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The existence of constrained minimizers for a class of nonlinear Kirchhoff–Schrödinger equations with doubly critical exponents in dimension four☆

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ABSTRACT

In this paper, the existence and nonexistence of energy minimizer of the Kirchhoff–Schrödinger energy function with prescribed L^2-norm in dimension four are considered. The energy infimum values are completely classified in terms of coefficient and exponent of the nonlinearity. The sharp existence results of global constraint minimizers for both the subcritical and critical exponent cases are obtained, and the criticality is in the sense of both Sobolev embedding and Gagliardo–Nirenberg inequality. Our results also show the delicate difference between the case without a trapping potential function and the one with potential function.

1. Introduction and main results

This paper is concerned with the existence of L^2-normalized minimizers of the nonlinear Schrödinger–Kirchhoff functional for a four-dimensional Bose–Einstein condensate:

$$E_V^c(u) = \frac{a}{2}\int_{\mathbb{R}^4} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^4} V(x)u^2 + \frac{b}{4}\int_{\mathbb{R}^4} |\nabla u|^4 - \frac{c}{p}\int_{\mathbb{R}^4} |u|^p, \quad u \in H^1(\mathbb{R}^4), \tag{1.1}$$

where $a, b > 0$ are constants, $c > 0$ is a parameter and V is a potential function from various physics applications. If $u \in H^1(\mathbb{R}^4)$ achieves the minimizer of the functional E_V^c with the constraint $\int_{\mathbb{R}^4} |u|^2 = 1$, then there exists a real number λ such that u satisfies

$$-\left(a + b\int_{\mathbb{R}^4} |\nabla u|^2\right) \Delta u + V(x)u = c|u|^{p-2}u + \lambda u, \quad x \in \mathbb{R}^4. \tag{1.2}$$

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Solutions of Eq. (1.2) are the stationary solutions of Kirchhoff wave equation

$$u_{tt} - \left(a + b\int_{\mathbb{R}^4} |\nabla u|^2\right) \Delta u = f(x,u), \quad x \in \mathbb{R}^4, \tag{1.3}$$

where f is a general nonlinearity. The problem (1.3) was proposed by Kirchhoff [10] in 1883 to describe the transversal oscillations of a stretched string. Comparing with the corresponding semilinear Schrödinger equations (i.e., setting $b = 0$ in the above two equations), it is much more challenging and interesting to investigate equations (1.2) and (1.3) in view of the presence of the nonlocal term $\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u$.

After the pioneering work of [14], much attention was paid to (1.2) and (1.3). For instance, replacing the term $|u|^{p-2}u$ with a general nonlinearity $f(x,u)$, there are many results on the existence of solutions for Eq. (1.2), one can refer, for example, [3,7,8] and the references therein. Eq. (1.2) can be viewed as an eigenvalue problem by taking λ as an unknown Lagrange multiplier. From this point of view, one can solve (1.2) by studying some constrained variational problem and obtain normalized solutions.

For spatial dimension $N = 3$, a lot of interests are paid to (1.2) and some existence results are obtained, see for example, [3–5,8,11–13,16,17]. Because of the additional Kirchhoff type nonlocal term, the nonlinearity is usually assumed that 4-suplinear when $N = 3$ and the critical Sobolev exponent is $2^*_S = 6$. Furthermore the ground state of the functional E^c_V with prescribed mass when $V = 0$ and $N = 3$ has been also considered [19–21]. Typically the subcritical case $p \in (2, 14/3)$ (thus $p < 2^*_S$) is assumed when considering the constrained minimizer of E^c_V, where $2^*_{GN} = 14/3$ is the Gagliardo–Nirenberg (G–N) critical exponent from the celebrated Gagliardo–Nirenberg inequality when $N = 3$. When $p \in [14/3, 2^*)$ and $N = 3$, the energy functional E^c_V is not bounded from below hence the minimization problem is not valid.

For the spatial dimension $N = 4$, the critical Sobolev exponent is $2^*_S = 4$ and the G–N critical exponent is also $2^*_{GN} = 4$. The existence of nontrivial solutions to Kirchhoff equation on bounded domain in R^4 was shown in [15] without constraint, see also [9]. In this paper, we consider the energy minimizer of the energy functional E^c_V with a prescribed mass when $N = 4$, $p \in (2, 4]$, and either $V = 0$ or $V \neq 0$. This include both subcritical ($p < 4$) and the double critical ($p = 4$) cases. When $p = 4$, the nonlocal term and the nonlinearity are both 4-linear growth, which generates an additional competition in the energy function E^c_V.

From now on we assume that $N = 4$. We consider the following minimization problem

$$e^c_V := \inf_{u \in S_V} E^c_V(u), \tag{1.4}$$

where $E^c_V(u)$ is defined in (1.1) for $u \in H_V := \{u \in H^1(\mathbb{R}^4) : \int_{\mathbb{R}^4} V(x)u^2 < \infty\}$ and $S_V := \{u \in H_V : |u|_2 = 1\}$. Here $|\cdot|_p$ denotes the norm of $L^p(\mathbb{R}^4)$ defined by $|u|_p^p = \int_{\mathbb{R}^4} |u|^p$. If $V = 0$, we denote the space H_V by H_0, the set S_V by S_0, the functional E^c_V by E^c_0, and e^c_V by e^c_0 respectively.

Before stating our main results, we recall the well-known Gagliardo–Nirenberg inequality with the best constant (see [18]): let $p \in [2, 4)$, then

$$|u|_p \leqslant \left(\frac{p}{2|Q_p|_2^{p-2}}\right)^{\frac{1}{p}} |\nabla u|_2^{\frac{2(p-2)}{p}} |u|_2^{1-\frac{2(p-2)}{p}}, \quad u \in H^1(\mathbb{R}^4) \tag{1.5}$$

with the equality only holds when $u = t^2 Q_p(tx)$, where up to translations, Q_p is the unique ground state solution of

$$-(p-2)\Delta Q_p + \left(1 - \frac{p-2}{2}\right) Q_p = |Q_p|^{p-2} Q_p \quad \text{in } \mathbb{R}^4. \tag{1.6}$$

If $|u|_2 = 1$, then

$$|u|_p \leqslant \left(\frac{p}{2|Q_p|_2^{p-2}}\right)^{\frac{1}{p}} |\nabla u|_2^{\frac{2(p-2)}{p}}, \quad p \in (2, 4), \tag{1.7}$$

with the equality only holds when $u = u_p(x) = t^2 Q_p(tx)/|Q_p|_2$ up to translations. We also recall the Sobolev inequality in $\mathbb{R}^4$:

$$S^2 \int_{\mathbb{R}^4} u^4 \leqslant \left(\int_{\mathbb{R}^4} |\nabla u|^2 \right)^2, \ u \in H^1(\mathbb{R}^4) \tag{1.8}$$

where

$$S^2 = \frac{|\nabla Q_4|_2^4}{|Q_4|_4^4}, \quad Q_4(x) = \frac{8^{\frac{1}{2}}}{1+|x|^2}, \tag{1.9}$$

and S^2 is the best Sobolev embedding constant satisfying $S^2 = |\nabla Q_4|_2^2 = |Q_4|_4^4$ (see [1]).

The main results of this paper, which we will prove in Section 3, read as follows.

Theorem 1.1. *Let $V = 0$ and $p \in (2,4]$. Then*

(a) There exists $c_* \in [0,\infty)$ defined by

$$c_* = \begin{cases} 0, & 2 < p < 3, \\ a|Q_p|_2, & p = 3, \\ a^{4-p}|Q_p|_2^{p-2}(4-p)^{p-4}(\frac{b}{2p-6})^{p-3}, & 3 < p < 4, \\ bS^2, & p = 4, \end{cases}$$

such that $e_0^c = 0$ for $0 < c \leqslant c_*$ and $e_0^c < 0$ for $c > c_*$. Furthermore, when $c > c_*$, $e_0^c > -\infty$ if $p \in (2,4)$ and $e_0^c = -\infty$ if $p = 4$.

(b) If $p \in (2,4)$, then $e_0^c = \min_{t \geqslant 0} f_{c,p}(t)$ where

$$f_{c,p}(t) = \frac{a}{2}t + \frac{b}{4}t^2 - \frac{c}{2|Q_p|_2^{p-2}} t^{p-2}. \tag{1.10}$$

Moreover, E_0^c has a minimizer if and only if $p \in (2,4)$ and $c > c_*$, or $p \in (3,4)$ and $c \geqslant c_*$. While e_0^c can be achieved, the minimizer equals to $(Q_p)_{t_p}$ up to translations, where $(Q_p)_t(x) = t^2 Q_p(tx)/|Q_p|_2$ and $t_p > 0$ is the unique minimum point of $f_{c,p}$.

(c) If $p = 4$, E_0^c has no energy minimizers for any $c > 0$.

Theorem 1.2. *Let $p \in (2,4]$, and $V(\cdot)$ satisfies the condition:*

(V) $V \in C(\mathbb{R}^4, [0,\infty))$, $\lim_{|x|\to\infty} V(x) = \infty$ and $\inf_{x\in\mathbb{R}^4} V(x) = 0$; there exists a small $\varepsilon_0 > 0$ such that $m(\{V(x) \leqslant \varepsilon_0\}) \leqslant \varepsilon_0$.

Then there exists a $c_* > 0$ such that

(a) if $p \in (2,4)$, then $e_V^c > 0$ for $c \in (0,c_*)$, $e_V^{c_*} = 0$ and $e_V^c \in (-\infty,0)$ for $c \in (c_*,\infty)$. Moreover, e_V^c can be achieved for all $c > 0$.

(b) if $p = 4$, then $c_* = bS^2$, $e_V^c \in (-\infty,0)$ for $c \in (0,c_*)$, $e_V^{c_*} = 0$ and $e_V^c = -\infty$ for $c \in (c_*,\infty)$. Moreover, E_V^c has an energy minimizer for $c < c_*$, and it has no minimizers for $c \geqslant c_*$.

Remark 1.3.

1. When $p = 4$, $c_* = bS^2$ which does not depend on the potential function V. For $p < 4$, this is not known unless $V = 0$ (see Theorem 1.1).
2. The functions $V(x) = |x|$ and $V(x) = (|x|-1)^2$ satisfy (V).

Theorems 1.1 and 1.2 provide the first existence results for the constraint minimization problem for the Kirchhoff–Schrödinger energy functional (1.1) when $N = 4$ and $p \in (2, 4]$. Similar results for $N = 3$, $p \in (2, 14/3)$ and $V = 0$ were obtained in [21]. Our results here consider the effect of potential function V and also the more delicate doubly critical exponent $p = 4$ case. When $V \neq 0$, the existence of ground state of E_V^c for $N = 2$, $p = 4$ and $b = 0$ was obtained in [6]. That is the Gagliardo–Nirenberg critical exponent case, but not the Sobolev critical exponent. In Section 2, we give some basic estimates, and we prove our main results in Section 3.

2. Preliminaries

In this section, we give some estimates of E_V^c which are required in the proof of the main results in next section.

2.1. The case of $V = 0$

For $p \in (2, 4)$, we conclude from (1.6) that

$$\int_{\mathbb{R}^4} |\nabla Q_p|^2 = \int_{\mathbb{R}^4} |Q_p|^2 = \frac{2}{p}\int_{\mathbb{R}^4} |Q_p|^p. \tag{2.1}$$

We define

$$g_{c,p}(t) = \frac{a}{2} + \frac{b}{4}t - \frac{c}{2|Q_p|_2^{p-2}}t^{p-3}, \quad f_{c,p}(t) = tg_{c,p}(t), \quad t \geqslant 0. \tag{2.2}$$

The following lemma characterizes the threshold value c_* and infimum energy level e_0^c when $2 < p < 4$.

Lemma 2.1. *If $p \in (2, 4)$, then $e_0^c = \min_{t\geqslant 0} f_{c,p}(t)$. Moreover, there exists a $c_* \geqslant 0$ such that*

1. *If $2 < p < 3$, then $c_* = 0$ and $e_0^c \in (-\infty, 0)$ for all $c > 0$.*
2. *If $3 \leqslant p < 4$, then $c_* \in (0, +\infty)$, $e_0^c = 0$ for $c \in (0, c_*]$, and $-\infty < e_0^c < 0$ for $c > c_*$; Moreover, $c_* = a|Q_p|_2$ if $p = 3$, $c_* = a^{4-p}|Q_p|_2^{p-2}(b/(2p-6))^{p-3}$ if $3 < p < 4$.*

Proof. Let $p \in (2, 4)$. For any $c > 0$ and $u \in S_0$, then by (1.7),

$$E_0^c(u) \geqslant \frac{a}{2}|\nabla u|_2^2 + \frac{b}{4}|\nabla u|_2^4 - \frac{c}{2|Q_p|_2^{p-2}}|\nabla u|_2^{2(p-2)}. \tag{2.3}$$

Because $2 < p < 4$, E_0^c is bounded from below on S_0; that is, $e_0^c > -\infty$ is well defined.

For $p \in (2, 4)$, let

$$(Q_p)_t(x) := \frac{t^2 Q_p(tx)}{|Q_p|_2}, \quad t > 0, \tag{2.4}$$

then $(Q_p)_t \in S_0$ and by (2.1), we have

$$e_0^c \leqslant E_0^c((Q_p)_t) = \frac{a}{2}t^2 + \frac{b}{4}t^4 - \frac{c}{2}t^{2(p-2)}|Q_p|_2^{2-p} = f_{c,p}(t) \to 0, \ t \to 0. \tag{2.5}$$

Therefore, $e_0^c \leqslant 0$ for all $c > 0$ when $p \in (2, 4)$. It follows from (2.3) that $E_0^c(u) \geqslant \inf_{t>0} f_{c,p}(t)$ for all $u \in S_0$. According to (2.3) and (2.5), $e_0^c = \min_{t\geqslant 0} f_{c,p}(t)$.

We complete the proof for the following three cases.

(a) If $p \in (2,3)$ and $c > 0$, then $f_{c,p}(t) < 0$ for $t > 0$ small. We set $c_* = 0$, then $e_0^c < 0$ for all $c > 0$. Therefore, $c_* = 0$ is well defined. According to (2.3), $e_0^c \in (-\infty, 0)$.

(b) If $p = 3$, then $g_{c,p}(t) = \frac{a}{2} + \frac{b}{4}t - \frac{c}{2|Q_p|_2^{p-2}}$. It is obvious that $g_{c,p}(t) \geqslant g_{c,p}(0) = \frac{a}{2} - \frac{c}{2|Q_p|_2^{p-2}}$. Let $c_* = a|Q_p|_2^{p-2}$. If $c \leqslant c_*$, then $g_{c,p}(0) \geqslant 0$. This together with $e_0^c \leqslant 0$ implies that $e_0^c = 0$. If $c > c_*$, then $f_{c,p}(t) = tg_{c,p}(t) < 0$ for $t > 0$ small. Therefore, we have $e_0^c < 0$.

(c) If $p \in (3,4)$, then there exists $t_* > 0$ such that $g'_{c,p}(t_*) = 0$ and $g_{c,p}(t_*) = \min_{t \geqslant 0} g_{c,p}(t)$. Indeed it is easy to calculate that t_* satisfies

$$\frac{b}{4} = \frac{c}{2|Q_p|_2^{p-2}}(p-3)t_*^{p-4}.$$

Let c_* be the unique value satisfying

$$g_{c_*,p}(t_*) = \frac{a}{2} + \frac{b}{4}t_* - \frac{c_*}{2|Q_p|_2^{p-2}}t_*^{p-3} = \frac{a}{2} - \frac{c_*}{2|Q_p|_2^{p-2}}(4-p)t_*^{p-3} = 0.$$

Then we have

$$c_* = a^{4-p}|Q_p|_2^{p-2}(4-p)^{p-4}\left(\frac{b}{2p-6}\right)^{p-3}.$$

If $c \leqslant c_*$, then $\min_{t \geqslant 0} g_{c,p} \geqslant 0$ and $e_0^c = 0$. If $c > c_*$, then $g_{c_*,p}(t_*) < 0$ and it follows from (2.5) that $e_0^c < 0$. $\square$

Next we show the related estimates for $p = 4$.

Lemma 2.2. *If $p = 4$, then $c_* = bS^2$, $e_0^c = 0$ for $c \in (0, c_*]$, and $e_0^c = -\infty$ for $c > c_*$. Moreover when $c \leqslant c_* = bS^2$, the infimum $e_0^c = 0$ is not achieved.*

Proof. If $p = 4$, we show that $c_* = bS^2$. In fact, for $c \leqslant bS^2$ and $u \in S_0$, since

$$E_0^c(u) = \frac{a}{2}|\nabla u|_2^2 + \frac{b}{4}|\nabla u|_2^4 - \frac{c}{4}|u|_4^4 \geqslant \frac{a}{2}|\nabla u|_2^2 + \frac{b}{4}|\nabla u|_2^4 - \frac{c}{4S^2}|\nabla u|_2^4 \geqslant \frac{b}{4}(1 - \frac{c}{c_*})|\nabla u|_2^4 \geqslant 0,$$

then we have $e_0^c \geqslant 0$ for $c \leqslant bS^2$. On the other hand, for $u \in S_0$, let $u_t(x) = t^2 u(tx)$, then $u_t \in S_0$ and

$$E_0^c(u_t) = \frac{a}{2}t^2|\nabla u|_2^2 + \frac{b}{4}t^4|\nabla u|_2^4 - \frac{c}{4}t^4|u|_4^4 \to 0,\ t \to 0.$$

Hence $e_0^c = 0$ for $c \leqslant bS^2$. For $c > bS^2$, since $Q_4 \notin L^2(\mathbb{R}^4)$, we cannot use the same method of $p < 4$ here and we use a cut-off function technique to obtain the estimates of e_0^c.

In what follows, we assume that $c > bS^2$. For $R > 1$, we can obtain by directly computing that

$$\int_{|x|>R} Q_4^4(x) = \int_{|x|>R} \frac{64}{(1+|x|^2)^4}dx = 64\omega_4 \int_R^\infty \frac{r^3}{(1+r^2)^4}dr < 64\omega_4 \int_R^\infty \frac{1}{r^5}dr = \frac{16\omega_4}{R^4}, \tag{2.6}$$

$$\int_{|x|>R} |\nabla Q_4(x)|^2 = 8\int_{|x|>R} \frac{|2x|^2}{(1+|x|^2)^4}dx = 32\omega_4 \int_R^\infty \frac{r^5}{(1+r^2)^4}dr < 32\omega_4 \int_R^\infty \frac{1}{r^3}dr < \frac{16\omega_4}{R^2}, \tag{2.7}$$

$$
\begin{aligned}
\int_{R<|x|<2R} Q_4^2(x) &= \int_{R<|x|<2R} \frac{8}{(1+|x|^2)^2} dx \\
&= 8\omega_4 \int_R^{2R} \frac{r^3}{(1+r^2)^2} dr \\
&= 4\omega_4 \int_{R^2}^{4R^2} \frac{t}{(1+t)^2} dt \\
&= 4\omega_4 \int_{R^2}^{4R^2} \left[\frac{1}{1+t} - \frac{1}{(1+t)^2}\right] dt \\
&= 4\omega_4 \left[\ln \frac{1+4R^2}{1+R^2} + \frac{1}{1+4R^2} - \frac{1}{1+R^2}\right],
\end{aligned} \tag{2.8}
$$

$$
\begin{aligned}
\int_{|x|<R} Q_4^2(x) &= \int_{|x|<R} \frac{8}{(1+|x|^2)^2} dx \\
&= 8\omega_4 \int_0^{R} \frac{r^3}{(1+r^2)^2} dr \\
&= 4\omega_4 \int_0^{R^2} \left[\frac{1}{1+t} - \frac{1}{(1+t)^2}\right] dt \\
&= 4\omega_4 \left[\ln(1+R^2) + \frac{1}{1+R^2} - 1\right],
\end{aligned} \tag{2.9}
$$

where $\omega_4 = 2\pi^2/\Gamma(2) = 2\pi^2$ is the surface area of unit sphere in $\mathbb{R}^4$. We choose a radially symmetric function $\phi \in C_0^\infty(\mathbb{R}^4)$ with $0 \leqslant \phi \leqslant 1$, $\phi = 1$ on $B_R := \{x \in \mathbb{R}^4 : |x| < R\}$ and $\phi = 0$ on $B_{2R}^c := \{x \in \mathbb{R}^4 : |x| > 2R\}$, and $|\nabla \phi| \leqslant 2/R$. Moreover, we may choose ϕ to be non-increasing on $|x|$. Let $U = \phi Q_4/|\phi Q_4|_2$ and $U_t(x) = t^2 U(tx)$, then $U, U_t \in S_0$. Since $\lim_{R\to\infty} \ln(1+R^2) = +\infty$ and $\lim_{R\to\infty}(1 - \frac{1}{1+R^2}) = 1$, then for $R > 0$ large,

$$
\frac{\ln(1+R^2)}{2} > 1 - \frac{1}{1+R^2}.
$$

Therefore, we have from (2.9)

$$
|\phi Q_4|_2^2 = \int_{\mathbb{R}^4} |\phi Q_4|^2 \geqslant \int_{|x|<R} |Q_4|^2 \geqslant \frac{\omega_4}{4} \ln(1+R^2). \tag{2.10}
$$

Since

$$
\lim_{R\to\infty} \left(\ln \frac{1+4R^2}{1+R^2} + \frac{1}{1+4R^2} - \frac{1}{1+R^2}\right) = \ln 4 < 2
$$

and (2.8),

$$
\int_{R<|x|<2R} Q_4^2 < 8\omega_4.
$$

This together with (2.6)–(2.10) and (1.9) implies that for R large,

$$
\begin{aligned}
&E_0^c(U_t) = \frac{a}{2} t^2 |\nabla U|_2^2 + \frac{b}{4} t^4 |\nabla U|_2^4 - \frac{c}{4} t^4 |U|_4^4 \\
&= \frac{at^2}{2|Q_4\phi|_2^2} \left[\int_{\mathbb{R}^4} |\nabla Q_4 \phi + Q_4 \nabla \phi|^2\right] + \frac{bt^4}{|Q_4\phi|_2^4} \left[\int_{\mathbb{R}^4} |\nabla Q_4 \phi + Q_4 \nabla \phi|^2\right]^2 - \frac{ct^4}{4|Q_4\phi|_2^4} \int_{\mathbb{R}^4} |Q_4\phi|^4 \\
&\leqslant \frac{at^2}{2|Q_4\phi|_2^2} \left[\int_{\mathbb{R}^4} |\nabla Q_4 \phi|^2 + \int_{\mathbb{R}^4} |Q_4|^2 |\nabla \phi|^2 + 2\int_{\mathbb{R}^4} |\nabla Q_4| |\nabla \phi| Q_4 \phi\right] \\
&\quad + \frac{bt^4}{4|Q_4\phi|_2^4} \left[\int_{\mathbb{R}^4} |\nabla Q_4 \phi|^2 + \int_{\mathbb{R}^4} |Q_4|^2 |\nabla \phi|^2 + 2\int_{\mathbb{R}^4} |\nabla Q_4| |\nabla \phi| Q_4 \phi\right]^2 - \frac{ct^4}{4|Q_4\phi|_2^4} \int_{\mathbb{R}^4} |Q_4\phi|^4
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{at^2}{2|Q_4\phi|_2^2}\left[\int_{|x|<2R}|\nabla Q_4|^2+\int_{R<|x|<2R}|Q_4|^2|\nabla\phi|^2+2\int_{R<|x|<2R}|\nabla Q_4||\nabla\phi|Q_4\phi\right]\\
&\quad+\frac{bt^4}{4|Q_4\phi|_2^4}\left[\int_{|x|<2R}|\nabla Q_4|^2+\int_{R<|x|<2R}|Q_4|^2|\nabla\phi|^2+2\int_{R<|x|<2R}|\nabla Q_4||\nabla\phi|Q_4\phi\right]^2\\
&\quad-\frac{ct^4}{4|Q_4\phi|_2^4}\int_{|x|<R}|Q_4|^4\\
&\leqslant \frac{at^2}{2|Q_4\phi|_2^2}\left[\int_{|x|<R}|\nabla Q_4|^2+\int_{R<|x|<2R}|\nabla Q_4|^2+\frac{4}{R^2}\int_{|R<|x|<2R}|Q_4|^2+\frac{4}{R}\int_{R<|x|<2R}|\nabla Q_4|Q_4\right]\\
&\quad+\frac{bt^4}{4|Q_4\phi|_2^4}\left[\int_{|x|<R}|\nabla Q_4|^2+\int_{R<|x|<2R}|\nabla Q_4|^2+\frac{4}{R^2}\int_{|R<|x|<2R}|Q_4|^2+\frac{4}{R}\int_{R<|x|<2R}|\nabla Q_4|Q_4\right]^2\\
&\quad-\frac{ct^4}{4|Q_4\phi|_2^4}\int_{|x|<R}|Q_4|^4\\
&\leqslant \frac{at^2}{2|Q_4\phi|_2^2}\left[\int_{|x|<R}|\nabla Q_4|^2+2\int_{|x|>R}|\nabla Q_4|^2+\frac{8}{R^2}\int_{R<|x|<2R}|Q_4|^2\right]\\
&\quad+\frac{bt^4}{4|Q_4\phi|_2^4}\left[\int_{|x|<R}|\nabla Q_4|^2+2\int_{|x|>R}|\nabla Q_4|^2+\frac{8}{R^2}\int_{R<|x|<2R}|Q_4|^2\right]^2-\frac{ct^4}{4|Q_4\phi|_2^4}\int_{|x|<R}|Q_4|^4\\
&\leqslant \frac{at^2}{2|Q_4\phi|_2^2}\left[\int_{|x|<R}|\nabla Q_4|^2+\frac{32\omega_4}{R^4}+\frac{64\omega_4}{R^2}\right]+\frac{bt^4}{4|Q_4\phi|_2^4}\left[\int_{|x|<R}|\nabla Q_4|^2+\frac{32\omega_4}{R^4}+\frac{64\omega_4}{R^2}\right]^2\\
&\quad-\frac{ct^4}{4|Q_4\phi|_2^4}\left(\int_{\mathbb{R}^4}|Q_4|^4-\int_{|x|>R}|Q_4|^4\right)\\
&\leqslant \frac{at^2}{2|Q_4\phi|_2^2}\left[\int_{|x|<R}|\nabla Q_4|^2+\frac{128\omega_4}{R^2}\right]+\frac{bt^4}{4|Q_4\phi|_2^4}\left[\int_{|x|<R}|\nabla Q_4|^2+\frac{128\omega_4}{R^2}\right]^2\\
&\quad-\frac{ct^4}{4|Q_4\phi|_2^4}\left(S^2-\frac{16\omega_4}{R^4}\right)\\
&\leqslant \frac{at^2}{2|Q_4\phi|_2^2}\left[S^2+\frac{128\omega_4}{R^2}\right]+\frac{t^4}{4|Q_4\phi|_2^4}\left[(bS^2-c)S^2+b\frac{256\omega_4}{R^2}S^2+b\frac{128^2\omega_4^2}{R^4}+c\frac{16\omega_4}{R^4}\right].
\end{aligned}
\tag{2.11}
$$

Since $c > bS^2$, we can choose $R > 0$ large such that

$$
b\frac{256\omega_4}{R^2}S^2+b\frac{128^2\omega_4^2}{R^4}+c\frac{16\omega_4}{R^4}<\frac{1}{2}(c-bS^2)S^2,
$$

then we have from (2.11) that $e_0^c \leqslant E_0^c(U_t) \to -\infty$ as $t \to \infty$. Therefore, c_* is well defined and $c_* = bS^2$.

If e_0^c is achieved by $u \in S_0$, then $E_0^c(u) = 0$. Then from the Sobolev inequality (1.8), we have

$$
\begin{aligned}
0 = E_0^c(u) &= \frac{a}{2}|\nabla u|_2^2+\frac{b}{4}|\nabla u|_2^4-\frac{c}{4}|u|_4^4 \geqslant \frac{a}{2}|\nabla u|_2^2+\frac{b}{4}|\nabla u|_2^4-\frac{c}{4}S^{-2}|\nabla u|_2^4\\
&= \frac{a}{2}|\nabla u|_2^2+(1-\frac{c}{c_*})\frac{b}{4}|\nabla u|_2^4.
\end{aligned}
$$

By the above inequality, $\frac{a}{2}|\nabla u|_2^2+(1-\frac{c}{c_*})\frac{b}{4}|\nabla u|_2^4 \leqslant E_0^c(u) = 0$, then we have from $c \leqslant c_*$ that $|\nabla u|_2 = 0$, which implies that $|u|_4 = 0$ from the Sobolev inequality, and then $u = 0$ a.e. $x \in \mathbb{R}^4$. This contradicts with $u \in S_0$. $\square$

2.2. The case of $V \neq 0$

In this subsection, we consider the case of $V \neq 0$ which is different from the case of $V = 0$. First we recall the following result about the embedding, and its proof is almost the same as that of [22, Lemma 5.1] or Section 3 of [2], hence we omit it here.

Lemma 2.3. *Suppose V satisfies the condition (V), then the embedding $H_V \hookrightarrow L^q(\mathbb{R}^4)$ is compact, for any $2 \leqslant q < 4$.*

Next we prove some properties of the energy infimum e_V^c.

Lemma 2.4. *Suppose V satisfies the condition (V) and $p \in (2,4)$, then the energy infimum e_V^c is continuous and non-increasing for $c \in (0,\infty)$.*

Proof. The non-increasing property is obvious since $E_V^{c_1}(u) \leqslant E_V^{c_2}(u)$ if $c_1 \geqslant c_2$. So we only need to prove the continuity. We first prove the continuity from left. Let $\{c_n\}$ be a sequence such that $c_n \leq c$. If $c_n \to c^-$, then for $\varepsilon > 0$, $c - \varepsilon \leqslant c_n \leqslant c$ and $e_V^c \leqslant e_V^{c_n} \leqslant e_V^{c-\varepsilon}$ as n large. Therefore, $E_V^c(u) \leqslant E_V^{c_n}(u) \leqslant E_V^c(u) + \frac{\varepsilon}{p}|u|_p^p$ for every $u \in S_V$. By the definition of e_V^c, there exists a $u \in S_V$ such that $E_V^c(u) \leqslant e_V^c + \varepsilon$. Hence

$$e_V^c \leqslant e_V^{c_n} \leqslant E_V^{c_n}(u) \leqslant E_V^c(u) + \frac{\varepsilon}{p}|u|_p^p \leqslant e_V^c + \varepsilon + \frac{\varepsilon}{p}|u|_p^p.$$

Therefore $e_V^{c_n} \to e_V^c$ if $c_n \to c^-$ as $n \to \infty$.

On the other hand, we prove that $e_V^{c_n} \to e_V^c$ if $c_n \to c^+$ where $\{c_n\}$ is a sequence satisfying $c_n \geq c$. In fact, it is easy to see that $e_V^{c_n} \leqslant e_V^c$ and $c \leqslant c_n \leqslant 2c$ for n large. According to the definition of $e_V^{c_n}$, there exists a sequence $\{u_n\} \subset S_V$ such that $E_V^{c_n}(u_n) \leqslant e_V^{c_n} + \frac{1}{n}$. From (1.7) and $u_n \in S_V$, we have

$$\begin{aligned} E_V^{c_n}(u_n) &= \frac{a}{2}|\nabla u_n|_2^2 + \frac{b}{4}|\nabla u_n|_2^4 + \frac{1}{2}\int_{\mathbb{R}^4} V(x)u_n^2 - \frac{c_n}{p}|u_n|_p^p \\ &\geqslant \frac{a}{2}|\nabla u_n|_2^2 + \frac{b}{4}|\nabla u_n|_2^4 + \frac{1}{2}\int_{\mathbb{R}^4} V(x)u_n^2 - \frac{2c}{2|Q_p|_2^{p-2}}|\nabla u_n|_2^{2(p-2)}, \end{aligned}$$

which implies that u_n is bounded in H_V. We may assume from Lemma 2.3 that $u_n \rightharpoonup u$ in H_V and $u_n \to u$ in $L^r(\mathbb{R}^4)$ for $r \in [2,4)$. Hence $u \in S_V$ and

$$e_V^c \leqslant E_V^c(u) \leqslant \liminf_{n\to\infty} E_V^c(u_n) = \liminf_{n\to\infty} E_V^{c_n}(u_n) \leqslant \liminf_{n\to\infty}(e_V^{c_n} + \frac{1}{n}) \leqslant e_V^c. \quad \square$$

Again we give the estimates of the threshold value c_* and infimum energy level e_0^c for $2 < p < 4$ and $p = 4$ separately.

Lemma 2.5. *Suppose V satisfies the condition (V) and $p \in (2,4)$, then there exists a $c_* > 0$ such that if $0 < c < c_*$, $e_V^c > 0$; if $c = c_*$, $e_V^c = 0$; and if $c > c_*$, $-\infty < e_V^c < 0$.*

Proof. It is obvious that $E_V^c(u) \geqslant E_0^c(u)$, therefore $\inf_{S_V} E_V^c(u) \geqslant e_0^c > -\infty$. However, if we compute $E_V^c(u_t)$, then $\int_{\mathbb{R}^4} V(x)u_t^2$ may not converge to 0 as $t \to 0$. That is, e_V^c may not be 0 as $e_0^c = 0$. In what follows, we use other methods to obtain the conclusion. For $p \in (2,4)$ and $u \in S_V$, let $C(u) = \frac{1}{p}|u|_p^p$. It follows from

(1.7) and the Young inequality that

$$\begin{aligned}
E_V^c(u) &= \frac{a}{2}|\nabla u|_2^2 + \frac{1}{2}\int_{\mathbb{R}^4} V(x)u^2 + \frac{b}{4}|\nabla u|_2^4 - \frac{c}{p}|u|_p^p \\
&\geqslant \frac{a}{2}\left(\frac{2}{p}\right)^{\frac{1}{p-2}} |Q_p|_2 |u|_p^{\frac{p}{p-2}} + \frac{b}{4}\left(\frac{2}{p}\right)^{\frac{2}{p-2}} |Q_p|_2^2 |u|_p^{\frac{2p}{p-2}} - \frac{c}{p}|u|_p^p + \frac{1}{2}\int_{\mathbb{R}^4} V(x)u^2 \\
&\geqslant \frac{c}{2} - \frac{1}{2}\int_{V(x)\leqslant c} (c-V(x))u^2 dx + 2^{\frac{3-p}{p-2}} a|Q_p|_2 C(u)^{\frac{1}{p-2}} + 4^{\frac{3-p}{p-2}} b|Q_p|_2^2 C(u)^{\frac{2}{p-2}} - cC(u) \\
&\geqslant \frac{c}{2} - \frac{p-2}{2p} c^{-\frac{2}{p-2}} \int_{V(x)\leqslant c} (c-V(x))^{\frac{p}{p-2}} + 4^{\frac{3-p}{p-2}} b|Q_p|_2^2 C(u)^{\frac{2}{p-2}} - 2cC(u).
\end{aligned}$$

Let $f(x) = 4^{\frac{3-p}{p-2}} b|Q_p|_2^2 x^{\frac{2}{p-2}} - 2cx$, then

$$\min_{x\geqslant 0} f(x) = -(4-p)c^{\frac{2}{4-p}} 4^{\frac{p-3}{4-p}} \left(\frac{p-2}{b|Q_p|_2^2}\right)^{\frac{p-2}{4-p}}.$$

If $c < c_1 = \min\left\{\frac{\varepsilon_0}{2}, \frac{b|Q_p|_2^2}{2(p-2)}\left(\frac{1}{p(4-p)}\right)^{\frac{4-p}{p-2}}\right\}$, then we have

$$\begin{aligned}
E_V^c(u) &\geqslant \frac{c}{2} - \frac{p-2}{2p} c^{-\frac{2}{p-2}} c^{\frac{p}{p-2}} m(V_c) - (4-p)c^{\frac{2}{4-p}} 4^{-\frac{3-p}{4-p}} \left(\frac{p-2}{b|Q_p|_2^2}\right)^{\frac{p-2}{4-p}} \\
&\geqslant \frac{c}{2} - \frac{p-2}{2p} c\varepsilon_0 - (4-p)c^{\frac{2}{4-p}} 4^{-\frac{3-p}{4-p}} \left(\frac{p-2}{b|Q_p|_2^2}\right)^{\frac{p-2}{4-p}} \\
&\geqslant \frac{c}{p} - (4-p)c^{\frac{2}{4-p}} 4^{-\frac{3-p}{4-p}} \left(\frac{p-2}{b|Q_p|_2^2}\right)^{\frac{p-2}{4-p}} \\
&= c\left[\frac{1}{p} - (4-p)c^{\frac{p-2}{4-p}} 4^{-\frac{3-p}{4-p}} \left(\frac{p-2}{b|Q_p|_2^2}\right)^{\frac{p-2}{4-p}}\right] \geqslant \frac{c}{2p},
\end{aligned}$$

where $V_c := \{x \in \mathbb{R}^4 : V(x) \leqslant c\}$ and $m(V_c)$ denotes the Lebesgue measure. This implies that $e_V^c > 0$ if $c < c_1$. Let $c_* = \sup\{c > 0 : e_V^c > 0\}$. According to the definition of c_*, $e_V^c > 0$ for $c < c_*$. Fix $u \in S_V$, then it follows from (2.1) that

$$E_V^c(u) = \frac{a}{2}|\nabla u|_2^2 + \frac{1}{2}\int_{\mathbb{R}^4} V(x)u^2 + \frac{b}{4}|\nabla u|_2^4 - \frac{c}{p}|u|_p^p,$$

this implies that there exists a $c_2 > c_*$ (for example, $c_2 > pE_V^0(u)/|u|_p^p$) such that $e_V^c \leqslant E_V^c(u) < 0$ for $c > c_2$. Moreover, by the continuity of e_V^c (Lemma 2.4) and definition of c_*, we have $e_V^{c_*} \geqslant 0$. In the following, we prove that $e_V^{c_*} = 0$ and $e_V^c < 0$ for $c > c_*$. Let $\{u_n\} \subset S_V$ be a minimizing sequence for $e_V^{c_*}$, then one has from (1.7) that

$$\begin{aligned}
E_V^{c_*}(u_n) &= \frac{a}{2}|\nabla u_n|_2^2 + \frac{b}{4}|\nabla u_n|_2^4 + \frac{1}{2}\int_{\mathbb{R}^4} V(x)u_n^2 - \frac{c_*}{p}|u_n|_p^p \\
&\geqslant \frac{a}{2}|\nabla u_n|_2^2 + \frac{b}{4}|\nabla u_n|_2^4 - \frac{c}{2|Q_p|_2^{p-2}}|\nabla u_n|_2^{2(p-2)}.
\end{aligned}$$

Therefore $\{u_n\}$ is bounded in H_V, and we may assume from Lemma 2.3 that there exists a $u \in H_V$ such that $u_n \rightharpoonup u$ in H_V and $u_n \to u$ in $L^r(\mathbb{R}^4)$ for $r \in [2, 4)$. This together with the lower semicontinuity of the norm implies that $u \in S_V$ and $e_V^{c_*} \leqslant E_V^{c_*}(u) \leqslant 0$. Therefore, $e_V^{c_*} = 0$. From direct computing, we can obtain that $e_V^c \leqslant E_V^c(u) < E_V^{c_*}(u) = 0$ if $c > c_*$. □

Next we deal with the case of $p = 4$.

Lemma 2.6. *Suppose V satisfies the condition (V) and $p = 4$. Let $c_* = bS^2$, then $e_V^c > 0$ for $c < c_*$, $e_V^{c^*} = 0$ and $e_V^c = -\infty$ for $c > c_*$.*

Proof. For $c < c_* = bS^2$, let $\lambda = \min\{\varepsilon_0, bS^2 - c\}$. Then we have from (1.8), Young inequality and the condition (V) that

$$\begin{aligned} E_V^c(u) &= \frac{a}{2}|\nabla u|_2^2 + \frac{1}{2}\int_{\mathbb{R}^4} V(x)u^2 + \frac{b}{4}|\nabla u|_2^4 - \frac{c}{4}|u|_4^4 \\ &\geqslant \frac{bS^2}{4}|u|_4^4 - \frac{c}{4}|u|_4^4 + \frac{1}{2}\int_{\mathbb{R}^4} V(x)u^2 \\ &\geqslant \frac{\lambda}{2} - \frac{1}{2}\int_{V(x)\leqslant\lambda} (\lambda - V(x))u^2 dx + \frac{1}{4}(bS^2 - c)|u|_4^4 \\ &\geqslant \frac{\lambda}{2} - \frac{1}{4(bS^2 - c)}\int_{V(x)\leqslant\lambda} (\lambda - V(x))^2 \\ &\geqslant \frac{\lambda}{2} - \frac{1}{4(bS^2 - c)}\lambda^2\varepsilon_0 \geqslant \frac{\lambda}{4}. \end{aligned}$$

Therefore, $e_V^c > 0$ for all $c < c_*$.

For $c > c_*$, according to the proof above, let U_t be defined in Lemma 2.1, then by (2.11),

$$\begin{aligned} E_V^c(U_t) \leqslant{}& \frac{a}{2|Q_4\phi|_2^2}t^2\left[S^2 + \frac{128\omega_4}{R^2}\right] \\ &- \frac{1}{4|Q_4\phi|_2^4}t^4\left[(c - c_*)S^2 - b\frac{256\omega_4}{R^2}S^2 - b\frac{128^2\omega_4^2}{R^4} - c\frac{16\omega_4}{R^4} - 2|Q_4\phi|_2^2\int_{\mathbb{R}^4} V(x)\phi^2(tx)Q_4^2(tx)\right]. \end{aligned}$$

Let $R > 0$ large be fixed such that

$$b\frac{256\omega_4}{R^2}S^2 + b\frac{128^2\omega_4^2}{R^4} + c\frac{16\omega_4}{R^4} < \frac{1}{2}(c - c^*)S^2.$$

Since $\phi(tx)Q_4(tx) \leqslant \phi(x)Q_4(x)$ when $t > 1$, then by the Lebesgue dominated convergence theorem, we have

$$\lim_{t\to\infty}\int_{\mathbb{R}^4} V(x)\phi^2(tx)Q_4^2(tx) = 0, \tag{2.12}$$

and together with (1.9), we have

$$e_V^c \leqslant \lim_{t\to\infty} E_V^c(U_t) = -\infty. \tag{2.13}$$

Finally for $c = c_*$ and $u \in S_V$, we have

$$\begin{aligned} E_V^c(u) &= \frac{a}{2}|\nabla u|_2^2 + \frac{1}{2}\int_{\mathbb{R}^4} V(x)u^2 + \frac{b}{4}|\nabla u|_2^4 - \frac{c}{4}|u|_4^4 \\ &\geqslant \frac{a}{2}|\nabla u|_2^2 + \frac{1}{2}\int_{\mathbb{R}^4} V(x)u^2 + \frac{1}{4}(c_* - c)|u|_4^4 = \frac{a}{2}|\nabla u|_2^2 + \frac{1}{2}\int_{\mathbb{R}^4} V(x)u^2. \end{aligned}$$

Hence, $e_V^{c^*} \geqslant 0$. On the other hand, let ϕ be the cut-off function defined in the proof of Lemma 2.2. For any $\varepsilon > 0$, we have from (2.8)–(2.11) that there exists $R_0 = R_0(\varepsilon) > 0$ such that

$$|Q_4\phi|_2^2 \geqslant \frac{\omega_4}{4}\ln(1 + R^2) \geqslant \frac{\omega_4}{4}\ln(1 + R_0^2) \geqslant \frac{1}{\varepsilon}, \quad |\nabla Q_4|_2^2 = S^2 \geqslant \frac{128\omega_4}{R^2}, \quad R \geqslant R_0, \tag{2.14}$$

and then

$$\frac{1}{4|Q_4\phi|_2^4}\left[b\frac{256\omega_4}{R^2}S^2 + b\frac{128^2\omega_4^2}{R^4} + c_*\frac{16\omega_4}{R^4}\right] \leqslant \frac{\varepsilon^2}{4}\left[b\frac{256\omega_4}{R^2}S^2 + b\frac{128^2\omega_4^2}{R^4} + c_*\frac{16\omega_4}{R^4}\right] \leqslant \varepsilon^3, \ R \geqslant R_0. \tag{2.15}$$

We choose $R = R_0$ in the definition of ϕ. Since $\phi(tx)Q_4(tx) \leqslant \phi(x)Q_4(x)$ when $t > 1$, then by the Lebesgue dominated convergence theorem, we have

$$\lim_{t\to\infty}\int_{\mathbb{R}^4} V(x)\phi^2(tx)Q_4^2(tx) = 0,$$

and hence there exists a $t_0 = t_0(\varepsilon) > 0$ from the definition of limit such that

$$\int_{\mathbb{R}^4} V(x)\phi^2(tx)Q_4^2(tx) \leqslant \varepsilon^2, \quad t \geqslant t_0. \tag{2.16}$$

It follows from (2.14)–(2.16) and the Young inequality that

$$\begin{aligned}
e_V^{c_*} &\leqslant E_V^{c_*}(U_t)\\
&\leqslant \frac{at^2}{2|Q_4\phi|_2^2}\left[S^2 + \frac{128\omega_4}{R^2}\right]\\
&\quad + \frac{t^4}{4|Q_4\phi|_2^4}\left[b\frac{256\omega_4}{R^2}S^2 + b\frac{128^2\omega_4^2}{R^4} + c_*\frac{16\omega_4}{R^4} + 2|Q_4\phi|_2^2\int_{\mathbb{R}^4} V(x)\phi^2(tx)Q_4^2(tx)\right]\\
&\leqslant C\varepsilon t^2 + \varepsilon t^4\int_{\mathbb{R}^4} V(x)\phi^2(tx)Q_4^2(tx) + \varepsilon^3 t^4 \leqslant C\varepsilon t^2 + 2\varepsilon^3 t^4,
\end{aligned}$$

where $C > 0$ is a positive constant independent of ε and R. Choosing $t = \max\{\varepsilon^{-1/4}, t_0\}$, we can obtain that $e_V^{c^*} \leqslant C\varepsilon^{\frac{1}{2}}$. This implies that $e_V^{c^*} = 0$. □

3. Proofs of main theorems

In this section, we prove our main results Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Part (a) follows directly from Lemma 2.1, and part (c) follows from Lemmas 2.1 and 2.2. So we only need to prove part (b) which is divided in the following cases.

Case 1. When $2 < p < 4$, E_0^c has a minimizer with $-\infty < e_0^c < 0$ if $c > c_*$.

Let $\{u_n\} \subset S_0$ such that $E_0^c(u_n) \to e_0^c$. Then

$$E_0^c(u_n) = \frac{a}{2}|\nabla u_n|_2^2 + \frac{b}{4}|\nabla u_n|_2^4 - \frac{c}{p}|u_n|_p^p \geqslant \frac{a}{2}|\nabla u_n|_2^2 + \frac{b}{4}|\nabla u_n|_2^4 - \frac{c}{2|Q_p|_2^{p-2}}|\nabla u_n|_2^{2(p-2)}.$$

The conditions $p \in (2,4)$ and $|u_n|_2 = 1$ imply that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^4)$. According to Schwarz rearrangement, there exists a radial symmetric sequence $\{\hat{u}_n\} \subset H^1(\mathbb{R}^4)$ such that

$$\int_{\mathbb{R}^4} |\nabla \hat{u}_n|^2 dx \leqslant \int_{\mathbb{R}^4} |\nabla u_n|^2 dx, \quad \int_{\mathbb{R}^4} |\hat{u}_n|^r dx = \int_{\mathbb{R}^4} |u_n|^r dx, \; r \in [2,4].$$

Hence $\{\hat{u}_n\} \subset S_0$ is bounded in $H_r^1(\mathbb{R}^4)$ which consists of radial functions in $H^1(\mathbb{R}^4)$. From the compactness of embedding $H_r^1(\mathbb{R}^4) \hookrightarrow L^p(\mathbb{R}^4)$ for $p \in (2,4)$, we may assume that $\hat{u}_n \rightharpoonup u$ in $H^1(\mathbb{R}^4)$, $\hat{u}_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^4$ and $\hat{u}_n \to u$ in $L^q(\mathbb{R}^4)$ for $p \in (2,4)$. From $\{\hat{u}_n\} \subset S_0$, we have

$$\begin{aligned}
e_0^c \leqslant E_0^c(\hat{u}_n) &= \frac{a}{2}|\nabla \hat{u}_n|_2^2 + \frac{b}{4}|\nabla \hat{u}_n|_2^4 - \frac{c}{p}|\hat{u}_n|_p^p\\
&\leqslant \frac{a}{2}|\nabla u_n|_2^2 + \frac{b}{4}|\nabla u_n|_2^4 - \frac{c}{p}|u_n|_p^p = E_0^c(u_n) \to e_0^c \;\; (n\to\infty).
\end{aligned}$$

Therefore, $\{\hat{u}_n\}$ is also a minimizing sequence. By the weak lower semicontinuity of the norm function, we have $|u|_2 \leqslant 1$. If $|u|_2 = 1$, then the weak lower semicontinuity of the norm and $\hat{u}_n \to u$ in $L^p(\mathbb{R}^4)$ imply that,

$$e_0^c \leqslant E_0^c(u) \leqslant \liminf_{n\to\infty} E_0^c(\hat{u}_n) = e_0^c. \tag{3.1}$$

Therefore, e_0^c is achieved by u. If $|u|_2 < 1$, by Lemma 2.5 and the second inequality of (3.1), $E_0^c(u) \leqslant e_0^c < 0$ and then $u \neq 0$. Let $u_t(x) = u(tx)$ for all $x \in \mathbb{R}^4$, then there exists a $t < 1$ such that $|u_t|_2 = t^{-4}|u|_2 = 1$. That is $u_t \in S_0$. Since $E_0^c(u) \leqslant e_0^c < 0$,

$$\begin{aligned} e_0^c \leqslant E_0^c(u_t) &= \frac{a}{2}t^{-2}|\nabla u|_2^2 + \frac{b}{4}t^{-4}|\nabla u|_2^4 - \frac{c}{p}t^{-4}|u|_p^p \\ &< \frac{a}{2}t^{-4}|\nabla u|_2^2 + \frac{b}{4}t^{-4}|\nabla u|_2^4 - \frac{c}{p}t^{-4}|u|_p^p = t^{-4}E_0^c(u) < E_0^c(u) \leqslant \liminf_{n\to\infty} E_0^c(\hat{u}_n) = e_0^c. \end{aligned}$$

This is a contradiction. Therefore, $|u|_2 = 1$ and $E_0^c(u) = e_0^c$. Moreover, since u is the limit of $\hat{u}_n$, u is also radial symmetric.

Case 2. When $p \in (2,3]$ and $c \leqslant c_*$, e_0^c cannot be achieved.

If $p \in (2,3)$, then $c_* = 0$. Therefore we have from Case 1 that e_0^c is achieved for all $c > 0$. If $p = 3$, then E_0^c has no minimizer for all $0 < c \leqslant c_*$. In fact if $u \in S_0$ with $e_0^c = E_0^c(u)$, then from (2.3), we have

$$0 = e_0^c = E_0^c(u) \geqslant (1 - \frac{c}{c_*})\frac{a}{2}|\nabla u|_2^2 + \frac{b}{4}|\nabla u|_2^4 \geqslant \frac{b}{4}|\nabla u|_2^4.$$

This implies that $u = 0$ in $D^{1,2}(\mathbb{R}^4)$ and then $u = 0$ in $L^2(\mathbb{R}^4)$, which is impossible.

Case 3. When $p \in (3,4)$ and $c \leqslant c_*$. In this case, if $u \in S_0$ satisfying $0 = e_0^c = E_0^c(u)$, then from (2.3), we have $f_{c,p}(|\nabla u|_2^2) \leqslant 0$. When $c < c_*$, $g_{c,p}(|\nabla u|_2^2) > 0$ and then $u = 0$ which is impossible. If $c = c_*$, then $f_{c,p}(t_p) = \min_{t\geqslant 0} f_{c,p}(t) = 0$. Then it follows from (2.5) and the definition of c_* in Lemma 2.1 that $E_0^c((Q_p)_{t_p}) = f_{c,p}(t_p) = 0$. Therefore, $e_0^{c_*}$ can be achieved by $(Q_p)_{t_p}$. Similar to the proof of Case 1, we can obtain that the minimizer is radially symmetric.

Finally we prove that minimizer is unique up to translations. It is easy to see that $(Q_p)_{t_p}$ can achieve e_0^c. If $e_0^c = E_0^c(u)$, then $f_{c,p}(t_p) = e_0^c$. According to (1.7) and (2.3), we have that $u = (Q_p)_t$ for some $t > 0$. By using (2.5), we have $f_{c,p}(t) = e_0^c$. From the uniqueness of t_p, we know that $t = t_p$. Therefore, $u = (Q_p)_{t_p}$ up to translations. □

Proof of Theorem 1.2. The range of value of e_V^c has been obtained in Lemmas 2.5 and 2.6. So we only need to show whether the infimum can be attained or not in the following cases.

Case 1. When $p = 4$ and $c < c_* = bS^2$, e_V^c can be achieved.

Let $\{u_n\} \subset S_V$ be a minimizing sequence. The boundedness of $\{u_n\}$ is obvious from the Sobolev inequality and

$$\begin{aligned} E_V^c(u_n) &= \frac{a}{2}|\nabla u_n|_2^2 + \frac{1}{2}\int_{\mathbb{R}^4} V(x)u_n^2 + \frac{b}{4}|\nabla u_n|_2^4 - \frac{c}{4}|u_n|_4^4 \\ &\geqslant \frac{a}{2}|\nabla u_n|_2^2 + \frac{bS^2}{4}|u_n|_4^4 - \frac{c}{4}|u_n|_4^4 + \frac{1}{2}\int_{\mathbb{R}^4} V(x)u_n^2. \end{aligned}$$

We may assume that $u_n \rightharpoonup u$ in H_V. By Lemma 2.3 and $u_n \in S_V$, we have $u \in S_V$ and there exist $\lambda_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty}[E_V^{c\,\prime}(u_n) - \lambda_n u_n] = 0.$$

Set $\lim_{n\to\infty}|\nabla u_n|_2^2 = A$. Then

$$\begin{aligned} \lim_{n\to\infty}\lambda_n &= \lim_{n\to\infty}(E_V^{c\,\prime}(u_n), u_n) = \lim_{n\to\infty}[(E_V^{c\,\prime}(u_n), u_n) - 4E_V^c(u_n) + 4e_V^c] \\ &= \lim_{n\to\infty}(4e_V^c - a|\nabla u_n|_2^2 - \int_{\mathbb{R}^4} V(x)u_n^2) := \lambda. \end{aligned}$$

On the other hand, we have for $\phi \in H_V$,

$$
\begin{aligned}
0 &= \lim_{n\to\infty} ({E_V^c}'(u_n) - \lambda_n u_n, \phi) = \lim_{n\to\infty} [({E_V^c}'(u_n), \phi) - \lambda_n \int_{\mathbb{R}^4} u_n \phi] \\
&= \lim_{n\to\infty} [(a + b|\nabla u_n|_2^2) \int_{\mathbb{R}^4} \nabla u_n \nabla \phi + \int_{\mathbb{R}^4} V(x) u_n \phi - c \int_{\mathbb{R}^4} u_n^3 \phi - \lambda_n \int_{\mathbb{R}^4} u_n \phi] \\
&= (a + bA) \int_{\mathbb{R}^4} \nabla u \nabla \phi + \int_{\mathbb{R}^4} V(x) u \phi - c \int_{\mathbb{R}^4} u^3 \phi - \lambda \int_{\mathbb{R}^4} u \phi, \quad \phi \in H_V.
\end{aligned}
$$

Let $\phi = u$, then

$$
\lambda = (a + bA)|\nabla u|_2^2 + \int_{\mathbb{R}^4} V(x) u^2 - c|u|^4.
$$

By $u \in S_V$ and $|\nabla u|_2^2 \leqslant \liminf_{n\to\infty} |\nabla u_n|_2^2 = A$, we obtain that

$$
\begin{aligned}
e_V^c \leqslant E_V^c(u) &= \frac{1}{2} a |\nabla u|_2^2 + \frac{1}{2} \int_{\mathbb{R}^4} V(x) u^2 + \frac{1}{4} b |\nabla u|_2^4 - \frac{c}{4} |u|_4^4 \\
&\leqslant \frac{1}{2} a |\nabla u|_2^2 + \frac{1}{2} \int_{\mathbb{R}^4} V(x) u^2 + \frac{1}{4} bA |\nabla u|_2^2 - \frac{c}{4} |u|_4^4 \\
&= \frac{1}{4} a |\nabla u|_2^2 + \frac{1}{4} \int_{\mathbb{R}^4} V(x) u^2 + \frac{1}{4} \lambda \\
&\leqslant \liminf_{n\to\infty} \left[\frac{1}{4} a |\nabla u_n|_2^2 + \frac{1}{4} \int_{\mathbb{R}^4} V(x) u_n^2 + \frac{1}{4} \lambda_n \right] \\
&= \liminf_{n\to\infty} \left(\frac{1}{4} a |\nabla u_n|_2^2 + \frac{1}{4} \int_{\mathbb{R}^4} V(x) u_n^2 + \frac{1}{4} [({E_V^c}'(u_n), u_n) - 4E_V^c(u_n) + 4e_V^c] \right) \\
&= e_V^c.
\end{aligned}
$$

Therefore, $E_V^c(u) = e_V^c$ and e_V^c can be achieved.

Case 2. When $p = 4$ and $c = c_*$, e_V^c is not achieved.

When $c = c_*$, since

$$
E_V^{c_*}(u) \geqslant \frac{a}{2} |\nabla u|_2^2 + \frac{1}{2} \int_{\mathbb{R}^4} V(x) u^2 + \frac{b}{4} (1 - \frac{c_*}{c_*}) |\nabla u|_2^4 \geqslant \frac{a}{2} |\nabla u|_2^2,
$$

then by using similar proof of that of the case $V = 0$, we can show that $e_V^{c_*} = 0$ is not achieved.

Case 3. When $2 < p < 4$, e_V^c is achieved for all $c > 0$.

If $2 < p < 4$, for all $c > 0$, let $\{u_n\} \subset S_V$ be a minimizing sequence for e_V^c. Then it can easily be seen that $\{u_n\}$ is bounded in $H_V(\mathbb{R}^4)$ such that $u_n \rightharpoonup u$ in $H_V(\mathbb{R}^4)$ and $u_n \to u$ in $L^q(\mathbb{R}^4)$ for $q \in [2, 4)$. Similarly, we can obtain $|u|_2 = 1$. So, we have

$$
\begin{aligned}
e_V^c \leqslant E_V^c(u) &= \frac{a}{2} |\nabla u|_2^2 + \frac{b}{4} |\nabla u|_2^4 + \frac{1}{2} \int_{\mathbb{R}^4} V(x) u^2 - \frac{c}{p} |u|_p^p \\
&\leqslant \liminf_{n\to\infty} [\frac{a}{2} |\nabla u_n|_2^2 + \frac{b}{4} |\nabla u_n|_2^4 + \frac{1}{2} \int_{\mathbb{R}^4} V(x) u_n^2 - \frac{c}{p} |u_n|_p^p] \\
&= \liminf_{n\to\infty} E_V^c(u_n) = e_V^c.
\end{aligned}
$$

Then, $E_V^c(u) = e_V^c$, i.e., E_V^c has a minimizer u. □

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