}$$

where $u_i(x,t)$ is the population density of ith biological species, $m_i \in C^\alpha(\overline{\Omega})$, $i = 1, \ldots, k$, represent the densities of non-uniform resources, the nonnegative function $a_{ij} \in C^\alpha(\overline{\Omega})$ is the strength of competition of species u_j against u_i at location x, and the function $d_i(x) \ge 0$ is the diffusion coefficient of u_i at location x. The spatial habitat $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega \in C^{2+\alpha}$, ν is the outward unit normal vector over $\partial\Omega$, and the homogeneous Neumann boundary condition indicates that this system is self-contained with zero population flux across the boundary.

If all of d_i, m_i and a_{ij} are positive constants (so the environment is spatially homogeneous), the global stability of positive constant equilibrium of (1.1) had been proved in the weak competition case, see [2,8] and the references therein. Lyapunov functional methods are used to prove the global stability of positive constant equilibrium of (1.1) in the homogeneous case. However the spatial heterogeneity of the environment may change the outcome of the competition, and it is an important biological question to understand how the spatially non-homogeneous environment affects the competition between species. When $k = 2$, d_i are constants, the two species u_1 and u_2 share the same spatially distributed resource $m_1(x) = m_2(x)$ and have the same competition coefficients $a_{ij} = 1$, it was shown in [7] that the species with smaller diffusion coefficient survives while the other one with larger diffusion coefficient becomes extinct, that is, the slower diffuser prevails. The same question with $k \ge 3$ species remains as an open question. The global dynamics of the two-species case of (1.1) was recently completely classified for the weak competition regime $a_{11}a_{22} > a_{12}a_{21}$ in [12–14], assuming d_i and a_{ij} are constants and $m_i(x)$ are spatially heterogeneous. It was shown that there is always a globally asymptotically stable non-negative equilibrium for the problem, and the dynamics can be completely determined according to the competition strength a_{ij}, the diffusion coefficients d_i and heterogeneous resource functions $m_i(x)$ by using linear stability analysis and monotone dynamical system theory.

The role of spatial heterogeneity in diffusive two-species competition system (1.1) have been explored in many work, see for example [3,5,10,11,18,19,22,25] and the references therein, in which various methods and mathematical tools have been applied to analyze the existence and stability of the equilibrium solutions. The additional effect of advection on the diffusive two-species competition models have been considered in [26,36] and the references therein, and the effect of nonlocal competition has been studied in [28]. Note that the diffusive two-species Lotka–Volterra competition model (1.1) generates a monotone dynamical system, so the powerful tools from monotone dynamical system theory can be applied [16,31]. However, when $k \ge 3$, the monotone dynamical system theory cannot be applied to problem (1.1). Our approach here does not rely on the monotone dynamical system methods, and the global stability proved in Theorem 3.6 for competition models with arbitrary number of species is perhaps the first such result for spatially heterogeneous models.

The main results (see Theorem 3.6) in this paper are the global stability of positive non-constant equilibrium solution or semi-trivial equilibrium solution of (1.1). For $d_i > 0$, to show the global stability of positive equilibrium solution denoted by $\mathbf{u}^* := (u_1^*, \ldots, u_k^*)$, the following two assumptions are needed:

(i) (1.1) admits a positive equilibrium solution $\mathbf{u}^*$;
(ii) for this (u_i^*), there exist some positive constants ξ_i such that the matrix $Q(x) = (q_{ij}(x) + q_{ji}(x))$ is positive definite for every $x \in \overline{\Omega}$, where

$$q_{ij}(x) = \frac{\xi_i u_i^*(x)}{d_i(x)} a_{ij}(x)$$

The same idea can be applied to obtain the global stability of semi-trivial equilibrium solution of (1.1) with $d_i > 0$. Moreover, in Theorem 3.6 we also consider the global stability of non-homogeneous equilibrium solution in the case that some diffusion coefficients are degenerate, namely $d_{i_0} \equiv 0$ for some $i_0 \in \{1, \ldots, k\}$. We also remark that the global stability results in Theorem 3.6 still hold when the diffusion terms Δu_i are replaced by a divergence form $\mathrm{div}(b_i(x)\nabla u_i)$, and the homogeneous Dirichlet boundary condition are changed to Robin boundary condition (see Remark 3.7 (i)), where $b_i(x) \in C^{1+\alpha}(\overline{\Omega})$ with $1 \leq i \leq k$ are positive functions.

A key ingredient of our work here is a new Lyapunov functional method. In [8,17], the global stability of positive equilibrium solution of (1.1) for homogeneous environment is proved using Lyapunov functional methods when d_i, m_i, a_{ij} are all constants. A more general equation, including Lotka–Volterra competitive models (the corresponding ODE problem to (1.1)) and chemostat systems as particular cases, is investigated by utilizing Lyapunov functional methods in [6]. The Lyapunov functional in [8,17] is constructed as $F_1(t) = \int_\Omega V(u_i(x,t))\mathrm{d}x$, where $V(u_i)$ is the Lyapunov function for the ordinary differential equation model, and the equilibrium solution is a constant one. The integral form of the Lyapunov functional can be viewed as an unweighted average of the ODE Lyapunov function on the spatial domain. However this simple construction does not work for the spatially heterogeneous situation, and the equilibrium solution in that case is a non-constant one. In this work, we use a new Lyapunov functional in form of $F_2(t) = \int_\Omega w_*(x)V(u_i(x,t))\mathrm{d}x$, which is a weighted average of the ODE Lyapunov function on the spatial domain, and the weight function $w_*(x)$ depends on the non-homogeneous functions d_i, m_i, a_{ij} and non-constant equilibrium solution (assuming it exists). Such construction is motivated by the method used in [24] for the the global stability of equilibrium solutions of coupled ordinary differential equation models on networks (which is patchy environment or discrete spatial domain). Such a Lyapunov function has also been used in [21] for a diffusive SIR epidemic model. To demonstrate this new method, we first prove the global stability of a non-constant equilibrium solution for a spatially heterogeneous diffusive logistic model (see Theorem 3.1). That result is well-known but we give a new proof for the spatial heterogeneous case.

This paper is organized as follows. In Sect. 2, we give some preliminaries. In Sect. 3, we apply Lyapunov functional method to show the main results Theorem 3.6 on the global stability of the equilibrium solutions of (1.1). In Sect. 4, making use of upper and lower solutions method, we present some applications of Theorem 3.6.

2 Preliminaries

When using the Lyapunov functional method to investigate the global stability of equilibrium of reaction–diffusion systems, the uniform estimates of solutions of parabolic equations play an important role. We first recall the following results on the uniform estimates for the second order parabolic equations.

Consider the initial-boundary value problem

$$\begin{cases} u_t + \mathcal{L}u = f(x,t,u), & x \in \Omega, \quad t > 0, \\ B[u] = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = \varphi(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where the domain $\Omega \subset \mathbb{R}^n$ is bounded with a smooth boundary $\partial\Omega \in C^{2+\alpha}$ with $0 < \alpha < 1$, operators $\mathcal{L}$ and B have the forms:

$$\mathcal{L}[u] = -a_{ij}(x,t)D_{ij}u + b_j(x,t)D_j u + c(x,t)u,$$
$$B[u] = u, \ \text{or} \ B[u] = \frac{\partial u}{\partial \nu} + b(x)u,$$

with $b \in C^{1+\alpha}(\partial\Omega)$ and $b \geq 0$. The initial condition $\varphi \in W_p^2(\Omega)$, $p > 1 + n/2$, satisfies $B[\varphi]\big|_{\partial\Omega} = 0$.

Denote $Q_\infty = \Omega \times [0,\infty)$. We make the following assumptions:

($\mathbf{L_1}$) $a_{ij},\ b_j,\ c \in C(\overline{\Omega} \times [0,\infty))$ and there are positive constants λ and Λ such that

$$\lambda|y|^2 \leq \sum_{1\leq i,j\leq n} a_{ij}(x,t)y_i y_j \leq \Lambda|y|^2, \quad |b_j(x,t)|,\ |c(x,t)| \leq \Lambda$$

for all $(x,t) \in Q_\infty,\ \ y \in \mathbb{R}^n$.

($\mathbf{L_2}$) For any fixed $m > 0$, there exists a positive constant $C(m)$ such that, for all $k \geq 1$,

$$||a_{ij}||_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[k,k+m])},\ ||b_j||_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[k,k+m])},\ ||c||_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[k,k+m])} \leq C(m).$$

($\mathbf{L_3}$)

(i) $f(x,0,0) = 0$ on $\partial\Omega$ when $B[u] = u$, $f \in L^\infty(Q_\infty \times [\sigma_1,\sigma_2])$ for some $\sigma_1 < \sigma_2$,

(ii) there exists $C(\sigma_1,\sigma_2) > 0$ such that

$$|f(x,t,u) - f(x,t,v)| \leq C(\sigma_1,\sigma_2)|u - v|, \quad \forall\ (x,t) \in Q_\infty,\ u, v \in [\sigma_1,\sigma_2]$$

and $f(\cdot,u) \in C^{\alpha,\alpha/2}(\overline{\Omega} \times [h,h+3])$ uniformly for $u \in [\sigma_1,\sigma_2]$ and $h \geq 0$, i.e., there exists a constant $C > 0$ so that

$$|f(x,t,u) - f(y,s,u)| \leq C(|x-y|^\alpha + |t-s|^{\alpha/2})$$

for all $(x,t),\ (y,s) \in \overline{\Omega} \times [h,h+3]$, $u \in [\sigma_1,\sigma_2]$ and $h \geq 0$.

Under the above assumptions, we have the following boundedness result for a globally defined solution $u(x,t)$ of (2.1).

Theorem 2.1 *Let $u(x,t)$ be a solution of* (2.1) *and $\sigma_1 < u < \sigma_2$ for some $\sigma_1,\ \sigma_2 \in \mathbb{R}$. Assume that f satisfies* ($\mathbf{L_3}$) (i) *for these σ_1, σ_2, and a_{ij}, b_j and c satisfy the assumption* ($\mathbf{L_1}$). *Then, for any given $m > 0$, there is a constant $C_1(m) > 0$ such that*

$$\|u\|_{W_p^{2,1}(\overline{\Omega}\times[\tau,\tau+m])} \leq C_1(m), \quad \forall\ \tau \geq 1.$$

If additionally the assumptions ($\mathbf{L_2}$) *and* ($\mathbf{L_3}$) *(ii) hold, then, for any given* $m \geq 1$*, there is a constant* $C_2(m) > 0$ *such that*

$$\max_{x\in\overline{\Omega}} \|u_t(x,\cdot)\|_{C^{\alpha/2}([m,\infty))} + \max_{t\geq m} \|u_t(\cdot,t)\|_{C(\overline{\Omega})} + \max_{t\geq m} \|u(\cdot,t)\|_{C^{2+\alpha}(\overline{\Omega})} \leq C_2(m).$$

For the idea of proof to Theorem 2.1, the interested readers can refer to the proofs of [33, Theorem 2.1] and [35, Theorem 2.2] for the details. We also recall the following calculus lemma which will be used to prove the global stability of equilibrium solution.

Lemma 2.2 [34, Theorem 1.1] *Let* $\delta > 0$ *be a constant, and let the two functions* $\psi, h \in C([\delta,\infty))$ *satisfy* $\psi(t) \geq 0$ *and* $\int_\delta^\infty h(t)dt < \infty$*, respectively. Assume that* $\varphi \in C^1([\delta,\infty))$ *is bounded from below and satisfies*

$$\varphi'(t) \leq -\psi(t) + h(t) \quad in \ [\delta,\infty).$$

If one of the following conditions holds:

(i) ψ *is uniformly continuous in* $[\delta,\infty)$*,*
(ii) $\psi \in C^1([\delta,\infty))$ *and* $\psi'(t) \leq K$ *in* $[\delta,\infty)$ *for some constant* $K > 0$*,*
(iii) $\psi \in C^\beta([\delta,\infty))$ *with* $0 < \beta < 1$*, and for* $\tau > 0$ *there exists* $K > 0$ *just depending on* τ *such that* $\|\psi\|_{C^\beta([x,x+\tau])} \leq K$ *for all* $x \geq \delta$*,*

then $\lim_{t\to\infty} \psi(t) = 0$.

In fact, the conditions (ii) and (iii) appear to be stronger than (i), however for the convenience of application we still list them there since the uniform continuity of a function on a unbounded domain may not be verified easily.

Another useful result concerning with convergence of function $\psi(x)$ as $x \to \infty$ is the well known Barbalat's Lemma [1].

Lemma 2.3 *Suppose that* $\psi : [0,\infty) \to \mathbb{R}$ *is uniformly continuous and that* $\lim_{t\to\infty} \int_0^t \psi(s)ds$ *exists. Then* $\lim_{t\to\infty} \psi(t) = 0$ *holds.*

In order to use the Lyapunov functional method to study the global asymptotic stability of spatially non-homogeneous equilibrium solutions, we now give a key integral inequality which plays a crucial role in the later analysis.

Lemma 2.4 *Let* $w,\ w_* \in C^2(\overline{\Omega})$ *be two positive functions,* $a \in C^1(\overline{\Omega})$ *with* $a(x) \geq 0$ *on* $\overline{\Omega}$*. If* Φ*,* g *and* h *satisfy*

(i) $\Phi \in C^{2,2}(\overline{\Omega} \times [0,\infty))$*,* $\Phi(x,0) = 0$ *and* $\Phi_u(x,u) > 0$ *for* $x \in \overline{\Omega}$ *and* $u > 0$*,*
(ii) $g \in C^{0,1}(\partial\Omega \times [0,\infty))$*, and for any* $x \in \partial\Omega$*, the function* $\dfrac{g(x,u)}{\Phi(x,u)}$ *is nonincreasing for* $u \in [0,\infty)$*,*
(iii) $h \in C^1([0,\infty))$ *and* $h'(u) \leq 0$*,*
(iv) $\dfrac{\partial \Phi(x,w)}{\partial \nu} = g(x,w)$*,* $\dfrac{\partial \Phi(x,w_*)}{\partial \nu} = g(x,w_*)$ *on* $\partial\Omega$*,*

then

$$\begin{aligned}&\int_\Omega \Phi(x,w_*)h\left(\frac{\Phi(x,w_*)}{\Phi(x,w)}\right)\left(\operatorname{div}[a(x)\nabla\Phi(x,w)] - \frac{\Phi(x,w)}{\Phi(x,w_*)}\operatorname{div}[a(x)\nabla\Phi(x,w_*)]\right)\mathrm{d}x \\ &\leq \int_\Omega a(x)[\Phi(x,w)]^2 h'\left(\frac{\Phi(x,w_*)}{\Phi(x,w)}\right)\left|\nabla\frac{\Phi(x,w_*)}{\Phi(x,w)}\right|^2 \mathrm{d}x \leq 0. \end{aligned} \tag{2.2}$$

Proof It follows from Green's Theorem that

$$\begin{aligned}
&\int_\Omega \Phi(x,w_*)h\left(\frac{\Phi(x,w_*)}{\Phi(x,w)}\right)\left(\mathrm{div}[a\nabla\Phi(x,w)]-\frac{\Phi(x,w)}{\Phi(x,w_*)}\mathrm{div}[a\nabla\Phi(x,w_*)]\right)\mathrm{d}x\\
&=\int_\Omega h\left(\frac{\Phi(x,w_*)}{\Phi(x,w)}\right)\mathrm{div}[a\Phi(x,w_*)\nabla\Phi(x,w)-a\Phi(x,w)\nabla\Phi(x,w_*)]\mathrm{d}x\\
&=\int_{\partial\Omega} h\left(\frac{\Phi(x,w_*)}{\Phi(x,w)}\right)\left(a\Phi(x,w_*)\frac{\partial\Phi(x,w)}{\partial\nu}-a\Phi(x,w)\frac{\partial\Phi(x,w_*)}{\partial\nu}\right)\mathrm{d}S\\
&\quad-\int_\Omega a\nabla h\left(\frac{\Phi(x,w_*)}{\Phi(x,w)}\right)[\Phi(x,w_*)\nabla\Phi(x,w)-\Phi(x,w)\nabla\Phi(x,w_*)]\mathrm{d}x\\
&=\int_{\partial\Omega} h\left(\frac{\Phi(x,w_*)}{\Phi(x,w)}\right)a\Phi(x,w)\Phi(x,w_*)\left(\frac{g(x,w)}{\Phi(x,w)}-\frac{g(x,w_*)}{\Phi(x,w_*)}\right)\mathrm{d}S\\
&\quad+\int_\Omega a[\Phi(x,w)]^2h'\left(\frac{\Phi(x,w_*)}{\Phi(x,w)}\right)\left|\nabla\frac{\Phi(x,w_*)}{\Phi(x,w)}\right|^2\mathrm{d}x\\
&\le\int_\Omega a[\Phi(x,w)]^2h'\left(\frac{\Phi(x,w_*)}{\Phi(x,w)}\right)\left|\nabla\frac{\Phi(x,w_*)}{\Phi(x,w)}\right|^2\mathrm{d}x\le 0.
\end{aligned}$$

The proof is finished. □

A simple example of g, h and Φ is

$$\Phi(x,\tau)=\tau,\ g(x,\tau)=c(x)-b(x)\tau,\ h(\tau)=1-\tau^\beta$$

for some constant $\beta\ge 1$ and nonnegative function $b\in C(\overline{\Omega})$. Then (iv) becomes $\frac{\partial w}{\partial\nu}=g(x,w)$, $\frac{\partial w_*}{\partial\nu}=g(x,w_*)$ on $\partial\Omega$, and the estimate (2.2) reduces to

$$\begin{aligned}
&\int_\Omega \frac{w_*(w^\beta-w_*^\beta)}{w^\beta}\left(\mathrm{div}[a(x)\nabla w]-\frac{w}{w_*}\mathrm{div}[a(x)\nabla w_*]\right)\mathrm{d}x\\
&\quad\le-\beta\int_\Omega a(x)w^2\left(\frac{w_*}{w}\right)^{\beta-1}\left|\nabla\frac{w_*}{w}\right|^2\mathrm{d}x\le 0.
\end{aligned}\tag{2.3}$$

3 Main results

3.1 Scalar equation

In this subsection, we study the following scalar parabolic equation

$$\begin{cases}u_t=d(x)\Delta u+uf(x,u), & x\in\Omega,\quad t>0,\\ \partial_\nu u=0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=\varphi(x)\ge\not\equiv 0, & x\in\Omega,\end{cases}\tag{3.1}$$

where $d\in C^\alpha(\overline{\Omega})$ and $f\in C^{\alpha,\alpha}(\overline{\Omega}\times[0,\infty))$ satisfy

$$\begin{cases}d(x)>0 \text{ on } \overline{\Omega}, \quad f(x,\tau) \text{ is strictly decreasing for } \tau\ge 0,\\ \text{there exists } K>0 \text{ such that } f(x,\tau)\le 0 \text{ for } (x,\tau)\in\overline{\Omega}\times[K,\infty).\end{cases}\tag{3.2}$$

When $d(x) \equiv 1$, the global stability of the positive equilibrium solution $u^*(x)$ (if exists) of (3.1) has been shown in [4, Proposition 3.2] by using the Lyapunov functional

$$V(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - F(x,u) \right) \mathrm{d}x,$$

where $F(x,u) = \int_0^u \tau f(x,\tau) \mathrm{d}\tau$. Here we consider a more general case that $d(x) > 0$ on $\overline{\Omega}$, and we use a different Lyapunov functional to prove the global stability of u^* with respect to (3.1).

Theorem 3.1 *Assume that* $u_0 \in C(\overline{\Omega})$ *and* $u_0(x) \geq, \not\equiv 0$. *If* $d \in C^\alpha(\overline{\Omega})$ *and* $f \in C^{\alpha,\alpha}(\overline{\Omega} \times [0,\infty))$ *satisfy* (3.2)*, then the problem* (3.1) *has a unique positive solution* $u(x,t)$ *for* $t \in (0,\infty)$. *Moreover, if* (3.1) *admits a positive equilibrium solution* u^**, then* $\lim_{t\to\infty} u(x,t) = u^*(x)$ *in* $C^2(\overline{\Omega})$.

Proof Denote

$$M = \max \left\{ K, \max_{x\in\bar{\Omega}} u_0(x) \right\},$$

where K is given by (3.2). Then $(M, 0)$ is a pair of ordered upper and lower solutions of problem (3.1). This implies that the problem (3.1) has a unique positive solution $u(x,t)$ satisfying $0 < u(x,t) \leq M$ for $(x,t) \in \overline{\Omega} \times (0,\infty)$. It then follows from Theorem 2.1 that there exists a constant $C > 0$ such that

$$\max_{t\geq 1} \|u_t(\cdot,t)\|_{C(\overline{\Omega})} + \max_{t\geq 1} \|u(\cdot,t)\|_{C^{2+\alpha}(\overline{\Omega})} \leq C, \tag{3.3}$$

since here the functions $a_{ij}(x) \equiv 0$ for $i \neq j$, $a_{ii}(x) = d(x)$, and $b_i = c \equiv 0$ satisfy $(\mathbf{L_1})$ and $(\mathbf{L_2})$, and the function $g(x,u) := uf(x,u)$ satisfies $(\mathbf{L_3})$ with $\sigma_1 = 0$ and $\sigma_2 = M$.

Define a function $E : [0,\infty) \to \mathbb{R}$ by

$$E(t) = \int_\Omega \int_{u^*(x)}^{u(x,t)} \frac{u^*(x)}{d(x)} \cdot \frac{s - u^*(x)}{s} \mathrm{d}s \mathrm{d}x.$$

Then $E(t) \geq 0$ for $t \geq 0$. From (2.3), we obtain

$$\begin{aligned}
\frac{dE(t)}{dt} &= \int_\Omega \frac{u^*(u-u^*)}{du} u_t \mathrm{d}x = \int_\Omega \frac{u^*(u-u^*)}{du} \left[d\Delta u + uf(x,u) \right] \mathrm{d}x \\
&= \int_\Omega \frac{u^*(u-u^*)}{du} \left(d\Delta u + uf(x,u) - \frac{u}{u^*} d\Delta u^* - \frac{u}{u^*} u^* f(x,u^*) \right) \mathrm{d}x \\
&= \int_\Omega \frac{u^*(u-u^*)}{du} \left(d\Delta u - \frac{u}{u^*} d\Delta u^* \right) \mathrm{d}x + \int_\Omega \frac{u^*(u-u^*)}{d} \left[f(x,u) - f(x,u^*) \right] \mathrm{d}x \\
&\leq \int_\Omega \left(-u^2 \left| \nabla \frac{u^*}{u} \right|^2 + \frac{u^*(u-u^*)}{d} \left[f(x,u) - f(x,u^*) \right] \right) \mathrm{d}x \\
&\leq - \int_\Omega \frac{u^*(u-u^*)}{d} \left[f(x,u) - f(x,u^*) \right] \mathrm{d}x =: \psi(t) \leq 0.
\end{aligned}$$

Taking advantages of (3.3), we have $||\psi'||_{C([1,\infty))} < C_1$ for some $C_1 > 0$. Then it follows from Lemma 2.2 that

$$\lim_{t\to\infty} \psi(t) = - \int_\Omega \frac{u^*(u-u^*)}{d} \left[f(x,u) - f(x,u^*) \right] \mathrm{d}x = 0. \tag{3.4}$$

The estimate (3.3) also implies that the set $\{u(\cdot, t) : t \geq 1\}$ is relatively compact in $C^2(\overline{\Omega})$. Therefore, we may assume that

$$\|u(x, t_k) - u_\infty(x)\|_{C^2(\overline{\Omega})} \to 0 \quad \text{as} \quad t_k \to \infty$$

for some function $u_\infty \in C^2(\overline{\Omega})$. Combining this with (3.4), we can conclude that $u_\infty(x) \equiv u^*(x)$ for $x \in \overline{\Omega}$. Thus $\lim\limits_{t\to\infty} u(x, t) = u^*(x)$ in $C^2(\overline{\Omega})$. The proof is finished. □

Assume $f(x, u) = m(x) - \phi(x)u$ with m and ϕ satisfying

$$m, \phi \in C^\alpha(\overline{\Omega}), \ \int_\Omega \frac{m(x)}{d(x)} \mathrm{d}x \geq 0, \ m(x) \not\equiv 0 \text{ and } \phi(x) > 0, \quad x \in \overline{\Omega}. \tag{3.5}$$

Then $f(x, u) = m(x) - \phi(x)u$ satisfies (3.2). Let $\theta_{d,m,\phi}$ be the unique positive solution of

$$\begin{cases} d(x)\Delta\theta + \theta[m(x) - \phi(x)\theta] = 0, & x \in \Omega, \\ \partial_\nu \theta = 0, & x \in \partial\Omega. \end{cases} \tag{3.6}$$

Indeed the existence of $\theta_{d,m,\phi}$ follows from [4, Proposition 3.2] and [10, Proposition 2.2], and the uniqueness of $\theta_{d,m,\phi}$ is a consequence of [4, Proposition 3.3].

Clearly, for the above defined $f(x, u)$, we could directly apply Theorem 3.1 to obtain the following conclusion.

Corollary 3.2 *Assume that $u_0 \in C(\overline{\Omega})$ and $d \in C^\alpha(\overline{\Omega})$ with $u_0(x) \geq, \not\equiv 0$, $d(x) > 0$ on $\overline{\Omega}$. Let $f(x, \tau) = m(x) - \phi(x)\tau$ with m and ϕ satisfying (3.5). Then the problem (3.1) has a unique positive solution $u(x, t)$ and a unique positive equilibrium solution $\theta_{d,m,\phi}$, and $\lim\limits_{t\to\infty} u(x, t) = \theta_{d,m,\phi}(x)$ in $C^2(\overline{\Omega})$.*

Remark 3.3 For the quasilinear parabolic problem with nonlinear diffusion and nonlinear boundary condition:

$$\begin{cases} u_t = d(x)\mathrm{div}[a(x)\nabla\Phi(x, u)] + f(x, u), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial\Phi(x, u)}{\partial\nu} = g(x, u), & x \in \partial\Omega, \ t > 0, \\ u(x, 0) = \varphi(x) \geq\not\equiv 0, & x \in \Omega, \end{cases} \tag{3.7}$$

where g and Φ satisfy (i) and (ii) in Lemma 2.4, $a \in C^{1+\alpha}(\overline{\Omega})$, $d \in C^\alpha(\overline{\Omega})$ with $a(x) > 0$ and $d(x) > 0$ on $\overline{\Omega}$, one may construct a similar Lyapunov functional

$$E(t) = \int_\Omega \int_{u_*(x)}^{u(x,t)} \frac{\Phi(x, u_*)}{d(x)} \cdot \frac{\Phi(x, s) - \Phi(x, u_*)}{\Phi(x, s)} \mathrm{d}s\mathrm{d}x$$

to prove the uniqueness and global stability of the positive equilibrium solution u_* with respect to (3.7). For more results about the problem (3.7), readers can refer to [27,30] and the references therein.

3.2 Competition models

We consider a Lotka–Volterra competition model with k species

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = d_i(x)\Delta u_i + u_i\left(m_i(x) - \displaystyle\sum_{j=1}^k a_{ij}(x)u_j\right), & x \in \Omega, \quad t > 0, \ 1 \leq i \leq k, \\ \partial_\nu u_i = 0, & x \in \partial\Omega, \ t > 0, \ 1 \leq i \leq k, \\ u_i(x, 0) = \varphi_i(x) \geq, \not\equiv 0, & x \in \Omega, \ 1 \leq i \leq k, \end{cases} \tag{3.8}$$

where d_i, a_{ij} and m_i satisfy

$$d_i, a_{ij}, m_i \in C^\alpha(\overline{\Omega}),\ a_{ij} \geq 0, a_{ii} > 0,\ d_i \geq 0 \text{ in } \overline{\Omega} \tag{3.9}$$

for some $0 < \alpha < 1$.

In the following, we always assume that for each $1 \leq i \leq k$, either $\inf_{x\in\overline{\Omega}} d_i(x) > 0$ or $d_i \equiv 0$. Without loss of generality, we assume for some integer $1 \leq i_0 \leq k$,

$$\begin{aligned} &\inf_{x\in\overline{\Omega}} d_i(x) > 0, \quad i \in A_1 := \{1, \ldots, i_0\}, \\ &d_i \equiv 0, \qquad\qquad i \in A_2 := \{1, \ldots, k\} \setminus A_1. \end{aligned}$$

Here $i_0 = k$ just means $\inf_{x\in\overline{\Omega}} d_i(x) > 0$ for all $1 \leq i \leq k$. For the case $i_0 < k$, it is clear that the species u_i for $i \in A_2$ is immobile in the habitat $\overline{\Omega}$.

We say that $\mathbf{u}^* := (u_1^*, \ldots, u_k^*)$ with

$$u_i^* \in C^{2+\alpha}(\overline{\Omega}),\ u_j^* \in C^\alpha(\overline{\Omega}) \quad \text{for } i \in A_1,\ j \in A_2$$

is a positive equilibrium solution of (3.8) if $u_i^*(x) > 0$ for $x \in \overline{\Omega}$, $1 \leq i \leq k$ satisfy

$$\begin{cases} d_i(x)\Delta u_i^* + u_i^*\left(m_i(x) - \sum_{j=1}^k a_{ij}u_j^*\right) = 0, & x \in \Omega,\quad i \in A_1, \\ m_i(x) - \sum_{j=1}^k a_{ij}u_j^* = 0, & x \in \Omega,\quad i \in A_2, \\ \partial_\nu u_i^* = 0, & x \in \partial\Omega,\ i \in A_1. \end{cases} \tag{3.10}$$

Moreover we will also consider the global stability of some semi-trivial equilibrium solutions $(v_i^*) := (v_1^*, \ldots, v_k^*)$ of (3.8) with v_i^* satisfying

$$v_i^* \in C^{2+\alpha}(\overline{\Omega}),\ v_j^* \in C^\alpha(\overline{\Omega}) \quad \text{for } i \in A_1,\ j \in A_2$$

and

$$v_i^* = \begin{cases} > 0, & x \in \overline{\Omega},\ i \in B_1 \subset A_1, \\ \equiv 0, & x \in \overline{\Omega},\ i \in B_2 \subset A_1, \\ > 0, & x \in \overline{\Omega},\ i \in B_3 \subset A_2, \\ \equiv 0, & x \in \overline{\Omega},\ i \in B_4 \subset A_2 \end{cases}$$

and

$$\begin{cases} d_i(x)\Delta v_i^* + v_i^*\left(m_i(x) - \sum_{j\in B_1\cup B_3} a_{ij}v_j^*\right) = 0, & x \in \Omega,\quad i \in B_1, \\ m_i(x) - \sum_{j\in B_1\cup B_3} a_{ij}v_j^* = 0, & x \in \Omega,\quad i \in B_3, \\ \partial_\nu v_i^* = 0, & x \in \partial\Omega,\ i \in B_1, \end{cases} \tag{3.11}$$

where the index set B_i satisfy $\cup_{i=1}^4 B_i = \{1, \ldots, k\}$ and $B_i \cap B_j = \emptyset$ for $i \neq j$. Clearly, $B_1 \cup B_2 = A_1 = \{1, \ldots, i_0\}$ and $B_3 \cup B_4 = A_2$.

If the resource functions m_i and other coefficient functions d_i and a_{ij} are all positive constants, the global stability conclusions of the following ordinary differential equation

$$\begin{cases} u_i'(t) = u_i(t)\left(m_i - \sum_{i=1}^{k} a_{ij}u_i(t)\right), \quad 1 \le i \le k, \\ u_i(0) > 0, \end{cases} \tag{3.12}$$

are well known (see [8, Page 138]). For convenience of the reader, we recall it as follows.

Theorem 3.4 [8, Page 138] *Assume that $m_i > 0$, $a_{ii} > 0$ for $1 \le i \le k$, and $a_{ij} \ge 0$ for $1 \le i, j \le k$. If (3.12) has a positive equilibrium $\mathbf{u}^* \in \mathbb{R}^k$ and there exists a diagonal matrix C with positive constant entries such that $CA + A^TC$ is positive definite where $A = (a_{ij})_{k\times k}$, then $\mathbf{u}^*$ is globally asymptotically stable with respect to (3.12).*

Moreover, a similar Lyapunov function as in [8] could be utilized to show the following global stability results regarding the semi-trivial equilibrium of (3.12).

Corollary 3.5 *Assume that $m_i > 0$, $a_{ii} > 0$ for $1 \le i \le k$, and $a_{ij} \ge 0$ for $1 \le i, j \le k$. Let $i_1 \in [1, k)$ be an integer. If*

(i) *the problem (3.12) has a semi-trivial equilibrium $\mathbf{v}^* = (v_1^*, \ldots, v_k^*) \in \mathbb{R}^k$ with $v_i^* > 0$ for $1 \le i \le i_1$ and $v_i^* = 0$ for $i_1 + 1 \le i \le k$,*

(ii) $m_i - \sum_{j=1}^{i_1} a_{ij}v_j^* \le 0$ *for $i_1 + 1 \le i \le k$,*

(iii) *there exists a diagonal matrix $C := \mathrm{diag}(c_1, \ldots, c_k)$ with positive constant entries such that $CA + A^TC$ is positive definite,*

then $\mathbf{v}^*$ is globally asymptotically stable with respect to (3.12), and $\int_0^\infty u_i^2(t)\mathrm{d}t < \infty$ for $i_1 + 1 \le i \le k$.

Proof It is clear that the problem (3.12) admits a unique nonnegative solution $(u_1(t), \ldots, u_k(t))$ satisfying $0 < u_i(t) \le m_i/a_{ii}$ for $t \in (0, \infty)$. Define a function $F : [0, \infty) \to \mathbb{R}$ by

$$F(t) := \sum_{i=1}^{i_1} c_i[u_i(t) - v_i^* \ln u_i(t)] + \sum_{i=i_1+1}^{k} c_iu_i(t).$$

Denote $U(t) := (u_1(t) - v_1^*, \ldots, u_k(t) - v_k^*)$. Making use of (3.12) and (i)–(iii), we deduce

$$\begin{aligned} F'(t) &= -\sum_{i=1}^{k}\sum_{j=1}^{k} c_ia_{ij}(u_i - v_i^*)(u_j - v_j^*) + \sum_{i=i_1+1}^{k} u_i\left(m_i - \sum_{j=1}^{i_1} a_{ij}v_j^*\right) \\ &\le -\sum_{i=1}^{k}\sum_{j=1}^{k} c_ia_{ij}(u_i - v_i^*)(u_j - v_j^*) \\ &= -\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}(c_ia_{ij} + a_{ij}c_j)(u_i - v_i^*)(u_j - v_j^*) \\ &= -\frac{1}{2}U(CA + A^TC)U^T \le -\epsilon\sum_{i=1}^{k}(u_i - v_i^*)^2 \le 0 \end{aligned}$$

for some small $\epsilon > 0$. Then the boundedness of $u_i'(t)$ allows us to apply Lemma 2.2 to conclude that $\sum_{i=1}^{k}[u_i(t) - v_i^*]^2 \to 0$ as $t \to \infty$. Consequently, $\lim_{t\to\infty} u_i(t) = v_i^*$.

Moreover, $F(\infty) := \lim_{t\to\infty} F(t)$ exists since $F(t)$ is non-increasing and has a lower bound for $t \geq 0$. This combined with the following inequality

$$F'(t) \leq -\epsilon \sum_{i=i_1+1}^{k} (u_i - v_i^*)^2 = -\epsilon \sum_{i=i_1+1}^{k} u_i^2(t)$$

implies that $\int_0^\infty u_i^2(t)\mathrm{d}t \leq (F(0) - F(\infty))/\epsilon < \infty$ for $i_1 + 1 \leq i \leq k$. □

The Lyapunov function in [8] is only for ordinary differential equation model without diffusion, but the integral of this function over spatial domain can be used as a Lyapunov function for reaction–diffusion models to prove the global stability of the constant equilibrium solutions with respect to (3.8) when d_i, m_i and a_{ij} are all constants. On the other hand, if one of the functions m_i and a_{ij} is not constant, then the equilibrium solutions of (3.8) may not be constants which brings difficulties to study the global stability of these non-constant equilibrium solutions if we use the same Lyapunov functions as that in [8].

Now we are ready to present our main result on the global stability.

Theorem 3.6 *Let (3.9) be satisfied, and let the sets A_i, B_i be defined as above, and additionally initial function $\varphi_i > 0$ for $i \in A_2$. Then the problem (3.8) admits a unique positive solution (u_i) satisfying $u_i \in C(\overline{\Omega} \times [0,\infty)) \cap C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{\Omega} \times (0,\infty))$ for $i \in A_1$ and $u_i \in C(\overline{\Omega} \times [0,\infty))$ for $i \in A_2$, and for fixed constant $T > 0$ there exists a constant $M := M(T) > 0$ such that*

$$\begin{cases} \sup_{s\geq 1} ||u_i(\cdot,\cdot)||_{C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}\times[s,s+T])} \leq M, & i \in A_1, \\ \max_{t\geq 1} ||u_j(\cdot,t)||_{C(\overline{\Omega})},\ \max_{t\geq 1} ||\partial_t u_j(\cdot,t)||_{C(\overline{\Omega})} \leq M, & j \in A_2. \end{cases} \tag{3.13}$$

Moreover, we have the following global stability conclusions.

(1) (Global stability of positive equilibrium solution) *Assume that*

  (i) (3.8) *has a positive equilibrium solution* $\mathbf{u}^*$*,*
  (ii) *for this $\mathbf{u}^*$, there exist some positive constants ξ_i with $i \in A_1$ and some positive functions $\xi_i \in C(\overline{\Omega})$ with $i \in A_2$ such that the matrix $Q(x) = (q_{ij}(x) + q_{ji}(x))$ is positive definite for every $x \in \overline{\Omega}$, where*

$$q_{ij}(x) = \begin{cases} \dfrac{\xi_i u_i^*(x)}{d_i(x)} a_{ij}(x), & i \in A_1, \\ \xi_i(x) a_{ij}(x), & i \in A_2. \end{cases}$$

Then

$$\lim_{t\to\infty} u_i(x,t) = u_i^*(x) \ \text{ in } \ C^1(\overline{\Omega}) \ \text{ for } i \in A_1,$$
$$\lim_{t\to\infty} u_i(x,t) = u_i^*(x) \ \text{ in } \ L^2(\Omega) \ \text{ for } i \in A_2,$$

which immediately implies that $\mathbf{u}^*$ is the unique positive equilibrium solution of (3.8).

(2) (Global stability of semi-trivial equilibrium solution) *Assume that*

(i) (3.8) *has a semi-trivial equilibrium solution* $\mathbf{v}^* = (v_1^*, \dots, v_k^*)$*, and*

$$m_i(x) - \sum_{1\le j\le k} a_{ij} v_i^*(x) = m_i(x) - \sum_{j\in B_1\cup B_3} a_{ij} v_i^*(x) < 0, \quad i \in B_2 \cup B_4, \quad (3.14)$$

(ii) *for this* $\mathbf{v}^*$*, there exist some positive constants* ξ_i *with* $i \in B_1 \cup B_2 = A_1$ *and some positive functions* $\xi_i \in C(\overline{\Omega})$ *with* $i \in B_3 \cup B_4 = A_2$ *such that the matrix* $Q(x) = (q_{ij}(x) + q_{ji}(x))$ *is positive definite for every* $x \in \overline{\Omega}$*, where*

$$q_{ij}(x) = \begin{cases} \dfrac{\xi_i a_{ij}(x) u_i^*(x)}{d_i(x)}, & i \in B_1, \\ \dfrac{\xi_i a_{ij}(x)}{d_i(x)}, & i \in B_2, \\ \xi_i(x) a_{ij}(x), & i \in B_3 \cup B_4. \end{cases}$$

Then

$$\lim_{t\to\infty} u_i(x,t) = v_i^*(x) \ \text{ in } \ C^1(\Omega) \ \text{ for } i \in B_1 \cup B_2 = A_1,$$
$$\lim_{t\to\infty} u_i(x,t) = v_i^*(x) \ \text{ in } \ L^2(\Omega) \ \text{ for } i \in B_3 \cup B_4 = A_2,$$

where $v_i^* \equiv 0$ *for* $i \in B_2 \cup B_4$.

Proof By the similar arguments as Theorem 3.1, we could apply upper and lower solutions method to show the existence and uniqueness of solution to the problem (3.8), and also

$$u_i \le \max\left\{\max_{x\in\bar{\Omega}} \frac{m_i(x)}{a_{ii}(x)}, \max_{x\in\bar{\Omega}} \varphi_i(x)\right\}.$$

Note that the growth rate per capita term $m_i(x) - \sum_{j=1}^k a_{ij} u_j^*$ in (3.8) is uniformly bounded in $\overline{\Omega} \times [0, \infty)$. Making use of Theorem 2.1 for each $i \in A_1$, we obtain that for fixed $T > 0$, $p \ge 1$, there is a constant $M_1 = M_1(p, T) > 0$ such that

$$\|u_i\|_{W_p^{2,1}(\overline{\Omega}\times[s,s+T])} \le M_1, \quad \forall\, s \ge 1, \ i \in A_1.$$

Then it follows from Sobolev embedding Theorem for large enough p that there is $M = M(T) > 0$ such that (3.13) holds for $i \in A_1$. For $i \in A_2$, the estimates in (3.13) are obvious just from the equation of $\partial_t u_i$ in (3.8).

(1) Define a function $F : [0, \infty) \to \mathbb{R}$ by

$$F(t) = \sum_{i\in A_1} \xi_i \int_\Omega \int_{u_i^*(x)}^{u_i(x,t)} \frac{u_i^*(x)}{d_i(x)} \times \frac{s - u_i^*(x)}{s} \mathrm{d}s\mathrm{d}x + \sum_{i\in A_2} \int_\Omega \int_{u_i^*(x)}^{u_i(x,t)} \xi_i(x) \frac{s - u_i^*(x)}{s} \mathrm{d}s\mathrm{d}x.$$

From (3.8) and (3.10), we obtain

$$\begin{aligned} \frac{dF}{dt} &= \sum_{i\in A_1} \xi_i \int_\Omega \frac{[u_i - u_i^*(x)]u_i^*(x)}{d_i(x)u_i} \partial_t u_i \mathrm{d}x + \sum_{i\in A_2} \int_\Omega \xi_i(x) \frac{u_i - u_i^*(x)}{u_i} \partial_t u_i \mathrm{d}x \\ &= \sum_{i\in A_1} \xi_i \int_\Omega \frac{u_i^*(u_i - u_i^*)}{d_i u_i} \left[d_i \Delta u_i + u_i \left(m_i - \sum_{j=1}^k a_{ij} u_j \right) \right] \mathrm{d}x \\ &\quad + \sum_{i\in A_2} \int_\Omega \frac{\xi_i (u_i - u_i^*)}{u_i} u_i \left(m_i - \sum_{j=1}^k a_{ij} u_j \right) \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i \in A_1} \xi_i \int_\Omega \frac{u_i^*(u_i - u_i^*)}{d_i u_i} \left[d_i \Delta u_i + u_i \left(m_i - \sum_{j=1}^k a_{ij} u_j \right) \right] \mathrm{d}x \\
&\quad - \sum_{i \in A_1} \xi_i \int_\Omega \frac{u_i^*(u_i - u_i^*)}{d_i u_i} \left[\frac{u_i}{u_i^*} d_i \Delta u_i^* + \frac{u_i}{u_i^*} u_i^* \left(m_i - \sum_{j=1}^k a_{ij} u_j^* \right) \right] \mathrm{d}x \\
&\quad + \sum_{i \in A_2} \int_\Omega \xi_i (u_i - u_i^*) \left(m_i - \sum_{j=1}^k a_{ij} u_j - m_i + \sum_{j=1}^k a_{ij} u_j^* \right) \mathrm{d}x \\
&= \sum_{i \in A_1} \xi_i \int_\Omega \left[\frac{u_i^*(u_i - u_i^*)}{u_i} \left(\Delta u_i - \frac{u_i}{u_i^*} \Delta u_i^* \right) - \frac{u_i^*(u_i - u_i^*)}{d_i} \sum_{j=1}^k a_{ij}(u_j - u_j^*) \right] \mathrm{d}x \\
&\quad - \sum_{i \in A_2} \int_\Omega \xi_i (u_i - u_i^*) \sum_{j=1}^k a_{ij}(u_j - u_i^*) \mathrm{d}x.
\end{aligned}
$$

Making use of (2.3) and the definition of q_{ij}, we deduce

$$
\begin{aligned}
\frac{dF}{dt} &\le -\int_\Omega \left(\sum_{i \in A_1} \xi_i u_i^2 \left| \nabla \frac{u_i^*}{u_i} \right|^2 + \sum_{i \in A_1, 1 \le j \le k} \frac{\xi_i a_{ij} u_i^*}{d_i} (u_i - u_i^*)(u_j - u_j^*) \right) \mathrm{d}x \\
&\quad - \int_\Omega \sum_{i \in A_2, 1 \le j \le k} \xi_i a_{ij} (u_i - u_i^*)(u_j - u_j^*) \mathrm{d}x \\
&\le -\int_\Omega \left(\sum_{i \in A_1, 1 \le j \le k} \frac{\xi_i a_{ij} u_i^*}{d_i} (u_i - u_i^*)(u_j - u_j^*) + \sum_{i \in A_2, 1 \le j \le k} \xi_i a_{ij} (u_i - u_i^*)(u_j - u_j^*) \right) \mathrm{d}x \\
&= -\int_\Omega \sum_{1 \le i, j \le k} q_{ij} (u_i - u_i^*)(u_j - u_j^*) \mathrm{d}x \\
&= -\frac{1}{2} \int_\Omega \sum_{1 \le i, j \le k} [(q_{ij} + q_{ji})(u_i - u_i^*)(u_j - u_j^*)] \mathrm{d}x.
\end{aligned}
$$

Note that the matrix $Q(x) = (q_{ij}(x) + q_{ji}(x))$ is positive definite and every function q_{ij} is continuous for $x \in \overline{\Omega}$, there is a constant $\epsilon > 0$ such that

$$
\frac{dF}{dt} < -\frac{\epsilon}{2} \int_\Omega \sum_{1 \le i \le k} (u_i - u_i^*)^2 \mathrm{d}x =: \psi(t) \le 0.
$$

By (3.13), $\psi(t)$ is uniformly continuous in $t \in [1, \infty)$. It then follows from Lemma 2.2 that

$$
\lim_{t \to \infty} ||u_i(\cdot, t) - u_i^*(\cdot)||_{L^2(\Omega)} = 0, \quad 1 \le i \le k. \tag{3.15}
$$

Clearly, it remains to show that for $i \in A_1$, $u_i(\cdot, t)$ converges to u_i^* in $C^1(\overline{\Omega})$ as $t \to \infty$. Making use of (3.13), we see that $\{u_i(\cdot, t)\}_{t \ge 1}$ is relatively compact in $C^1(\overline{\Omega})$, and for any convergent subsequence of $\{u_i(\cdot, t)\}_{t \ge 1}$, denoted by $\{u_i(\cdot, t_k)\}_{k=1}^\infty$ with $t_k \to \infty$, there exists a $\tilde{u}_i^* \in C^1(\overline{\Omega})$ such that

$$\lim_{t_k \to \infty} ||u_i(\cdot, t_k) - \tilde{u}_i^*(\cdot)||_{C^1(\overline{\Omega})} = 0, \quad i \in A_1.$$

Recalling (3.15), the uniqueness of the limit implies $u_i^* = \tilde{u}_i^*$. Therefore, $\lim_{t\to\infty} u_i(x,t) = u_i^*(x)$ in $C^1(\overline{\Omega})$ for $i \in A_1$.

(2) Define a function $F : [0, \infty) \to \mathbb{R}$ by

$$\begin{aligned} F(t) = &\sum_{i\in B_1} \xi_i \int_\Omega \int_{v_i^*(x)}^{u_i(x,t)} \frac{v_i^*(x)}{d_i(x)} \times \frac{s - v_i^*(x)}{s} \mathrm{d}s\mathrm{d}x + \sum_{i\in B_2}^{k} \xi_i \int_\Omega \frac{u_i(x,t)}{d_i}\mathrm{d}x \\ &\sum_{i\in B_3} \int_\Omega \int_{v_i^*(x)}^{u_i(x,t)} \xi_i(x) \frac{s - v_i^*(x)}{s} \mathrm{d}s\mathrm{d}x + \sum_{i\in B_4}^{k} \int_\Omega \xi_i(x) u_i(x,t)\mathrm{d}x \\ =:&\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4. \end{aligned}$$

Similar calculation as in (1) shows

$$\begin{aligned} \frac{d(\Theta_1+\Theta_3)}{dt} &\le -\int_\Omega \left(\sum_{i\in B_1, 1\le j\le k} \frac{\xi_i a_{ij} v_i^*}{d_i} (u_i - v_i^*)(u_j - v_j^*) + \sum_{i\in B_3, 1\le j\le k} \xi_i a_{ij}(u_i - v_i^*)(u_j - v_j^*) \right) \mathrm{d}x \\ &= -\int_\Omega \sum_{i\in B_1\cup B_3, 1\le j\le k} q_{ij}(u_i - v_i^*)(u_j - v_j^*)\mathrm{d}x, \end{aligned}$$

where we have used $v_j^* = 0$ for $j \in B_2 \cup B_4$. From (3.8) and (3.11), we deduce

$$\begin{aligned} \frac{d(\Theta_2+\Theta_4)}{dt} &= \sum_{i\in B_2} \xi_i \int_\Omega \frac{\partial_t u_i}{d_i}\mathrm{d}x + \sum_{i\in B_4} \int_\Omega \xi_i \partial_t u_i \mathrm{d}x \\ &= \sum_{i\in B_2} \xi_i \int_\Omega \left[\Delta u_i + \frac{1}{d_i} u_i \left(m_i - \sum_{j=1}^k a_{ij} u_j \right) \right] \mathrm{d}x \\ &\quad + \sum_{i\in B_4} \int_\Omega \xi_i u_i \left(m_i - \sum_{j=1}^k a_{ij} u_j \right) \mathrm{d}x \\ &= \sum_{i\in B_2} \int_\Omega \frac{\xi_i}{d_i} u_i \left(m_i - \sum_{j=1}^k a_{ij} v_j^* + \sum_{j=1}^k a_{ij} v_j^* - \sum_{j=1}^k a_{ij} u_j \right) \mathrm{d}x \\ &\quad + \sum_{i\in B_4} \int_\Omega \xi_i u_i \left(m_i - \sum_{j=1}^k a_{ij} v_j^* + \sum_{j=1}^k a_{ij} v_j^* - \sum_{j=1}^k a_{ij} u_j \right) \mathrm{d}x. \end{aligned}$$

Note that $v_j^* = 0$ for $j \in B_2 \cup B_4$. In view of (3.14) and the definition of q_{ij}, we deduce

$$\begin{aligned} \frac{d(\Theta_2+\Theta_4)}{dt} &\le -\sum_{i\in B_2} \int_\Omega \frac{\xi_i}{d_i} u_i \sum_{j=1}^k a_{ij}(u_j - v_j^*)\mathrm{d}x - \sum_{i\in B_4} \int_\Omega \xi_i u_i \sum_{j=1}^k a_{ij}(u_j - v_j^*)\mathrm{d}x \\ &= -\sum_{i\in B_2} \int_\Omega \frac{\xi_i}{d_i} \sum_{j=1}^k a_{ij}(u_i - v_i^*)(u_j - v_j^*)\mathrm{d}x - \sum_{i\in B_4} \int_\Omega \xi_i \sum_{j=1}^k a_{ij}(u_i - v_i^*)(u_j - v_j^*)\mathrm{d}x \end{aligned}$$

$$= -\int_\Omega \sum_{i\in B_2\cup B_4, 1\le j\le k} q_{ij}(u_i - v_i^*)(u_j - v_j^*)\mathrm{d}x.$$

Therefore,

$$\frac{dF}{dt} \le -\int_\Omega \sum_{1\le i,j\le k} q_{ij}(u_i - v_i^*)(u_j - v_j^*)\mathrm{d}x = -\frac{1}{2}\int_\Omega \sum_{1\le i,j\le k} (q_{ij} + q_{ij})(u_i - v_i^*)(u_j - v_j^*)\mathrm{d}x.$$

By the similar arguments as (1), we obtain the desired conclusion. □

Remark 3.7 (i) If the boundary condition of (3.8) is Robin boundary condition taking the form $\partial_\nu u_i = b(x)u_i + c(x)$ with nonnegative functions $b, c \in C^\alpha(\overline{\Omega})$, one sees that by (2.3) that the conclusions in Theorem 3.6 still hold.

(ii) If $A_2 = \emptyset$, then the convergence conclusions in Theorem 3.6 could be enhanced, say in $C^2(\overline{\Omega})$. See also Proposition 3.8 below for a strengthened version.

(iii) In Theorem 3.6(1), we could rewrite the matrix Q as

$$Q = CA + A^T C, \tag{3.16}$$

where the matrices A and C are defined as $A = (a_{ij}(x))$ and

$$C = \mathrm{diag}\left(\frac{u_1^*}{d_1}\xi_1, \ldots, \frac{u_{i_0}^*}{d_{i_0}}\xi_{i_0}, \xi_{i_0+1}, \ldots, \xi_k\right).$$

Then the condition in Theorem 3.6 (1) (ii) becomes that $CA + AC^T$ is positive definite, which coincides with the condition in Theorem 3.4.

Finally we show that when all diffusion coefficients are positive, the convergence to the semi-trivial solution can be shown under a condition on an $i_1 \times i_1$ matrix Q_1 instead of on the full $k \times k$ matrix Q.

Proposition 3.8 *Assume that $d_i > 0$, a_{ij} and m_i satisfy (3.9). Let $v_i^* = (v_i)$ with $v_i > 0$ for $1 \le i \le i_1$ and $v_i \equiv 0$ for $i_1 + 1 \le i \le k$ be a semi-trivial equilibrium solution of (3.8) for some $1 \le i_1 < k$. Suppose*

$$\int_0^\infty ||u_i(\cdot, t)||^2_{L^2(\Omega)}\mathrm{d}t < \infty, \quad i_1 + 1 \le i \le k \tag{3.17}$$

and there exist some positive constants ξ_i for $1 \le i \le i_1$ such that the $i_1 \times i_1$ matrix $Q_1(x) = (q_{ij}(x) + q_{ji}(x))$ with $q_{ij}(x) := \xi_i \frac{v_i^*(x)}{d_i(x)} a_{ij}(x)$ is positive definite for every $x \in \overline{\Omega}$. Then $\lim_{t\to\infty} u_i(x, t) = v_i^*(x)$ in $C^2(\overline{\Omega})$ for $1 \le i \le k$.

Proof From (3.17) and the uniformly boundedness of $||u_i(\cdot, t)||_{C^{2+\alpha}(\overline{\Omega})}$ for $t \ge 1$, we could apply Babarlat's Lemma to show that $\lim_{t\to\infty} ||u_i(\cdot, t)||_{L^2(\Omega)} = 0$ for $i_1 + 1 \le i \le k$. Then the relative compactness of $\{u_i(\cdot, t) : t \ge 1\}$ in $C^2(\overline{\Omega})$ implies

$$\lim_{t\to\infty} ||u_i(\cdot, t)||_{C^2(\overline{\Omega})} = 0, \quad i_1 + 1 \le i \le k.$$

Define

$$F(t) = \sum_{i=1}^{i_1} \xi_i \int_\Omega \int_{v_i^*(x)}^{u_i(x,t)} \frac{v_i^*(x)}{d_i(x)} \times \frac{s - v_i^*(x)}{s}\mathrm{d}s\mathrm{d}x.$$

Denote $q_{ij}(x) = \frac{\xi_i v_i^*(x)}{d_i(x)} a_{ij}(x)$ for $1 \le i \le i_1$ and $i_1 + 1 \le j \le k$. Then by the similar computation as Theorem 3.6 (1) with $A_2 = \emptyset$, we obtain

$$\frac{dF}{dt} \le -\int_\Omega \sum_{i=1}^{i_1} \sum_{j=1}^{k} q_{ij}(u_i - v_i^*)(u_j - v_j^*)\mathrm{d}x.$$

Since $Q_1(x) = (q_{ij}(x) + q_{ji}(x))$ with $1 \le i, j \le i_1$ is positive definite, then there is $\delta > 0$ such that

$$-\int_\Omega \sum_{i=1}^{i_1} \sum_{j=1}^{i_1} q_{ij}(u_i - v_i^*)(u_j - v_j^*)\mathrm{d}x \le -\delta \int_\Omega \sum_{i=1}^{i_1} (u_i - v_i^*)^2 \mathrm{d}x,$$

and hence for $\epsilon = \delta/(2k)$,

$$\begin{aligned}
\frac{dF}{dt} &\le -\int_\Omega \left[\sum_{i=1}^{i_1} \delta(u_i - v_i^*)^2 + \sum_{i=1}^{i_1} \sum_{j=i_1+1}^{k} q_{ij}(u_i - v_i^*)(u_j - v_j^*) \right] \mathrm{d}x \\
&\le -\int_\Omega \left[\sum_{i=1}^{i_1} \delta(u_i - v_i^*)^2 - \sum_{i=1}^{i_1} \sum_{j=i_1+1}^{k} \left(\epsilon(u_i - v_i^*)^2 + \frac{q_{ij}^2}{4\epsilon}(u_j - v_j^*)^2 \right) \right] \mathrm{d}x \\
&\le -\int_\Omega \left[\sum_{i=1}^{i_1} (\delta - k\epsilon)(u_i - v_i^*)^2 - \sum_{i=1}^{i_1} \sum_{j=i_1+1}^{k} \frac{q_{ij}^2}{4\epsilon}(u_j - v_j^*)^2 \right] \mathrm{d}x \\
&= -\int_\Omega \sum_{i=1}^{i_1} \frac{\delta}{2}(u_i - v_i^*)^2 \mathrm{d}x + \int_\Omega \sum_{j=i_1+1}^{k} \left(\sum_{i=1}^{i_1} \frac{q_{ij}^2}{4\epsilon} \right) (u_j - v_j^*)^2 \mathrm{d}x =: -\psi(t) + h(t).
\end{aligned}$$

Making use of (3.17) and Lemma 2.2, we obtain $\lim_{t\to\infty} \psi(t) = 0$, and therefore $\lim_{t\to\infty} ||u_i - v_i^*||_{L^2(\Omega)} = 0$ for $1 \le i \le i_1$. Then the similar arguments as the proof of Theorem 3.1 yield $\lim_{t\to\infty} u_i(x, t) = v_i^*(x)$ in $C^2(\overline{\Omega})$ for $1 \le i \le i_1$. □

4 Applications

In this section, we consider the following two special cases of d_i and a_{ij}:

$$\begin{aligned}
&\text{Case 1} \;\; d_i(x) \ge 0, a_{ij}(x) \ge 0 \text{ when } k = 2, \\
&\text{Case 2} \;\; d_i > 0 \text{ and } a_{ij} \ge 0 \text{ with } a_{ii} = 1 \text{ are all constants, and } m_{ij} \ge 0.
\end{aligned} \tag{4.1}$$

To show the global stability of equilibrium solutions (u_i^*) with respect to (3.8), we need to find some positive constants (or functions) ξ_i such that the matrix-valued function $Q(x)$ defined in Theorem 3.6 is positive definite. As Q depends on u_i^*, the estimates of the equilibrium solution u_i^* are crucial to achieve this goal. With the help of upper and lower solutions method, we could obtain a rough estimate of the equilibrium solutions. And then, by verifying the conditions in Theorem 3.6 we can show the global stability of equilibrium solutions of (3.8).

4.1 Two species

Theorem 4.1 *Suppose that the functions d_i, m_i, a_{ij} satisfy (3.9) with $k = 2$.*

(1) *Suppose that* $\inf_{x\in\overline{\Omega}} d_i(x) > 0$. *If* (3.8) *has a positive equilibrium solution* $\mathbf{u}^* = (u_1^*, u_2^*)$ *and there are two constants* $\xi_1 > 0$ *and* $\xi_2 > 0$ *such that*

$$Q = CA + A^T C \text{ is positive definite} \tag{4.2}$$

with $A = (a_{ij})$ *and* $C := \mathrm{diag}(\xi_1 u_1^*/d_1, \xi_2 u_2^*/d_2)$, *then* $\lim_{t\to\infty} u_i(x,t) = u_i^*(x)$ *in* $C^2(\overline{\Omega})$.

(2) *Suppose that* $\inf_{x\in\overline{\Omega}} d_1(x) > 0$, $d_2(x) \equiv 0$ *and*

$$0 < a_{12}(x)a_{21}(x) < a_{11}(x)a_{22}(x), \quad x \in \bar{\Omega}. \tag{4.3}$$

Then

(i) *the positive equilibrium solution* $(u_1^*(x), u_2^*(x))$ *of* (3.8), *if exists, is globally asymptotically stable, and* $\lim_{t\to\infty} u_1(x,t) = u_1^*(x)$ *in* $C^1(\overline{\Omega})$ *and* $\lim_{t\to\infty} u_2(x,t) = u_2^*(x)$ *in* $L^2(\Omega)$.

(ii) *If*

$$\frac{m_2(x)}{a_{21}(x)} \le \theta_{d_1,m_1,a_{11}}(x), \quad x \in \bar{\Omega}, \tag{4.4}$$

then $\lim_{t\to\infty} u_1(x,t) = \theta_{d_1,m_1,a_{11}}(x)$ *in* $C^1(\overline{\Omega})$ *and* $\lim_{t\to\infty} u_2(x,t) = 0$ *in* $L^2(\Omega)$, *where* $\theta_{d_1,m_1,a_{11}}$ *is defined as in* (3.6).

(iii) *If*

$$\frac{a_{22}(x)}{a_{12}(x)} \le \frac{m_2(x)}{m_1(x)}, \quad x \in \bar{\Omega}, \tag{4.5}$$

then $\lim_{t\to\infty} u_1(x,t) = 0$ *in* $C^1(\overline{\Omega})$ *and* $\lim_{t\to\infty} u_2(x,t) = \dfrac{m_2(x)}{a_{22}(x)}$ *in* $L^2(\Omega)$.

Proof (1) It follows from Theorem 3.6 (1) and Remark 3.7 that the statement in (1) is valid.

(2) Let $\xi_1 = 1$ and $\xi_2(x) = \frac{a_{12}(x)u_1^*(x)}{a_{21}(x)d_1(x)}$. Then it follows from (3.16) with $i_0 = 1$ that

$$\begin{aligned}Q = CA + A^T C &= \begin{pmatrix} 2\frac{\xi_1 u_1^*}{d_1}a_{11} & \frac{\xi_1 u_1^*}{d_1}a_{12} + \xi_2 a_{21} \\ \frac{\xi_1 u_1^*}{d_1}a_{12} + \xi_2 a_{21} & 2\xi_2 a_{22} \end{pmatrix} = \begin{pmatrix} 2\frac{u_1^*}{d_1}a_{11} & 2\frac{u_1^*}{d_1}a_{12} \\ 2\frac{u_1^*}{d_1}a_{12} & 2\frac{a_{12}u_1^*}{a_{21}d_1}a_{22} \end{pmatrix} \\ &= 2\frac{u_1^*}{d_1}\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & \frac{a_{12}}{a_{21}}a_{22} \end{pmatrix}\end{aligned}$$

Since $a_{11} > 0, a_{22} > 0$ and $\det Q = 2\frac{u_1^*}{d_1}(a_{11}a_{22}a_{12}/a_{21} - a_{12}^2) = 2\frac{u_1^* a_{12}}{d_1 a_{21}}(a_{11}a_{22} - a_{12}a_{21}) > 0$ by (4.3), we easily see that Q is positive definite. Hence, by Theorem 3.6 (1), the conclusion of (i) holds.

(ii) Clearly, $(u_1^*, 0)$ is a semi-trivial equilibrium solution of (3.8), where $u_1^* = \theta_{d_1,m_1,a_{11}}$ is the unique positive solution of (3.6). We also use the notations B_i and q_{ij} as in Theorem 3.6. Then $B_1 = \{1\}$, $B_2 = \emptyset$, $B_3 = \emptyset$ and $B_4 = \{2\}$, and

$$q_{ij}(x) = \begin{cases} \dfrac{\xi_1 u_1^*(x)}{d_1(x)} a_{1j}(x), & i = 1, \\ \xi_2(x) a_{2j}(x), & i = 2. \end{cases}$$

Let $\xi_1 = 1$ and $\xi_2(x) = \frac{a_{12}(x)u_1^*(x)}{a_{21}(x)d_1(x)}$. Then

$$Q = \begin{pmatrix} 2\frac{\xi_1 u_1^*}{d_1}a_{11} & \frac{\xi_1 u_1^*}{d_1}a_{12} + \xi_2 a_{21} \\ \frac{\xi_1 u_1^*}{d_1}a_{12} + \xi_2 a_{21} & 2\xi_2 a_{22} \end{pmatrix} = 2\frac{u_1^*}{d_1}\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & \frac{a_{12}}{a_{21}}a_{22} \end{pmatrix}$$

is positive definite by (4.3). In view of (4.4), we could apply Theorem 3.6 (2) to obtain the conclusion (ii).

(iii) Clearly, $(v_1^*(x), v_2^*(x)) = \left(0, \dfrac{m_2(x)}{a_{22}(x)}\right)$ is a semi-trivial equilibrium solution. Let B_i and q_{ij} be defined as in Theorem 3.6. Then $B_1 = \emptyset$, $B_2 = \{1\}$, $B_3 = \{2\}$ and $B_4 = \emptyset$, and

$$q_{ij}(x) = \begin{cases} \dfrac{\xi_1}{d_1(x)}a_{1j}(x), & i = 1, \\ \xi_2(x)a_{2j}(x), & i = 2, \end{cases}$$

Let $\xi_1 = 1$ and $\xi_2(x) = \dfrac{a_{12}(x)}{d_1(x)a_{21}(x)}$. Then

$$Q = \begin{pmatrix} 2\frac{\xi_1}{d_1}a_{11} & \frac{\xi_1}{d_1}a_{12} + \xi_2 a_{21} \\ \frac{\xi_1}{d_1}a_{12} + \xi_2 a_{21} & 2\xi_2 a_{22} \end{pmatrix} = \frac{2}{d_1}\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & \frac{a_{12}}{a_{21}}a_{22} \end{pmatrix}$$

is positive definite by (4.3). This combined with (4.5) allows us to show the conclusion (iii) by Theorem 3.6 (2). □

Remark 4.2 (1) A simple calculation indicates that (4.2) is equivalent to $a_{ii} > 0$ and

$$\frac{a_{12}(x)a_{21}(x)}{a_{11}(x)a_{22}(x)} < \frac{4\tilde{\xi}_1(x)\tilde{\xi}_2(x)}{[\tilde{\xi}_1(x) + \tilde{\xi}_2(x)]^2}, \quad x \in \overline{\Omega},$$

where $\tilde{\xi}_i(x) = \xi_i u_i^*(x)/d_i(x)$.

(2) Clearly, when $d_2 \equiv 0$, the problem (3.8) admits a positive equilibrium solution $(u_1^*(x), u_2^*(x))$ if and only if $(u_1^*(x), u_2^*(x))$ satisfies

$$\begin{cases} -d_1(x)\Delta u_1 = u_1\left[m_1(x) - \dfrac{a_{12}}{a_{22}}m_2(x) - \left(a_{11} - \dfrac{a_{12}a_{21}}{a_{22}}\right)u_1\right], & x \in \Omega, \\ \partial_\nu u_1 = 0, & x \in \partial\Omega, \end{cases} \tag{4.6}$$

and $u_2^* = \dfrac{m_2 - a_{21}u_1^*}{a_{22}} > 0$ on $\overline{\Omega}$. From [4, Proposition 3.2], [10, Proposition 2.2] and [4, Proposition 3.3], one possible condition leading to the existence of $(u_1^*(x), u_2^*(x))$ is that

$$\int_\Omega \frac{a_{22}(x)m_1(x) - a_{12}(x)m_2(x)}{d_1(x)a_{22}(x)}\mathrm{d}x > 0, \quad m_2(x) - a_{21}(x)\theta(x) > 0, \quad x \in \overline{\Omega},$$

where $\theta := \theta_{d_1, m_1 - a_{12}/a_{22}, a_{11} - a_{12}a_{21}/a_{22}}$ is defined as in (3.6).

Example 4.3 For the problem (3.8), let $k = 2$, $d_1(x) = rd_2(x) > 0$ for some constant $r > 0$. If the positive constant vectors $(\bar{u}_1, \bar{u}_2)$ and $(\underline{u}_1, \underline{u}_2)$ are the upper and lower solution of (3.10) [or the corresponding elliptic boundary value problem of (3.8)], and

$$\max_{x\in\overline{\Omega}} \frac{a_{12}(x)a_{21}(x)}{a_{11}(x)a_{22}(x)} < \frac{\underline{u}_1\underline{u}_2}{\bar{u}_1\bar{u}_2},$$

then (3.10) has a unique positive equilibrium solution (u_1^*, u_2^*) bounded by $(\underline{u}_1, \underline{u}_2)$ and $(\bar{u}_1, \bar{u}_2)$, and $\lim_{t\to\infty} u_i(x,t) = u_i^*(x)$ in $C^2(\overline{\Omega})$.

4.2 *k* species

We assume that (4.1) holds throughout this subsection.

4.2.1 Positive equilibrium solution

For the simplicity of notations, we define, for $1 \le i \le k$,

$$\begin{cases} m_i^- = \min_{x\in\overline{\Omega}} m_i(x), \quad m_i^+ = \max_{x\in\overline{\Omega}} m_i(x), \\ \mathbf{m}^- = (m_1^-, \ldots, m_k^-)^T, \quad \mathbf{m}^+ = (m_1^+, \ldots, m_k^+)^T, \\ A = (a_{ij})_{k\times k}, \quad B = A - I_k, \end{cases} \tag{4.7}$$

where I_k is the $k \times k$ identity matrix. Clearly, the diagonal entries of B are 0 because of (4.1) and (4.7).

To study the global stability of positive equilibrium solution of the problem (3.8), we make the following assumptions:

($\mathbf{F_1}$) The determinant $\det[A(2I_k - A)] \ne 0$, and the algebraic equations

$$\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \mathbf{C}_*^T = \begin{bmatrix} \mathbf{m}^- + (2I_k - A)^{-1}(\mathbf{m}^+ - \mathbf{m}^-) \\ \mathbf{m}^+ - (2I_k - A)^{-1}(\mathbf{m}^+ - \mathbf{m}^-) \end{bmatrix} \tag{4.8}$$

has a unique positive solution $\mathbf{C}_* := (\bar{c}_1, \ldots, \bar{c}_k, \underline{c}_1, \ldots, \underline{c}_k) \in \mathbb{R}^{2k}$.

($\mathbf{F_2}$) There exists a $2k \times 2k$ diagonal matrix Q_1 with positive constant entries such that the matrix

$$Q_1 \begin{bmatrix} I_k & B \\ B & I_k \end{bmatrix} + \begin{bmatrix} I_k & B \\ B & I_k \end{bmatrix}^T Q_1 \text{ is positive definite.}$$

($\mathbf{F_3}$) There exists a $k \times k$ diagonal matrix Q_2 with positive constant entries, such that $Q_2(I_k - B - \mathbf{c}_1) + (I_k - B - \mathbf{c}_1)^T Q_2$ is positive definite, where

$$\mathbf{c}_1 = \mathrm{diag}\left(\frac{\bar{c}_1 - \underline{c}_1}{\bar{c}_1}, \frac{\bar{c}_2 - \underline{c}_2}{\bar{c}_2}, \ldots, \frac{\bar{c}_k - \underline{c}_k}{\bar{c}_k}\right), \tag{4.9}$$

and $\bar{c}_i$ and $\underline{c}_i$ are given by ($\mathbf{F_1}$).

For the assumptions ($\mathbf{F_2}$) and ($\mathbf{F_3}$), we have the following conclusion

$$(\mathbf{F_3}) \text{ implies } (\mathbf{F_2}), \tag{4.10}$$

which in fact is a direct consequence of Proposition 4.5. We recall that the system (3.8) is uniformly persistent (see, e.g., [9, Page 390]) if all solutions satisfy $\liminf_{t\to\infty} u_i(x,t) > 0$ for all $1 \le i \le k$ and $x \in \bar{\Omega}$, and it is permanent (see, e.g., [5,20]) if it also satisfies $\limsup_{t\to\infty} u_i(x,t) \le M$ for some $M > 0$. Now we prove the following result which concerns with the permanence property of (3.8), and also the global stability of the positive equilibrium solution of (3.8).

Theorem 4.4 *Let* (3.9), (4.1) *and* $(\mathbf{F_1})$ *be satisfied.*

(i) *Assume* $(\mathbf{F_2})$ *holds. Then the problem* (3.8) *has a positive equilibrium solution* $(u_1^*(x), \ldots, u_k^*(x))$ *which satisfies*

$$0 < \underline{c}_i \le u_i^*(x) \le \bar{c}_i, \quad \forall\, x \in \overline{\Omega},\ 1 \le i \le k, \tag{4.11}$$

where $\mathbf{C}_* = (\bar{c}_1, \ldots, \bar{c}_k, \underline{c}_1, \ldots, \underline{c}_k) \in \mathbb{R}^{2k}$ *is given by* $(\mathbf{F_1})$*. Moreover, the solution* $(u_1, \ldots, u_k)$ *of* (3.8) *satisfies*

$$0 < \underline{c}_i \le \liminf_{t\to\infty} u_i(x,t) \le \limsup_{t\to\infty} u_i(x,t) \le \bar{c}_i, \quad \forall\, x \in \overline{\Omega},\ 1 \le i \le k, \tag{4.12}$$

which immediately implies that the problem (3.8) *is permanent.*

(ii) *If* $(\mathbf{F_3})$ *holds, then* $\lim_{t\to\infty} u_i(x,t) = u_i^*(x)$ *in* $C^2(\overline{\Omega})$ *for* $1 \le i \le k$.

Proof (i) From the assumption $(\mathbf{F_1})$, we know that $\bar{c}_i > 0$ and $\underline{c}_i > 0$. Next we show $\bar{c}_i \ge \underline{c}_i$ and (4.12) by considering an auxiliary problem

$$\begin{cases} \bar{u}_i' = \bar{u}_i \left(m_i^+ - \bar{u}_i - \sum\limits_{1\le j\le k,\ j\ne i} a_{ij}\underline{u}_j \right), & t > 0,\ i = 1, \ldots, k, \\ \underline{u}_i' = \underline{u}_i \left(m_i^- - \underline{u}_i - \sum\limits_{1\le j\le k,\ j\ne i} a_{ij}\bar{u}_j \right), & t > 0,\ i = 1, \ldots, k, \\ \bar{u}_i(0) = \max\limits_{x\in\overline{\Omega}} \varphi_i(x),\ \underline{u}_i(0) = \min\limits_{x\in\overline{\Omega}} \varphi_i(x), & i = 1, \ldots, k. \end{cases}$$

Denote $U(t) = (\bar{u}_1(t), \ldots, \bar{u}_k(t), \underline{u}_1(t), \ldots, \underline{u}_k(t))^T$. Then the above ordinary differential equations could be rewritten as

$$U'(t) = \left(\begin{bmatrix} \mathbf{m}^+ \\ \mathbf{m}^- \end{bmatrix} - \begin{bmatrix} I_k & B \\ B & I_k \end{bmatrix} \right) U(t). \tag{4.13}$$

Here, without loss of generality, we can assume $\varphi_i(x) > 0$ on $\overline{\Omega}$ since the solution $u_i(x,t)$ of (3.8) is positive for any $t > 0$ which can be easily obtained by applying upper and lower solutions method [29, Theorem 8.1] and Hopf's Lemma for parabolic equations. Then $(\bar{u}_1(t), \ldots, \bar{u}_k(t))$ and $(\underline{u}_1(t), \ldots, \underline{u}_k(t))$ are a pair of coupled ordered upper and lower solutions of (3.8) and so

$$0 < \underline{u}_i(t) \le u_i(x,t) \le \bar{u}_i(t), \quad \forall\, x \in \overline{\Omega},\ t > 0. \tag{4.14}$$

Note that (4.14) holds. In order to prove $\bar{c}_i \ge \underline{c}_i$ and (4.12), it suffices to show that

$$\lim_{t\to\infty} \bar{u}_i(t) = \bar{c}_i, \quad \lim_{t\to\infty} \underline{u}_i(t) = \underline{c}_i,$$

which could be proved by using Theorem 3.4 and the assumption $(\mathbf{F_2})$ if $\mathbf{C}_* = (\bar{c}_1, \ldots, \bar{c}_k, \underline{c}_1, \ldots, \underline{c}_k)$ is a positive equilibrium of (4.13), namely, $\mathbf{C}_*$ satisfies

$$\begin{bmatrix} I_k & B \\ B & I_k \end{bmatrix} \mathbf{C}_*^T = \begin{bmatrix} \mathbf{m}^+ \\ \mathbf{m}^- \end{bmatrix}, \tag{4.15}$$

where I_k, B, $\mathbf{m}^-$ and $\mathbf{m}^+$ are given by (4.7). Denote

$$B_1 = \begin{bmatrix} (I_k - B)^{-1} & -(I_k - B)^{-1} + I_k \\ -(I_k - B)^{-1} + I_k & (I_k - B)^{-1} \end{bmatrix}.$$

It follows from $\det[(I_k+B)(I_k-B)] = \det[A(2I_k-A)] \neq 0$ that $\det B_1 = \det(I_k-B)^{-1}\det(I_k+B) \neq 0$. Then multiplying the equation (4.15) by B_1 on the left, we have

$$\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \mathbf{C}_*^T = \begin{bmatrix} \mathbf{m}^- + (I_k-B)^{-1}(\mathbf{m}^+ - \mathbf{m}^-) \\ \mathbf{m}^+ - (I_k-B)^{-1}(\mathbf{m}^+ - \mathbf{m}^-) \end{bmatrix}.$$

Thanks to $\det B_1 \neq 0$ and $(\mathbf{F_1})$, we know that (4.15) holds and $\underline{c}_i,\ \bar{c}_i > 0$.

In the following, we show that (3.8) admits a positive equilibrium solution $(u_1^*(x),\dots,u_k^*(x))$ satisfying (4.11). Since $(\bar{c}_1,\dots,\bar{c}_k,\underline{c}_1,\dots,\underline{c}_k)$ with $\underline{c}_i \leq \bar{c}_i$ is the unique positive equilibrium of (4.13), then $(\bar{c}_1,\dots,\bar{c}_k)$ and $(\underline{c}_1,\dots,\underline{c}_k)$ are a pair of coupled ordered upper and lower solutions of (3.10) under the condition (4.1). It follows from [29, Theorem 10.2, Page 440] that the problem (3.10) has a positive solution $\mathbf{u}^*$ which is a positive equilibrium solution of (3.8), and also (4.11) holds.

(ii) Note that $(\mathbf{F_3})$ implies $(\mathbf{F_2})$ by (4.10). From (i), the problem (3.8) has a positive equilibrium solution $\mathbf{u}^*$ satisfying (4.11). In view of Theorem 3.6 (1) and Remark 3.7, we just need to show that there are positive constants ξ_i such that

$$CA + A^T C \tag{4.16}$$

is positive definite, where $C = \operatorname{diag}\left(\frac{u_1^*}{d_1}\xi_1,\dots,\frac{u_k^*}{d_k}\xi_k\right)$. Let $\mathbf{x}_1 = (x_1,\dots,x_k) \in \mathbb{R}^k$. Thanks to (4.11), we deduce

$$\begin{aligned}
\mathbf{x}_1(CA + A^TC)\mathbf{x}_1^T =& 2\sum_{i=1}^k\sum_{j=1}^k \frac{u_i^*}{d_i}\xi_i a_{ij}x_ix_j \geq 2\sum_{i=1}^k \frac{\underline{c}_i}{d_i}\xi_i a_{ii}x_i^2 - 2\sum_{i\neq j}\frac{\bar{c}_i}{d_i}\xi_i a_{ij}|x_ix_j| \\
=& 2\sum_{i=1}^k\sum_{j=1}^k \frac{\bar{c}_i}{d_i}\xi_i a_{ij}|x_ix_j| - 2\sum_{i=1}^k \frac{\bar{c}_i}{d_i}\frac{(\bar{c}_i-\underline{c}_i)}{\bar{c}_i}\xi_i a_{ii}x_i^2 \\
=& -\mathbf{x}_2[E(I_k - B - \mathbf{c}_1) + (I_k - B - \mathbf{c}_1)^T E]\mathbf{x}_2^T
\end{aligned}$$

where $\mathbf{c}_1$ is defined as in (4.9), and

$$\mathbf{x}_2 = (|x_1|,\dots,|x_k|),\quad E = \operatorname{diag}\left(\frac{\bar{c}_i}{d_1}\xi_1,\dots,\frac{\bar{c}_k}{d_k}\xi_k\right).$$

Choose suitable $\xi_i > 0$ such that $E = Q_2$, where Q_2 is given by $(\mathbf{F_3})$. Then $CA + A^TC$ is positive definite by $(\mathbf{F_3})$, and the desired conclusion follows directly from Theorem 3.6. □

Proposition 4.5 *If one of the following holds,*

(iii) *there exist two $k\times k$ diagonal matrices Q_3, Q_4 with positive constant entries such that both Q_3 and $4Q_4 - (Q_4B + B^TQ_3){Q_3}^{-1}(B^TQ_4 + Q_3B)$ are positive definite,*

(iii) *there exists a $k\times k$ diagonal matrix Q_5 with positive constant entries, such that $Q_5(I_k - B) + (I_k-B)^TQ_5$ is positive definite,*

then $(\mathbf{F_2})$ *is satisfied.*

The proof of Proposition 4.5 is placed in "Appendix". Clearly, (4.10) can be shown directly by Proposition 4.5.

Note that the condition $(\mathbf{F_2})$ is weaker than $(\mathbf{F_3})$. With the condition $(\mathbf{F_2})$, the system (3.8) is permanent and has a positive equilibrium, but it is not clear whether the positive equilibrium is unique and globally asymptotically stable. The condition $(\mathbf{F_3})$ ensures the uniqueness and global stability of the positive equilibrium. We give an application of Theorem 4.4 to more specific resource functions.

Corollary 4.6 *Assume $m_i(x) = 1 + \varepsilon f_i(x)$ with $f_i \in C^\alpha(\overline{\Omega})$ satisfies $|f_i(x)| \le 1$ on $\overline{\Omega}$. If*

(i) *there is a $k \times k$ diagonal matrix Q with positive constant entries, such that $Q(I_k - B) + (I_k - B)^T Q$ is positive definite,*
(ii) *The vector $A^{-1}\mathbf{v}^T$ has positive entries, where $\mathbf{v} = (1, 1, \dots, 1) \in \mathbb{R}^n$.*

Then there exists a positive constant ε_0 such that for any $0 < \varepsilon < \varepsilon_0$, the problem (3.8) has a unique positive equilibrium solution which is globally asymptotically stable.

Proof From (i) and Proposition 4.5, we see that $(\mathbf{F_2})$ holds. Next we show that $(\mathbf{F_1})$ holds for small $\epsilon > 0$. By (i), we know

$$Q(I_k + B) + (I_k + B)^T Q \text{ is positive definite,} \tag{4.17}$$

which leads to $\det(I_k - B) \neq 0$ and $\det(A) = \det(I_k + B) \neq 0$. In fact, in the proof of Corollary 5.2 (see "Appendix") we will verify (4.17). Since $I_k - B$ and A are non-degenerate, there is a unique $\mathbf{C}_* := (\bar{c}_1, \dots, \bar{c}_k, \underline{c}_1, \dots, \underline{c}_k) \in \mathbb{R}^{2k}$ such that (4.8) holds. It remains to show that

$$\bar{c}_i > 0, \ \underline{c}_i > 0, \quad 1 \le i \le k.$$

Denote $\bar{\mathbf{c}} = (\bar{c}_1, \dots, \bar{c}_k)$, $\underline{\mathbf{c}} = (\underline{c}_1, \dots, \underline{c}_k)$ and $\mathbf{C}_* = (\bar{\mathbf{c}}, \underline{\mathbf{c}})$. From (4.8), we deduce

$$\begin{aligned}
\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \mathbf{C}_*^T &= \begin{bmatrix} \mathbf{m}^- + (I_k - B)^{-1}(\mathbf{m}^+ - \mathbf{m}^-) \\ \mathbf{m}^+ - (I_k - B)^{-1}(\mathbf{m}^+ - \mathbf{m}^-) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}(\mathbf{m}^+ + \mathbf{m}^-) - \frac{1}{2}(\mathbf{m}^+ - \mathbf{m}^-) + (I_k - B)^{-1}(\mathbf{m}^+ - \mathbf{m}^-) \\ \frac{1}{2}(\mathbf{m}^+ + \mathbf{m}^-) + \frac{1}{2}(\mathbf{m}^+ - \mathbf{m}^-) - (I_k - B)^{-1}(\mathbf{m}^+ - \mathbf{m}^-) \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} \mathbf{m}^+ + \mathbf{m}^- + A(I_k - B)^{-1}(\mathbf{m}^+ - \mathbf{m}^-) \\ \mathbf{m}^+ + \mathbf{m}^- - A(I_k - B)^{-1}(\mathbf{m}^+ - \mathbf{m}^-) \end{bmatrix},
\end{aligned}$$

and so

$$\begin{cases} \bar{\mathbf{c}} = \frac{1}{2}A^{-1}(\mathbf{m}^- + \mathbf{m}^+) + \frac{1}{2}(I_k - B)^{-1}(\mathbf{m}^+ - \mathbf{m}^-), \\ \underline{\mathbf{c}} = \frac{1}{2}A^{-1}(\mathbf{m}^- + \mathbf{m}^+) - \frac{1}{2}(I_k - B)^{-1}(\mathbf{m}^+ - \mathbf{m}^-), \\ \bar{\mathbf{c}} - \underline{\mathbf{c}} = (I_k - B)^{-1}(\mathbf{m}^+ - \mathbf{m}^-). \end{cases} \tag{4.18}$$

Since $m_i(x) = 1 + \varepsilon f_i(x)$ with $-1 \le f_i(x) \le 1$ on $\overline{\Omega}$, we have $\mathbf{m}^+ = (1 + \varepsilon)\mathbf{v}^T$, $\mathbf{m}^- = (1 - \varepsilon)\mathbf{v}^T$, where $\mathbf{v} = (1, 1, \dots, 1)$. Then by (4.18),

$$\begin{cases} \bar{\mathbf{c}} = A^{-1}\mathbf{v}^T + \varepsilon(I_k - B)^{-1}\mathbf{v}^T, \\ \underline{\mathbf{c}} = A^{-1}\mathbf{v}^T - \varepsilon(I_k - B)^{-1}\mathbf{v}^T, \\ \bar{\mathbf{c}} - \underline{\mathbf{c}} = 2\varepsilon(I_k - B)^{-1}\mathbf{v}^T. \end{cases} \tag{4.19}$$

Clearly, the condition (ii) implies that $\bar{c}_i > 0$ and $\underline{c}_i > 0$ for small ϵ. Thus, $(\mathbf{F_1})$ is satisfied. Furthermore, we see from Theorem 4.4 (i) that

$$\underline{c}_i \le \bar{c}_i, \quad \forall\, 1 \le i \le k, \tag{4.20}$$

as $(\mathbf{F_2})$ holds, and (3.8) has a positive equilibrium solution $(u_1^*(x), \dots, u_k^*(x))$ satisfying (4.11). Moreover, (4.20) implies that the vector $(I_k - B)^{-1}\mathbf{v}^T$ also has positive entries.

In the following, we apply Theorem 4.4 to show the global stability of $(u_1^*(x), \dots, u_k^*(x))$. Clearly, we just need to verify $(\mathbf{F_3})$. Using (4.19) and (4.20), we see that $\bar{c}_i - \underline{c}_i > 0$ and $\bar{c}_i$

for $1 \le i \le k$ are linear increasing with respect to $\varepsilon > 0$. Meanwhile, it can be verified that $\frac{\bar{c}_i - \underline{c}_i}{\bar{c}_i}$ for $1 \le i \le k$ are linear increasing with respect to $\varepsilon > 0$. Recalling that the matrix $Q(I_k - B) + (I_k - B)^T Q$ is positive definite, by (5.1) we get the positive definiteness of the matrix $Q(I_k - B - \mathbf{c}_1) + (I_k - B - \mathbf{c}_1)^T Q$ for $0 < \varepsilon < \varepsilon_0$ provided $\varepsilon_0 > 0$ small, where $\mathbf{c}_1$ is defined in (4.9). Therefore $(\mathbf{F_3})$ holds. □

The global stability of the positive coexistence stated in Corollary 4.6 is achieved under a weak competition condition on the competition matrix A and the resource function being a small perturbation from homogeneous one. We end this subsection by giving another two examples of competition with 2 and 4 species.

Example 4.7 Let $k = 2$ and

$$A = \begin{bmatrix} 1 & a_{12} \\ a_{21} & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix}, \quad \mathbf{m}^- = \begin{pmatrix} m_1^- \\ m_2^- \end{pmatrix}, \quad \mathbf{m}^+ = \begin{pmatrix} m_1^+ \\ m_2^+ \end{pmatrix}.$$

Then the conclusions in Theorem 4.4 hold if

$$a_{12} < \frac{m_1^-}{m_2^+} \le \frac{m_1^+}{m_2^-} < \frac{1}{a_{21}}, \tag{4.21}$$

$$a_{12}a_{21} < \left(1 - \frac{m_1^+ - m_1^- + a_{12}(m_2^+ - m_2^-)}{m_1^+ - a_{12}m_2^-}\right)\left(1 - \frac{a_{21}(m_1^+ - m_1^-) + m_2^+ - m_2^-}{m_2^+ - a_{21}m_1^-}\right). \tag{4.22}$$

We verify $(\mathbf{F_1})$ and $(\mathbf{F_3})$ under the conditions (4.21) and (4.22). A simple calculation gives

$$A^{-1} = \frac{1}{1 - a_{12}a_{12}} \begin{bmatrix} 1 & -a_{12} \\ -a_{21} & 1 \end{bmatrix}, \qquad (I_2 - B)^{-1} = \frac{1}{1 - a_{12}a_{21}} \begin{bmatrix} 1 & a_{12} \\ a_{21} & 1 \end{bmatrix}.$$

Then from (4.18), we see

$$\bar{\mathbf{c}} = \frac{1}{1 - a_{21}a_{12}}(m_1^+ - a_{12}m_2^-, m_2^+ - a_{21}m_1^-)^T,$$
$$\underline{\mathbf{c}} = \frac{1}{1 - a_{21}a_{12}}(m_1^- - a_{12}m_2^+, m_2^- - a_{21}m_1^+)^T,$$
$$\bar{\mathbf{c}} - \underline{\mathbf{c}} = \frac{1}{1 - a_{21}a_{12}}(m_1^+ - m_1^- + a_{12}(m_2^+ - m_2^-), a_{21}(m_1^+ - m_1^-) + m_2^+ - m_2^-)^T.$$

By (4.21), any element in the vectors $\bar{\mathbf{c}}, \underline{\mathbf{c}}$ is positive, and each element in $\bar{\mathbf{c}} - \underline{\mathbf{c}}$ is nonnegative, which implies that $(\mathbf{F_1})$ holds. Using the above formulas, we deduce

$$I_k - B - \mathbf{c}_1 = \begin{bmatrix} 1 - \dfrac{m_1^+ - m_1^- + a_{12}(m_2^+ - m_2^-)}{m_1^+ - a_{12}m_2^-} & -a_{12} \\ -a_{21} & 1 - \dfrac{a_{21}(m_1^+ - m_1^-) + m_2^+ - m_2^-}{m_2^+ - a_{21}m_1^-} \end{bmatrix},$$

where $\mathbf{c}_1$ is defined in (4.9). Clearly, $(\mathbf{F_3})$ holds if and only if (4.22) is satisfied.

If both m_1 and m_2 are positive constants, then the two conditions (4.21) and (4.22) become $a_{12} < \frac{m_1}{m_2} < \frac{1}{a_{21}}$ which coincides with the weak competition condition in the two species diffusive competitive problem in an homogeneous environment [2,8]. On the other hand, for the nonhomogeneous environment case, the result here is not as optimal as the ones in [12], but our proof is completely different: we use Lyapunov functional method, and we do not use the monotone dynamical system method.

Example 4.8 Suppose that $m_i(x)$ for $1 \le i \le k$ satisfy the condition in Corollary 4.6. If $k = 4$ and

$$A = \begin{bmatrix} 1 & 0.2 & 0.1 & 0.1 \\ 0.2 & 1 & 0.2 & 0.15 \\ 0.1 & 0.2 & 1 & 0.1 \\ 0.1 & 0.15 & 0.1 & 1 \end{bmatrix},$$

then for $0 < \varepsilon \le 0.1$, the results in Corollary 4.6 hold true.

4.2.2 Semi-trivial equilibrium solution

We investigate the global stability of the semi-trivial equilibrium solution $\mathbf{v}^* := (v_1^*, \ldots, v_k^*)$ with

$$v_i^* > 0 \text{ for } 1 \le i \le i_1, \text{ and } v_i^* \equiv 0 \text{ for } i_1 + 1 \le i \le k,$$

where $i_1 \in [1, k)$ is an integer.

For the simplicity of notations, similarly to (4.7), we denote

$$\widetilde{A} = (a_{ij})_{i_1 \times i_1},\ \widetilde{B} = \widetilde{A} - I_{i_1},\ \widetilde{\mathbf{m}}^- = (m_1^-, \ldots, m_{i_1}^-)^T,\ \widetilde{\mathbf{m}}^+ = (m_1^+, \ldots, m_{i_1}^+)^T, \tag{4.23}$$

where I_{i_1} is the $i_1 \times i_1$ identity matrix. The diagonal entries of $\widetilde{B}$ are 0.

To study the global stability of the semi-trivial equilibrium solution $\mathbf{v}^*$ of the problem (3.8), we make the following assumptions:

($\mathbf{G_1}$) The determinant $\det[\widetilde{A}(2I_{i_1} - \widetilde{A})] \neq 0$, and the algebraic equations

$$\begin{bmatrix} \widetilde{A} & 0 \\ 0 & \widetilde{A} \end{bmatrix} (\mathbf{c}_{i_1}^*)^T = \begin{bmatrix} \widetilde{\mathbf{m}}^- + (I_{i_1} - \widetilde{B})^{-1}(\widetilde{\mathbf{m}}^+ - \widetilde{\mathbf{m}}^-) \\ \widetilde{\mathbf{m}}^+ - (I_{i_1} - \widetilde{B})^{-1}(\widetilde{\mathbf{m}}^+ - \widetilde{\mathbf{m}}^-) \end{bmatrix} \tag{4.24}$$

has a unique positive solution $\mathbf{c}_{i_1}^* := (\bar{c}_1, \ldots, \bar{c}_{i_1}, \underline{c}_1, \ldots, \underline{c}_{i_1})$, and

$$m_i^+ - \sum_{j=1}^{i_1} a_{ij}\underline{c}_i \le 0,\ \ m_i^- - \sum_{j=1}^{i_1} a_{ij}\bar{c}_i \le 0,\ \ \forall\, i_1 + 1 \le i \le k, \tag{4.25}$$

where I_{i_1}, $\widetilde{A}$, $\widetilde{B}$, $\widetilde{\mathbf{m}}^-$ and $\widetilde{\mathbf{m}}^+$ are given by (4.23).

($\mathbf{G_2}$) There exists an $i_1 \times i_1$ diagonal matrix Q_6 with positive constant entries such that $Q_6(I_{i_1} - \widetilde{B} - \mathbf{c}_2) + (I_{i_1} - \widetilde{B} - \mathbf{c}_2)^T Q_6$ is positive definite, where

$$\mathbf{c}_2 = \operatorname{diag}\left(\frac{\bar{c}_1 - \underline{c}_1}{\bar{c}_1}, \frac{\bar{c}_2 - \underline{c}_2}{\bar{c}_2}, \ldots, \frac{\bar{c}_{i_1} - \underline{c}_{i_1}}{\bar{c}_{i_1}} \right)$$

and $\bar{c}_i$, $\underline{c}_i$ are given by ($\mathbf{G_1}$).

Theorem 4.9 *Suppose that* (3.9)*,* (4.1) *and the assumptions* ($\mathbf{G_1}$)*,* ($\mathbf{F_2}$) *hold.*

(i) *The problem* (3.8) *has a semi-trivial solution* $\mathbf{v}^* = (v_1^*, \ldots, v_k^*)$ *which satisfies*

$$\underline{c}_i \le v_i^*(x) \le \bar{c}_i,\ \ \forall\, x \in \overline{\Omega},\ 1 \le i \le i_1, \tag{4.26}$$

and $v_i^* \equiv 0$ *for* $i_1+1 \le i \le k$, *and the solution of* (3.8) *satisfies*

$$\begin{cases} \underline{c}_i \le \liminf_{t\to\infty} u_i(x,t) \le \limsup_{t\to\infty} u_i(x,t) \le \bar{c}_i, & \forall\, x \in \overline{\Omega},\ 1 \le i \le i_1, \\ \lim_{t\to\infty} u_i(x,t) = 0 \text{ uniformly on } \overline{\Omega}, & \forall\, i_1+1 \le i \le k. \end{cases} \tag{4.27}$$

(ii) *Moreover, if* $(\mathbf{G_2})$ *is satisfied, then*

$$\begin{cases} \lim_{t\to\infty} u_i(x,t) = u_i^*(x) & \text{in } C^1(\overline{\Omega}),\ 1 \le i \le i_1, \\ \lim_{t\to\infty} u_i(x,t) = 0 & \text{in } C^1(\overline{\Omega}),\ i_1+1 \le i \le k. \end{cases} \tag{4.28}$$

Proof (i) From $(\mathbf{G_1})$, we know that $\bar{c}_i > 0$ and $\underline{c}_i > 0$ for $1 \le i \le i_1$. Next we show that $\bar{c}_i \ge \underline{c}_i$ and

$$\lim_{t\to\infty} \bar{u}_i = \bar{c}_i, \quad \lim_{t\to\infty} \underline{u}_1 = \underline{c}_i, \tag{4.29}$$

where $(\bar{u}_1, \ldots, \bar{u}_k, \underline{u}_1, \ldots, \underline{u}_k)^T$ is the solution of (4.13), and the positive constants $\bar{c}_i, \underline{c}_i$ for $1 \le i \le i_1$ are given by $(\mathbf{G_1})$, and $\bar{c}_i = \underline{c}_i = 0$ for $i_1+1 \le i \le k$. As $(\mathbf{G_1})$ holds, by the similar arguments as Theorem 4.4 (i) we know that $(\bar{c}_1, \ldots, \bar{c}_k, \underline{c}_1, \ldots, \underline{c}_k)$ is a semi-trivial equilibrium of (4.13). Then we could apply Corollary 3.5 to prove (4.29) since (4.25) and $(\mathbf{F_2})$ hold. Moreover, (4.29), combined with (4.14), gives (4.27).

Clearly, $(\bar{c}_1, \ldots, \bar{c}_{i_1})$ and $(\underline{c}_1, \ldots, \underline{c}_{i_1})$ are a pair of coupled ordered upper and lower solutions of (3.10) with $A_1 = \{1, \ldots, i_1\}$ and $A_2 = \emptyset$. By [29, Theorem 10.2, Page 440] we know that the problem (3.10) has a positive solution $(v_1^*(x), \ldots, v_{i_1}^*(x))$ and (4.26) holds. Moreover, $\mathbf{v}^* = (v_1^*, \ldots, v_k^*)$ with $v_i^* \equiv 0$ for $i_1+1 \le i \le k$ is a semi-trivial equilibrium solution of (3.8).

(ii) Note that $\int_0^\infty \bar{u}_i^2(t)\mathrm{d}t < \infty$ for $i_1+1 \le i \le k$ by Corollary 3.5. We see that

$$\int_0^\infty ||u_i(\cdot,t)||^2_{L^2(\Omega)}\mathrm{d}t \le |\Omega|^2 \int_0^\infty \bar{u}_i^2(t)\mathrm{d}t < \infty, \quad i_1+1 \le i \le k.$$

Moreover, we can show that the $i_1 \times i_1$ matrix $Q(x) = (q_{ij}(x)+q_{ji}(x))$ defined in Proposition 3.8 is positive definite for every $x \in \overline{\Omega}$ by using $(\mathbf{G_2})$ and the similar calculation as Theorem 4.4, where $q_{ij}(x) = \xi_i a_{ij} \frac{v_i^*(x)}{d_i}$ for $1 \le i, j \le i_1$. Thus, (4.28) holds from Proposition 3.8. □

5 Appendix

Before giving the proof of Proposition 4.5, we recall some preliminary results about matrices. For any $k \times k$ matrices M, N, P and R, the following results hold (see, e.g., [15, Page 104 and 149–150]):

$$\det \begin{bmatrix} M & N \\ P & R \end{bmatrix} = \det(M)\det(R - PM^{-1}N) = \det(R)\det(M - NR^{-1}P),$$

$$\begin{bmatrix} M & N \\ N^T & R \end{bmatrix} \text{ is positive definite } \Longleftrightarrow \text{ both } M \text{ and } R - N^TM^{-1}N \text{ are positive definite,}$$

$$\text{If } M \text{ is positive definite, then } xMx^T \ge \varepsilon x x^T \ \textit{for all } x \in \mathbb{R}^k \textit{ and some } \varepsilon > 0. \tag{5.1}$$

Moreover, we have the following elementary conclusion.

Lemma 5.1 *Let M and N be two $k \times k$ symmetric matrices. Then $M + N$ and $M - N$ are positive definite if and only if $H = \begin{bmatrix} M & N \\ N & M \end{bmatrix}$ is positive definite.*

Proof (i) Suppose that $M + N$ and $M - N$ are positive definite. We will show that H is positive definite. Set $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^k$, $X = (\mathbf{x}_1, \mathbf{x}_2)$ and $X \neq 0$, then we have

$$\begin{aligned}
XHX^T &= \mathbf{x}_1 M \mathbf{x}_1^T + \mathbf{x}_2 M \mathbf{x}_2^T + 2\mathbf{x}_1 N \mathbf{x}_2^T \\
&= \mathbf{x}_1 (M+N)\mathbf{x}_1^T + \mathbf{x}_2 (M+N)\mathbf{x}_2^T + 2\mathbf{x}_1 N \mathbf{x}_2^T - \mathbf{x}_1 N \mathbf{x}_1^T - \mathbf{x}_2 N \mathbf{x}_2^T \\
&= \mathbf{x}_1 (M+N)\mathbf{x}_1^T + \mathbf{x}_2 (M+N)\mathbf{x}_2^T + \mathbf{x}_1 N (\mathbf{x}_2 - \mathbf{x}_1)^T + (\mathbf{x}_1 - \mathbf{x}_2) N \mathbf{x}_2^T \\
&= \mathbf{x}_1 (M+N)\mathbf{x}_1^T + \mathbf{x}_2 (M+N)\mathbf{x}_2^T - (\mathbf{x}_2 - \mathbf{x}_1) N (\mathbf{x}_2 - \mathbf{x}_1)^T \\
&= \mathbf{x}_1 \frac{M+N}{2}\mathbf{x}_1^T + \mathbf{x}_2 \frac{M+N}{2}\mathbf{x}_2^T - 2\mathbf{x}_1 \frac{M+N}{2}\mathbf{x}_2^T + \mathbf{x}_1 \frac{M+N}{2}\mathbf{x}_1^T \\
&\quad + \mathbf{x}_2 \frac{M+N}{2}\mathbf{x}_2^T + 2\mathbf{x}_1 \frac{M+N}{2}\mathbf{x}_2^T - (\mathbf{x}_2 - \mathbf{x}_1) N (\mathbf{x}_2 - \mathbf{x}_1)^T \\
&= (\mathbf{x}_2 - \mathbf{x}_1) \frac{M+N}{2} (\mathbf{x}_2 - \mathbf{x}_1)^T + (\mathbf{x}_2 + \mathbf{x}_1) \frac{M+N}{2} (\mathbf{x}_2 + \mathbf{x}_1)^T \\
&\quad - (\mathbf{x}_2 - \mathbf{x}_1) N (\mathbf{x}_2 - \mathbf{x}_1)^T \\
&= (\mathbf{x}_2 - \mathbf{x}_1) \frac{M-N}{2} (\mathbf{x}_2 - \mathbf{x}_1)^T + (\mathbf{x}_2 + \mathbf{x}_1) \frac{M+N}{2} (\mathbf{x}_2 + \mathbf{x}_2)^T > 0.
\end{aligned}$$

(ii) Assume that the matrix H is positive definite. Suppose on the contrary that $M + N$ or $M - N$ is not positive definite. Without loss of generality, we assume that $M - N$ is not positive definite. Then there exists $0 \neq \mathbf{Q} \in \mathbb{R}^k$ such that $\tilde{\mathbf{x}}(M - N)\tilde{\mathbf{x}}^T \leq 0$. Let $\tilde{\mathbf{X}} = (-\tilde{\mathbf{x}}, \tilde{\mathbf{x}})$. From (5.2) we have

$$\begin{aligned}
\tilde{\mathbf{X}} H \tilde{\mathbf{X}}^T &= (\tilde{\mathbf{x}} + \tilde{\mathbf{x}}) \frac{M-N}{2} (\tilde{\mathbf{x}} + \tilde{\mathbf{x}})^T + (\tilde{\mathbf{x}} - \tilde{\mathbf{x}}) \frac{M+N}{2} (\tilde{\mathbf{x}} - \tilde{\mathbf{x}})^T \\
&= 4\tilde{\mathbf{x}} \frac{M-N}{2} \tilde{\mathbf{x}}^T \leq 0,
\end{aligned}$$

which contradicts to the fact that H is positive definite. Thus $M + N$ and $M - N$ are positive definite. The proof is finished. □

Corollary 5.2 *If there is a $k \times k$ diagonal matrix Q_1 with positive constant entries such that $Q_1(I_k - B) + (I_k - B)^T Q_1$ is positive definite, then $Q_2 \begin{bmatrix} I_k & B \\ B & I_k \end{bmatrix} + \begin{bmatrix} I_k & B \\ B & I_k \end{bmatrix}^T Q_2$ is positive definite, where $Q_2 = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_1 \end{bmatrix}$.*

Proof Denote $\mathbf{x} := (x_1, \ldots, x_n)$, $\hat{\mathbf{x}} := (|x_1|, \ldots, |x_n|)$ and $\mathbf{x} \neq \mathbf{0}$. Then

$$\begin{aligned}
\mathbf{x} Q_1 (I + B) \mathbf{x}^T + \mathbf{x}(I + B)^T Q_1 \mathbf{x}^T &= 2\mathbf{x} Q_1 \mathbf{x}^T + \mathbf{x} Q_1 B \mathbf{x}^T + \mathbf{x} B^T Q_1 \mathbf{x}^T \\
&\geq 2\mathbf{x} Q_1 \mathbf{x}^T - \hat{\mathbf{x}} Q_1 B \hat{\mathbf{x}}^T - \hat{\mathbf{x}} B^T Q_1 \hat{\mathbf{x}}^T \\
&= \hat{\mathbf{x}} Q_1 (I - B) \hat{\mathbf{x}}^T + \hat{\mathbf{x}}(I - B)^T Q_1 \hat{\mathbf{x}}^T > 0.
\end{aligned}$$

This, combined with Lemma 5.1, gives the desired conclusion. □

Proof of Proposition 4.5 When (i) holds, let $Q = \begin{bmatrix} Q_3 & 0 \\ 0 & Q_4 \end{bmatrix}$. Then Q is a $2k \times 2k$ diagonal matrix with positive constant entries. The direct calculation yields

$$E := Q \begin{bmatrix} I_k & B \\ B & I_k \end{bmatrix} + \begin{bmatrix} I_k & B \\ B & I_k \end{bmatrix}^T Q = \begin{bmatrix} 2Q_3 & Q_3 B + B^T Q_4 \\ Q_4 B + B^T Q_3 & 2Q_4 \end{bmatrix}.$$

By (5.1), we obtain that the matrix E is positive definite.

When (ii) holds, the desired conclusion follows immediately from Corollary 5.2. □

Acknowledgements The authors would like to thank the anonymous reviewer for very helpful comments and suggestions that significantly improve the initial version of this paper.

References

1. Barbălat, I.: Systèmes d'équations différentielles d'oscillations non linéaires. Rev. Math. Pures Appl. **4**, 267–270 (1959)
2. Brown, P.N.: Decay to uniform states in competitive systems. SIAM J. Math. Anal. **14**(4), 659–673 (1983)
3. Cantrell, R.S., Cosner, C.: The effects of spatial heterogeneity in population dynamics. J. Math. Biol. **29**(4), 315–338 (1991)
4. Cantrell, R.S., Cosner, C.: Spatial Ecology Via Reaction–Diffusion Equations. Wiley Series in Mathematical and Computational Biology. Wiley, Chichester (2003)
5. Cantrell, R.S., Cosner, C., Hutson, V.: Permanence in ecological systems with spatial heterogeneity. Proc. R. Soc. Edinb. Sect. A **123**(3), 533–559 (1993)
6. Champagnat, N., Jabin, P.-E., Raoul, G.: Convergence to equilibrium in competitive Lotka–Volterra and chemostat systems. C. R. Math. Acad. Sci. Paris **348**(23–24), 1267–1272 (2010)
7. Dockery, J., Hutson, V., Mischaikow, K., Pernarowski, M.: The evolution of slow dispersal rates: a reaction diffusion model. J. Math. Biol. **37**(1), 61–83 (1998)
8. Goh, B.S.: Global stability in many-species systems. Am. Nat. **111**(977), 135–143 (1977)
9. Hale, J.K., Waltman, P.: Persistence in infinite-dimensional systems. SIAM J. Math. Anal. **20**(2), 388–395 (1989)
10. He, X.Q., Ni, W.-M.: The effects of diffusion and spatial variation in Lotka–Volterra competition–diffusion system I: heterogeneity vs. homogeneity. J. Differ. Equ. **254**(2), 528–546 (2013)
11. He, X.Q., Ni, W.-M.: The effects of diffusion and spatial variation in Lotka–Volterra competition–diffusion system II: the general case. J. Differ. Equ. **254**(10), 4088–4108 (2013)
12. He, X.Q., Ni, W.-M.: Global dynamics of the Lotka–Volterra competition–diffusion system: diffusion and spatial heterogeneity I. Commun. Pure Appl. Math. **69**(5), 981–1014 (2016)
13. He, X.Q., Ni, W.-M.: Global dynamics of the Lotka–Volterra competition–diffusion system with equal amount of total resources, II. Calc. Var. Partial Differ. Equ. **55**(2), 25 (2016)
14. He, X.Q., Ni, W.-M.: Global dynamics of the Lotka–Volterra competition–diffusion system with equal amount of total resources, III. Calc. Var. Partial Differ. Equ. **56**(5), 132 (2017)
15. Heesterman, A.R.G.: Matrices and their Roots. A Textbook of Matrix Algebra. World Scientific Publishing Co., Inc, Teaneck (1990)
16. Hess, P.: Periodic-Parabolic Boundary Value Problems and Positivity, Volume 247 of Pitman Research Notes in Mathematics Series. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley and Sons, Inc., New York (1991)
17. Hsu, S.-B.: A survey of constructing Lyapunov functions for mathematical models in population biology. Taiwan. J. Math. **9**(2), 151–173 (2005)
18. Hsu, S.-B., Waltman, P.: On a system of reaction–diffusion equations arising from competition in an unstirred chemostat. SIAM J. Appl. Math. **53**(4), 1026–1044 (1993)
19. Hutson, V., Lou, Y., Mischaikow, K.: Spatial heterogeneity of resources versus Lotka–Volterra dynamics. J. Differ. Equ. **185**(1), 97–136 (2002)
20. Hutson, V., Schmitt, K.: Permanence and the dynamics of biological systems. Math. Biosci. **111**(1), 1–71 (1992)
21. Kuniya, T., Wang, J.L.: Lyapunov functions and global stability for a spatially diffusive SIR epidemic model. Appl. Anal. **96**(11), 1935–1960 (2017)

22. Lam, K.-Y., Ni, W.-M.: Uniqueness and complete dynamics in heterogeneous competition–diffusion systems. SIAM J. Appl. Math. **72**(6), 1695–1712 (2012)
23. Levin, S.A.: Population dynamic models in heterogeneous environments. Annu. Rev. Ecol. Evol. Syst. **7**(1), 287–310 (1976)
24. Li, M.Y., Shuai, Z.S.: Global-stability problem for coupled systems of differential equations on networks. J. Differ. Equ. **248**(1), 1–20 (2010)
25. Lou, Y.: On the effects of migration and spatial heterogeneity on single and multiple species. J. Differ. Equ. **223**(2), 400–426 (2006)
26. Lou, Y., Zhao, X.Q., Zhou, P.: Global dynamics of a Lotka–Volterra competition–diffusion–advection system in heterogeneous environments. J. Math. Pures Appl. **9**(121), 47–82 (2019)
27. Morales-Rodrigo, C., Suárez, A.: Uniqueness of solution for elliptic problems with non-linear boundary conditions. Commun. Appl. Nonlinear Anal. **13**(3), 69–78 (2006)
28. Ni, W.J., Shi, J.P., Wang, M.X.: Global stability and pattern formation in a nonlocal diffusive Lotka–Volterra competition model. J. Differ. Equ. **264**(11), 6891–6932 (2018)
29. Pao, C.V.: Nonlinear Parabolic and Elliptic Equations. Plenum Press, New York (1992)
30. Pao, C.V.: Quasilinear parabolic and elliptic equations with nonlinear boundary conditions. Nonlinear Anal. **66**(3), 639–662 (2007)
31. Smith, H.L.: Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems. Mathematical Surveys and Monographs, vol. 41. American Mathematical Society, Providence (1995)
32. Tilman, D.: Competition and biodiversity in spatially structured habitats. Ecology **75**(1), 2–16 (1994)
33. Wang, M.X.: A diffusive logistic equation with a free boundary and sign-changing coefficient in time-periodic environment. J. Funct. Anal. **270**(2), 483–508 (2016)
34. Wang, M.X.: Note on the Lyapunov functional method. Appl. Math. Lett. **75**, 102–107 (2018)
35. Wang, M.X., Zhang, Y.: Dynamics for a diffusive prey–predator model with different free boundaries. J. Differ. Equ. **264**(5), 3527–3558 (2018)
36. Zhou, P., Xiao, D.-M.: Global dynamics of a classical Lotka–Volterra competition–diffusion–advection system. J. Funct. Anal. **275**(2), 356–380 (2018)