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Formulation of the normal form of Turing-Hopf bifurcation in partial functional differential equations ☆

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Abstract

Turing-Hopf bifurcation is considered as an important mechanism for generating complex spatio-temporal patterns in dynamical systems. In this work, the normal form up to the third order for the Hopf-steady state bifurcation, which includes the Turing-Hopf bifurcation, of a general system of partial functional differential equations (PFDEs) in a bounded spatial region of any dimension, is derived based on the center manifold and the normal form theories of PFDEs. The explicit formula for the coefficients in the normal form of Hopf-steady state bifurcation are presented concisely in a matrix form, which makes it more convenient in not only symbolic derivation but also numerical implementation. This provides an approach of showing the existence and stability of stationary and time-periodic solutions with spatial heterogeneity. The derived formulas are also applicable to partial differential equations and functional differential equations.
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1. Introduction

In a dynamical model, the asymptotic behavior of the system often changes when some parameter moves across certain threshold values and such phenomenon is called a bifurcation. The

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method of the normal form is a standard and effective tool to analyze and simplify bifurcation problems, see [12,21,60]. The main idea is to transform the differential equations to a topologically conjugate normal form near the singularity. For ordinary differential equations (ODEs), the methods of computing the normal form have been developed in, for example, [5,12,13,20, 21], and for functional differential equations (FDEs), similar methods have also been developed in, for example, [18,19,25]. In FDEs, due to the effect of time-delays, Hopf bifurcation occurs more frequently which destabilizes a stable equilibrium and produces temporal oscillatory patterns [25]. Often a center manifold reduction reduces a higher dimensional problem to a lower dimensional one, and the normal form can be computed on the center manifold [9,26,40, 57]. The method proposed in [18,19] has been applied to the bifurcation problems in ODEs or FDEs, such as codimension-one Hopf bifurcation, and codimension-two Hopf-zero bifurcation and Bogdanov-Takens bifurcation, etc., see [7,28,30,43,58,59,65,70].

Methods of center manifolds and normal forms have also been extended to partial differential equations (PDEs). For example, the existence of center manifolds or other invariant manifolds for semilinear parabolic equations have been proved in, for example, [2,3,14,27]. For parabolic equations with time-delay or functional partial differential equations (FPDEs), the existence and smoothness of center manifolds have also been established in [15–17,34,56,63,64]. In particular, the calculation of normal form on the center manifold of FPDEs was provided in [15,16]. These theories can be applied to reaction-diffusion systems (with or without time-delays) which appear in many applications from physics, chemistry and biology. For example, the existence of Hopf bifurcations and associated stability switches have been considered in many recent work [6,10, 22–24,44,53,54,66,68,69,71]. More recently with the integrated semigroup theory, the center manifold and normal form theory for semilinear equations with non-dense domain have also been developed in [35–38].

An important application of the normal form theory for PDEs and FPDEs is the formation and bifurcation of spatiotemporal patterns in reaction-diffusion systems (with delays) from various physical, chemical and biological models. In the pioneer work of Turing [55], it was shown that diffusion could destabilize an otherwise stable spatially homogeneous equilibrium of a reaction-diffusion system, which leads to the spontaneous formation of spatially inhomogeneous pattern. This phenomenon is often called the Turing instability or diffusion-driven instability, and associated Turing bifurcation could lead to spatially inhomogeneous steady states [32,50,62]. Such Turing type pattern formation mechanisms have been verified in several recent chemical or biological studies [31,33,42,48].

In many reaction-diffusion models, temporal oscillation caused by Hopf bifurcation and spatial patterns from Turing mechanism can occur simultaneously producing Turing-Hopf patterns which oscillate in both space and time [4,39,41,45,46]. Mathematically the complex spatiotemporal Turing-Hopf pattern involves the interaction of the dynamical properties of two Fourier modes, and it can be analyzed through unfolding a codimension-two Turing-Hopf bifurcation, see [49,51,52,61] and references therein. It is also a Hopf-steady state bifurcation with the zero eigenvalue corresponding to a spatially inhomogeneous eigenfunction.

The aim of the present paper is to provide the computation of the normal form up to the third order at a known steady state solution for a reaction-diffusion system with time-delay. This normal form can be used to unfold the complex spatiotemporal dynamics near a Turing-Hopf bifurcation point. We follow the framework of Faria [15,17] to reduce the general PFDEs with perturbation parameters to three-dimensional systems of ODEs up to third order, restricted on the local center manifold near a Hopf-steady state type of singularity, and the unfolding parameters can be expressed by those original perturbation parameters. Usually the third order normal form is sufficient for analyzing the bifurcation phenomena in most applications. The reduced three-dimensional ODE system can be further transformed to a two-dimensional amplitude system and the bifurcation analysis can be carried out following [21] to provide precise dynamical behavior of the system using the two-dimensional unfolding parameters. Furthermore we give an explicit formula of the coefficients in the truncated normal form up to third order for the Hopf-steady state bifurcation of delayed reaction-diffusion equations with Neumann boundary condition, which includes the important application to the Turing-Hopf bifurcation.

Our approach in this paper has several new features compared to previous work. First, our basic setup of PFDE systems follows the assumptions $(H1)$-$(H4)$ in [15] with slight changes to fit our situation, but we remove the more restrictive assumption $(H5)$ which was used in [15]. Hence our computation of the normal form can be applied to more general situations. Second, the normal form formulas here are directly expressed by the Fréchet derivatives up to the third orders and characteristic functions of the original PFDEs, not the reduced ODEs. Hence one can apply our results directly to the original PFDEs without the reduction steps. Also our formulas of the normal form are presented in a concise matrix notation which also eases the applications. Third, we neglect the higher order (≥ 2) dependence of the perturbation parameters on the system, as in practical applications, the influence of the small perturbation parameter on the dynamics of the system is mainly linear. This again simplifies the normal form but still fulfills the need in most applications. Finally we remark that the normal form formulas developed in this paper for PFDEs are also applicable to the general reaction-diffusion equations without delay and the delay differential equations without diffusion with obvious adaptions, and the unfolding parameters are not necessarily the time-delay or diffusion coefficients.

Because the coefficients of the computed normal form can be explicitly expressed using the information from the original system, our algorithm enables us to draw conclusions on the impact of original system parameters on the dynamical behavior near the Turing-Hopf singularity. To illustrate our normal form computation procedure, we apply our methods to the Turing-Hopf bifurcation in a diffusive Schnakenberg type chemical reaction system with gene expression time delay, proposed in [47]. Turing and Hopf bifurcations for this system have been considered in [11,67], and Turing-Hopf bifurcation for the system in a different set of parameters was recently considered in [29].

The rest of the paper is organized as follows. In Section 2, the framework of the system of PFDEs at a Hopf-steady state singularity and the phase space decomposition are given, and the reduction of the original equations to a three-dimensional ODE system is introduced. The formulas of the normal form up to third order are presented in Section 3 while the proof is postponed to Section 6. In Section 4, the precise formulas of the normal form with the Neumann boundary condition and the spatial domain $\Omega = (0, l\pi)$ are given. The application of abstract formulas to the example of diffusive Schnakenberg system with gene expression time delay is shown in Section 5, and some concluding remarks are given in Section 7.

2. Reduction based on phase space decomposition

In this section, we discuss the reduction and normal form for PFDEs subject to homogeneous Neumann or Dirichlet boundary condition at a Hopf-steady state singularity with original perturbation parameters following the theoretical framework in [15,17]. We will show that the system of PFDEs can be reduced to a system of three ordinary differential equations defined on its center manifold.

Assume that Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary, and X is a Hilbert space of complex-valued functions defined on $\bar{\Omega}$ with inner product $\langle \cdot, \cdot \rangle$. Denote by $\mathbb{N}_B$ the set of nonnegative integers or positive integers, depending on the boundary condition:

$$\mathbb{N}_B = \begin{cases} \mathbb{N} \cup \{0\}, & \text{for homogeneous Neumann boundary condition,} \\ \mathbb{N}, & \text{for homogeneous Dirichlet boundary condition.} \end{cases}$$

Let $\{\mu_k : k \in \mathbb{N}_B\}$ be the set of eigenvalues of $-\Delta$ on Ω subject to homogeneous Neumann or Dirichlet boundary condition, that is

$$\Delta \beta_k + \mu_k \beta_k = 0, \;\; x \in \Omega, \;\; \frac{\partial u}{\partial n} = 0 \;\text{ or }\; u = 0, \;\; x \in \partial\Omega.$$

Then we have

$$\begin{aligned} &0 = \mu_0 < \mu_1 \leq \cdots \leq \mu_k \leq \cdots \to \infty, \text{ for Neumann boundary condition, or} \\ &0 < \mu_1 < \mu_2 \leq \cdots \leq \mu_k \leq \cdots \to \infty, \text{ for Dirichlet boundary condition,} \end{aligned}$$

and the corresponding eigenfunctions $\{\beta_k : k \in \mathbb{N}_B\}$ form an orthonormal basis of X. Fixing $m \in \mathbb{N}$ and $r > 0$, define $\mathcal{C} = C([-r, 0]; X^m)$ $(r > 0)$ to be the Banach space of continuous maps from $[-r, 0]$ to X^m with the sup norm. We consider an abstract PFDE with parameters in the phase space $\mathcal{C}$ defined as

$$\dot{u}(t) = D(\alpha)\Delta u(t) + L(\alpha)u_t + G(u_t, \alpha), \tag{2.1}$$

where $u_t \in \mathcal{C}$ is defined by $u_t(\theta) = u(t + \theta)$ for $-r \leq \theta \leq 0$, $D(\alpha) = \mathrm{diag}(d_1(\alpha), d_2(\alpha), \ldots, d_m(\alpha))$ with $d_i(0) > 0$ for $1 \leq i \leq m$; the domain of Δ is defined by $dom(\Delta) = Y^m \subseteq X^m$ where Y is defined as

$$\begin{aligned} &Y = \left\{ u \in W^{2,2}(\Omega) : \frac{\partial u}{\partial n} = 0, \; x \in \partial\Omega \right\} \text{ for Neumann boundary condition, or} \\ &Y = \left\{ u \in W^{2,2}(\Omega) : u = 0, \; x \in \partial\Omega \right\} \text{ for Dirichlet boundary condition;} \end{aligned}$$

the parameter vector $\alpha = (\alpha_1, \alpha_2)$ is in a neighborhood $V \subset \mathbb{R}^2$ of $(0, 0)$, $L : \; V \to L(\mathcal{C}, \mathbb{R}^m)$ (the set of linear mappings) is C^1 smooth, $G : \mathcal{C} \times V \to \mathbb{R}^m$ is C^k smooth for $k \geq 3$, $G(0, 0) = 0$, and the Jacobian matrix $D_\varphi G(0, 0) = 0$ with $\varphi \in \mathcal{C}$.

Let $L_0 = L(0)$ and $D_0 = D(0)$. Then the linearized equation about the zero equilibrium of (2.1) can be written as

$$\dot{u}(t) = D_0 \Delta u(t) + L_0 u_t. \tag{2.2}$$

We impose the following hypotheses (similar to [15]):

(H1) $D_0\Delta$ generates a C_0 semigroup $\{T(t)\}_{t \geq 0}$ on X^m with $|T(t)| \leq Me^{\omega t}$ for all $t \geq 0$, where $M \geq 1$, $\omega \in \mathbb{R}$, and $T(t)$ is a compact operator for each $t > 0$;

(H2) L_0 can be extended to a bounded linear operator from BC to X^m, where $BC = \{\psi : [-r, 0] \to X^m \mid \psi$ is continuous on $[-r, 0)$, $\lim_{\theta\to 0^-} \psi(\theta)$ exists$\}$ with the sup norm.

(H3) the subspaces $\mathcal{B}_k = \{\langle v(\cdot), \beta_k\rangle \beta_k : v \in \mathcal{C}\} \subset \mathcal{C}$ $(k \in \mathbb{N}_B)$ satisfy $L_0(\mathcal{B}_k) \subseteq span\{e_i \beta_k : 1 \le i \le m\}$, where $\{e_i : 1 \le i \le m\}$ is the canonical basis of $\mathbb{R}^m$, and

$$\langle v, \beta_k\rangle = (\langle v_1, \beta_k\rangle,\ \langle v_2, \beta_k\rangle,\ \ldots,\ \langle v_m, \beta_k\rangle)^{\mathrm{T}},\ k \in \mathbb{N}_B,\ \text{for } v = (v_1, v_2, \ldots, v_m)^{\mathrm{T}} \in \mathcal{C}.$$

Let A be the infinitesimal generator associated with the semiflow of the linearized equation (2.2). It is known that A is given by

$$(A\phi)(\theta) = \dot{\phi}(\theta),\ dom(A) = \{\phi \in \mathcal{C} : \dot{\phi} \in \mathcal{C}, \phi(0) \in dom(\Delta), \dot{\phi}(0) = D_0\Delta\phi(0) + L_0\phi\},$$

and the spectrum $\sigma(A)$ of A coincides with its point spectrum $\sigma_P(A)$. Moreover $\lambda \in \mathbb{C}$ is in $\sigma_P(A)$ if and only if there exists $y \in dom(\Delta) \setminus \{0\}$ such that λ satisfies

$$\triangle(\lambda)y = 0, \quad \text{with}\ \ \triangle(\lambda) = \lambda I - D_0\Delta - L_0(e^{\lambda\cdot} I). \tag{2.3}$$

By using the decomposition of X by $\{\beta_k\}_{k\in\mathbb{N}_B}$ and $\mathcal{B}_k$, the equation $\triangle(\lambda)y = 0$, for some $y \in dom(\Delta) \setminus \{0\}$, is equivalent to a sequence of characteristic equations

$$\det\triangle_k(\lambda) = 0, \quad \text{with}\ \ \triangle_k(\lambda) = \lambda I + \mu_k D_0 - L_0(e^{\lambda\cdot} I),\ \ k \in \mathbb{N}_B. \tag{2.4}$$

Here $L_0 : C \to \mathbb{C}^m$ where $C \triangleq C([-r, 0]; \mathbb{C}^m)$. Then for any $k \in \mathbb{N}_B$, on $\mathcal{B}_k$, the linear equation (2.2) is equivalent to a FDE on $\mathbb{C}^m$:

$$\dot{z}(t) = -\mu_k D_0 z(t) + L_0 z_t, \tag{2.5}$$

with characteristic equation (2.4), where $z_t(\cdot) = \langle u_t(\cdot), \beta_k\rangle \in C$. For any $k \in \mathbb{N}_B$, we also denote by $\eta_k \in BV([-r, 0], \mathbb{C}^m)$ to be the $m \times m$ matrix-valued function of bounded variation defined on $[-r, 0]$ such that

$$-\mu_k D_0\psi(0) + L_0\psi = \int_{-r}^{0} \mathrm{d}\eta_k(\theta)\psi(\theta), \quad \psi \in C. \tag{2.6}$$

The adjoint bilinear form on $C^* \times C$, where $C^* \triangleq C([0, r];\ \mathbb{C}^{m*})$, is defined by

$$(\psi,\ \varphi)_k = \psi(0)\varphi(0) - \int_{-r}^{0}\int_{0}^{\theta} \psi(\xi - \theta)\mathrm{d}\eta_k(\theta)\varphi(\xi)\mathrm{d}\xi,\ \ \psi \in C^*,\ \ \varphi \in C. \tag{2.7}$$

We make the following basic assumption on a Hopf-steady state bifurcation point:

(H4) There exists a neighborhood $V \subset \mathbb{R}^2$ of zero such that for $\alpha := (\alpha_1, \alpha_2) \in V$, the characteristic equation (2.4) with $k = k_1 \in \mathbb{N}_B$ has a simple real eigenvalue $\gamma(\alpha)$ with $\gamma(0) = 0$,

$\frac{\partial \gamma}{\partial \alpha_2}(0) \neq 0$, and (2.4) with $k = k_2 \in \mathbb{N}_B$ has a pair of simple complex conjugate eigenvalues $\nu(\alpha) \pm \mathrm{i}\omega(\alpha)$ with $\nu(0) = 0$, $\omega(0) = \omega_0 > 0$, $\frac{\partial \nu}{\partial \alpha_1}(0) \neq 0$, all other eigenvalues of (2.3) have non-zero real parts for $\alpha \in V$.

Definition 2.1. We say that a $(k_1, k_2)-$mode Hopf-steady state bifurcation occurs for (2.1) near the trivial equilibrium at $\alpha = (0, 0)$ if assumptions **(H1)-(H4)** are satisfied, or briefly, a Hopf-steady state bifurcation occurs. Moreover, if $k_1 \neq 0$, we say that a $(k_1, k_2)-$mode Turing-Hopf bifurcation occurs, or briefly, a Turing-Hopf bifurcation occurs.

Remark 2.2. A k_1-mode Turing bifurcation or a k_2-mode Hopf bifurcation can be defined in the same manner as that in Definition 2.1. In short, a k_1-mode Turing (k_2-mode Hopf) bifurcation is referred to the characteristic equation (2.4) with $k = k_1 \neq 0$ ($k = k_2$) having a zero root (a pair of purely imaginary roots), while the other roots have non-zero real parts, and the corresponding transversal conditions are satisfied.

Let $\Lambda_1 = \{0\}$, $\Lambda_2 = \{\pm \mathrm{i}\omega_0\}$, and $\Lambda = \Lambda_1 \cup \Lambda_2$. By [25], the phase space C is decomposed by Λ_i:

$$C = P_i \oplus Q_i,$$

where $Q_i = \{\varphi \in C : (\psi, \varphi)_{k_i} = 0, \text{ for all } \psi \in P_i^*\}$, and P_i^*, associated with Λ_i, is the generalized eigenspace of the formal adjoint differential equation of Eq. (2.5) with $k = k_i$, $i = 1, 2$. We choose the basis

$$\Phi_1 = \phi_1,\ \Psi_1 = \psi_1,\ \Phi_2 = (\phi_2, \bar{\phi_2}),\ \Psi_2 = \begin{pmatrix} \psi_2 \\ \bar{\psi_2} \end{pmatrix} \tag{2.8}$$

in P_1, P_1^*, P_2, P_2^* respectively, such that $(\Psi_i, \Phi_i)_{k_i} = I$, $i = 1, 2$, (I is the identity matrix), and

$$\dot{\Phi}_i = \Phi_i B_i \quad \text{and} \quad -\dot{\Psi}_i = B_i \Psi_i,\ i = 1, 2, \quad \text{with} \quad B_1 = 0,\ B_2 = \mathrm{diag}(\mathrm{i}\omega_0, -\mathrm{i}\omega_0).$$

We know from [25] that

$$\begin{aligned} &\phi_1(\theta) \equiv \phi_1(0),\ \ \phi_2(\theta) = \phi_2(0)e^{\mathrm{i}\omega_0\theta},\ \ \theta \in [-r, 0], \\ &\psi_1(s) \equiv \psi_1(0),\ \ \psi_2(s) = \psi_2(0)e^{-\mathrm{i}\omega_0 s},\ \ s \in [0, r], \end{aligned} \tag{2.9}$$

and

$$\begin{gathered} \triangle_{k_1}(0)\phi_1(0) = 0,\ \ \triangle_{k_2}(\mathrm{i}\omega_0)\phi_2(0) = 0, \\ \psi_1(0)\triangle_{k_1}(0) = 0,\ \ \psi_2(0)\triangle_{k_2}(\mathrm{i}\omega_0) = 0, \\ (\psi_i,\ \phi_i)_{k_i} = 1,\ i = 1, 2. \end{gathered} \tag{2.10}$$

Now we use the above definition to decompose $\mathcal{C}$ by Λ:

$$\mathcal{C} = \mathcal{P} \oplus \mathcal{Q}, \quad \mathcal{P} = \mathrm{Im}\pi, \quad \mathcal{Q} = \mathrm{Ker}\pi,$$

where $\dim\mathcal{P} = 3$ and $\pi : \mathcal{C} \to \mathcal{P}$ is the projection defined by

$$\pi\phi = \sum_{i=1,2} \Phi_i (\Psi_i, \langle \phi(\cdot), \beta_{k_i} \rangle)_{k_i} \beta_{k_i} . \tag{2.11}$$

We project the infinite-dimensional flow on $\mathcal{C}$ to the one on a finite-dimensional manifold $\mathcal{P}$. Following the ideas in [15], we consider the enlarged phase space $B\mathcal{C}$ introduced in (**H2**). This space can be identified as $\mathcal{C} \times X^m$, with elements in the form $\phi = \varphi + X_0 c$, where $\varphi \in \mathcal{C}$, $c \in X^m$, and X_0 is the $m \times m$ matrix-valued function defined by $X_0(\theta) = 0$ for $\theta \in [-r, 0)$ and $X_0(0) = I$.

In $B\mathcal{C}$, we consider an extension of the infinitesimal generator, still denoted by A,

$$A : \mathcal{C}_0^1 \subset B\mathcal{C} \to B\mathcal{C}, \quad A\phi = \dot{\phi} + X_0[L_0\phi + D_0\Delta\phi(0) - \dot{\phi}(0)], \tag{2.12}$$

defined on $\mathcal{C}_0^1 \triangleq \{\phi \in \mathcal{C} : \ \dot{\phi} \in \mathcal{C}, \ \phi(0) \in \mathrm{dom}(\Delta)\}$, and $(\cdot, \cdot)_{k_i}$ can be also defined by the same expression (2.7), $i = 1, 2$, on $C^* \times BC$, where

$$BC = \left\{ \psi : [-r, 0] \to \mathbb{C}^m | \psi \ \text{ is continuous on } \ [-r, 0), \ \lim_{\theta \to 0-} \psi(\theta) \text{ exists} \right\}.$$

Thus it is easy to see that π, as defined in (2.11), is extended to a continuous projection (which we still denote by π) $\pi : B\mathcal{C} \to \mathcal{P}$. In particular, for $c \in X^m$ we have

$$\pi(X_0 c) = \sum_{i=1,2} \Phi_i \Psi_i(0) \langle c, \beta_{k_i} \rangle \beta_{k_i} . \tag{2.13}$$

The projection π leads to the topological decomposition

$$B\mathcal{C} = \mathcal{P} \oplus \mathrm{Ker}\pi, \tag{2.14}$$

with the property $\mathcal{Q} \subsetneqq \mathrm{Ker}\pi$.

In the space $B\mathcal{C}$, (2.1) becomes an abstract ODE

$$\frac{\mathrm{d}v}{\mathrm{d}t} = Av + X_0 F(v, \alpha), \tag{2.15}$$

where

$$F(v, \alpha) = (L_\alpha - L_0)v + (D(\alpha) - D(0))\Delta v(0) + G(v, \alpha), \tag{2.16}$$

for $v \in \mathcal{C}$, $\alpha \in V$. We decompose $v \in \mathcal{C}_0^1$ according to (2.14) by

$$v(t) = \phi_1 z_1(t) \beta_{k_1} + (\phi_2 z_2(t) + \bar{\phi}_2 \bar{z}_2(t))\beta_{k_2} + y(t),$$

where $z_i(t) = (\psi_i, \langle v(t)(\cdot), \beta_{k_i} \rangle)_{k_i}$ $(i = 1, 2)$, and $y(t) \in \mathcal{C}_0^1 \cap \mathrm{Ker}\pi = \mathcal{C}_0^1 \cap \mathcal{Q} \triangleq \mathcal{Q}^1$.

Since π commutes with A in $\mathcal{C}_0^1$, we see that in BC, the abstract ODE (2.15) is equivalent to ODEs:

$$\begin{aligned}
\dot{z}_1 &= \psi_1(0)\langle F(\phi_1 z_1\beta_{k_1} + (\phi_2 z_2 + \bar{\phi}_2\bar{z}_2)\beta_{k_2} + y, \alpha), \beta_{k_1}\rangle,\\
\dot{z}_2 &= i\omega_0 z_2 + \psi_2(0)\langle F(\phi_1 z_1\beta_{k_1} + (\phi_2 z_2 + \bar{\phi}_2\bar{z}_2)\beta_{k_2} + y, \alpha), \beta_{k_2}\rangle,\\
\dot{\bar{z}}_2 &= -i\omega_0 \bar{z}_2 + \bar{\psi}_2(0)\langle F(\phi_1 z_1\beta_{k_1} + (\phi_2 z_2 + \bar{\phi}_2\bar{z}_2)\beta_{k_2} + y, \alpha), \beta_{k_2}\rangle,\\
\frac{\mathrm{d}}{\mathrm{d}t} y &= A_1 y + (I-\pi)X_0 F(\phi_1 z_1\beta_{k_1} + (\phi_2 z_2 + \bar{\phi}_2\bar{z}_2)\beta_{k_2} + y, \alpha),
\end{aligned} \tag{2.17}$$

for $z = (z_1, z_2, \bar{z}_2) \in P \subset \mathbb{C}^3$, $y \in \mathcal{Q}^1 \subset \mathrm{Ker}\pi$, where A_1 is the restriction of A as an operator from $\mathcal{Q}^1$ to the Banach space $\mathrm{Ker}\pi$: $A_1 : \mathcal{Q}^1 \subset \mathrm{Ker}\pi \to \mathrm{Ker}\pi,\ A_1\phi = A\phi$ for $\phi \in \mathcal{Q}^1$.

3. Formulas of the second and third terms in the normal form

In this section, we present the formulas of the second order and third order terms of the normal form of (2.1), and the proofs will be postponed to Section 6.

First we neglect the dependence on the higher order (≥ 2) terms of small parameters α_1, α_2 in the normal forms of (2.1). By doing the Taylor expansion formally for the operator $L(\alpha)$ and diagonal matrix $D(\alpha)$ at $\alpha = 0$, we have

$$\begin{aligned}
L(\alpha)v =& L(0)v + \frac{1}{2}L_1(\alpha)v + \cdots, \quad \text{for } v \in \mathcal{C},\\
D(\alpha) =& D(0) + \frac{1}{2}D_1(\alpha) + \cdots,
\end{aligned} \tag{3.1}$$

where $L_1:\ V \to L(\mathcal{C}, X^m)$, and $D_1 : V \to \mathbb{R}^m \times \mathbb{R}^m$ are linear. As in [26], we write G in (2.16) in the form of

$$G(v, 0) = \frac{1}{2!}Q(v, v) + \frac{1}{3!}C(v, v, v) + O(|v|^4),\ v \in \mathcal{C}, \tag{3.2}$$

where Q, C are symmetric multilinear forms. For simplicity, we also write $Q(X, Y)$ as Q_{XY}, $Q_X Y$ or $Q_Y X$, and $C(X, Y, Z)$ as C_{XYZ}.

The formulas of the third order normal form of (2.1) are given in the following theorem, and the detailed proof of the theorem is provided in Section 6.

Theorem 3.1. *Assume that* **(H1)**-**(H4)** *are satisfied. Ignoring the effect of the perturbation parameters in high-order items* (≥ 3)*, then the normal form of* (2.1) *restricted on the center manifold up to the third order is*

$$\dot{z} = Bz + \frac{1}{2}g_2^1(z, 0, \alpha) + \frac{1}{3!}g_3^1(z, 0, 0) + h.o.t., \tag{3.3}$$

or equivalently

$$\begin{aligned}\dot{z}_1 =& a_1(\alpha)z_1 + a_{11}z_1^2 + a_{23}z_2\bar{z}_2 + +a_{111}z_1^3 + a_{123}z_1z_2\bar{z}_2 + h.o.t.,\\ \dot{z}_2 =& \mathrm{i}\omega_0 z_2 + b_2(\alpha)z_2 + b_{12}z_1z_2 + b_{112}z_1^2z_2 + b_{223}z_2^2\bar{z}_2 + h.o.t.,\\ \dot{\bar{z}}_2 =& -\mathrm{i}\omega_0\bar{z}_2 + \overline{b_2(\alpha)}\bar{z}_2 + \overline{b_{12}}z_1\bar{z}_2 + \overline{b_{112}}z_1^2\bar{z}_2 + \overline{b_{223}}z_2\bar{z}_2^2 + h.o.t.,\end{aligned} \tag{3.4}$$

Here a_{ij}, b_{ij}, a_{ijk}, b_{ijk} are given by

$$a_{11} = \frac{1}{2}\psi_1(0)Q_{\phi_1\phi_1}\langle\beta_{k_1}^2, \beta_{k_1}\rangle,\ a_{23} = \psi_1(0)Q_{\phi_2\bar{\phi}_2}\langle\beta_{k_2}^2, \beta_{k_1}\rangle,\ b_{12} = \psi_2(0)Q_{\phi_1\phi_2}\langle\beta_{k_1}\beta_{k_2}, \beta_{k_2}\rangle, \tag{3.5}$$

$$\begin{aligned}a_{111} =& \frac{1}{6}\psi_1(0)C_{\phi_1\phi_1\phi_1}\langle\beta_{k_1}^3, \beta_{k_1}\rangle + \psi_1(0)\langle Q_{\phi_1 h_{200}}\beta_{k_1}, \beta_{k_1}\rangle\\ &+ \frac{1}{2i\omega_0}\psi_1(0)[-Q_{\phi_1\phi_2}\psi_2(0) + Q_{\phi_1\bar{\phi}_2}\bar{\psi}_2(0)]Q_{\phi_1\phi_1}\langle\beta_{k_1}\beta_{k_2}, \beta_{k_1}\rangle\langle\beta_{k_1}^2, \beta_{k_2}\rangle,\\ a_{123} =& \psi_1(0)C_{\phi_1\phi_2\bar{\phi}_2}\langle\beta_{k_1}\beta_{k_2}^2, \beta_{k_1}\rangle\\ &+ \frac{1}{i\omega_0}\psi_1(0)\{[-Q_{\phi_1\phi_2}\psi_2(0) + Q_{\phi_1\bar{\phi}_2}\bar{\psi}_2(0)]Q_{\phi_2\bar{\phi}_2}\langle\beta_{k_1}\beta_{k_2}, \beta_{k_1}\rangle\\ &\langle\beta_{k_2}^2, \beta_{k_2}\rangle + \frac{1}{2}[-Q_{\phi_2\phi_2}\psi_2(0)Q_{\phi_1\bar{\phi}_2} + Q_{\bar{\phi}_2\bar{\phi}_2}\bar{\psi}_2(0)Q_{\phi_1\phi_2}]\langle\beta_{k_1}\beta_{k_2}, \beta_{k_2}\rangle\langle\beta_{k_2}^2, \beta_{k_1}\rangle]\}\\ &+ \psi_1(0)(\langle Q_{\phi_1 h_{011}}\beta_{k_1}, \beta_{k_1}\rangle + \langle Q_{\phi_2 h_{101}}\beta_{k_2}, \beta_{k_1}\rangle + \langle Q_{\bar{\phi}_2 h_{110}}\beta_{k_2}, \beta_{k_1}\rangle),\end{aligned} \tag{3.6}$$

$$\begin{aligned}b_{112} =& \frac{1}{2}\psi_2(0)C_{\phi_1\phi_1\phi_2}\langle\beta_{k_1}^2\beta_{k_2}, \beta_{k_2}\rangle + \frac{1}{2i\omega_0}\psi_2(0)\{[2Q_{\phi_1\phi_1}\psi_1(0)\langle\beta_{k_1}\beta_{k_2}, \beta_{k_1}\rangle\langle\beta_{k_1}^2, \beta_{k_2}\rangle\\ &+ Q_{\phi_1\bar{\phi}_2}\bar{\psi}_2(0)\langle\beta_{k_1}\beta_{k_2}, \beta_{k_2}\rangle^2]Q_{\phi_1\phi_2}\\ &+ [-Q_{\phi_2\phi_2}\psi_2(0) + Q_{\phi_2\bar{\phi}_2}\bar{\psi}_2(0)]Q_{\phi_1\phi_1}\langle\beta_{k_1}^2, \beta_{k_2}\rangle\langle\beta_{k_2}^2, \beta_{k_2}\rangle\}\\ &+ \psi_2(0)(\langle Q_{\phi_1 h_{110}}\beta_{k_1}, \beta_{k_2}\rangle + \langle Q_{\phi_2 h_{200}}\beta_{k_2}, \beta_{k_2}\rangle),\\ b_{223} =& \frac{1}{2}\psi_2(0)C_{\phi_2\phi_2\bar{\phi}_2}\langle\beta_{k_2}^3, \beta_{k_2}\rangle + \frac{1}{4i\omega_0}\psi_2(0)\{Q_{\phi_1\bar{\phi}_2}\psi_1(0)Q_{\phi_2\phi_2}\langle\beta_{k_1}\beta_{k_2}, \beta_{k_2}\rangle\langle\beta_{k_2}^2, \beta_{k_1}\rangle\\ &+ \frac{2}{3}Q_{\bar{\phi}_2\bar{\phi}_2}\bar{\psi}_2(0)Q_{\phi_2\phi_2}\langle\beta_{k_2}^2, \beta_{k_2}\rangle^2 + [-2Q_{\phi_2\phi_2}\psi_2(0) + 4Q_{\phi_2\bar{\phi}_2}\bar{\psi}_2(0)]Q_{\phi_2\bar{\phi}_2}\langle\beta_{k_2}^2, \beta_{k_2}\rangle^2\}\\ &+ \psi_2(0)(\langle Q_{\phi_2 h_{011}}\beta_{k_2}, \beta_{k_2}\rangle + \langle Q_{\bar{\phi}_2 h_{020}}\beta_{k_2}, \beta_{k_2}\rangle).\end{aligned} \tag{3.7}$$

respectively, and h_{ijk} $(i + j + k = 2,\ i, j, k \in \mathbb{N}_0)$ are determined by (6.32) *and* (6.34).

Remark 3.2. For the normal form up to the third order, we only need to calculate the eigenvectors which are given by (2.9) and (2.10), the linear parts $L_1(\alpha)$ and $D_1(\alpha)$ in (3.1), and the multilinear forms Q and C which are given in (3.2).

For the reduced system (3.4), which is a system of ODEs, the bifurcation structure can be distinguished into the two main types: Hopf-transcritical type and Hopf-pitchfork type, which we will discuss separately below.

3.1. Hopf-transcritical type

Now, the reduced system (3.4) has the same form as Eqs. (22) of [28]. According to Proposition 3.2 in [28], we give the following definition.

Definition 3.3. Assume that **(H1)-(H4)** are satisfied, $a_{11} \neq 0$, $a_{23} \neq 0$, $\mathrm{Re}(b_{12}) \neq 0$, and $a_{11} - \mathrm{Re}(b_{12}) \neq 0$. Then we say that a Hopf-steady state bifurcation with Hopf-transcritical type occurs for (2.1) (or referred to as a Hopf-transcritical bifurcation) at the trivial equilibrium when $\alpha = 0$.

Applying coordinate transformations (23) and (25) in [28], (3.4) can be reduced to the planar system (see [28])

$$\begin{aligned} \dot{r} =& r(\varepsilon_1(\alpha) + az + cr^2 + dz^2), \\ \dot{z} =& \varepsilon_2(\alpha)z + br^2 - z^2 + er^2 z + fz^3, \end{aligned} \tag{3.8}$$

where

$$\varepsilon_1(\alpha) = \mathrm{Re}(b_2(\alpha)),\ \varepsilon_2(\alpha) = a_1(\alpha),\ a = -\frac{\mathrm{Re}(b_{12})}{a_{11}},\ b = -\mathrm{sign}(a_{11}a_{23}),$$
$$c = \frac{\mathrm{Re}(b_{223})}{|a_{11}a_{23}|},\ d = \frac{\mathrm{Re}(b_{112})}{a_{11}{}^2},\ e = \frac{a_{123}}{|a_{11}a_{23}|},\ f = \frac{a_{111}}{a_{11}{}^2}.$$

Now (3.8) has the same form as (36) in [28]. By [21] and [28], there are four different topological structures for (3.8) with the Hopf-transcritical bifurcation depending on the signs of a and b:

$$\begin{aligned} &\text{Case I}: \quad b = 1, a > 0; \qquad &&\text{Case II}: \quad b = 1, a < 0; \\ &\text{Case III}: \quad b = -1, a > 0; \qquad &&\text{Case IV}: \quad b = -1, a < 0. \end{aligned}$$

The results in [28] can be directly applied to analyze the equations (3.8), and the dynamical properties for the original system (2.1) can be revealed with the help of the analysis in [1, Section 4].

3.2. Hopf-pitchfork type

When the conditions in the following definition are satisfied, the reduced system (3.4) has the same structure as Eqs.(12) of [59]. Then, based on [59], we define it as a Hopf-pitchfork bifurcation.

Definition 3.4. Assume that **(H1)-(H4)** are satisfied, $a_{11} = a_{23} = b_{12} = 0$, $a_{111} \neq 0$, $a_{123} \neq 0$, $\mathrm{Re}(b_{112}) \neq 0$, $\mathrm{Re}(b_{223}) \neq 0$, and $a_{111}\mathrm{Re}(b_{223}) - a_{123}\mathrm{Re}(b_{112}) \neq 0$. Then we say that a Hopf-steady state bifurcation with Hopf-pitchfork type occurs for (2.1) (or referred to as a Hopf-pitchfork bifurcation) at the trivial equilibrium when $\alpha = 0$.

By using the same coordinate transformation in (23) of [28] and

$$\sqrt{|\mathrm{Re}b_{223}|}\, r \to r,\ \sqrt{|a_{111}|}\, z \to z,\ \mathrm{sign}(\mathrm{Re}(b_{223}))t \to t, \tag{3.9}$$

Table 1
The twelve unfoldings of (3.10), see [21].

Case	Ia	Ib	II	III	IVa	IVb	V	VIa	VIb	VIIa	VIIb	VIII
d_0	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	-1	-1	-1	-1	-1	-1
b_0	$+$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$-$	$-$	$-$
c_0	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$-$	$+$	$+$	$-$
$d_0 - b_0c_0$	$+$	$-$	$+$	$+$	$+$	$-$	$-$	$+$	$-$	$+$	$-$	$-$

we obtain a planar system (see [21] and [59]):

$$\begin{aligned}\dot{r} =& r(\varepsilon_1(\alpha) + r^2 + b_0 z^2),\\ \dot{z} =& z(\varepsilon_2(\alpha) + c_0 r^2 + d_0 z^2),\end{aligned} \tag{3.10}$$

where

$$\varepsilon_1(\alpha) = \mathrm{Re}(b_2(\alpha))\mathrm{sign}(\mathrm{Re}(b_{223})),\ \ \varepsilon_2(\alpha) = a_1(\alpha)\mathrm{sign}(\mathrm{Re}(b_{223})),$$
$$b_0 = \frac{\mathrm{Re}(b_{112})}{|a_{111}|}\mathrm{sign}(\mathrm{Re}(b_{223})),\ \ c_0 = \frac{a_{123}}{|\mathrm{Re}(b_{223})|}\mathrm{sign}(\mathrm{Re}(b_{223})),\ \ d_0 = \mathrm{sign}(a_{111}\mathrm{Re}(b_{223})).$$

For (3.10), there are possibly four equilibrium points as follows:

$$\begin{aligned}
&E_1 = (0,0), && \text{for all } \ \varepsilon_1, \varepsilon_2,\\
&E_2 = (\sqrt{-\varepsilon_1}, 0), && \text{for } \varepsilon_1 < 0,\\
&E_3^{\pm} = (0, \pm\sqrt{-\frac{\varepsilon_2}{d_0}}), && \text{for } \ \frac{\varepsilon_2}{d_0} < 0,\\
&E_4^{\pm} = (\sqrt{\frac{b_0\varepsilon_2 - d_0\varepsilon_1}{d_0 - b_0c_0}}, \pm\sqrt{\frac{c_0\varepsilon_1 - \varepsilon_2}{d_0 - b_0c_0}}), && \text{for } \frac{b_0\varepsilon_2 - d_0\varepsilon_1}{d_0 - b_0c_0}, \frac{c_0\varepsilon_1 - \varepsilon_2}{d_0 - b_0c_0} > 0.
\end{aligned} \tag{3.11}$$

Based on [21, §7.5], by the different signs of $b_0, c_0, d_0, d_0 - b_0c_0$ in Table 1, Eqs. (3.10) has twelve distinct types of unfoldings, corresponding to twelve essentially distinct types of phase portraits and bifurcation diagrams. The results in [21] can be directly applied to analyze the equations (3.10) and the dynamical properties for the original system (2.1) can be revealed with the help of the analysis in [1, Section 4]. For the convenience of application to the example in Section 5, we give the bifurcation diagram in parameters $(\varepsilon_1, \varepsilon_2)$ and phase portraits of Case III of Hopf-pitchfork type (see Table 1) in Fig. 1.

In the next section, we will give the explicit expressions of h_{ijk} in (3.5), (3.6) and (3.7) under the Neumann boundary condition and $\Omega = (0, l\pi)$ for $l > 0$.

4. Explicit formulas for Neumann boundary condition

In this section, we give more exact formulas of coefficients h_{ijk} in (3.5), (3.6) and (3.7) of the normal form (3.4) restricted on the center manifold in the case of spatial dimension $n = 1$ and $\Omega = (0, l\pi)$ for some $l > 0$, and we consider (2.1) with Neumann boundary condition.

It is well known that the eigenvalue problem

$$\beta'' + \mu\beta = 0, \quad x \in (0, l\pi), \quad \beta'(0) = \beta'(l\pi) = 0,$$

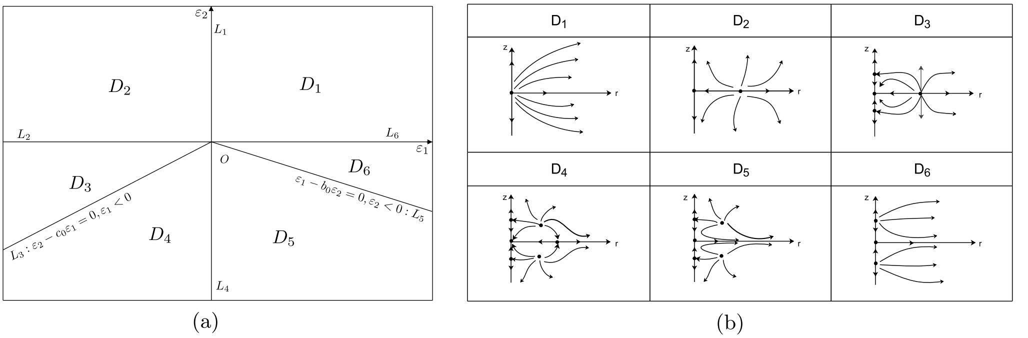


Fig. 1. The bifurcation set (a) and the phase portraits (b) for Case III of Hopf-pitchfork type.

has eigenvalues $\mu_n = n^2/l^2$ for $n \in \mathbb{N}_0$ with corresponding normalized eigenfuctions

$$\beta_0(x) = 1, \ \ \beta_n(x) = \sqrt{2}\cos\frac{n}{l}x, \quad n \in \mathbb{N},$$

where $\langle \beta_m(x), \beta_n(x)\rangle = \frac{1}{l\pi}\int_0^{l\pi} \beta_m(x)\beta_n(x)\mathrm{d}x = \delta_{mn}$. In what follows, we denote

$$h_q^k = \langle h_q, \beta_k\rangle, \ \ q \in \mathbb{N}_0^3, \ \ |q| = 2, \ \ k \in \mathbb{N}_0.$$

According to the different situations of $k_1,\ k_2$ in (**H4**), we will give the explicit formulas of $a_{11},\ a_{23},\ a_{111},\ a_{123},\ b_{12},\ b_{112},\ b_{223}$ in (3.4) for the following five cases.

Case (1) $k_1 = k_2 = 0$**.**

In this case,

$$\begin{aligned}
\langle \beta_{k_1}^2, \beta_{k_1}\rangle = &\ \ \langle \beta_{k_2}^2, \beta_{k_1}\rangle = \langle \beta_{k_1}\beta_{k_2}, \beta_{k_2}\rangle = \langle \beta_0, \beta_0\rangle = 1,\\
\langle \beta_{k_1}^2, \beta_{k_2}\rangle = &\ \ \langle \beta_{k_2}^2, \beta_{k_2}\rangle = \langle \beta_{k_1}\beta_{k_2}, \beta_{k_1}\rangle = \langle \beta_0, \beta_0\rangle = 1,\\
\langle \beta_{k_1}^3, \beta_{k_1}\rangle = &\ \ \langle \beta_{k_2}^3, \beta_{k_2}\rangle = \langle \beta_{k_1}\beta_{k_2}^2, \beta_{k_1}\rangle = \langle \beta_{k_1}^2\beta_{k_2}, \beta_{k_2}\rangle = \langle \beta_0, \beta_0\rangle = 1,\\
\langle Q_{\phi h_q}\beta_{k_i}, \beta_{k_j}\rangle = &\ \ Q_\phi h_q^0, \ \ \phi \in \{\phi_1, \phi_2, \bar{\phi}_2\}, \ \ i, j = 1, 2, \ \ q \in \mathbb{N}_0^3, \ \ |q| = 2.
\end{aligned}$$

By (6.32) and (6.34), $h_q^{k_i} = h_q^0$ for $i = 1, 2$ and $q \in \mathbb{N}_0^3$ with $|q| = 2$, which is needed in (3.5), (3.6) and (3.7), are as follows:

$$\begin{aligned}
h_{200}^0(\theta) = &\ \ \tfrac{1}{2}[\theta\phi_1(0)\psi_1(0) + \tfrac{1}{i\omega_0}(\phi_2(\theta)\psi_2(0) - \bar{\phi}_2(\theta)\bar{\psi}_2(0))]Q_{\phi_1\phi_1} + E_{200},\\
h_{011}^0(\theta) = &\ \ [\theta\phi_1(0)\psi_1(0) + \tfrac{1}{i\omega_0}(\phi_2(\theta)\psi_2(0) - \bar{\phi}_2(\theta)\bar{\psi}_2(0))]Q_{\phi_2\bar{\phi}_2} + E_{011},\\
h_{020}^0(\theta) = &\ \ -\tfrac{1}{2i\omega_0}[\tfrac{1}{2}\phi_1(0)\psi_1(0) + \phi_2(\theta)\psi_2(0) + \tfrac{1}{3}\bar{\phi}_2(\theta)\bar{\psi}_2(0))]Q_{\phi_2\phi_2} + E_{020}e^{2i\omega_0\theta},\\
h_{110}^0(\theta) = &\ \ \tfrac{1}{i\omega_0}[-\phi_1(0)\psi_1(0) + i\omega_0\theta\phi_2(\theta)\psi_2(0) - \tfrac{1}{2}\bar{\phi}_2(\theta)\bar{\psi}_2(0))]Q_{\phi_1\phi_2} + E_{110}e^{i\omega_0\theta},\\
h_{002}^0(\theta) = &\ \ \overline{h_{020}^0(\theta)}, \quad h_{101}^0(\theta) = \overline{h_{110}^0(\theta)},
\end{aligned} \tag{4.1}$$

where $\theta \in [-r, 0]$ and the constant vectors E_q for $q \in \mathbb{N}_0^3$ with $|q| = 2$, satisfy the following conditions

$$\begin{aligned}
[\textstyle\int_{-r}^{0} \mathrm{d}\eta_0(\theta)]E_{200} = & \ \tfrac{1}{2}[-I + (I - \textstyle\int_{-r}^{0} \theta \mathrm{d}\eta_0(\theta))\phi_1(0)\psi_1(0)]Q_{\phi_1\phi_1}, \\
[\textstyle\int_{-r}^{0} \mathrm{d}\eta_0(\theta)]E_{011} = & \ [-I + (I - \textstyle\int_{-r}^{0} \theta \mathrm{d}\eta_0(\theta))\phi_1(0)\psi_1(0)]Q_{\phi_2\bar{\phi}_2}, \\
E_{020} = & \ \tfrac{1}{2}[2i\omega_0 I - \textstyle\int_{-r}^{0} e^{2i\omega_0\theta}\mathrm{d}\eta_0(\theta)]^{-1}Q_{\phi_2\phi_2}, \\
[i\omega_0 I - \textstyle\int_{-r}^{0} e^{i\omega_0\theta}\mathrm{d}\eta_0(\theta)]E_{110} = & \ [I - \phi_2(0)\psi_2(0) + \textstyle\int_{-r}^{0} \theta \mathrm{d}\eta_0(\theta)\phi_2(\theta)\psi_2(0)]Q_{\phi_1\phi_2}.
\end{aligned} \tag{4.2}$$

Thus we have the following result.

Proposition 4.1. *For $k_1 = k_2 = 0$ and Neumann boundary condition on spatial domain $\Omega = (0, l\pi)$, $l > 0$, the parameters a_{11}, a_{23}, a_{111}, a_{123}, b_{12}, b_{112}, b_{223} in (3.4) are given by*

$$\begin{aligned}
a_{11} = & \ \tfrac{1}{2}\psi_1(0)Q_{\phi_1\phi_1}, \ \ a_{23} = \psi_1(0)Q_{\bar{\phi}_2\phi_2}, \ \ b_{12} = \psi_2(0)Q_{\phi_1\phi_2}, \\
a_{111} = & \ \tfrac{1}{6}\psi_1(0)C_{\phi_1\phi_1\phi_1} + \tfrac{1}{\omega_0}\psi_1(0)\mathrm{Re}(\mathrm{i}Q_{\phi_1\phi_2}\psi_2(0))Q_{\phi_1\phi_1} + \psi_1(0)Q_{\phi_1}h_{200}^0, \\
a_{123} = & \ \psi_1(0)C_{\phi_1\phi_2\bar{\phi}_2} + \tfrac{2}{\omega_0}\psi_1(0)\mathrm{Re}(\mathrm{i}Q_{\phi_1\phi_2}\psi_2(0))Q_{\phi_2\bar{\phi}_2} \\
& + \tfrac{1}{\omega_0}\psi_1(0)\mathrm{Re}(\mathrm{i}Q_{\phi_2\phi_2}\psi_2(0)Q_{\phi_1\bar{\phi}_2}) + \psi_1(0)(Q_{\phi_1}h_{011}^0 + Q_{\phi_2}h_{101}^0 + Q_{\bar{\phi}_2}h_{110}^0), \\
b_{112} = & \ \tfrac{1}{2}\psi_2(0)C_{\phi_1\phi_1\phi_2} + \tfrac{1}{2i\omega_0}\psi_2(0)\{[2Q_{\phi_1\phi_1}\psi_1(0) + Q_{\phi_1\bar{\phi}_2}\bar{\psi}_2(0)]Q_{\phi_1\phi_2} \\
& + [-Q_{\phi_2\phi_2}\psi_2(0) + Q_{\phi_2\bar{\phi}_2}\bar{\psi}_2(0)]Q_{\phi_1\phi_1}\} + \psi_2(0)(Q_{\phi_1}h_{110}^0 + Q_{\phi_2}h_{200}^0), \\
b_{223} = & \ \tfrac{1}{2}\psi_2(0)C_{\phi_2\phi_2\bar{\phi}_2} + \tfrac{1}{4i\omega_0}\psi_2(0)\{Q_{\phi_1\bar{\phi}_2}\psi_1(0)Q_{\phi_2\phi_2} + \tfrac{2}{3}Q_{\bar{\phi}_2\bar{\phi}_2}\bar{\psi}_2(0)Q_{\phi_2\phi_2} \\
& + [-2Q_{\phi_2\phi_2}\psi_2(0) + 4Q_{\phi_2\bar{\phi}_2}\bar{\psi}_2(0)]Q_{\phi_2\bar{\phi}_2}\} + \psi_2(0)(Q_{\phi_2}h_{011}^0 + Q_{\bar{\phi}_2}h_{020}^0).
\end{aligned} \tag{4.3}$$

Case (2) $k_1 = k_2 \neq 0$.

Here we have

$$\begin{aligned}
\langle \beta_{k_1}^2, \beta_{k_1} \rangle = & \ \langle \beta_{k_2}^2, \beta_{k_1} \rangle = \langle \beta_{k_1}\beta_{k_2}, \beta_{k_2} \rangle = 0, \\
\langle \beta_{k_1}^2, \beta_{k_2} \rangle = & \ \langle \beta_{k_2}^2, \beta_{k_2} \rangle = \langle \beta_{k_1}\beta_{k_2}, \beta_{k_1} \rangle = 0, \\
\langle \beta_{k_1}^3, \beta_{k_1} \rangle = & \ \langle \beta_{k_2}^3, \beta_{k_2} \rangle = \langle \beta_{k_1}\beta_{k_2}^2, \beta_{k_1} \rangle = \langle \beta_{k_1}^2\beta_{k_2}, \beta_{k_2} \rangle = \tfrac{3}{2}, \\
\langle Q_{\phi h_q}\beta_{k_i}, \beta_{k_j} \rangle = & \ Q_\phi(h_q^0 + \tfrac{1}{\sqrt{2}}h_q^{2k_1}), \ \ \phi \in \{\phi_1, \phi_2, \bar{\phi}_2\}, \ \ i = 1, 2, \ \ q \in \mathbb{N}_0^3, \ \ |q| = 2.
\end{aligned}$$

It also follows from (6.32) and (6.34) that $h_q^{k_i} = h_q^0$ for $i = 1, 2$, $q \in \mathbb{N}_0^3$ with $|q| = 2$, are given by

$$\begin{aligned}
h_{200}^0(\theta) \equiv & \ -\tfrac{1}{2}[\textstyle\int_{-r}^{0} \mathrm{d}\eta_0(\theta)]^{-1}Q_{\phi_1\phi_1}, \ \ h_{200}^{2k_1}(\theta) \equiv -\tfrac{1}{2\sqrt{2}}[\textstyle\int_{-r}^{0} \mathrm{d}\eta_{2k_1}(\theta)]^{-1}Q_{\phi_1\phi_1}, \\
h_{011}^0(\theta) \equiv & \ -[\textstyle\int_{-r}^{0} \mathrm{d}\eta_0(\theta)]^{-1}Q_{\phi_2\bar{\phi}_2}, \ \ h_{011}^{2k_1}(\theta) \equiv -\tfrac{1}{\sqrt{2}}[\textstyle\int_{-r}^{0} \mathrm{d}\eta_{2k_1}(\theta)]^{-1}Q_{\phi_2\bar{\phi}_2}, \\
h_{020}^0(\theta) = & \ \tfrac{1}{2}[2i\omega_0 I - \textstyle\int_{-r}^{0} e^{2i\omega_0\theta}\mathrm{d}\eta_0(\theta)]^{-1}Q_{\phi_2\phi_2}e^{2i\omega_0\theta}, \\
h_{020}^{2k_1}(\theta) = & \ \tfrac{1}{2\sqrt{2}}[2i\omega_0 I - \textstyle\int_{-r}^{0} e^{2i\omega_0\theta}\mathrm{d}\eta_{2k_1}(\theta)]^{-1}Q_{\phi_2\phi_2}e^{2i\omega_0\theta},
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
h^0_{110}(\theta)= &\ [i\omega_0 I-\textstyle\int_{-r}^0 e^{i\omega_0\theta}\mathrm{d}\eta_0(\theta)]^{-1}Q_{\phi_1\phi_2}e^{i\omega_0\theta},\\
h^{2k_1}_{110}(\theta)= &\ \tfrac{1}{\sqrt{2}}[i\omega_0 I-\textstyle\int_{-r}^0 e^{i\omega_0\theta}\mathrm{d}\eta_{2k_1}(\theta)]^{-1}Q_{\phi_1\phi_2}e^{i\omega_0\theta},\\
h^0_{002}(\theta)= &\ \overline{h^0_{020}(\theta)},\quad h^{2k_1}_{002}(\theta)=\overline{h^{2k_1}_{020}(\theta)},\quad h^0_{101}(\theta)=\overline{h^0_{110}(\theta)},\quad h^{2k_1}_{101}(\theta)=\overline{h^{2k_1}_{110}(\theta)}.
\end{aligned}$$

where $\theta\in[-r,0]$. Thus we have the following result.

Proposition 4.2. *For $k_1=k_2\neq 0$ and Neumann boundary condition on spatial domain $\Omega=(0,l\pi)$, $l>0$, the parameters a_{11}, a_{23}, a_{111}, a_{123}, b_{12}, b_{112}, b_{223} in (3.4) are*

$$\begin{aligned}
a_{11}= &\ a_{23}=b_{12}=0,\\
a_{111}= &\ \tfrac14\psi_1(0)C_{\phi_1\phi_1\phi_1}+\psi_1(0)Q_{\phi_1}(h^0_{200}+\tfrac{1}{\sqrt2}h^{2k_1}_{200}),\\
a_{123}= &\ \tfrac32\psi_1(0)C_{\phi_1\phi_2\bar\phi_2}+\psi_1(0)[Q_{\phi_1}(h^0_{011}+\tfrac{1}{\sqrt2}h^{2k_1}_{011})+Q_{\phi_2}(h^0_{101}+\tfrac{1}{\sqrt2}h^{2k_1}_{101})\\
&+Q_{\bar\phi_2}(h^0_{110}+\tfrac{1}{\sqrt2}h^{2k_1}_{110})],\\
b_{112}= &\ \tfrac34\psi_2(0)C_{\phi_1\phi_1\phi_2}+\psi_2(0)[Q_{\phi_1}(h^0_{110}+\tfrac{1}{\sqrt2}h^{2k_1}_{110})+Q_{\phi_2}(h^0_{200}+\tfrac{1}{\sqrt2}h^{2k_1}_{200})],\\
b_{223}= &\ \tfrac34\psi_2(0)C_{\phi_2\phi_2\bar\phi_2}+\psi_2(0)[Q_{\phi_2}(h^0_{011}+\tfrac{1}{\sqrt2}h^{2k_1}_{011})+Q_{\bar\phi_2}(h^0_{020}+\tfrac{1}{\sqrt2}h^{2k_1}_{020})].
\end{aligned}\tag{4.5}$$

Case (3) $k_2=0$, $k_1\neq 0$.

Here we have

$$\begin{aligned}
\langle\beta^2_{k_1},\beta_{k_1}\rangle= &\ \langle\beta^2_{k_2},\beta_{k_1}\rangle=\langle\beta_{k_1}\beta_{k_2},\beta_{k_2}\rangle=0,\\
\langle\beta^2_{k_1},\beta_{k_2}\rangle= &\ \langle\beta^2_{k_2},\beta_{k_2}\rangle=\langle\beta_{k_1}\beta_{k_2},\beta_{k_1}\rangle=1,\\
\langle\beta^3_{k_1},\beta_{k_1}\rangle= &\ \tfrac32,\ \langle\beta^3_{k_2},\beta_{k_2}\rangle=\langle\beta_{k_1}\beta^2_{k_2},\beta_{k_1}\rangle=\langle\beta^2_{k_1}\beta_{k_2},\beta_{k_2}\rangle=1,\\
\langle Q_{\phi_1h_{200}}\beta_{k_1},\beta_{k_1}\rangle= &\ Q_{\phi_1}(h^0_{200}+\tfrac{1}{\sqrt2}h^{2k_1}_{200}),\quad \langle Q_{\phi_1h_{011}}\beta_{k_1},\beta_{k_1}\rangle=Q_{\phi_1}(h^0_{011}+\tfrac{1}{\sqrt2}h^{2k_1}_{011}),\\
\langle Q_{\phi_2h_{101}}\beta_{k_2},\beta_{k_1}\rangle= &\ Q_{\phi_2}h^{k_1}_{101},\quad \langle Q_{\bar\phi_2h_{110}}\beta_{k_2},\beta_{k_1}\rangle=Q_{\bar\phi_2}h^{k_1}_{110},\\
\langle Q_{\phi_1h_{101}}\beta_{k_1},\beta_{k_2}\rangle= &\ Q_{\phi_1}h^{k_1}_{101},\quad \langle Q_{\phi_1h_{110}}\beta_{k_1},\beta_{k_2}\rangle=Q_{\phi_1}h^{k_1}_{110},\\
\langle Q_{\phi_2h_{200}}\beta_{k_2},\beta_{k_2}\rangle= &\ Q_{\phi_2}h^0_{200},\quad \langle Q_{\phi_2h_{011}}\beta_{k_2},\beta_{k_2}\rangle=Q_{\phi_2}h^0_{011},\\
\langle Q_{\bar\phi_2h_{200}}\beta_{k_2},\beta_{k_2}\rangle= &\ Q_{\bar\phi_2}h^0_{200},\quad \langle Q_{\bar\phi_2h_{011}}\beta_{k_2},\beta_{k_2}\rangle=Q_{\bar\phi_2}h^0_{011},\\
\langle Q_{\bar\phi_2h_{020}}\beta_{k_2},\beta_{k_2}\rangle= &\ Q_{\bar\phi_2}h^0_{020},\quad \langle Q_{\phi_2h_{002}}\beta_{k_2},\beta_{k_2}\rangle=Q_{\phi_2}h^0_{002},
\end{aligned}$$

and

$$\begin{aligned}
h^0_{200}(\theta)= &\ -\tfrac12[\textstyle\int_{-r}^0\mathrm{d}\eta_0(\theta)]^{-1}Q_{\phi_1\phi_1}+\tfrac{1}{2i\omega_0}(\phi_2(\theta)\psi_2(0)-\bar\phi_2(\theta)\bar\psi_2(0))]Q_{\phi_1\phi_1},\\
h^{2k_1}_{200}(\theta)\equiv &\ -\tfrac{1}{2\sqrt2}[\textstyle\int_{-r}^0\mathrm{d}\eta_{2k_1}(\theta)]^{-1}Q_{\phi_1\phi_1},\\
h^0_{011}(\theta)= &\ -[\textstyle\int_{-r}^0\mathrm{d}\eta_0(\theta)]^{-1}Q_{\phi_2\bar\phi_2}+\tfrac{1}{i\omega_0}(\phi_2(\theta)\psi_2(0)-\bar\phi_2(\theta)\bar\psi_2(0))]Q_{\phi_2\bar\phi_2},\\
h^{2k_1}_{011}(\theta)= &\ 0,
\end{aligned}\tag{4.6}$$

$$
\begin{aligned}
h^0_{020}(\theta) = &\ \tfrac{1}{2}[2i\omega_0 I - \textstyle\int_{-r}^{0} e^{2i\omega_0\theta}\mathrm{d}\eta_0(\theta)]^{-1} Q_{\phi_2\phi_2} e^{2i\omega_0\theta} - \tfrac{1}{2i\omega_0}[\phi_2(\theta)\psi_2(0) \\
&+ \tfrac{1}{3}\bar{\phi}_2(\theta)\bar{\psi}_2(0)] Q_{\phi_2\phi_2}, \\
h^{k_1}_{110}(\theta) = &\ [i\omega_0 I - \textstyle\int_{-r}^{0} e^{i\omega_0\theta}\mathrm{d}\eta_{k_1}(\theta)]^{-1} Q_{\phi_1\phi_2} e^{i\omega_0\theta} - \tfrac{1}{i\omega_0}\phi_1(0)\psi_1(0) Q_{\phi_1\phi_2}, \\
h^0_{002}(\theta) = &\ \overline{h^0_{020}(\theta)}, \quad h^{k_1}_{101}(\theta) = \overline{h^{k_1}_{110}(\theta)}.
\end{aligned}
$$

where $\theta \in [-r, 0]$. Then we have the following result.

Proposition 4.3. *For $k_2 = 0$, $k_1 \neq 0$ and Neumann boundary condition on spatial domain $\Omega = (0, l\pi)$, $l > 0$, the parameters a_{11}, a_{23}, a_{111}, a_{123}, b_{12}, b_{112}, b_{223} in (3.4) are*

$$
\begin{aligned}
a_{11} = &\ a_{23} = b_{12} = 0, \\
a_{111} = &\ \tfrac{1}{4}\psi_1(0)C_{\phi_1\phi_1\phi_1} + \tfrac{1}{\omega_0}\psi_1(0)\mathrm{Re}(\mathrm{i}Q_{\phi_1\phi_2}\psi_2(0))Q_{\phi_1\phi_1} + \\
&\ \psi_1(0)Q_{\phi_1}(h^0_{200} + \tfrac{1}{\sqrt{2}}h^{2k_1}_{200}), \\
a_{123} = &\ \psi_1(0)C_{\phi_1\phi_2\bar{\phi}_2} + \tfrac{2}{\omega_0}\psi_1(0)\mathrm{Re}(\mathrm{i}Q_{\phi_1\phi_2}\psi_2(0))Q_{\phi_2\bar{\phi}_2} + \\
&\ \psi_1(0)[Q_{\phi_1}(h^0_{011} + \tfrac{1}{\sqrt{2}}h^{2k_1}_{011}) + Q_{\phi_2}h^{k_1}_{101} + Q_{\bar{\phi}_2}h^{k_1}_{110}], \\
b_{112} = &\ \tfrac{1}{2}\psi_2(0)C_{\phi_1\phi_1\phi_2} + \tfrac{1}{2i\omega_0}\psi_2(0)\{2Q_{\phi_1\phi_1}\psi_1(0)Q_{\phi_1\phi_2} + [-Q_{\phi_2\phi_2}\psi_2(0) \\
&+ Q_{\phi_2\bar{\phi}_2}\bar{\psi}_2(0)]Q_{\phi_1\phi_1}\} + \psi_2(0)(Q_{\phi_1}h^{k_1}_{110} + Q_{\phi_2}h^0_{200}), \\
b_{223} = &\ \tfrac{1}{2}\psi_2(0)C_{\phi_2\phi_2\bar{\phi}_2} + \tfrac{1}{4i\omega_0}\psi_2(0)\{\tfrac{2}{3}Q_{\bar{\phi}_2\bar{\phi}_2}\bar{\psi}_2(0)Q_{\phi_2\phi_2} + [-2Q_{\phi_2\phi_2}\psi_2(0) \\
&+ 4Q_{\phi_2\bar{\phi}_2}\bar{\psi}_2(0)]Q_{\phi_2\bar{\phi}_2}\} + \psi_2(0)(Q_{\phi_2}h^0_{011} + Q_{\bar{\phi}_2}h^0_{020}).
\end{aligned}
\tag{4.7}
$$

Case (4) $k_1 = 0$, $k_2 \neq 0$.

Here we have

$$
\begin{aligned}
\langle \beta^2_{k_1}, \beta_{k_1}\rangle = &\ \langle \beta^2_{k_2}, \beta_{k_1}\rangle = \langle \beta_{k_1}\beta_{k_2}, \beta_{k_2}\rangle = 1, \\
\langle \beta^2_{k_1}, \beta_{k_2}\rangle = &\ \langle \beta^2_{k_2}, \beta_{k_2}\rangle = \langle \beta_{k_1}\beta_{k_2}, \beta_{k_1}\rangle = 0, \\
\langle \beta^3_{k_1}, \beta_{k_1}\rangle = &\ 1,\ \langle \beta^3_{k_2}, \beta_{k_2}\rangle = \tfrac{3}{2},\ \langle \beta_{k_1}\beta^2_{k_2}, \beta_{k_1}\rangle = \langle \beta^2_{k_1}\beta_{k_2}, \beta_{k_2}\rangle = 1, \\
\langle Q_{\phi_1 h_{200}}\beta_{k_1}, \beta_{k_1}\rangle = &\ Q_{\phi_1}h^0_{200}, \quad \langle Q_{\phi_1 h_{011}}\beta_{k_1}, \beta_{k_1}\rangle = Q_{\phi_1}h^0_{011}, \\
\langle Q_{\phi_2 h_{101}}\beta_{k_2}, \beta_{k_1}\rangle = &\ Q_{\phi_2}h^{k_2}_{101}, \quad \langle Q_{\bar{\phi}_2 h_{110}}\beta_{k_2}, \beta_{k_1}\rangle = Q_{\bar{\phi}_2}h^{k_2}_{110}, \\
\langle Q_{\phi_1 h_{101}}\beta_{k_1}, \beta_{k_2}\rangle = &\ Q_{\phi_1}h^{k_2}_{101}, \quad \langle Q_{\phi_1 h_{110}}\beta_{k_1}, \beta_{k_2}\rangle = Q_{\phi_1}h^{k_2}_{110}, \\
\langle Q_{\phi_2 h_{200}}\beta_{k_2}, \beta_{k_2}\rangle = &\ Q_{\phi_2}(h^0_{200} + \tfrac{1}{\sqrt{2}}h^{2k_2}_{200}), \quad \langle Q_{\phi_2 h_{011}}\beta_{k_2}, \beta_{k_2}\rangle = Q_{\phi_2}(h^0_{011} + \tfrac{1}{\sqrt{2}}h^{2k_2}_{011}), \\
\langle Q_{\bar{\phi}_2 h_{200}}\beta_{k_2}, \beta_{k_2}\rangle = &\ Q_{\bar{\phi}_2}(h^0_{200} + \tfrac{1}{\sqrt{2}}h^{2k_2}_{200}), \quad \langle Q_{\bar{\phi}_2 h_{011}}\beta_{k_2}, \beta_{k_2}\rangle = Q_{\bar{\phi}_2}(h^0_{011} + \tfrac{1}{\sqrt{2}}h^{2k_2}_{011}), \\
\langle Q_{\bar{\phi}_2 h_{020}}\beta_{k_2}, \beta_{k_2}\rangle = &\ Q_{\bar{\phi}_2}(h^0_{020} + \tfrac{1}{\sqrt{2}}h^{2k_2}_{020}), \quad \langle Q_{\phi_2 h_{002}}\beta_{k_2}, \beta_{k_2}\rangle = Q_{\phi_2}(h^0_{002} + \tfrac{1}{\sqrt{2}}h^{2k_2}_{002}),
\end{aligned}
$$

and

$$\begin{aligned}
h_{200}^{0}(\theta) = &\ \tfrac{1}{2}\theta\phi_1(0)\psi_1(0)Q_{\phi_1\phi_1} + E_{200},\ \ h_{200}^{2k_2}(\theta) \equiv 0,\\
h_{011}^{0}(\theta) = &\ \theta\phi_1(0)\psi_1(0)Q_{\phi_2\bar{\phi}_2} + E_{011},\ \ h_{011}^{2k_2}(\theta) \equiv -\tfrac{1}{\sqrt{2}}[\textstyle\int_{-r}^{0} \mathrm{d}\eta_{2k_2}(\theta)]^{-1}Q_{\phi_2\bar{\phi}_2},\\
h_{020}^{0}(\theta) = &\ -\tfrac{1}{4i\omega_0}\phi_1(0)\psi_1(0)Q_{\phi_2\phi_2} + E_{020}e^{2i\omega_0\theta},\\
h_{020}^{2k_2}(\theta) = &\ \tfrac{1}{2\sqrt{2}}[2i\omega_0 I - \textstyle\int_{-r}^{0} e^{2i\omega_0\theta}\mathrm{d}\eta_{2k_2}(\theta)]^{-1}Q_{\phi_2\phi_2}e^{2i\omega_0\theta},\\
h_{110}^{k_2}(\theta) = &\ \tfrac{1}{i\omega_0}[i\omega_0\theta\phi_2(\theta)\psi_2(0) - \tfrac{1}{2}\bar{\phi}_2(\theta)\bar{\psi}_2(0))]Q_{\phi_1\phi_2} + E_{110}e^{i\omega_0\theta},\\
h_{002}^{0}(\theta) = &\ \overline{h_{020}^{0}(\theta)},\quad h_{002}^{2k_2}(\theta) = \overline{h_{020}^{2k_2}(\theta)},\quad h_{101}^{k_2}(\theta) = \overline{h_{110}^{k_2}(\theta)},
\end{aligned} \tag{4.8}$$

where $\theta \in [-r, 0]$ and the constant vectors E_q^0 for $q \in \mathbb{N}_0^3$ with $|q| = 2$, satisfy the following equations

$$\begin{aligned}
[\textstyle\int_{-r}^{0} \mathrm{d}\eta_0(\theta)]E_{200} = &\ \tfrac{1}{2}[-I + (I - \textstyle\int_{-r}^{0} \theta\mathrm{d}\eta_0(\theta))\phi_1(0)\psi_1(0)]Q_{\phi_1\phi_1},\\
[\textstyle\int_{-r}^{0} \mathrm{d}\eta_0(\theta)]E_{011} = &\ [-I + (I - \textstyle\int_{-r}^{0} \theta\mathrm{d}\eta_0(\theta))\phi_1(0)\psi_1(0)]Q_{\phi_2\bar{\phi}_2},\\
E_{020} = &\ \tfrac{1}{2}[2i\omega_0 I - \textstyle\int_{-r}^{0} e^{2i\omega_0\theta}\mathrm{d}\eta_0(\theta)]^{-1}Q_{\phi_2\phi_2},\\
[i\omega_0 I - \textstyle\int_{-r}^{0} e^{i\omega_0\theta}\mathrm{d}\eta_{k_2}(\theta)]E_{110} = &\ [I - \phi_2(0)\psi_2(0) + \textstyle\int_{-r}^{0} \theta\mathrm{d}\eta_{k_2}(\theta)\phi_2(\theta)\psi_2(0)]Q_{\phi_1\phi_2}.
\end{aligned} \tag{4.9}$$

Then we have the following result.

Proposition 4.4. *For $k_1 = 0$, $k_2 \neq 0$ and Neumann boundary condition on spatial domain $\Omega = (0, l\pi)$, $l > 0$, the parameters a_{11}, a_{23}, a_{111}, a_{123}, b_{12}, b_{112}, b_{223} in (3.4) are*

$$\begin{aligned}
a_{11} = &\ \tfrac{1}{2}\psi_1(0)Q_{\phi_1\phi_1},\ \ a_{23} = \psi_1(0)Q_{\bar{\phi}_2\phi_2},\ \ b_{12} = \psi_2(0)Q_{\phi_1\phi_2},\\
a_{111} = &\ \tfrac{1}{6}\psi_1(0)C_{\phi_1\phi_1\phi_1} + \psi_1(0)Q_{\phi_1}h_{200}^{0},\\
a_{123} = &\ \psi_1(0)C_{\phi_1\phi_2\bar{\phi}_2} + \tfrac{1}{\omega_0}\psi_1(0)\mathrm{Re}(\mathrm{i}Q_{\phi_2\phi_2}\psi_2(0)Q_{\phi_1\bar{\phi}_2})\\
&+\psi_1(0)(Q_{\phi_1}h_{011}^{0} + Q_{\phi_2}h_{101}^{k_2} + Q_{\bar{\phi}_2}h_{110}^{k_2}),\\
b_{112} = &\ \tfrac{1}{2}\psi_2(0)C_{\phi_1\phi_1\phi_2} + \tfrac{1}{2i\omega_0}\psi_2(0)Q_{\phi_1\bar{\phi}_2}\bar{\psi}_2(0)Q_{\phi_1\phi_2}\\
&+\psi_2(0)[Q_{\phi_1}h_{110}^{k_2} + Q_{\phi_2}(h_{200}^{0} + \tfrac{1}{\sqrt{2}}h_{200}^{2k_2})],\\
b_{223} = &\ \tfrac{3}{4}\psi_2(0)C_{\phi_2\phi_2\bar{\phi}_2} + \tfrac{1}{4i\omega_0}\psi_2(0)Q_{\phi_1\bar{\phi}_2}\psi_1(0)Q_{\phi_2\phi_2} + \psi_2(0)[Q_{\phi_2}(h_{011}^{0} + \tfrac{1}{\sqrt{2}}h_{011}^{2k_2})\\
&+Q_{\bar{\phi}_2}(h_{020}^{0} + \tfrac{1}{\sqrt{2}}h_{020}^{2k_2})].
\end{aligned} \tag{4.10}$$

Case (5) $k_1 \neq k_2$**,** $k_1, k_2 \neq 0$**.**

Here we have

$$\begin{aligned}
\langle \beta_{k_1}^2, \beta_{k_1} \rangle = &\ 0,\ \langle \beta_{k_2}^2, \beta_{k_1} \rangle = \langle \beta_{k_1}\beta_{k_2}, \beta_{k_2} \rangle = \tfrac{1}{\sqrt{2}}\delta(k_1 - 2k_2), \\
\langle \beta_{k_2}^2, \beta_{k_2} \rangle = &\ 0,\ \langle \beta_{k_1}^2, \beta_{k_2} \rangle = \langle \beta_{k_1}\beta_{k_2}, \beta_{k_1} \rangle = \tfrac{1}{\sqrt{2}}\delta(k_2 - 2k_1), \\
\langle \beta_{k_1}^3, \beta_{k_1} \rangle = &\ \langle \beta_{k_2}^3, \beta_{k_2} \rangle = \tfrac{3}{2},\ \langle \beta_{k_1}\beta_{k_2}^2, \beta_{k_1} \rangle = \langle \beta_{k_1}^2\beta_{k_2}, \beta_{k_2} \rangle = 1, \\
\langle Q_{\phi_1 h_{200}}\beta_{k_1}, \beta_{k_1} \rangle = &\ \tfrac{1}{\sqrt{2}} Q_{\phi_1} h_{200}^{2k_1} + Q_{\phi_1} h_{200}^{0},\ \langle Q_{\phi_1 h_{011}}\beta_{k_1}, \beta_{k_1} \rangle = \tfrac{1}{\sqrt{2}} Q_{\phi_1} h_{011}^{2k_1} + Q_{\phi_1} h_{011}^{0}, \\
\langle Q_{\phi_2 h_{101}}\beta_{k_2}, \beta_{k_1} \rangle = &\ Q_{\phi_2}(\tfrac{1}{\sqrt{2}} h_{101}^{|k_1-k_2|} + \tfrac{1}{\sqrt{2}} h_{101}^{k_1+k_2} + h_{101}^{0}), \\
\langle Q_{\bar{\phi}_2 h_{110}}\beta_{k_2}, \beta_{k_1} \rangle = &\ Q_{\bar{\phi}_2}(\tfrac{1}{\sqrt{2}} h_{110}^{|k_1-k_2|} + \tfrac{1}{\sqrt{2}} h_{110}^{k_1+k_2} + h_{110}^{0}), \\
\langle Q_{\phi_1 h_{101}}\beta_{k_1}, \beta_{k_2} \rangle = &\ Q_{\phi_1}(\tfrac{1}{\sqrt{2}} h_{101}^{|k_1-k_2|} + \tfrac{1}{\sqrt{2}} h_{101}^{k_1+k_2} + h_{101}^{0}), \\
\langle Q_{\phi_1 h_{110}}\beta_{k_1}, \beta_{k_2} \rangle = &\ Q_{\phi_1}(\tfrac{1}{\sqrt{2}} h_{110}^{|k_1-k_2|} + \tfrac{1}{\sqrt{2}} h_{110}^{k_1+k_2} + h_{110}^{0}), \\
\langle Q_{\phi_2 h_{200}}\beta_{k_2}, \beta_{k_2} \rangle = &\ \tfrac{1}{\sqrt{2}} Q_{\phi_2} h_{200}^{2k_2} + Q_{\phi_2} h_{200}^{0},\ \langle Q_{\phi_2 h_{011}}\beta_{k_2}, \beta_{k_2} \rangle = \tfrac{1}{\sqrt{2}} Q_{\phi_2} h_{011}^{2k_2} + Q_{\phi_2} h_{011}^{0}, \\
\langle Q_{\bar{\phi}_2 h_{200}}\beta_{k_2}, \beta_{k_2} \rangle = &\ \tfrac{1}{\sqrt{2}} Q_{\bar{\phi}_2} h_{200}^{2k_2} + Q_{\bar{\phi}_2} h_{200}^{0},\ \langle Q_{\bar{\phi}_2 h_{011}}\beta_{k_2}, \beta_{k_2} \rangle = \tfrac{1}{\sqrt{2}} Q_{\bar{\phi}_2} h_{011}^{2k_2} + Q_{\bar{\phi}_2} h_{011}^{0}, \\
\langle Q_{\bar{\phi}_2 h_{020}}\beta_{k_2}, \beta_{k_2} \rangle = &\ \tfrac{1}{\sqrt{2}} Q_{\bar{\phi}_2} h_{020}^{2k_2} + Q_{\bar{\phi}_2} h_{020}^{0},\ \langle Q_{\phi_2 h_{002}}\beta_{k_2}, \beta_{k_2} \rangle = \tfrac{1}{\sqrt{2}} Q_{\phi_2} h_{002}^{2k_2} + Q_{\phi_2} h_{002}^{0},
\end{aligned}$$

where $\delta(x) = 1,$ for $x = 0$ and $\delta(x) = 0,$ for $x \neq 0$. And $h_{200}^0,\ h_{200}^{2k_1},\ h_{200}^{2k_2},\ h_{011}^{0},\ h_{011}^{2k_1},\ h_{011}^{2k_2},$ $h_{101}^0,\ h_{101}^{|k_1-k_2|},\ h_{101}^{k_1+k_2}, h_{110}^0,\ h_{110}^{|k_1-k_2|},\ h_{110}^{k_1+k_2},\ h_{020}^0,\ h_{020}^{2k_2},\ h_{002}^{2k_2}$ are given by

$$\begin{aligned}
h_{200}^0(\theta) \equiv &\ -\tfrac{1}{2}[\textstyle\int_{-r}^0 \mathrm{d}\eta_0(\theta)]^{-1} Q_{\phi_1\phi_1},\ \ h_{200}^{2k_1}(\theta) \equiv -\tfrac{1}{2\sqrt{2}}[\textstyle\int_{-r}^0 \mathrm{d}\eta_{2k_1}(\theta)]^{-1} Q_{\phi_1\phi_1}, \\
h_{200}^{2k_2}(\theta) \equiv &\ 0, \\
h_{011}^0(\theta) \equiv &\ -[\textstyle\int_{-r}^0 \mathrm{d}\eta_0(\theta)]^{-1} Q_{\phi_2\bar{\phi}_2},\quad h_{011}^{2k_1}(\theta) \equiv 0,\ \ h_{011}^{2k_2}(\theta) = -\tfrac{1}{\sqrt{2}}[\textstyle\int_{-r}^0 \mathrm{d}\eta_{2k_2}(\theta)]^{-1} Q_{\phi_2\bar{\phi}_2}, \\
h_{020}^0(\theta) = &\ \tfrac{1}{2}[2i\omega_0 I - \textstyle\int_{-r}^0 e^{2i\omega_0\theta}\mathrm{d}\eta_0(\theta)]^{-1} Q_{\phi_2\phi_2} e^{2i\omega_0\theta}, \\
h_{020}^{2k_2}(\theta) = &\ \tfrac{1}{2\sqrt{2}}[2i\omega_0 I - \textstyle\int_{-r}^0 e^{2i\omega_0\theta}\mathrm{d}\eta_{2k_2}(\theta)]^{-1} Q_{\phi_2\phi_2} e^{2i\omega_0\theta}, \\
h_{110}^{|k_1-k_2|}(\theta) = &\ \tfrac{1}{\sqrt{2}}[i\omega_0 I - \textstyle\int_{-r}^0 e^{i\omega_0\theta}\mathrm{d}\eta_{|k_1-k_2|}(\theta)]^{-1} Q_{\phi_1\phi_2} e^{i\omega_0\theta}, \\
h_{110}^{k_1+k_2}(\theta) = &\ \tfrac{1}{\sqrt{2}}[i\omega_0 I - \textstyle\int_{-r}^0 e^{i\omega_0\theta}\mathrm{d}\eta_{k_1+k_2}(\theta)]^{-1} Q_{\phi_1\phi_2} e^{i\omega_0\theta},\ \ h_{110}^0(\theta) \equiv 0 \\
h_{101}^{|k_1-k_2|}(\theta) = &\ \overline{h_{110}^{|k_1-k_2|}(\theta)},\ \ h_{101}^{k_1+k_2}(\theta) = \overline{h_{110}^{k_1+k_2}(\theta)},\ \ h_{101}^0(\theta) \equiv 0,\quad h_{002}^{2k_2}(\theta) = \overline{h_{020}^{2k_2}(\theta)}.
\end{aligned} \tag{4.11}$$

where $\theta \in [-r, 0]$. Thus we have the following result.

Proposition 4.5. *For $k_1 \neq k_2,\ k_1, k_2 \neq 0$ and Neumann boundary condition on spatial domain $\Omega = (0, l\pi),\ l > 0,$ the parameters $a_{11},\ a_{23},\ a_{111},\ a_{123},\ b_{12},\ b_{112},\ b_{223}$ in* (3.4) *are*

$$
\begin{aligned}
a_{11} = & \ 0,\ a_{23} = \tfrac{1}{\sqrt{2}}\delta(k_1-2k_2)\psi_1(0)Q_{\bar{\phi}_2\phi_2},\ b_{12} = \tfrac{1}{\sqrt{2}}\delta(k_1-2k_2)\psi_2(0)Q_{\phi_1\phi_2},\\
a_{111} = & \ \tfrac{1}{4}\psi_1(0)C_{\phi_1\phi_1\phi_1} + \tfrac{1}{2\omega_0}\delta(k_2-2k_1)\psi_1(0)\mathrm{Re}(\mathrm{i}Q_{\phi_1\phi_2}\psi_2(0))Q_{\phi_1\phi_1} + \tfrac{1}{\sqrt{2}}\psi_1(0)Q_{\phi_1}h_{200}^{2k_1},\\
a_{123} = & \ \psi_1(0)C_{\phi_1\phi_2\bar{\phi}_2} + \tfrac{1}{2\omega_0}\psi_1(0)\delta(k_1-2k_2)\mathrm{Re}(\mathrm{i}Q_{\phi_2\phi_2}\psi_2(0)Q_{\phi_1\bar{\phi}_2}) + \psi_1(0)[\tfrac{1}{\sqrt{2}}Q_{\phi_1}h_{011}^{2k_1}+\\
& Q_{\phi_2}(\tfrac{1}{\sqrt{2}}h_{101}^{|k_1-k_2|} + \tfrac{1}{\sqrt{2}}h_{101}^{k_1+k_2} + h_{101}^{0}) + Q_{\bar{\phi}_2}(\tfrac{1}{\sqrt{2}}h_{110}^{|k_1-k_2|} + \tfrac{1}{\sqrt{2}}h_{110}^{k_1+k_2} + h_{110}^{0})],\\
b_{112} = & \ \tfrac{1}{2}\psi_2(0)C_{\phi_1\phi_1\phi_2} + \tfrac{1}{4i\omega_0}\psi_2(0)[2\delta(k_2-2k_1)Q_{\phi_1\phi_1}\psi_1(0)\\
& +\delta(k_1-2k_2)Q_{\phi_1\bar{\phi}_2}\bar{\psi}_2(0)]Q_{\phi_1\phi_2}\\
& +\psi_2(0)[Q_{\phi_1}(\tfrac{1}{\sqrt{2}}h_{110}^{|k_1-k_2|} + \tfrac{1}{\sqrt{2}}h_{110}^{k_1+k_2} + h_{110}^{0}) + \tfrac{1}{\sqrt{2}}Q_{\phi_2}h_{200}^{2k_2}],\\
b_{223} = & \ \tfrac{3}{4}\psi_2(0)C_{\phi_2\phi_2\bar{\phi}_2} + \tfrac{1}{8i\omega_0}\delta(k_1-2k_2)\psi_2(0)Q_{\phi_1\bar{\phi}_2}\psi_1(0)Q_{\phi_2\phi_2}\\
& +\psi_2(0)[Q_{\phi_2}(h_{011}^{0} + \tfrac{1}{\sqrt{2}}h_{011}^{2k_2}) + Q_{\bar{\phi}_2}(h_{020}^{0} + \tfrac{1}{\sqrt{2}}h_{020}^{2k_2})].
\end{aligned}
\tag{4.12}
$$

5. An example

In this section we apply our above results to the Turing-Hopf bifurcation of a diffusive Schnakenberg chemical reaction system with gene expression time delay in the following form (see [47]):

$$
\begin{cases}
u_t(x,t) = \varepsilon d u_{xx}(x,t) + a - u(x,t) + u^2(x,t-\tau)v(x,t-\tau), & x\in(0,1),\ t>0,\\
v_t(x,t) = d v_{xx}(x,t) + b - u^2(x,t-\tau)v(x,t-\tau), & x\in(0,1),\ t>0,\\
u_x(0,t) = u_x(1,t) = v_x(0,t) = v_x(1,t) = 0, & t\ge 0,\\
u(x,t) = \phi(x,t)\ge 0, v(x,t) = \varphi(x,t)\ge 0, & (x,t)\in[0,1]\times[-\tau,0],
\end{cases}
\tag{5.1}
$$

System (5.1) has a unique positive constant steady state solution $E_* = (u_*, v_*)$, where

$$
u_* = a+b, \qquad v_* = \frac{b}{(a+b)^2}, \tag{5.2}
$$

Recall that $\mu_k = k^2\pi^2$, $k\in\mathbb{N}_0$ are the eigenvalues of the $-\Delta$ in the one dimensional spatial domain $(0,1)$. Then, a straightforward analysis shows that the eigenvalues of the linearized system are given by the roots of

$$
D_k(\lambda) := \lambda^2 + p_k\lambda + r_k + (s_k\lambda + q_k)e^{-\lambda\tau} = 0,\ \ k\in\mathbb{N}_0, \tag{5.3}
$$

where,

$$
\begin{aligned}
p_k =& (\varepsilon+1)dk^2\pi^2 + 1, \quad r_k = \varepsilon d^2k^4\pi^4 + dk^2\pi^2,\\
s_k =& u_*^2 - 2u_*v_*, \qquad\quad q_k = (\varepsilon u_*^2 - 2u_*v_*)dk^2\pi^2 + u_*^2.
\end{aligned}
\tag{5.4}
$$

By analyzing the characteristic equation (5.3) with $a=1,\ b=2,\ d=4$ (see [29, Theorem 2.12 and 2.15] for details on general results), we have

Theorem 5.1. *For system (5.1) with $a = 1$, $b = 2$, $d = 4$, there is a positive constant steady state $(u_*, v_*) = (3, 2/9)$, and there exists $\tau_* \approx 0.2014$, $\varepsilon_* \approx 0.0022$, $\omega_* \approx 7.6907$ such that*

1. *when $\tau = \tau_*$, $\varepsilon = \varepsilon_*$, $D_0(\lambda)$ has a pair of purely imaginary roots $\pm \mathrm{i}\omega_*$, $D_1(\lambda)$ has a simple zero root, with all other roots of $D_k(\lambda)$ having negative real parts $k \in \mathbb{N}_0$.*
2. *the system (5.1) undergoes $(1, 0)-$mode Turing-Hopf bifurcation near the constant steady state (u_*, v_*) at $\tau = \tau_*$, $\varepsilon = \varepsilon_*$.*
3. *the constant steady state (u_*, v_*) is locally asymptotically stable for the system (5.1) with $\tau \in [0, \tau_*)$ and $\varepsilon > \varepsilon_*$, and unstable for $0 < \varepsilon < \varepsilon_*$ or $\tau > \tau_*$.*

It follows from Theorem 5.1 that $k_1 = 1$ and $k_2 = 0$ at $\tau = \tau_*$, $\varepsilon = \varepsilon_*$, which corresponds to **Case (3)** in Section 4. We normalize the time delay τ in system (5.1) by time-rescaling $t \to t/\tau$, and translate (u_*, v_*) into the origin. We also introduce two bifurcation parameters $\alpha = (\alpha_1, \alpha_2)$ by setting

$$\tau = \tau_* + \alpha_1, \quad \varepsilon = \varepsilon_* + \alpha_2. \tag{5.5}$$

Then, system (5.1) is transformed into an abstract equation in $C([-1, 0], X)$:

$$\begin{aligned}\frac{d}{dt}U(t) = {} & L_0 U_t + D_0 \Delta U(t) + \frac{1}{2} L_1(\alpha) U_t + \frac{1}{2} D_1(\alpha) \Delta U(t) + \frac{1}{2!} Q(U_t, U_t) \\ & + \frac{1}{3!} C(U_t, U_t, U_t) + \dots,\end{aligned} \tag{5.6}$$

where

$$D_0 = d\tau_* \begin{pmatrix} \varepsilon_* & 0 \\ 0 & 1 \end{pmatrix}, \quad D_1(\alpha) = 2d \begin{pmatrix} \alpha_1 \varepsilon_* + \alpha_2 \tau_* & 0 \\ 0 & \alpha_1 \end{pmatrix},$$

$$L_0 X = \tau_* \begin{pmatrix} -x_1(0) + 2u_* v_* x_1(-1) + u_*^2 x_2(-1) \\ -(2u_* v_* x_1(-1) + u_*^2 x_2(-1)) \end{pmatrix},$$

$$L_1(\alpha) X = 2\alpha_1 \begin{pmatrix} -x_1(0) + 2u_* v_* x_1(-1) + u_*^2 x_2(-1) \\ -(2u_* v_* x_1(-1) + u_*^2 x_2(-1)) \end{pmatrix},$$

and

$$Q_{XY} = 2\tau_* [v_* x_1(-1) y_1(-1) + u_* (x_1(-1) y_2(-1) + x_2(-1) y_1(-1))] \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$C_{XYZ} = 2\tau_* [x_1(-1) y_1(-1) z_2(-1) + x_1(-1) y_2(-1) z_1(-1) + x_2(-1) y_1(-1) z_1(-1)] \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

with $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$.

By (2.9) and (2.10), we have

$$\begin{aligned}
&\phi_1(0)=\begin{pmatrix}1\\-0.0274\end{pmatrix},\quad \phi_2(0)=\begin{pmatrix}1\\-1+0.1298\mathrm{i}\end{pmatrix},\\
&\psi_1(0)=\frac{1}{1.1734}(1,0.1849),\quad \psi_2(0)=\frac{1}{-8.1518-6.9779\mathrm{i}}(1,6.7502-0.8761\mathrm{i}).
\end{aligned}\tag{5.7}$$

By (4.6), we obtain that

$$\begin{aligned}
&h^0_{200}(0)=\begin{pmatrix}-0.0062\\0.0004\end{pmatrix},\ h^0_{200}(-1)=\begin{pmatrix}-0.0055\\-0.0018\end{pmatrix},\ h^2_{200}(0)=h^2_{200}(-1)=\begin{pmatrix}0.4506\\-0.0038\end{pmatrix}\\
&h^0_{011}(0)=\begin{pmatrix}1.2336\\-0.0877\end{pmatrix},h^0_{011}(-1)=\begin{pmatrix}1.0906\\0.3504\end{pmatrix},h^2_{011}(0)=\begin{pmatrix}0\\0\end{pmatrix},h^2_{011}(-1)=\begin{pmatrix}0\\0\end{pmatrix},
\end{aligned}\tag{5.8}$$

and

$$\begin{aligned}
&h^0_{020}(0)=\begin{pmatrix}0.0761+0.0358\mathrm{i}\\-0.0748+0.0093\mathrm{i}\end{pmatrix},\ h^0_{020}(-1)=\begin{pmatrix}0.2954-0.1131\mathrm{i}\\-0.2848+0.1679\mathrm{i}\end{pmatrix},\\
&h^1_{110}(0)=\begin{pmatrix}0.1171+0.1850\mathrm{i}\\-0.0029-0.1255\mathrm{i}\end{pmatrix},\ h^1_{110}(-1)=\begin{pmatrix}-0.3783-0.5733\mathrm{i}\\-0.1100+0.0128\mathrm{i}\end{pmatrix},\\
&h^1_{101}=\overline{h^1_{110}},\quad h^0_{002}=\overline{h^0_{020}}.
\end{aligned}\tag{5.9}$$

Substituting the above calculated values into the expression (4.7), we obtain the coefficients of the normal form (3.4) as follows

$$\begin{aligned}
&a_1(\alpha)=-0.0009\alpha_1-6.7762\alpha_2,\ b_2(\alpha)=(3.5818+2.2515\mathrm{i})\alpha_1,\\
&a_{11}=a_{23}=b_{12}=0,\\
&a_{111}=-9.4377\times10^{-4},\qquad b_{112}=0.0403+0.1213\mathrm{i},\\
&a_{123}=-0.4782,\qquad b_{223}=-0.2553-0.7712\mathrm{i}.
\end{aligned}\tag{5.10}$$

Thus, in the corresponding planar system (3.10), we have that

$$\begin{aligned}
&\varepsilon_1(\alpha)=3.5818\alpha_1,\ \varepsilon_2(\alpha)=-0.0009\alpha_1-6.7762\alpha_2,\\
&b_0=-42.7011,\ c_0=1.8735,\ d_0=1,\ \mathrm{sign}(\mathrm{Re}(b_{223}))=-1.
\end{aligned}\tag{5.11}$$

Therefore the Case III in Table 1 occurs, and we find that the bifurcation critical lines in Fig. 1 are, respectively,

$$\begin{aligned}
&L_1:\tau=\tau_*,\ \varepsilon>\varepsilon_*,\ L_2:\varepsilon=\varepsilon_*-0.00013(\tau-\tau_*),\ \tau>\tau_*,\\
&L_3:\varepsilon=\varepsilon_*-0.9916(\tau-\tau_*),\ \tau>\tau_*,\ L_4:\tau=\tau_*,\ \varepsilon<\varepsilon_*,\\
&L_5:\varepsilon=\varepsilon_*+0.0111(\tau-\tau_*),\ \tau<\tau_*,\ L_6:\varepsilon=\varepsilon_*-0.00013(\tau-\tau_*),\ \tau<\tau_*.
\end{aligned}$$

Taking notice of $\mathrm{sign}(\mathrm{Re}(b_{223})) = -1$ in the coordinate transformation (3.9), and from phase portraits in Fig. 1 and the analysis in [1, Section 4], we have the following results.

Theorem 5.2. *Let $a = 1$, $b = 2$ and $d = 4$. At the constant positive steady state $(u_*, v_*) = (3, 2/9)$, near the $(1, 0)$-mode Turing-Hopf bifurcation point $(\tau_*, \varepsilon_*) \approx (0.2014, 0.0022)$, with frequency $\omega_* = 7.6907$, the system (5.1) has the following dynamical behavior when the parameter pair (τ, ε) is sufficiently close to (τ_*, ε_*): (see Fig. 1)*

(1) *When $\varepsilon > \varepsilon_* - 0.0013(\tau - \tau_*)$ and $\tau < \tau_*$ (that is $(\tau, \varepsilon) \in D_1$), the constant steady state (u_*, v_*) is locally asymptotically stable; and a $0-$mode Hopf bifurcation occurs at (u_*, v_*) when $(\tau,\ \varepsilon)$ crosses L_1 transversally.*
(2) *When $\varepsilon > \varepsilon_* - 0.0013(\tau - \tau_*)$ and $\tau > \tau_*$ (that is $(\tau, \varepsilon) \in D_2$), the constant steady state (u_*, v_*) is unstable and there exists a locally asymptotically stable spatially homogeneous periodic orbit which bifurcates from (u_*, v_*); and a $1-$mode Turing bifurcation occurs at (u_*, v_*) when $(\tau,\ \varepsilon)$ crosses L_2 transversally.*
(3) *When $\varepsilon_* - 0.0013(\tau - \tau_*) > \varepsilon > \varepsilon_* - 0.9916(\tau - \tau_*)$ and $\tau > \tau_*$ (that is $(\tau, \varepsilon) \in D_3$), the constant steady state (u_*, v_*) is unstable, there are two unstable spatially non-homogeneous steady states which bifurcate from (u_*, v_*), and the spatially homogeneous periodic orbit is locally asymptotically stable; and a $1-$mode Turing bifurcation occurs at the spatially homogeneous periodic orbit when $(\tau,\ \varepsilon)$ crosses L_3 transversally.*
(4) *When $\varepsilon < \varepsilon_* - 0.9916(\tau - \tau_*)$ and $\tau > \tau_*$ (that is $(\tau, \varepsilon) \in D_4$), the constant steady state (u_*, v_*) and the two spatially non-homogeneous steady state solutions are all unstable, the spatially homogeneous periodic orbit is also unstable, and there are two locally asymptotically stable spatially non-homogeneous periodic orbits which bifurcate from the spatially homogeneous periodic orbit, whose linear main parts are approximately*

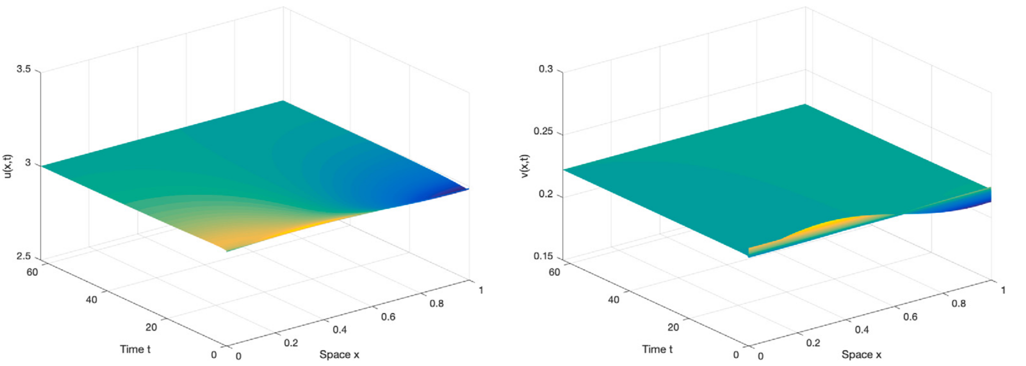
$$E_* + \rho\phi_2(0)e^{\mathrm{i}\tau_*\omega_* t} + \bar{\rho}\bar{\phi}_2(0)e^{-\mathrm{i}\tau_*\omega_* t} \pm h\phi_1(0)\cos(\pi x), \tag{5.12}$$

where ρ and h are some constants; and a $0-$mode Hopf bifurcation occurs at (u_*, v_*) when $(\tau,\ \varepsilon)$ crosses L_4 transversally.
(5) *When $\varepsilon < \varepsilon_* + 0.0111(\tau - \tau_*)$ and $\tau < \tau_*$ (that is $(\tau, \varepsilon) \in D_5$), the constant steady state (u_*, v_*) and the two spatially non-homogeneous steady state solutions are all unstable, there is no spatially homogeneous periodic orbit (disappearing through the Hopf bifurcation on L_4), and two spatially non-homogeneous periodic orbits are locally asymptotically stable; and a $0-$mode Hopf bifurcation occurs at each of two spatially non-homogeneous steady state solutions when $(\tau,\ \varepsilon)$ crosses L_5 transversally.*
(6) *When $\varepsilon_* - 0.0013(\tau - \tau_*) > \varepsilon > \varepsilon_* + 0.0111(\tau - \tau_*)$ and $\tau < \tau_*$ (that is $(\tau, \varepsilon) \in D_6$), the constant steady state (u_*, v_*) is unstable, the two spatially non-homogeneous steady state solutions are locally asymptotically stable, and there is no spatially non-homogeneous periodic orbits (disappearing through the Hopf bifurcations on L_5); and a $1-$mode Turing bifurcation occurs at (u_*, v_*) when $(\tau,\ \varepsilon)$ crosses L_6 transversally.*

We summarize the numbers of spatialtemporal patterned solutions (steady states or periodic orbits) in each parameter region D_i $(1 \le i \le 6)$ in Table 2. The Morse index of a steady state solution is defined to be the number of positive eigenvalues of associated linearized equation, and the Morse index of a periodic orbit is defined to be the number of Floquet multipliers which are greater than 1. A steady state or a periodic orbit is locally asymptotically stable if its Morse

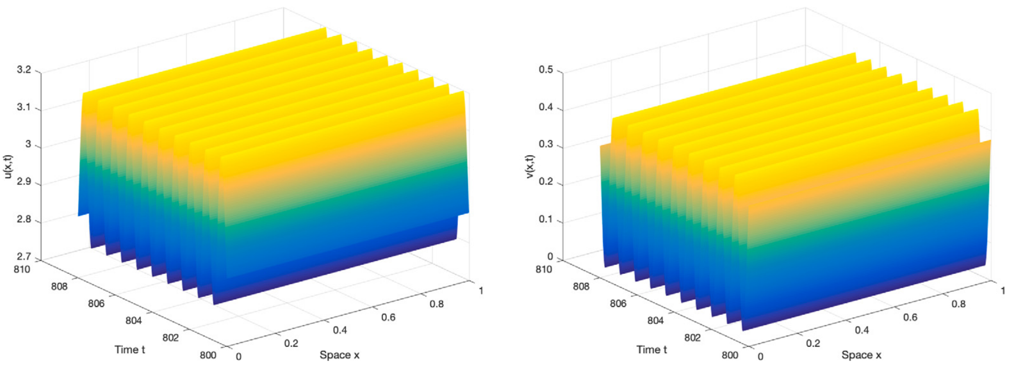
Table 2
The number of patterned solutions of (5.1) in each parameter regions D_i $(1 \leq i \leq 6)$. Here $j(k)$ means the number of specific patterned solutions is j, and the Morse index of each such patterned solution is k.

	D_1	D_2	D_3	D_4	D_5	D_6
homogeneous steady state	1(0)	1(2)	1(3)	1(3)	1(1)	1(1)
non-homogeneous steady state	0	0	2(2)	2(2)	2(2)	2(0)
homogeneous periodic orbit	0	1(0)	1(0)	1(1)	0	0
non-homogeneous periodic orbit	0	0	0	2(0)	2(0)	0



(a) $u(x,t)$ (b) $v(x,t)$

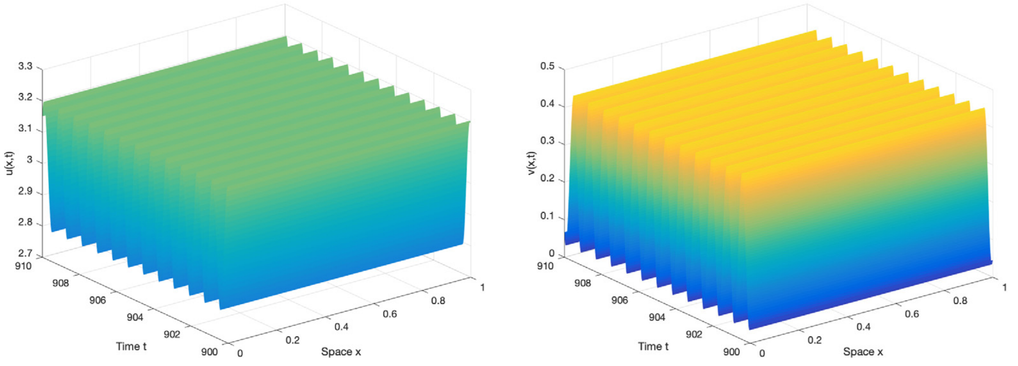
Fig. 2. The constant steady state is locally asymptotically stable for $(\tau, \varepsilon) = (0.1, 0.004) \in D_1$. The initial functions are $\phi(x,t) = u_* + 0.01 \sin 5x$, $\psi(x,t) = v_* + 0.01 \sin 5x$, for $(x,t) \in [0,1] \times [-0.1, 0]$.



(a) $u(x,t)$ (b) $v(x,t)$

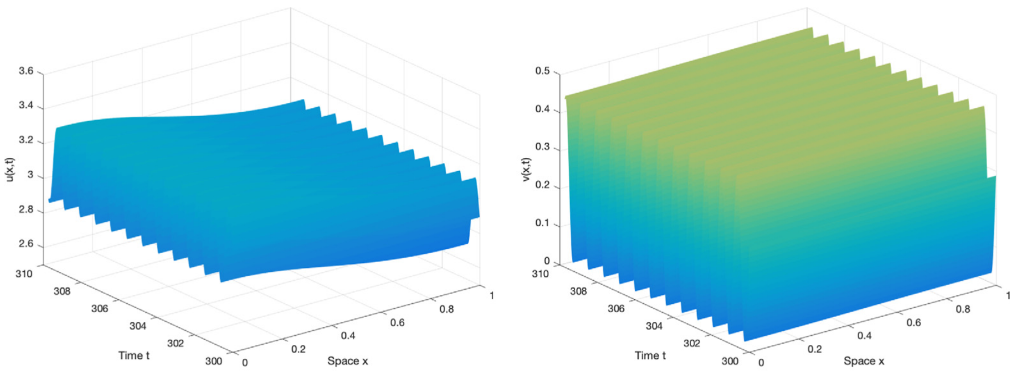
Fig. 3. There is a stable spatially homogeneous periodic orbit for $(\tau, \varepsilon) = (0.202, 0.0024) \in D_2$. The initial functions are $\phi(x,t) = u_* + 0.01 \sin 5x$, $\psi(x,t) = v_* + 0.01 \sin 5x$, for $(x,t) \in [0,1] \times [-0.202, 0]$.

index is 0. Hence the stable pattern for D_1 is the constant steady state (u_*, v_*); the stable pattern for D_2 and D_3 is the spatially homogeneous periodic orbit; a pair of spatially non-homogeneous periodic orbits are the stable patterns for D_4 and D_5; and a pair of spatially non-homogeneous steady state solutions are the stable patterns for D_6, see Figures 2–7.

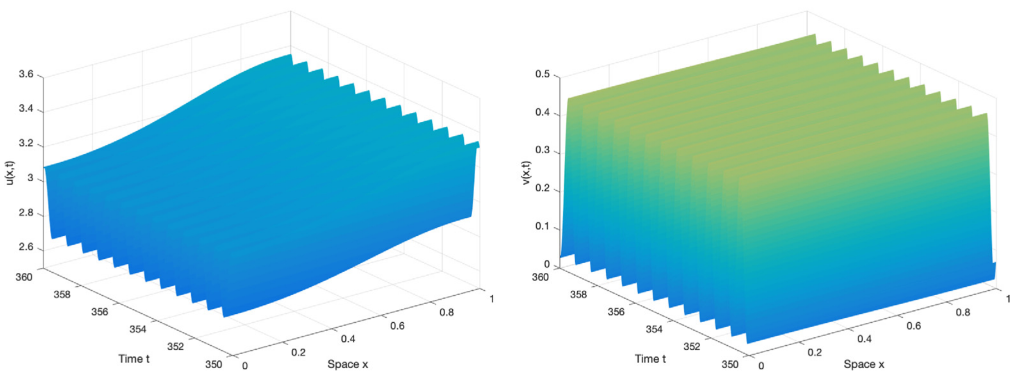


(a) $u(x,t)$ (b) $v(x,t)$

Fig. 4. There is a stable spatially homogeneous periodic orbit for $(\tau, \varepsilon) = (0.202, 0.002) \in D_3$. The initial functions are $\phi(x,t) = u_* + 0.01 \sin 5x$, $\psi(x,t) = v_* + 0.01 \sin 5x$, for $(x,t) \in [0,1] \times [-0.202, 0]$.

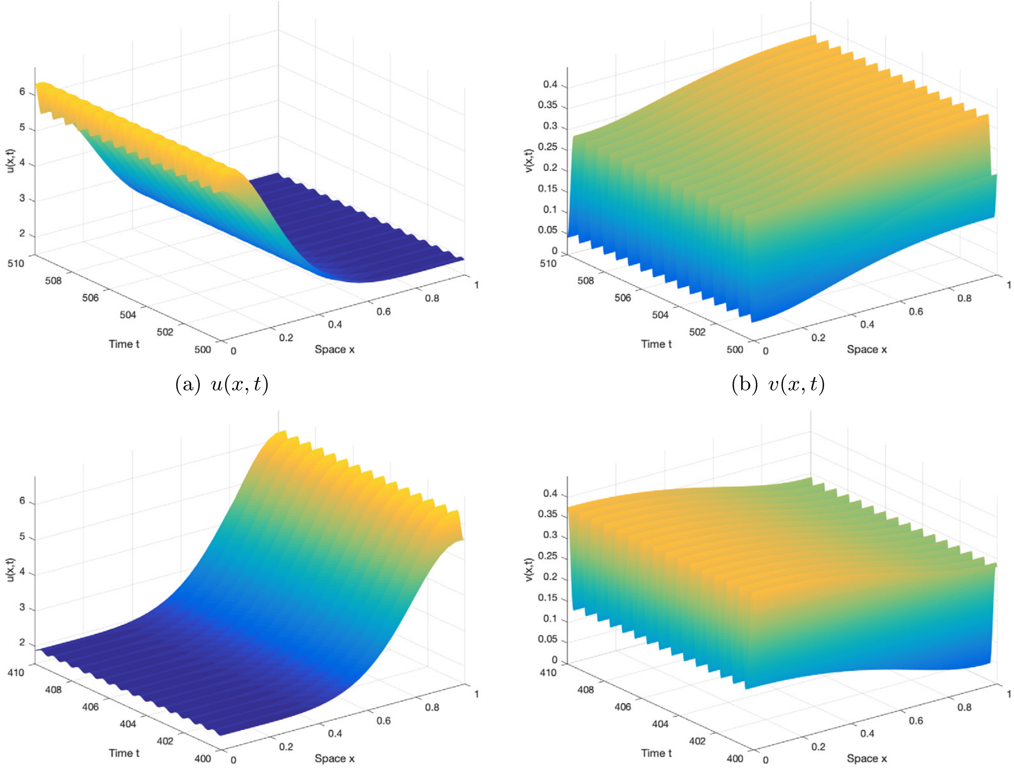


(a) $u(x,t)$ (b) $v(x,t)$



(c) $u(x,t)$ (d) $v(x,t)$

Fig. 5. There are a pair of stable spatially non-homogeneous periodic orbits for $(\tau, \varepsilon) = (0.202, 0.0015) \in D_4$. The initial functions of $(a) - (b)$ are $\phi(x,t) = u_* + 0.01 \sin 5x$, $\psi(x,t) = v_* + 0.01 \sin 5x$, and $(c) - (d)$ are $\phi(x,t) = u_* - 0.01 \sin 5x$, $\psi(x,t) = v_* - 0.01 \sin 5x$, for $(x,t) \in [0,1] \times [-0.202, 0]$.



(a) $u(x,t)$ (b) $v(x,t)$ (c) $u(x,t)$ (d) $v(x,t)$

Fig. 6. There are a pair of stable spatially non-homogeneous periodic orbits for $(\tau, \varepsilon) = (0.18, 0.0015) \in D_5$. The initial functions of $(a) - (b)$ are $\phi(x,t) = u_* + 0.01 \sin 5x$, $\psi(x,t) = v_* + 0.01 \sin 5x$, and $(c) - (d)$ are $\phi(x,t) = u_* - 0.01 \sin 5x$, $\psi(x,t) = v_* - 0.01 \sin 5x$, for $(x,t) \in [0,1] \times [-0.18, 0]$.

6. Proof of Theorem 3.1

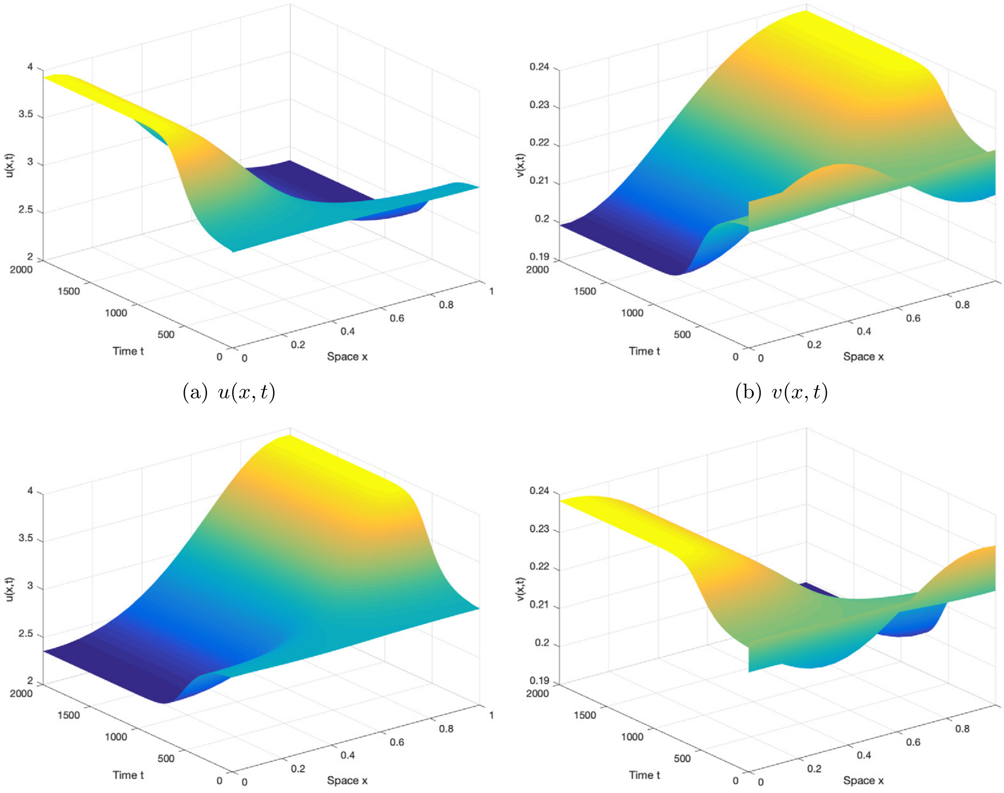
In this section we give the proof of the main result Theorem 3.1. From (3.1) and (3.2), we denote that, for $v \in \mathcal{C}$,

$$\begin{aligned} &F_2(v,\alpha) = L_1(\alpha)v + D_1(\alpha)\Delta v(0) + Q(v,v), \\ &F_3(v,0) = C(v,v,v). \end{aligned} \tag{6.1}$$

By doing Taylor expansions for the nonlinear terms in (2.17) at $(z,y,\alpha) = (0,0,0)$, we have

$$\begin{aligned} \dot{z} &= Bz + \frac{1}{2!} f_2^1(z,y,\alpha) + \frac{1}{3!} f_3^1(z,y,\alpha) + \cdots, \\ \frac{\mathrm{d}}{\mathrm{d}t} y &= A_1 y + \frac{1}{2!} f_2^2(z,y,\alpha) + \frac{1}{3!} f_3^2(z,y,\alpha) + \cdots, \end{aligned} \tag{6.2}$$

where $B = \mathrm{diag}(0, i\omega_0, -i\omega_0)$, $f_j^i(z,y,\alpha)$ $(i = 1,2)$ are the homogeneous polynomials of degree j in variables (z,y,α), $z = (z_1, z_2, \bar{z}_2) \in \mathbb{C}^3$, $y \in \mathcal{Q}^1$, $\alpha \in V$, $j \geq 2$, $j \in \mathbb{N}$. For the



(a) $u(x,t)$ (b) $v(x,t)$

(c) $u(x,t)$ (d) $v(x,t)$

Fig. 7. There are a pair of stable spatially non-homogeneous steady state solutions for $(\tau, \varepsilon) = (0.1, 0.00225) \in D_6$. The initial functions of $(a)-(b)$ are $\phi(x,t) = u_* + 0.01\sin 5x$, $\psi(x,t) = v_* + 0.01\sin 5x$, and $(c)-(d)$ are $\phi(x,t) = u_* - 0.01\sin 5x$, $\psi(x,t) = v_* - 0.01\sin 5x$, for $(x,t) \in [0,1] \times [-0.1, 0]$.

purposes of this article, we are interested in three terms:

$$f_2^1(z,y,\alpha) \triangleq \begin{pmatrix} f_2^{11}(z,y,\alpha) \\ f_2^{12}(z,y,\alpha) \\ \overline{f_2^{12}}(z,y,\alpha) \end{pmatrix}, \quad f_3^1(z,0,0) \triangleq \begin{pmatrix} f_3^{11}(z,0,0) \\ f_3^{12}(z,0,0) \\ \overline{f_3^{12}}(z,0,0) \end{pmatrix}, \tag{6.3}$$

$$f_2^2(z,0,0) = (X_0 - \Phi\Psi(0))F_2(\phi_1 z_1 \beta_{k_1} + (\phi_2 z_2 + \bar{\phi}_2 \bar{z}_2)\beta_{k_2}, 0). \tag{6.4}$$

where

$$f_2^{1i}(z,y,\alpha) = \psi_i(0)\langle F_2(\phi_1 z_1 \beta_{k_1} + (\phi_2 z_2 + \bar{\phi}_2 \bar{z}_2)\beta_{k_2} + y, \alpha), \beta_{k_i}\rangle, \quad i = 1, 2, \tag{6.5}$$

$$f_3^{1i}(z,0,0) = \psi_i(0)\langle F_3(\phi_1 z_1 \beta_{k_1} + (\phi_2 z_2 + \bar{\phi}_2 \bar{z}_2)\beta_{k_2}, 0), \beta_{k_i}\rangle, \quad i = 1, 2. \tag{6.6}$$

Noticing that $L_1(\alpha)$, $D_1(\alpha)$ are linear, Q, C are symmetric multilinear, and together with $\Delta\beta_{k_i} = -\mu_{k_i}\beta_{k_i}$, for $i = 1, 2$, we obtain that

$$
\begin{aligned}
f_2^{11}(z,y,\alpha)= \;& \psi_1(0)[L_1(\alpha)\phi_1 z_1 - \mu_{k_1} D_1(\alpha)\phi_1(0)z_1 + Q_{\phi_1\phi_1} z_1^2\langle \beta_{k_1}^2, \beta_{k_1}\rangle + 2(Q_{\phi_1\phi_2} z_1 z_2 \\
&+ Q_{\phi_1\bar{\phi}_2} z_1\bar{z}_2)\langle \beta_{k_1}\beta_{k_2}, \beta_{k_1}\rangle + (Q_{\phi_2\phi_2} z_2^2 + 2Q_{\phi_2\bar{\phi}_2} z_2\bar{z}_2 + Q_{\bar{\phi}_2\bar{\phi}_2}\bar{z}_2^2)\langle \beta_{k_2}^2, \beta_{k_1}\rangle \\
&+ \langle L_1(\alpha)y, \beta_{k_1}\rangle + \langle 2Q(\phi_1 z_1\beta_{k_1} + (\phi_2 z_2 + \bar{\phi}_2\bar{z}_2)\beta_{k_2}, y) + Q(y,y), \beta_{k_1}\rangle \\
&+ \langle D_1(\alpha)\Delta y(0), \beta_{k_1}\rangle],
\end{aligned}
\tag{6.7}
$$

$$
\begin{aligned}
f_2^{12}(z,y,\alpha)= \;& \psi_2(0)[L_1(\alpha)\phi_2 z_2 + L_1(\alpha)\bar{\phi}_2\bar{z}_2 - \mu_{k_2} D_1(\alpha)(\phi_2(0)z_2 + \bar{\phi}_2(0)\bar{z}_2) \\
&+ Q_{\phi_1\phi_1} z_1^2\langle \beta_{k_1}^2, \beta_{k_2}\rangle + 2(Q_{\phi_1\phi_2} z_1 z_2 + Q_{\phi_1\bar{\phi}_2} z_1\bar{z}_2)\langle \beta_{k_1}\beta_{k_2}, \beta_{k_2}\rangle + (Q_{\phi_2\phi_2} z_2^2 \\
&+ 2Q_{\phi_2\bar{\phi}_2} z_2\bar{z}_2 + Q_{\bar{\phi}_2\bar{\phi}_2}\bar{z}_2^2)\langle \beta_{k_2}^2, \beta_{k_2}\rangle + \langle L_1(\alpha)y, \beta_{k_2}\rangle + \langle 2Q(\phi_1 z_1\beta_{k_1} \\
&+ (\phi_2 z_2 + \bar{\phi}_2\bar{z}_2)\beta_{k_2}, y) + Q(y,y), \beta_{k_2}\rangle + \langle D_1(\alpha)\Delta y(0), \beta_{k_2}\rangle],
\end{aligned}
\tag{6.8}
$$

$$
\begin{aligned}
f_3^{1i}(z,0,0)= \;& \psi_i(0)[C_{\phi_1\phi_1\phi_1} z_1^3\langle \beta_{k_1}^3, \beta_{k_i}\rangle + (C_{\phi_2\phi_2\phi_2} z_2^3 + C_{\bar{\phi}_2\bar{\phi}_2\bar{\phi}_2}\bar{z}_2^3 + 3C_{\phi_2\phi_2\bar{\phi}_2} z_2^2\bar{z}_2 \\
&+ 3C_{\phi_2\bar{\phi}_2\bar{\phi}_2} z_2\bar{z}_2^2)\langle \beta_{k_2}^3, \beta_{k_i}\rangle + 3(C_{\phi_1\phi_1\phi_2} z_1^2 z_2 + C_{\phi_1\phi_1\bar{\phi}_2} z_1^2\bar{z}_2)\langle \beta_{k_1}^2\beta_{k_2}, \beta_{k_i}\rangle \\
&+ 3(C_{\phi_1\phi_2\phi_2} z_1 z_2^2 + C_{\phi_1\bar{\phi}_2\bar{\phi}_2} z_1\bar{z}_2^2 + 2C_{\phi_1\phi_2\bar{\phi}_2} z_1 z_2\bar{z}_2)\langle \beta_{k_1}\beta_{k_2}^2, \beta_{k_i}\rangle],\ i=1,2,
\end{aligned}
\tag{6.9}
$$

$$
\begin{aligned}
f_2^2(z,0,0)(\theta)= \;& \delta(\theta)[Q_{\phi_1\phi_1} z_1^2\beta_{k_1}^2 + 2(Q_{\phi_1\phi_2} z_1 z_2 + Q_{\phi_1\bar{\phi}_2} z_1\bar{z}_2)\beta_{k_1}\beta_{k_2} \\
&+ (Q_{\phi_2\phi_2} z_2^2 + 2Q_{\phi_2\bar{\phi}_2} z_2\bar{z}_2 + Q_{\bar{\phi}_2\bar{\phi}_2}\bar{z}_2^2)\beta_{k_2}^2] - \{\phi_1(\theta)\psi_1(0)[Q_{\phi_1\phi_1} z_1^2\langle \beta_{k_1}^2, \beta_{k_1}\rangle \\
&+ 2(Q_{\phi_1\phi_2} z_1 z_2 + Q_{\phi_1\bar{\phi}_2} z_1\bar{z}_2)\langle \beta_{k_1}\beta_{k_2}, \beta_{k_1}\rangle + (Q_{\phi_2\phi_2} z_2^2 + 2Q_{\phi_2\bar{\phi}_2} z_2\bar{z}_2 \\
&+ Q_{\bar{\phi}_2\bar{\phi}_2}\bar{z}_2^2)\langle \beta_{k_2}^2, \beta_{k_1}\rangle]\beta_{k_1} + (\phi_2(\theta)\psi_2(0) + \bar{\phi}_2(\theta)\bar{\psi}_2(0))[Q_{\phi_1\phi_1} z_1^2\langle \beta_{k_1}^2, \beta_{k_2}\rangle \\
&+ 2(Q_{\phi_1\phi_2} z_1 z_2 + Q_{\phi_1\bar{\phi}_2} z_1\bar{z}_2)\langle \beta_{k_1}\beta_{k_2}, \beta_{k_2}\rangle + (Q_{\phi_2\phi_2} z_2^2 + 2Q_{\phi_2\bar{\phi}_2} z_2\bar{z}_2 \\
&+ Q_{\bar{\phi}_2\bar{\phi}_2}\bar{z}_2^2)\langle \beta_{k_2}^2, \beta_{k_2}\rangle]\beta_{k_2}\}, \qquad \text{for } \theta \in [-r,0],
\end{aligned}
\tag{6.10}
$$

where $\delta(\theta) = 0$, for $\theta \in [-r, 0)$, $\delta(0) = 1$.

Now we first obtain the normal form of (2.1) up to the quadratic terms.

Lemma 6.1. *Assume that* **(H1)-(H4)** *are satisfied. Then the normal form of* (2.1) *up to the quadratic terms on the center manifold at* $\alpha = 0$ *is*

$$
\dot{z} = Bz + \frac{1}{2} g_2^1(z,0,\alpha) + h.o.t.. \tag{6.11}
$$

Here

$$
\begin{aligned}
g_2^1(z,0,\alpha)= \;& 2a_1(\alpha)z_1 + \psi_1(0)[Q_{\phi_1\phi_1} z_1^2\langle \beta_{k_1}^2, \beta_{k_1}\rangle + 2Q_{\phi_2\bar{\phi}_2} z_2\bar{z}_2\langle \beta_{k_2}^2, \beta_{k_1}\rangle]e_1 \\
&+ 2b_2(\alpha)z_2 + \psi_2(0)[2Q_{\phi_1\phi_2} z_1 z_2\langle \beta_{k_1}\beta_{k_2}, \beta_{k_2}\rangle]e_2 \\
&+ \overline{b_2(\alpha)}z_2 + \bar{\psi}_2(0)[2Q_{\phi_1\bar{\phi}_2} z_1\bar{z}_2\langle \beta_{k_1}\beta_{k_2}, \beta_{k_2}\rangle]e_3,
\end{aligned}
\tag{6.12}
$$

with

$$\begin{aligned} a_1(\alpha) = & \quad \tfrac{1}{2}\psi_1(0)(L_1(\alpha)\phi_1 - \mu_{k_1} D_1(\alpha)\phi_1(0)), \\ b_2(\alpha) = & \quad \tfrac{1}{2}\psi_2(0)(L_1(\alpha)\phi_2 - \mu_{k_2} D_1(\alpha)\phi_2(0)), \end{aligned} \tag{6.13}$$

and $h.o.t.$ stands for higher order terms.

Proof. Let M_2^1 denote the operator

$$M_2^1 : V_2^5(\mathbb{C}^3) \to V_2^5(\mathbb{C}^3), \text{ and } (M_2^1 p)(z, \alpha) = D_z p(z, \alpha)Bz - Bp(z, \alpha), \tag{6.14}$$

where $V_2^5(\mathbb{C}^3)$ is the linear space of the second order homogeneous polynomials in five variables $(z_1, z_2, \bar{z}_2, \alpha_1, \alpha_2)$ with coefficients in $\mathbb{C}^3$, $z = (z_1, z_2, \bar{z}_2)$, $\alpha = (\alpha_1, \alpha_2)$ and $B = diag(0, \mathrm{i}\omega_0, -\mathrm{i}\omega_0)$. One may choose the decomposition

$$V_2^5(\mathbb{C}^3) = \mathrm{Im}(M_2^1) \oplus \mathrm{Im}(M_2^1)^c$$

with complementary space $\mathrm{Im}(M_2^1)^c$ spanned by the elements

$$z_1^2 e_1, \ z_2\bar{z}_2 e_1, \ z_1\alpha_i e_1, \ z_1 z_2 e_2, \ z_2\alpha_i e_2, \ z_1\bar{z}_2 e_3, \ \bar{z}_2\alpha_i e_3, \ i = 1, 2, \tag{6.15}$$

where e_1, e_2, e_3 denote the natural basis of $\mathbb{R}^3$. By the projection mapping which was presented in [15],

$$g_2^1(z, 0, \alpha) = Proj_{(\mathrm{Im}(M_2^1))^c} f_2^1(z, 0, \alpha), \tag{6.16}$$

we get (6.12), (6.13) and that completes the proof. □

Let $V_2^3(\mathbb{C}^3 \times \mathrm{Ker}\pi)$ be the space of homogeneous polynomials of degree 2 in the variables $z = (z_1, z_2, \bar{z}_2)$ with coefficients in $\mathbb{C}^3 \times \mathrm{Ker}\pi$. Let the operator M_2^1 defined in (6.14) be restricted in $V_2^3(\mathbb{C}^3)$, as

$$M_2^1 : V_2^3(\mathbb{C}^3) \longmapsto V_2^3(\mathbb{C}^3), \text{ and } (M_2^1 p)(z) = D_z p(z)Bz - Bp(z), \tag{6.17}$$

and define the operator M_2^2 by

$$M_2^2 : \ V_2^3(\mathcal{Q}^1) \subset V_2^3(\mathrm{Ker}\pi) \longmapsto V_2^3(\mathrm{Ker}\pi), \text{ and } (M_2^2 h)(z) = D_z h(z)Bz - A_1(h(z)), \tag{6.18}$$

then we have the following decompositions:

$$\begin{aligned} V_2^3(\mathbb{C}^3) = \mathrm{Im}(M_2^1) \oplus \mathrm{Im}(M_2^1)^c, \ \ V_2^3(\mathbb{C}^3) = \mathrm{Ker}(M_2^1) \oplus \mathrm{Ker}(M_2^1)^c, \\ V_2^3(\mathrm{Ker}\pi) = \mathrm{Im}(M_2^2) \oplus \mathrm{Im}(M_2^2)^c, \ \ V_2^3(\mathcal{Q}^1) = \mathrm{Ker}(M_2^2) \oplus \mathrm{Ker}(M_2^2)^c. \end{aligned} \tag{6.19}$$

The projection associated with the preceding decomposition of $V_2^3(\mathbb{C}^3) \times V_2^3(\mathrm{Ker}\pi)$ over $\mathrm{Im}(M_2^1) \times \mathrm{Im}(M_2^2)$ is denoted by $P_{I,2} = (P_{I,2}^1, P_{I,2}^2)$.

Following [15], we set

$$U_2(z) = \begin{pmatrix} U_2^1 \\ U_2^2 \end{pmatrix} = M_2^{-1} P_{I,2} f_2(z, 0, 0), \tag{6.20}$$

and by a transformation of variables

$$(z, y) = (\hat{z}, \hat{y}) + \frac{1}{2!} U_2(\hat{z}), \tag{6.21}$$

the first equation of (6.2) becomes, after dropping the hats,

$$\dot{z} = Bz + \frac{1}{2!} g_2^1(z, 0, \alpha) + \frac{1}{3!} \overline{f}_3^1(z, 0, 0) + \cdots, \tag{6.22}$$

where

$$\overline{f}_3^1(z, 0, 0) = f_3^1(z, 0, 0) + \frac{3}{2}[(D_{(z,y)} f_2^1(z, y, 0))_{y=0} U_2(z) - D_z U_2^1(z) g_2^1(z, 0, 0)]. \tag{6.23}$$

To complete the proof of Theorem 3.1, we only need to calculate the third order term $g_3^1(z, 0, 0)$ in the normal form (3.3). It is divided into three steps.

Step 1. Computations of U_2^1.

Lemma 6.2. *Assume that* **(H1)-(H4)** *are satisfied. Then the formula of U_2^1 in (6.20) is*

$$\begin{aligned} & i\omega_0 U_2^1(z) \\ =& \psi_1(0)[2(Q_{\phi_1\phi_2} z_1 z_2 - Q_{\phi_1\bar{\phi}_2} z_1 \bar{z}_2)\langle \beta_{k_1}\beta_{k_2}, \beta_{k_1}\rangle + \frac{1}{2}(Q_{\phi_2\phi_2} z_2^2 - Q_{\bar{\phi}_2\bar{\phi}_2} \bar{z}_2^2)\langle \beta_{k_2}^2, \beta_{k_1}\rangle] e_1 \\ & + \psi_2(0)[-Q_{\phi_1\phi_1} z_1^2 \langle \beta_{k_1}^2, \beta_{k_2}\rangle - Q_{\phi_1\bar{\phi}_2} z_1 \bar{z}_2 \langle \beta_{k_1}\beta_{k_2}, \beta_{k_2}\rangle \\ & + (Q_{\phi_2\phi_2} z_2^2 - 2Q_{\phi_2\bar{\phi}_2} z_2\bar{z}_2 - \frac{1}{3} Q_{\bar{\phi}_2\bar{\phi}_2} \bar{z}_2^2)\langle \beta_{k_2}^2, \beta_{k_2}\rangle] e_2 \\ & - \bar{\psi}_2(0)[-Q_{\phi_1\phi_1} z_1^2 \langle \beta_{k_1}^2, \beta_{k_2}\rangle - Q_{\phi_1\phi_2} z_1 z_2 \langle \beta_{k_1}\beta_{k_2}, \beta_{k_2}\rangle \\ & + (Q_{\bar{\phi}_2\bar{\phi}_2} \bar{z}_2^2 - 2Q_{\phi_2\bar{\phi}_2} z_2\bar{z}_2 - \frac{1}{3} Q_{\phi_2\phi_2} z_2^2)\langle \beta_{k_2}^2, \beta_{k_2}\rangle] e_3. \end{aligned} \tag{6.24}$$

Proof. Since $U_2^1 \in \mathrm{Ker}(M_2^1)^c$, and $\mathrm{Ker}(M_2^1)^c$ is spanned by

$$\begin{aligned} & z_2^2 e_1,\ \bar{z}_2^2 e_1,\ z_1 z_2 e_1,\ z_1 \bar{z}_2 e_1,\ z_1^2 e_2,\ z_2^2 e_2,\ \bar{z}_2^2 e_2, \\ & z_1 \bar{z}_2 e_2,\ z_2 \bar{z}_2 e_2,\ z_1^2 e_3,\ z_2^2 e_3,\ \bar{z}_2^2 e_3,\ z_1 z_2 e_3,\ z_2 \bar{z}_2 e_3, \end{aligned} \tag{6.25}$$

The above elements are mapped by M_2^1 to, respectively,

$$\begin{aligned} & 2i\omega_0 z_2^2 e_1, \quad -2i\omega_0 \bar{z}_2^2 e_1, \quad i\omega_0 z_1 z_2 e_1, \quad -i\omega_0 z_1 \bar{z}_2 e_1, \quad -i\omega_0 z_1^2 e_2, \quad i\omega_0 z_2^2 e_2, \quad -3i\omega_0 \bar{z}_2^2 e_2, \\ & -2i\omega_0 z_1 \bar{z}_2 e_2, \quad -i\omega_0 z_2 \bar{z}_2 e_2, \quad i\omega_0 z_1^2 e_3, \quad 3i\omega_0 z_2^2 e_3, \quad -i\omega_0 \bar{z}_2^2 e_3, \quad 2i\omega_0 z_1 z_2 e_3, \quad i\omega_0 z_2 \bar{z}_2 e_3. \end{aligned}$$

Then, by (6.20) and (6.7), the expression (6.24) of U_2^1 is obtained and the proof is completed. □

Step 2. Computations of U_2^2.

We know that

$$\frac{1}{2!}U_2^2 \triangleq h(z) = (h^{(1)}(z),\ h^{(2)}(z), \cdots,\ h^{(m)}(z))^{\mathrm{T}} \in V_2^3(\mathcal{Q}^1) \tag{6.26}$$

is the unique solution of the equation

$$(M_2^2 h)(z) = \frac{1}{2!} f_2^2(z, 0, 0). \tag{6.27}$$

Thus, by (6.18) and the definition of A_1, we have

$$D_z h(z)Bz - \dot{h}(z) + X_0[\dot{h}(z)(0) - L_0 h(z) - D_0 \Delta h(z)(0)] = \frac{1}{2!} f_2^2(z, 0, 0). \tag{6.28}$$

where $\dot{h}$ denotes the derivative of $h(z)(\theta)$ respective to θ. Expressing $h(z)$ in the general monomial form, we have

$$\begin{aligned} h(z)(\theta) =& h_{200}(\theta)z_1^2 + h_{020}(\theta)z_2^2 + h_{002}(\theta)\bar{z}_2^2 + h_{110}(\theta)z_1 z_2 \\ &+ h_{101}(\theta)z_1\bar{z}_2 + h_{011}(\theta)z_2\bar{z}_2, \quad \theta \in [-r, 0]. \end{aligned} \tag{6.29}$$

Hence (6.28) is equivalent to, for $\theta \in [-r, 0]$,

$$\begin{aligned} &-\dot{h}_{200}(\theta)z_1^2 - \dot{h}_{011}(\theta)z_2\bar{z}_2 + [2i\omega_0 h_{020}(\theta) - \dot{h}_{020}(\theta)]z_2^2 + [-2i\omega_0 h_{002}(\theta) - \dot{h}_{002}(\theta)]\bar{z}_2^2 \\ &+ [i\omega_0 h_{110}(\theta) - \dot{h}_{110}(\theta)]z_1 z_2 + [-i\omega_0 h_{101}(\theta) - \dot{h}_{101}(\theta)]z_1\bar{z}_2 = \frac{1}{2!} f_2^2(z, 0, 0)(\theta). \end{aligned} \tag{6.30}$$

For $\theta \in [-r, 0)$, comparing the coefficients of z^q for $|q| = 2$, $q \in \mathbb{N}_0^3$ in (6.10) and (6.30), we obtain that

$$\begin{aligned} \dot{h}_{200}(\theta) =\ & \tfrac{1}{2}[\phi_1(\theta)\psi_1(0)Q_{\phi_1\phi_1}\langle\beta_{k_1}^2, \beta_{k_1}\rangle\beta_{k_1} \\ & +(\phi_2(\theta)\psi_2(0) + \bar{\phi}_2(\theta)\bar{\psi}_2(0))Q_{\phi_1\phi_1}\langle\beta_{k_1}^2, \beta_{k_2}\rangle\beta_{k_2}], \\ \dot{h}_{011}(\theta) =\ & \phi_1(\theta)\psi_1(0)Q_{\phi_2\bar{\phi}_2}\langle\beta_{k_2}^2, \beta_{k_1}\rangle\beta_{k_1} \\ & +(\phi_2(\theta)\psi_2(0) + \bar{\phi}_2(\theta)\bar{\psi}_2(0))Q_{\phi_2\bar{\phi}_2}\langle\beta_{k_2}^2, \beta_{k_2}\rangle\beta_{k_2}, \\ -2\mathrm{i}\omega_0 h_{020}(\theta) + \dot{h}_{020}(\theta) =\ & \tfrac{1}{2}[\phi_1(\theta)\psi_1(0)Q_{\phi_2\phi_2}\langle\beta_{k_2}^2, \beta_{k_1}\rangle\beta_{k_1} \\ & +(\phi_2(\theta)\psi_2(0) + \bar{\phi}_2(\theta)\bar{\psi}_2(0))Q_{\phi_2\phi_2}\langle\beta_{k_2}^2, \beta_{k_2}\rangle\beta_{k_2}], \\ -\mathrm{i}\omega_0 h_{110}(\theta) + \dot{h}_{110}(\theta) =\ & \phi_1(\theta)\psi_1(0)Q_{\phi_1\phi_2}\langle\beta_{k_1}\beta_{k_2}, \beta_{k_1}\rangle\beta_{k_1} \\ & +(\phi_2(\theta)\psi_2(0) + \bar{\phi}_2(\theta)\bar{\psi}_2(0))Q_{\phi_1\phi_2}\langle\beta_{k_1}\beta_{k_2}, \beta_{k_2}\rangle\beta_{k_2}. \end{aligned} \tag{6.31}$$

Solving these equations, we obtain that

$$\begin{aligned}
h_{200}(\theta)= \quad & h_{200}(0)+\tfrac{1}{2}\{\theta\phi_1(0)\psi_1(0)Q_{\phi_1\phi_1}\langle\beta_{k_1}^2,\beta_{k_1}\rangle\beta_{k_1}\\
&+\tfrac{1}{i\omega_0}[(e^{i\omega_0\theta}-1)\phi_2(0)\psi_2(0)-(e^{-i\omega_0\theta}-1)\bar{\phi}_2(0)\bar{\psi}_2(0)]Q_{\phi_1\phi_1}\langle\beta_{k_1}^2,\beta_{k_2}\rangle\beta_{k_2}\},\\
h_{011}(\theta)= \quad & h_{011}(0)+\theta\phi_1(0)\psi_1(0)Q_{\phi_2\bar{\phi}_2}\langle\beta_{k_2}^2,\beta_{k_1}\rangle\beta_{k_1}\\
&+\tfrac{1}{i\omega_0}[(e^{i\omega_0\theta}-1)\phi_2(0)\psi_2(0)-(e^{-i\omega_0\theta}-1)\bar{\phi}_2(0)\bar{\psi}_2(0)]Q_{\phi_2\bar{\phi}_2}\langle\beta_{k_2}^2,\beta_{k_2}\rangle\beta_{k_2},\\
h_{020}(\theta)= \quad & h_{020}(0)e^{2i\omega_0\theta}+\tfrac{1}{2i\omega_0}\{\tfrac{1}{2}(e^{2i\omega_0\theta}-1)\phi_1(0)\psi_1(0)Q_{\phi_2\phi_2}\langle\beta_{k_2}^2,\beta_{k_1}\rangle\beta_{k_1}\\
&+[(e^{2i\omega_0\theta}-e^{i\omega_0\theta})\phi_2(0)\psi_2(0)\\
&+\tfrac{1}{3}(e^{2i\omega_0\theta}-e^{-i\omega_0\theta})\bar{\phi}_2(0)\bar{\psi}_2(0)]Q_{\phi_2\phi_2}\langle\beta_{k_2}^2,\beta_{k_2}\rangle\beta_{k_2}\},\\
h_{110}(\theta)= \quad & h_{110}(0)e^{i\omega_0\theta}+\tfrac{1}{i\omega_0}(e^{i\omega_0\theta}-1)\phi_1(0)\psi_1(0)Q_{\phi_1\phi_2}\langle\beta_{k_1}\beta_{k_2},\beta_{k_1}\rangle\beta_{k_1}\\
&+[\theta e^{i\omega_0\theta}\phi_2(0)\psi_2(0)+\tfrac{1}{2i\omega_0}(e^{i\omega_0\theta}-e^{-i\omega_0\theta})\bar{\phi}_2(0)\bar{\psi}_2(0)]Q_{\phi_1\phi_2}\langle\beta_{k_1}\beta_{k_2},\beta_{k_2}\rangle\beta_{k_2},\\
h_{002}(\theta)= \quad & h_{020}(\theta),\quad h_{101}(\theta)=h_{110}(\theta),\quad \theta\in[-r,0].
\end{aligned} \tag{6.32}$$

And at $\theta=0$, by (6.32) we have

$$\begin{aligned}
&-(L_0(h_{200})+D_0\Delta h_{200}(0))z_1^2+[2i\omega_0h_{020}(0)-(L_0(h_{020})+D_0\Delta h_{020}(0))]z_2^2\\
&+[-2i\omega_0h_{002}(0)-(L_0(h_{002})+D_0\Delta h_{002}(0))]\bar{z}_2^2+[i\omega_0h_{110}(0)\\
&-(L_0(h_{110})+D_0\Delta h_{110}(0))]z_1z_2\\
&+[-i\omega_0h_{101}(0)-(L_0(h_{101})+D_0\Delta h_{101}(0))]z_1\bar{z}_2-(L_0(h_{011})+D_0\Delta h_{011}(0))z_2\bar{z}_2\\
=\ &\tfrac{1}{2!}f_2^2(z,0,0)(0),
\end{aligned} \tag{6.33}$$

Again expanding the above sum and comparing the coefficients, we obtain that

$$\begin{aligned}
L_0(h_{200})+D_0\Delta h_{200}(0)&=\tfrac{1}{2}[-Q_{\phi_1\phi_1}\beta_{k_1}^2+\phi_1(0)\psi_1(0)Q_{\phi_1\phi_1}\langle\beta_{k_1}^2,\beta_{k_1}\rangle\beta_{k_1}\\
&\quad+(\phi_2(0)\psi_2(0)+\bar{\phi}_2(0)\bar{\psi}_2(0))Q_{\phi_1\phi_1}\langle\beta_{k_1}^2,\beta_{k_2}\rangle\beta_{k_2}],\\
L_0(h_{011})+D_0\Delta h_{011}(0)&=-Q_{\phi_2\bar{\phi}_2}\beta_{k_2}^2+\phi_1(0)\psi_1(0)Q_{\phi_2\bar{\phi}_2}\langle\beta_{k_2}^2,\beta_{k_1}\rangle\beta_{k_1}\\
&\quad+(\phi_2(0)\psi_2(0)+\bar{\phi}_2(0)\bar{\psi}_2(0))Q_{\phi_2\bar{\phi}_2}\langle\beta_{k_2}^2,\beta_{k_2}\rangle\beta_{k_2},\\
-2\mathrm{i}\omega_0h_{020}(0)+L_0(h_{020})+D_0\Delta h_{020}(0)&=\tfrac{1}{2}[-Q_{\phi_2\phi_2}\beta_{k_2}^2+\phi_1(0)\psi_1(0)Q_{\phi_2\phi_2}\langle\beta_{k_2}^2,\beta_{k_1}\rangle\beta_{k_1}\\
&\quad+(\phi_2(0)\psi_2(0)+\bar{\phi}_2(0)\bar{\psi}_2(0))Q_{\phi_2\phi_2}\langle\beta_{k_2}^2,\beta_{k_2}\rangle\beta_{k_2}],\\
-\mathrm{i}\omega_0h_{110}(0)+L_0(h_{110})+D_0\Delta h_{110}(0)&=-Q_{\phi_1\phi_2}\beta_{k_1}\beta_{k_2}+\phi_1(0)\psi_1(0)Q_{\phi_1\phi_2}\langle\beta_{k_1}\beta_{k_2},\beta_{k_1}\rangle\beta_{k_1}\\
&\quad+(\phi_2(0)\psi_2(0)+\bar{\phi}_2(0)\bar{\psi}_2(0))Q_{\phi_1\phi_2}\langle\beta_{k_1}\beta_{k_2},\beta_{k_2}\rangle\beta_{k_2}.
\end{aligned} \tag{6.34}$$

Therefore U_2^2 are determined by (6.32) and (6.34). For later computation of the third order normal form, note that for any $q\in\mathbb{N}_0^3$, $|q|=2$, we have

$$h_q(\theta)=\langle h_q(\theta),\beta_{k_1}\rangle\beta_{k_1}+\langle h_q(\theta),\beta_{k_2}\rangle\beta_{k_2}+\sum_{k\geq 0,k\neq k_1,k_2}\langle h_q(\theta),\beta_k\rangle\beta_k. \tag{6.35}$$

Then $\langle h_q(0), \beta_k\rangle$ ($k \in \mathbb{N}_0$) can be obtained from (6.34), and $\langle h_q(\theta), \beta_k\rangle$ for $\theta \in [-r, 0]$ is determined by (6.32). In fact, we do not need to find $\langle h_q(0), \beta_k\rangle$ for all $q \in \mathbb{N}_0^3$, $|q| = 2$ and $k \in \mathbb{N}_0$, but only need to find the ones appearing in $g_3^1(z, 0, 0)$.

Step 3. Computations of g_3^1.

Now we have all the components for computing the third order normal form. Let M_3 be the operator defined in $V_3^3(\mathbb{C}^3 \times Ker\pi)$, with

$$M_3^1 : V_3^3(\mathbb{C}^3) \to V_3^3(\mathbb{C}^3) \;\text{ and }\; (M_3^1 p)(z) = D_z p(z)Bz - Bp(z),$$

where $V_3^3(\mathbb{C}^3)$ denotes the linear space of homogeneous polynomials of degree 3 in the variables $z = (z_1, z_2, \bar{z}_2)$ with coefficients in $\mathbb{C}^3$. Then one may choose the decomposition

$$V_3^3(\mathbb{C}^3) = \mathrm{Im}(M_3^1) \oplus \mathrm{Im}(M_3^1)^c$$

with the complementary space $(\mathrm{Im}(M_3^1))^c$ spanned by the elements

$$z_1^3 e_1,\; z_1 z_2 \bar{z}_2 e_1,\; z_1^2 z_2 e_2,\; z_2^2 \bar{z}_2 e_2,\; z_1^2 \bar{z}_2 e_3,\; z_2 \bar{z}_2^2 e_3, \tag{6.36}$$

where e_1, e_2, e_3 denote the natural basis of $\mathbb{R}^3$. Now we have the normal form up to the third order

$$\dot{z} = Bz + \frac{1}{2!} g_2^1(z, 0, \alpha) + \frac{1}{3!} g_3^1(z, 0, 0) + h.o.t., \tag{6.37}$$

where

$$\frac{1}{3!} g_3^1(z, 0, 0) = \frac{1}{3!} Proj_{(\mathrm{Im}(M_3^1))^c} \overline{f}_3^1(z, 0, 0). \tag{6.38}$$

From (6.23) denoting

$$\begin{aligned}
g_{31}(z) = &\quad \frac{1}{6} Proj_{(\mathrm{Im}(M_3^1))^c} f_3^1(z, 0, 0), \\
g_{32}(z) = &\quad -\frac{1}{4} Proj_{(\mathrm{Im}(M_3^1))^c} D_z U_2^1(z) g_2^1(z, 0, 0), \\
g_{33}(z) = &\quad \frac{1}{4} Proj_{(\mathrm{Im}(M_3^1))^c} (D_z f_2^1(z, y, 0))_{y=0} U_2^1(z), \\
g_{34}(z) = &\quad \frac{1}{4} Proj_{(\mathrm{Im}(M_3^1))^c} (D_y f_2^1(z, y, 0))_{y=0} U_2^2(z),
\end{aligned} \tag{6.39}$$

then

$$\frac{1}{3!} g_3^1(z, 0, 0) = g_{31}(z) + g_{32}(z) + g_{33}(z) + g_{34}(z). \tag{6.40}$$

From (6.9) and (6.39), we obtain

$$\begin{aligned} g_{31}(z)= &\ \tfrac{1}{6}\psi_1(0)[C_{\phi_1\phi_1\phi_1}z_1^3\langle\beta_{k_1}^3,\beta_{k_1}\rangle+6C_{\phi_1\phi_2\bar{\phi}_2}z_1z_2\bar{z}_2\langle\beta_{k_1}\beta_{k_2}^2,\beta_{k_1}\rangle]e_1\\ &+\tfrac{1}{2}\psi_2(0)[C_{\phi_2\phi_2\bar{\phi}_2}z_2^2\bar{z}_2\langle\beta_{k_2}^3,\beta_{k_2}\rangle+C_{\phi_1\phi_1\phi_2}z_1^2z_2\langle\beta_{k_1}^2\beta_{k_2},\beta_{k_2}\rangle]e_2\\ &+\tfrac{1}{2}\bar{\psi}_2(0)[C_{\phi_2\bar{\phi}_2\bar{\phi}_2}z_2\bar{z}_2^2\langle\beta_{k_2}^3,\beta_{k_2}\rangle+C_{\phi_1\phi_1\bar{\phi}_2}z_1^2\bar{z}_2\langle\beta_{k_1}^2\beta_{k_2},\beta_{k_2}\rangle]e_3. \end{aligned} \tag{6.41}$$

For g_{32}, since $U_2^1(z)\in \mathrm{Ker}(M_2^1)^c$ and $g_2^1(z,0,0)\in \mathrm{Im}(M_2^1)^c$, by (6.25) and (6.15) we set

$$\begin{aligned} U_2^1(z)= &\ (a_{020}^{(1)}z_2^2+a_{002}^{(1)}\bar{z}_2^2+a_{110}^{(1)}z_1z_2+a_{101}^{(1)}z_1\bar{z}_2)e_1\\ &+(a_{200}^{(2)}z_1^2+a_{020}^{(2)}z_2^2+a_{002}^{(2)}\bar{z}_2^2+a_{101}^{(2)}z_1\bar{z}_2+a_{011}^{(2)}z_2\bar{z}_2)e_2\\ &+(a_{200}^{(3)}z_1^2+a_{020}^{(3)}z_2^2+a_{002}^{(3)}\bar{z}_2^2+a_{110}^{(3)}z_1z_2+a_{011}^{(3)}z_2\bar{z}_2)e_3, \end{aligned}$$

$$g_2^1(z,0,0)=(b_{200}^{(1)}z_1^2+b_{011}^{(1)}z_2\bar{z}_2)e_1+b_{110}^{(2)}z_1z_2e_2+b_{101}^{(3)}z_1\bar{z}_2e_3,$$

then

$$\begin{aligned} D_zU_2^1(z)g_2^1(z,0,0)= &\ [(a_{110}^{(1)}z_2+a_{101}^{(1)}\bar{z}_2)(b_{200}^{(1)}z_1^2+b_{011}^{(1)}z_2\bar{z}_2)+(2a_{020}^{(1)}z_2+a_{110}^{(1)}z_1)b_{110}^{(2)}z_1z_2\\ &+(2a_{002}^{(1)}\bar{z}_2+a_{101}^{(1)}z_1)b_{101}^{(3)}z_1\bar{z}_2]e_1+[(2a_{200}^{(2)}z_1+a_{101}^{(2)}\bar{z}_2)(b_{200}^{(1)}z_1^2+b_{011}^{(1)}z_2\bar{z}_2)\\ &+(2a_{020}^{(2)}z_2+a_{011}^{(2)}\bar{z}_2)b_{110}^{(2)}z_1z_2+(2a_{020}^{(2)}\bar{z}_2+a_{101}^{(2)}z_1+a_{011}^{(2)}z_2)b_{101}^{(3)}z_1\bar{z}_2]e_2\\ &+[(2a_{200}^{(3)}z_1+a_{110}^{(3)}z_2)(b_{200}^{(1)}z_1^2+b_{011}^{(1)}z_2\bar{z}_2)+(2a_{020}^{(3)}z_2+a_{110}^{(3)}z_1\\ &+a_{011}^{(3)}\bar{z}_2)b_{110}^{(2)}z_1z_2+(2a_{020}^{(3)}\bar{z}_2+a_{011}^{(3)}z_2)b_{101}^{(3)}z_1\bar{z}_2]e_3\\ &\in \mathrm{Im}(M_3^1). \end{aligned}$$

This implies that

$$g_{32}(z)=-\frac{1}{4}Proj_{(I_m(M_3^1))^c}DU_2^1(z)g_2^1(z,0,0)=0. \tag{6.42}$$

By using (6.39), (6.5), (6.24) and (6.36), we obtain

$$g_{33}(z)=g_{33}^{(1)}(z)e_1+g_{33}^{(2)}(z)e_2+\overline{g_{33}^{(2)}(z)}e_3, \tag{6.43}$$

with

$$\begin{aligned} g_{33}^{(1)}(z)= &\ \tfrac{1}{2i\omega_0}\psi_1(0)[-Q_{\phi_1\phi_2}\psi_2(0)+Q_{\phi_1\bar{\phi}_2}\bar{\psi}_2(0)]Q_{\phi_1\phi_1}\langle\beta_{k_1}\beta_{k_2},\beta_{k_1}\rangle\langle\beta_{k_1}^2,\beta_{k_2}\rangle z_1^3\\ &+\tfrac{1}{i\omega_0}\psi_1(0)\{[-Q_{\phi_1\phi_2}\psi_2(0)+Q_{\phi_1\bar{\phi}_2}\bar{\psi}_2(0)]Q_{\phi_2\bar{\phi}_2}\langle\beta_{k_1}\beta_{k_2},\beta_{k_1}\rangle\langle\beta_{k_2}^2,\beta_{k_2}\rangle\\ &+\tfrac{1}{2}[-Q_{\phi_2\phi_2}\psi_2(0)Q_{\phi_1\bar{\phi}_2}+Q_{\bar{\phi}_2\bar{\phi}_2}\bar{\psi}_2(0)Q_{\phi_1\phi_2}]\langle\beta_{k_1}\beta_{k_2},\beta_{k_2}\rangle\langle\beta_{k_2}^2,\beta_{k_1}\rangle]\}z_1z_2\bar{z}_2, \end{aligned} \tag{6.44}$$

$$\begin{aligned} g_{33}^{(2)}(z) = \ & \tfrac{1}{2i\omega_0}\psi_2(0)\{[2Q_{\phi_1\phi_1}\psi_1(0)\langle\beta_{k_1}\beta_{k_2},\beta_{k_1}\rangle\langle\beta_{k_1}^2,\beta_{k_2}\rangle + Q_{\phi_1\bar{\phi}_2}\bar{\psi}_2(0)\langle\beta_{k_1}\beta_{k_2},\beta_{k_2}\rangle^2]Q_{\phi_1\phi_2} \\ & +[-Q_{\phi_2\phi_2}\psi_2(0) + Q_{\phi_2\bar{\phi}_2}\bar{\psi}_2(0)]Q_{\phi_1\phi_1}\langle\beta_{k_1}^2,\beta_{k_2}\rangle\langle\beta_{k_2}^2,\beta_{k_2}\rangle\}z_1^2 z_2 \\ & +\tfrac{1}{4i\omega_0}\psi_2(0)\{Q_{\phi_1\bar{\phi}_2}\psi_1(0)Q_{\phi_2\phi_2}\langle\beta_{k_1}\beta_{k_2},\beta_{k_2}\rangle\langle\beta_{k_2}^2,\beta_{k_1}\rangle \\ & +\tfrac{2}{3}Q_{\bar{\phi}_2\bar{\phi}_2}\bar{\psi}_2(0)Q_{\phi_2\phi_2}\langle\beta_{k_2}^2,\beta_{k_2}\rangle^2 \\ & +[-2Q_{\phi_2\phi_2}\psi_2(0) + 4Q_{\phi_2\bar{\phi}_2}\bar{\psi}_2(0)]Q_{\phi_2\bar{\phi}_2}\langle\beta_{k_2}^2,\beta_{k_2}\rangle^2\}z_2^2\bar{z}_2. \end{aligned} \tag{6.45}$$

Finally from (6.29) and the symmetric multilinearity of Q, we obtain

$$Q_{\phi h}Q_{\phi h_{200}}z_1^2 + Q_{\phi h_{020}}z_2^2 + Q_{\phi h_{002}}\bar{z}_2^2 + Q_{\phi h_{110}}z_1z_2 + Q_{\phi h_{101}}z_1\bar{z}_2 + Q_{\phi h_{011}}z_2\bar{z}_2, \tag{6.46}$$

with $\phi \in \{\phi_1, \phi_2, \bar{\phi}_2\}$. From (6.39), (6.7), (6.26) and (6.36), we obtain

$$\begin{aligned} g_{34}(z) = \ & \psi_1(0)[\langle Q_{\phi_1 h_{200}}\beta_{k_1},\beta_{k_1}\rangle z_1^3 + (\langle Q_{\phi_1 h_{011}}\beta_{k_1},\beta_{k_1}\rangle + \langle Q_{\phi_2 h_{101}}\beta_{k_2},\beta_{k_1}\rangle \\ & +\langle Q_{\bar{\phi}_2 h_{110}}\beta_{k_2},\beta_{k_1}\rangle)z_1z_2\bar{z}_2]e_1 + \psi_2(0)[(\langle Q_{\phi_1 h_{110}}\beta_{k_1},\beta_{k_2}\rangle + \langle Q_{\phi_2 h_{200}}\beta_{k_2},\beta_{k_2}\rangle)z_1^2z_2 \\ & +(\langle Q_{\phi_2 h_{011}}\beta_{k_2},\beta_{k_2}\rangle + \langle Q_{\bar{\phi}_2 h_{020}}\beta_{k_2},\beta_{k_2}\rangle)z_2^2\bar{z}_2]e_2 + \bar{\psi}_2(0)[(\langle Q_{\phi_1 h_{101}}\beta_{k_1},\beta_{k_2}\rangle \\ & +\langle Q_{\bar{\phi}_2 h_{200}}\beta_{k_2},\beta_{k_2}\rangle)z_1^2\bar{z}_2 + (\langle Q_{\bar{\phi}_2 h_{011}}\beta_{k_2},\beta_{k_2}\rangle + \langle Q_{\phi_2 h_{002}}\beta_{k_2},\beta_{k_2}\rangle)z_2\bar{z}_2^2]e_3. \end{aligned} \tag{6.47}$$

The conclusion of Theorem 3.1 follows directly from Lemma 6.1, (6.41), (6.42), (6.43) and (6.47).

7. Conclusion

In this paper the normal forms up to the third order for a Hopf-steady state bifurcation of a general system of partial functional differential equations (PFDEs) is derived based on the center manifold and normal form theories of PFDEs. This is a codimension-two bifurcation with the characteristic equation having a pair of simple purely imaginary roots and a simple zero root. The PFDEs are reduced to a three-dimensional system of ordinary differential equations, and precise dynamics near the bifurcation point can be revealed by two unfolding parameters which can be expressed by original perturbation parameters. Usually, the third order normal form is sufficient for analyzing bifurcation phenomena in most of the applications.

The normal form for the Hopf-steady state bifurcation in a general PFDE has been investigated within the framework of Faria [15,17]. However, in [15] an important conclusion is that the normal forms up to a certain finite order for both the PFDEs and its associated FDEs are the same under the assumption $(H5)$. And when $(H5)$ is not satisfied, the associated FDEs may not provide complete information, and further general results on the normal forms are not given in [15]. In fact, the assumption $(H5)$ is not satisfied when a Hopf-steady state bifurcation occurs. Our results on computing the normal form on center manifolds, that is (3.4)-(3.7), (6.32) and (6.34), do not require $(H5)$, which makes the approach applicable to a wider class of systems.

For more concrete expressions, we provide explicit formulas of the coefficients in the third order normal form in the Hopf-steady state bifurcation for delayed reaction-diffusion equations with Neumann boundary condition, that is (4.1)-(4.11), and this includes the important case of

Turing-Hopf bifurcation (with $k_1 \neq 0$). The formulas are user-friendly as they are expressed directly by the Frèchet derivatives of the functions up to third orders and the characteristic functions of the original systems. Moreover they are shown in concise matrix form which is convenient for computer implementation. These results can also be applied to reaction diffusion equations without delay, for example, see [8], and functional differential equations without diffusion, see [28].

Our general results are applied to the diffusive Schnakenberg system of biochemical reactions with gene expression time delay to demonstrate how our formulas can be applied in practical examples. In particular we provide specific conditions on the parameters for the existence and stability of spatially nonhomogeneous steady state solutions and time-periodic solutions near the Turing-Hopf bifurcation point. Our specific example is for a one-dimensional spatial domain with Neumann boundary condition, but our general framework is broad enough for high-dimensional spatial domains and Dirichlet boundary condition. More specific computations for these cases will be done in the future.

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