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GROUND STATE SOLUTIONS OF NEHARI-POHOZAEV TYPE FOR THE PLANAR SCHRÖDINGER-POISSON SYSTEM WITH GENERAL NONLINEARITY

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Abstract. It is shown that the planar Schrödinger-Poisson system with a general nonlinear interaction function has a nontrivial solution of mountain-pass type and a ground state solution of Nehari-Pohozaev type. The conditions on the nonlinear functions are much weaker and flexible than previous ones, and new variational and analytic techniques are used in the proof.

1. **Introduction.** In this paper, we study the ground state solutions of the following planar Schrödinger-Poisson system with a general nonlinearity:

$$\begin{cases} -\Delta u + u + \phi u = f(u), & x \in \mathbb{R}^2, \\ \Delta \phi = u^2, & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where the nonlinear function f satisfies the following basic assumptions:

(F1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, and there exist constants $\mathcal{C}_0 > 0$ and $p \in (2, \infty)$ such that

$$|f(u)| \leq \mathcal{C}_0 \left(1 + |u|^{p-1}\right), \quad \forall\ u \in \mathbb{R};$$

(F2) $f(u) = o(|u|)$ as $u \to 0$;

(F3) $\inf_{u \neq 0} \dfrac{F(u)}{|u|^2} > -\infty$, where $F(u) := \displaystyle\int_0^u f(s)\mathrm{d}s$.

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System (1.1) is a special form of the following nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(u), & x \in \mathbb{R}^N, \\ \Delta\phi = u^2, & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $\lambda \in \mathbb{R}$, $V \in \mathcal{C}(\mathbb{R}^N, (0, \infty))$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$. It is well known that the solutions of (1.2) are related to the solitary wave solutions of the form $\psi(x, t) = e^{-i\mu t}u(x), \mu \in \mathbb{R}$ to the following nonlinear Schrödinger-Poisson system

$$\begin{cases} -i\psi_t - \Delta\psi + E(x)\psi + \lambda\phi\psi = f(\psi), & x \in \mathbb{R}^N,\ t > 0, \\ \Delta\phi = |\psi|^2, & x \in \mathbb{R}^N,\ t > 0, \end{cases} \tag{1.3}$$

where $\psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ is the wave function, $E(x) = V(x) - \mu$ with $\mu \in \mathbb{R}$ is a real-valued external potential, $\lambda \in \mathbb{R}$ is a parameter, ϕ represents an internal potential for a nonlocal self-interaction of the wave function and the nonlinear term f describes the interaction effect among particles. System (1.3) arises from quantum mechanics (see e.g. [5,7,24]) and in semiconductor theory [4,26,27]. For more details in the physical aspects, we refer the readers to [3,4].

The solution ϕ of the Poisson equation in (1.2) can be solved by $\phi = \Gamma_N * u^2$, where $*$ is the convolution in $\mathbb{R}^N$, Γ_N is the fundamental solution of the Laplacian, which is given by

$$\Gamma_N(x) = \begin{cases} \frac{1}{2\pi}\ln|x|, & N = 2, \\ \frac{1}{N(2-N)\omega_N}|x|^{2-N}, & N \geq 3, \end{cases}$$

and ω_N is the volume of the unit N-ball. With this formal inversion, system (1.2) is converted into an equivalent integro-differential equation

$$-\Delta u + V(x)u + \lambda(\Gamma_N * u^2)u = f(u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

Denote by $\phi_{N,u}(x) = (\Gamma_N * u^2)(x)$. Then at least formally, the energy functional associated with (1.2) is

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x + \frac{\lambda}{4}\int_{\mathbb{R}^N} \phi_{N,u}u^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

If u is a critical point of I_λ, then the pair $(u, \phi_{N,u})$ is a weak solution of (1.2). For the sake of simplicity, in many cases we just say u, instead of $(u, \phi_{N,u})$, is a weak solution of (1.2).

In recent years, the existence of nontrivial solutions, ground state solutions and multiple solutions to (1.2) (or (1.4)) have been investigated extensively. The majority of the literature focuses on the study of (1.2) with $N = 3$ and $\lambda < 0$. In this case, by the Hardy-Littlewood-Sobolev inequality, I_λ is a well-defined $\mathcal{C}^1$ functional on a weighted Sobolev space, and the mountain pass geometry can be verified provided $f(t)$ is superlinear at $t = 0$ and super-cubic at $t = \infty$. In this situation, the existence, multiplicity and concentration of solutions of (1.2) was obtained under various assumptions on V and f, see e.g. [1–4,8–12,14,15,18,19,21,30,31,37–39] and so on. If $f(t)$ is super-quadratic at $t = \infty$, by using the Nehari-Pohozaev manifold introduced in [28], the existence of Nehari-Pohozaev type ground state solutions of (1.2) were established, see e.g. [2,28,31,35,41] and so on.

Unlike the case of $N = 3$, there are only a few papers dealing with (1.2) with $N = 2$. The approach for the $N = 3$ case cannot be easily adapted to $N = 2$ because that the logarithmic integral kernel $1/(2\pi)\ln|x|$ is sign-changing and is neither bounded from above nor from below, and I_λ is not well defined on $H^1(\mathbb{R}^2)$ even if $V \in L^\infty(\mathbb{R}^2)$ and $\inf_{\mathbb{R}^2} V > 0$.

A new variational framework for (1.2) with $N = 2$ within the functional space

$$E = \left\{ u \in H^1(\mathbb{R}^2) \ : \ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x|)u^2(x)\mathrm{d}x < +\infty \right\}$$

was introduced in [29]. Considering the case $N = 2, V(x) = a \in \mathbb{R}, \lambda < 0$ and $f(u) = 0$ in (1.2), by using strict rearrangement inequalities, Stubbe [29] proved that there exists, for any $a \geq 0$, a unique ground state, which is a positive spherically symmetric decreasing function. In the same case, Bonheure, Cingolani and Van Schaftingen [6] derived the asymptotic decay of the unique positive, radially symmetric solution, and also established its nondegeneracy. Cingolani and Weth [13] developed a variational framework for the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{p-2}u, & x \in \mathbb{R}^2, \\ \Delta \phi = u^2, & x \in \mathbb{R}^2, \end{cases} \tag{1.5}$$

where $V \in L^\infty(\mathbb{R}^2)$ (i.e., $N = 2, \lambda > 0$ and $f(u) = |u|^{p-2}u$ in (1.2)). In particular, when $V(x)$ is 1-periodic in x_1 and x_2 and $p \geq 4$, they proved that (1.5) admits high energy solutions, and a ground state solution of Nehari type which is a minimizer of I_1 on the corresponding Nehari manifold. The key tool is a surprisingly strong compactness condition for Cerami sequences which is not available for the corresponding problem in higher space dimensions. Based on this strong compactness condition, Du and Weth [16] provided a counterpart of the results in [13] in the case where $2 < p < 4$ and V is a positive constant. They showed that (1.5) with $V \equiv 1$ admits a nontrivial solution of mountain-pass type if $p > 2$, and a ground state solution of Nehari-Pohozaev type which is a minimizer of I_1 on the Nehari-Pohozaev manifold (see definition below) if $p \geq 3$. However the approach in [16] relies heavily on the algebraic form $f(u) = |u|^{p-2}u$ with $p \geq 3$, see [16, Lemma 4.1], and it is difficult to generalize the results on existence of ground state solutions for (1.5) to (1.1) with a general interaction function $f(u)$. Also the smoothness of $f(u)$ in [16] is necessary for applying the Implicit Function Theorem to obtain certain results.

In this paper, we consider the existence of mountain-pass type solutions and also ground state solutions of (1.1) under much weaker and more general assumptions on f. As in [13], we define, for any measurable function $u : \mathbb{R}^2 \to \mathbb{R}$,

$$\|u\|_*^2 = \int_{\mathbb{R}^2} \ln(1+|x|)u^2(x)\mathrm{d}x \in [0, \infty].$$

Then the set

$$E = \left\{ u \in H^1(\mathbb{R}^2) \ : \ \|u\|_* < +\infty \right\}$$

is a Hilbert space equipped with the norm

$$\|u\|_E = \left(\|u\|^2 + \|u\|_*^2\right)^{1/2}.$$

We consider the system (1.1), the associated scalar equation

$$-\Delta u + u + \phi_{2,u}u = f(u), \qquad x \in \mathbb{R}^2, \tag{1.6}$$

and the associated energy functional $\Phi : E \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2\right)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^2} \phi_{2,u}(x)u^2\mathrm{d}x - \int_{\mathbb{R}^2} F(u)\mathrm{d}x, \tag{1.7}$$

where

$$\phi_{2,u}(x) = (\Gamma_2 * u^2)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x-y|\ u^2(y)\mathrm{d}y.$$

Similar to [16, Lemma 2.4], we define the Pohozaev functional of (1.6):

$$\mathcal{P}(u) := \|u\|_2^2 + \int_{\mathbb{R}^2} \phi_{2,u}(x)u^2\mathrm{d}x + \frac{1}{8\pi}\|u\|_2^4 - 2\int_{\mathbb{R}^2} F(u)\mathrm{d}x. \tag{1.8}$$

It is well-known that any solution u of (1.1) satisfies $\mathcal{P}(u) = 0$. Motivated by this fact, we define the following functional on E:

$$\begin{aligned} J(u) =& 2\langle \Phi'(u), u\rangle - \mathcal{P}(u) \\ =& 2\|\nabla u\|_2^2 + \|u\|_2^2 + \int_{\mathbb{R}^2} \phi_{2,u}(x)u^2\mathrm{d}x - \frac{1}{8\pi}\|u\|_2^4 - 2\int_{\mathbb{R}^2} [f(u)u - F(u)]\mathrm{d}x, \end{aligned} \tag{1.9}$$

and define the Nehari-Pohozaev manifold of Φ by

$$\mathcal{M} := \{u \in E \setminus \{0\} : \ J(u) = 0\}. \tag{1.10}$$

Then every non-trivial solution of (1.1) is contained in $\mathcal{M}$. In particular we call a solution $\bar{u}$ of (1.1) to be a ground state solution if $\bar{u} \neq 0$ satisfies $\Phi(\bar{u}) = \inf_{u\in\mathcal{M}} \Phi(u)$. Also a solution $\bar{u}$ is a least energy solution of (1.1) if $\Phi(\bar{u})$ is the smallest among all non-trivial solutions of (1.1). Finally a solution $\bar{u}$ is called a solution of Mountain-Pass type of (1.1) if $\Phi(\bar{u}) = c$ where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)), \qquad \Gamma := \{\gamma \in \mathcal{C}([0,1], E) : \gamma(0) = 0,\ \Phi(\gamma(1)) < 0\}.$$

To state our main results, in addition to (F1)-(F3), we introduce the following assumptions:

(F4) there exist constants $\alpha_0, \beta_0, c_0 > 0$ and $\kappa > 1$ such that

$$f(u)u - 3F(u) + \alpha_0 u^2 \geq 0, \qquad \forall\ u \in \mathbb{R},$$

and

$$\left|\frac{f(u)}{u}\right| \geq \beta_0 \quad \Rightarrow \quad \left|\frac{f(u)}{u}\right|^{\kappa} \leq c_0\left[f(u)u - 3F(u) + \alpha_0 u^2\right];$$

(F5) $p = p_0 \in (2, 4)$ in (F1), and there exist constants $\alpha_1 > 0$ and $p_1, p_2 \in [2, 6-p_0)$ such that

$$f(u)u - 3F(u) \geq -\alpha_1(|u|^{p_1} + |u|^{p_2}), \qquad \forall\ u \in \mathbb{R};$$

(F6) the function $\dfrac{f(u)u - F(u) - \frac{1}{2}u^2}{u^3}$ is nondecreasing on both $(-\infty, 0)$ and $(0, \infty)$.

Now, we state our results of this paper.

Theorem 1.1. *Assume that f satisfies (F1)-(F3) and (F4) or (F5). Then (1.1) or (1.6) has a solution of mountain-pass type $u_0 \in E$ such that $\Phi(u_0) > 0$. Moreover, (1.1) or (1.6) has a least energy solution $\hat{u} \in E \setminus \{0\}$.*

Theorem 1.2. *Assume that f satisfies (F1)-(F3) and (F6). Then (1.1) or (1.6) has a ground state solution $\bar{u} \in E \setminus \{0\}$ such that $\Phi(\bar{u}) = \inf_{u\in\mathcal{M}} \Phi(u)$.*

We remark that the assumptions (F4)-(F6) are weaker than some assumptions which are easier to state and verify:

(F7) $f(u)u - 3F(u) \geq 0$, $\forall\ u \in \mathbb{R}$;

(F8) there exist constants $\alpha_2 > 0$ and $p_3, p_4 \in [2, 3)$ such that

$$-\alpha_2(|u|^{p_3} + |u|^{p_4}) \le f(u)u - 3F(u) \le 0, \qquad \forall\, u \in \mathbb{R}.$$

(F9) the function $\dfrac{f(u)u - F(u)}{u^3}$ is nondecreasing on both $(-\infty, 0)$ and $(0, \infty)$.

It is easy to see that (F7) implies (F4), (F8) implies (F5) and (F9) implies (F6). Results in Theorems 1.1 and 1.2 in [16] are special cases of Theorems 1.1 and 1.2 as the function $f(u) = |u|^{p-2}u$ satisfies (F7) and (F9) when $p \ge 3$, and it satisfies (F8) when $2 < p < 3$.

Our more general conditions (F4)-(F6) on the function $f(u)$ allow for many other examples other than the pure power function as in [16]. Table 1 below lists some examples satisfying (F4) or (F5), and possibly (F6). Note that other than $f_1(u)$, all other functions are not pure power functions, and some of them allow logarithmic growth. The function $f_2(u)$ has different growth rates at $u = 0$ and $u = \infty$, and the function $f_3(u)$ is asymptotically linear as $u \to \infty$. This demonstrates that our results can be applied to much more general situations compared to the special case in [16]. The condition (F6) is a monotonicity one which is usually more restrictive. But we note that (F6) does not always imply (F4). Indeed for $f_7(u)$ in Table 1, $F_7(u) = \int_0^u f_7(t)dt = u^4 \int_0^u |s|^{1+\sin s} s ds$. Since $f(u)u \ge 4F(u)$, then f satisfies (F4). But it is easy to see that f does not satisfy (F6). The function $f_6(u)$ and $f_2(u)$ with $2 < q < 3 \le p$ and $0 \le b \le b_0$ are two examples that satisfy (F6) but do not satisfy (F9), while $f_5(u)$ satisfies (F9).

$f(u)$	(F4)	(F5)	(F6)
$f_1(u) = \|u\|^{p-2}u$	$3 \le p$	$2 < p \le 3$	$3 \le p$
$f_2(u) = \left(\|u\|^{p-2} + b\|u\|^{q-2}\right)u$	$2 < q < 3 < p$	$2 < q < p \le 3$	$2 < q < 3 \le p$, $0 \le b \le b_0$
$f_3(u) = u\left[1 - \frac{1}{\ln(e+u^2)}\right]$	YES	YES	NO
$f_4(u) = u\ln(1+u^2)$	NO	YES	NO
$f_5(u) = \|u\|u\ln(1+u^2)$	YES	NO	YES
$f_6(u) = 3\|u\|u\ln\left(1+u^2\right) + \frac{2\|u\|^3 u}{1+u^2}$	YES	NO	YES
$f_7(u) = 4u^3\int_0^u \|s\|^{1+\sin s} s ds + \|u\|^{5+\sin u}u$	YES	NO	NO

Table 1. Examples of nonlinear functions $f(u)$ satisfying conditions in Theorems 1.1 and 1.2. Here $b_0 = \frac{q(p-2)}{(q-1)(3-q)}\left[\frac{(p-1)(p-3)}{p(q-2)}\right]^{\frac{q-2}{p-2}}[2(p-q)]^{\frac{q-p}{p-2}}$.

For reader's convenience, we choose f_2, f_3 and f_4 of Table 1 as examples and furnish some details as follows.

(1) $f_2(u) = \left(|u|^{p-2} + b|u|^{q-2}\right)u$. Then $F_2(u) = \frac{1}{p}|u|^p + \frac{b}{q}|u|^q$.

- Case i) $2 < q < 3 < p$. Let

$$\alpha_0 = \frac{p-q}{p-2}\left(\frac{(3-q)|b|}{q}\right)^{\frac{p-2}{p-q}}\left[\frac{p(q-2)}{(p-2)(q-3)}\right]^{\frac{p-2}{p-q}}.$$

By Young's inequality, one has

$$f_2(u)u - 3F_2(u) + \alpha_0 u^2 = u^2\left(\frac{p-3}{p}|u|^{p-2} + \alpha_0 - \frac{(3-q)b}{q}|u|^{q-2}\right) \ge 0, \qquad \forall u \in \mathbb{R}.$$

Let $\kappa = \frac{p}{p-1}$. By an elemental calculation, one can derive that there exist $\beta_0, c_0 > 0$ such that

$$\left|\frac{f_2(u)}{u}\right| \geq \beta_0 \quad \Rightarrow \quad \left|\frac{f_2(u)}{u}\right|^{\kappa} \leq c_0\left[f_2(u)u - 3F_2(u) + \alpha_0 u^2\right].$$

- Case ii) $2 < q < p \leq 3$. Note that

$$f(u)u - 3F(u) = -\left(\frac{3-p}{p}|u|^p + \frac{(3-q)b}{q}|u|^q\right).$$

When $p = 3$, f_2 satisfies (F5) with $p_0 = 3$, $p_1 = p_2 = q$ and $c_1 = 1$. When $p < 3$, f_2 satisfies (F5) with $p_0 = p_1 = p$, $p_2 = q$ and $c_1 = 1 + |b|$.

- Case iii) $2 < q < 3 \leq p$ and $0 \leq b \leq b_0$. By a simple calculation, we can verify that f_2 satisfies (F6).

(2) $f_3(u) = u\left[1 - \frac{1}{\ln(e+u^2)}\right]$. Then

$$F_3(u) = \frac{u^2}{2}\left[1 - \frac{1}{\ln(e+u^2)}\right] - \int_0^u \frac{s^3}{\ln^2(e+s^2)}\mathrm{d}s.$$

Let $\alpha_0 = \frac{1}{2}$. Then

$$f_3(u)u - 3F_3(u) + \alpha_0 u^2 = \frac{u^2}{2\ln(e+u^2)} + 3\int_0^u \frac{s^3}{\ln^2(e+s^2)}\mathrm{d}s \geq 0.$$

Let $\kappa = 2$. By an elemental calculation, one can derive that there exist $\beta_0, c_0 > 0$ such that

$$\left|\frac{f_3(u)}{u}\right| \geq \beta_0 \quad \Rightarrow \quad \left|\frac{f_3(u)}{u}\right|^{\kappa} \leq c_0\left[f_3(u)u - 3F_3(u) + \alpha_0 u^2\right].$$

Hence, f_3 satisfies (F4). It is easy to see that f_3 satisfies (F5) with $\alpha_1 = 1$, $p_0 = 3$ and $p_1 = p_2 = 2$. Noting that

$$\frac{f_3(u)u - F_3(u) - \frac{1}{2}u^2}{u^3} = \frac{-\frac{u^2}{2\ln(e+u^2)} - \int_0^u \frac{s^3}{\ln^2(e+s^2)}\mathrm{d}s}{u^3},$$

it follows that f_3 does not satisfy (F6).

(3) $f_4(u) = u\ln(1+u^2)$. Then $F_4(u) = \frac{1}{2}u^2\ln(1+u^2) - \frac{1}{2}u^2 + \frac{1}{2}\ln(1+u^2)$, and so

$$f_4(u)u - F_4(u) = -\frac{1}{2}u^2\ln(1+u^2) + \frac{3}{2}u^2 - \frac{3}{2}\ln(1+u^2). \tag{1.11}$$

Noting that

$$\lim_{|u|\to\infty} \frac{f_4(u)u - F_4(u)}{u^2} = -\infty,$$

it follows that f_4 does not satisfy (F4). For $\delta \in (2,3)$, by (1.11), one has

$$\lim_{|u|\to\infty} \frac{f_4(u)u - F_4(u)}{|u|^{\delta}} = 0. \tag{1.12}$$

Letting $p_0 = p_1 = \delta$ and $p_2 = 2$, one can deduce from (1.12) that there exists $\alpha_1 > 0$ such that

$$f_4(u)u - 3F_4(u) \geq -\alpha_1(|u|^{p_1} + |u|^{p_2}), \quad \forall\, u \in \mathbb{R},$$

and so f_4 satisfies (F5). Noting that

$$\frac{f_4(u)u - F_4(u) - \frac{1}{2}u^2}{u^3} = -\frac{1}{2}\ln(1+u^2)\left(\frac{1}{u} + \frac{3}{u^3}\right) + \frac{1}{u},$$

it follows that f_4 does not satisfy (F6).

To prove Theorem 1.1, based on the variational approach developed in [13, 16], first we construct a Cerami sequence $\{u_n\}$ of Φ with the extra property that $J(u_n) \to 0$, this idea goes back to [17]. Then we prove the boundedness of $\{u_n\}$ in $H^1(\mathbb{R}^2)$ in the two cases that (F4) or (F5) holds (see Lemmas 2.3 and 2.4). The proof of boundedness result in [16] can be modified to the case that (F4) holds, but it does not work for the case that (F5) holds. For that case, we introduce sequences $t_n = \|\nabla u_n\|_2^{-1/2}$ and $v_n = t_n^2 (u_n)_{t_n}$ where $u_t(x) = u(tx)$ to deduce that $\|v_n\|_2 \to 0$ and $\|v_n\|_2^4 \ln t_n \to 0$ if $\|\nabla u_n\|_2 \to \infty$, and combining the fact that $J(u_n) \to 0$, the Gagliardo-Nirenberg inequality and subtle analysis, we obtain the boundedness of $\{u_n\}$ in $H^1(\mathbb{R}^2)$ by estimating $\Phi(v_n)$ from two different directions (see Lemma 2.4). Our proof of Theorem 1.2 is inspired by the approach used in [34, 35]. We first establish a crucial inequality

$$\Phi(u) \geq \Phi(t^2 u_t) + \frac{1-t^4}{4} J(u), \quad u \in E, \quad t > 0.$$

With this inequality in hand, then we can find a minimizing Cerami sequence for Φ on $\mathcal{M}$ and show its boundedness in $H^1(\mathbb{R}^2)$.

The paper is organized as follows. In Section 2, we give the variational setting and preliminaries. We complete the proofs of Theorems 1.1 and 1.2 in Sections 3 and 4 respectively. Throughout this paper, we let $u_t(x) := u(tx)$ for $t > 0$, $H^1(\mathbb{R}^2)$ is the usual Sobolev space with the standard scalar product and norm

$$(u, v) = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + uv)\, \mathrm{d}x, \quad \|u\| = \left(\int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2 \right) \mathrm{d}x \right)^{1/2}.$$

and denote the norm of $L^s(\mathbb{R}^2)$ by $\|u\|_s = \left(\int_{\mathbb{R}^2} |u|^s \mathrm{d}x \right)^{1/s}$ for $s \in [2, \infty)$, $B_r(x) = \{y \in \mathbb{R}^2 : |y - x| < r\}$, and positive constants possibly different in different places, by $C_1, C_2, \cdots$.

2. **Variational setting and preliminaries.**

We define the following symmetric bilinear forms

$$(u, v) \mapsto A_1(u, v) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(1 + |x - y|)\, u(x) v(y) \mathrm{d}x \mathrm{d}y, \tag{2.1}$$

$$(u, v) \mapsto A_2(u, v) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln\left(1 + \frac{1}{|x - y|}\right) u(x) v(y) \mathrm{d}x \mathrm{d}y, \tag{2.2}$$

$$(u, v) \mapsto A_0(u, v) = A_1(u, v) - A_2(u, v) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x - y| u(x) v(y) \mathrm{d}x \mathrm{d}y, \tag{2.3}$$

where the definition is restricted, in each case, to measurable functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ such that the corresponding double integral is well defined in Lebesgue sense. Noting that $0 \leq \ln(1 + r) \leq r$ for $r \geq 0$, it follows from the Hardy-Littlewood-Sobolev inequality (see [22] or [23, page 98]) that

$$|A_2(u, v)| \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x - y|} |u(x) v(y)| \mathrm{d}x \mathrm{d}y \leq \mathcal{C}_1 \|u\|_{4/3} \|v\|_{4/3} \tag{2.4}$$

with a constant $\mathcal{C}_1 > 0$. Using (2.1), (2.2) and (2.3), we define the functionals as follows:

$$
\begin{aligned}
I_1 : H^1(\mathbb{R}^2) \to [0,\infty], \qquad I_1(u) &= A_1(u^2,u^2)\\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|)\, u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y,\\
I_2 : L^{8/3}(\mathbb{R}^2) \to [0,\infty), \qquad I_2(u) &= A_2(u^2,u^2)\\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{1}{|x-y|}\right) u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

and

$$
\begin{aligned}
I_0 : H^1(\mathbb{R}^2) \to \mathbb{R}\cup\{\infty\}, \qquad I_0(u) &= A_0(u^2,u^2)\\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x-y|)\, u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Here I_2 only takes finite values on $L^{8/3}(\mathbb{R}^2)$. Indeed, (2.4) implies

$$
|I_2(u)| \le \mathcal{C}_1\|u\|_{8/3}^4, \qquad \forall\, u \in L^{8/3}(\mathbb{R}^2). \tag{2.5}
$$

Recall the definition of function space E in the introduction. It is easy to see that E is compactly embedded in $L^s(\mathbb{R}^2)$ for all $s \in [2,\infty)$. Moreover, since

$$
\ln(1+|x-y|) \le \ln(1+|x|+|y|) \le \ln(1+|x|) + \ln(1+|y|), \qquad \forall\, x,y \in \mathbb{R}^2, \tag{2.6}
$$

we have

$$
\begin{aligned}
|A_1(uv,wz)| &\le \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} [\ln(1+|x|)+\ln(1+|y|)]\,|u(x)v(x)||w(y)z(y)|\mathrm{d}x\mathrm{d}y\\
&\le \|u\|_*\|v\|_*\|w\|_2\|z\|_2 + \|u\|_2\|v\|_2\|w\|_*\|z\|_*, \quad \forall\, u,v,w,z \in E
\end{aligned} \tag{2.7}
$$

According to [13, Lemma 2.2], we have I_0, I_1 and I_2 are of class $\mathcal{C}^1$ on E, and

$$
\langle I_i'(u), v\rangle = 4A_i(u^2,v), \qquad \forall\, u,v \in E,\ i = 0,1,2. \tag{2.8}
$$

Then, (F1), (F2) and (2.8) imply that $\Phi \in \mathcal{C}^1(E,\mathbb{R})$, and that

$$
\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2+u^2)\mathrm{d}x + \frac{1}{4}[I_1(u)-I_2(u)] - \int_{\mathbb{R}^2}F(u)\mathrm{d}x, \tag{2.9}
$$

$$
\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^2}(\nabla u\cdot\nabla v + uv)\,\mathrm{d}x + A_1(u^2,uv) - A_2(u^2,uv) - \int_{\mathbb{R}^2} f(u)v\mathrm{d}x, \tag{2.10}
$$

$$
J(u) = 2\|\nabla u\|_2^2 + \|u\|_2^2 + I_1(u) - I_2(u) - \frac{1}{8\pi}\|u\|_2^4 - 2\int_{\mathbb{R}^2}[f(u)u - F(u)]\mathrm{d}x. \tag{2.11}
$$

Hence, the solutions of (1.1) are the critical points of the reduced functional (2.9).

To prove the existence of nontrivial solutions, we shall use the following general minimax principle [20, Proposition 2.8], which is a somewhat stronger variant of [40, Theorem 2.8].

Lemma 2.1. *Let X be a Banach space. Let M_0 be a closed subspace of the metric space M and $\Gamma_0 \subset \mathcal{C}(M_0,X)$. Define*

$$
\tilde{\Gamma} := \left\{\gamma \in \mathcal{C}(M,X) : \gamma\big|_{M_0} \in \Gamma_0\right\}.
$$

If $\varphi \in \mathcal{C}^1(X,\mathbb{R})$ satisfies

$$
\infty > c := \inf_{\gamma\in\Gamma}\sup_{u\in M}\varphi(\gamma(u)) > a := \sup_{\gamma_0\in\Gamma_0}\sup_{u\in M_0}\varphi(\gamma_0(u)),
$$

then, for every $\varepsilon \in (0,(c-a)/2)$, $\delta > 0$ and $\gamma \in \tilde{\Gamma}$ such that

$$\sup_M \varphi \circ \gamma \leq c + \varepsilon,$$

there exists $u \in X$ such that

(a) $c - 2\varepsilon \leq \varphi(u) \leq c + 2\varepsilon$,
(b) $\operatorname{dist}(u, \gamma(M)) \leq 2\delta$,
(c) $\|\varphi'(u)\| \leq 8\varepsilon/\delta$.

Similar to [16, Lemma 3.2], we will apply Lemma 2.1 to obtain a Cerami sequence for the functional Φ with $J(u_n) \to 0$. This idea goes back to Jeanjean [17].

Lemma 2.2. *Assume that (F1)-(F3) hold. Then there exists a sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to c > 0, \quad \|\Phi'(u_n)\|_{E^*}(1 + \|u_n\|_E) \to 0 \quad \text{and} \quad J(u_n) \to 0, \tag{2.12}$$

where

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)), \qquad \Gamma := \{\gamma \in \mathcal{C}([0,1], E) : \gamma(0) = 0, \ \Phi(\gamma(1)) < 0\}.$$

Proof. First, we prove that $0 < c < \infty$. Note that for any fixed $u \in E$ with $u \neq 0$,

$$\begin{aligned} I_0\left(t^2 u_t\right) &= \frac{t^4}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| u^2(tx) u^2(ty) \mathrm{d}(tx) \mathrm{d}(ty) \\ &= \frac{t^4}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\ln|tx - ty| - \ln t\right) u^2(tx) u^2(ty) \mathrm{d}(tx) \mathrm{d}(ty) \\ &= \frac{t^4}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\ln|x-y| - \ln t\right) u^2(x) u^2(y) \mathrm{d}x \mathrm{d}y \\ &= t^4 I_0(u) - \frac{t^4 \ln t}{2\pi} \|u\|_2^4, \qquad \forall\, t > 0. \end{aligned} \tag{2.13}$$

Combining (2.9) with (2.13), one has

$$\begin{aligned} \Phi\left(t^2 u_t\right) &= \frac{t^4}{2} \|\nabla u\|_2^2 + \frac{t^2}{2} \|u\|_2^2 + \frac{t^4}{4} I_0(u) - \frac{t^4 \ln t}{8\pi} \|u\|_2^4 \\ &\quad - \frac{1}{t^2} \int_{\mathbb{R}^2} F\left(t^2 u\right) \mathrm{d}x, \qquad \forall\, t > 0. \end{aligned} \tag{2.14}$$

By (F1)-(F3), there exists a constant $\Lambda > 0$ such that

$$F\left(t^2 u\right) \geq -\Lambda |t^2 u|^2, \qquad \forall\, t > 0. \tag{2.15}$$

Then, it follows from (2.14) and (2.15) that

$$\lim_{t \to 0} \Phi\left(t^2 u_t\right) = 0, \qquad \sup_{t > 0} \Phi\left(t^2 u_t\right) < \infty, \qquad \Phi\left(t^2 u_t\right) \to -\infty \quad \text{as} \quad t \to +\infty. \tag{2.16}$$

Now, we choose $T > 0$ large enough such that $\Phi(T^2 u_T) < 0$. Let $\gamma_T(t) = (tT)^2 u_{tT}$ for $t \in [0,1]$. Then $\gamma_T \in \mathcal{C}([0,1], E)$ such that $\gamma_T(0) = 0$, $\Phi(\gamma_T(1)) < 0$ and $\max_{t \in [0,1]} \Phi(\gamma_T(t)) < \infty$. This shows that $\Gamma \neq \emptyset$ and $c < \infty$.

By (F1) and (F2), for every $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that

$$f(u)u \leq \varepsilon u^2 + C(\varepsilon)|u|^p, \qquad F(u) \leq \varepsilon u^2 + C(\varepsilon)|u|^p, \qquad \forall\, u \in \mathbb{R}. \tag{2.17}$$

Choosing $\varepsilon = 1/4$, by (2.5), (2.9), (2.17) and Sobolev embedding inequality, one has

$$\begin{aligned}\Phi(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\mathcal{C}_1}{4}\|u\|_{8/3}^4 - \frac{1}{4}\|u\|_2^2 - C_1\|u\|_p^p \\ &\geq \frac{1}{4}\|u\|^2 - C_2\|u\|^4 - C_3\|u\|^p, \quad \forall\, u \in E. \end{aligned} \tag{2.18}$$

From (2.18), it is easy to see that there exist constants $\rho_0 > 0$ and $a_0 > 0$ such that

$$\Phi(u) \geq 0, \quad \forall\, \|u\| \leq \rho_0 \quad \text{and} \quad \Phi(u) \geq a_0, \ \forall\, \|u\| = \rho_0. \tag{2.19}$$

For every $\gamma \in \Gamma$, since $\gamma(0) = 0$ and $\Phi(\gamma(1)) < 0$, then it follows from (2.19) that $\|\gamma(1)\| > \rho_0$. By the continuity of $\gamma(t)$ and the intermediate value theorem, there exists $t_\gamma \in (0,1)$ such that $\|\gamma(t_\gamma)\| = \rho_0$. Thus, we have

$$\sup_{t\in[0,1]} \Phi(\gamma(t)) \geq \Phi(\gamma(t_\gamma)) \geq a_0 > 0, \quad \forall\, \gamma \in \Gamma,$$

which yields

$$\infty > c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)) \geq a_0 > 0. \tag{2.20}$$

Similar to [16, Lemma 3.2], we define a continuous map

$$h : \tilde{E} := \mathbb{R} \times E \to E, \quad h(s,v)(x) = e^{2s}v(e^s x) \ \text{ for } s \in \mathbb{R},\ v \in E \text{ and } x \in \mathbb{R}^2,$$

where $\tilde{E}$ is a Banach space equipped with the product norm $\|(s,v)\|_{\tilde{E}} := \left(|s|^2 + \|v\|_E^2\right)^{1/2}$. We consider the following auxiliary functional:

$$\begin{aligned}\Psi(s,v) = \Phi(h(s,v)) &= \frac{1}{2}\int_{\mathbb{R}^2} \left[|\nabla h(s,v)|^2 + |h(s,v)|^2\right] \mathrm{d}x \\ &\quad + \frac{1}{4}[I_1(h(s,v)) - I_2(h(s,v))] - \int_{\mathbb{R}^2} F(h(s,v))\mathrm{d}x \\ &= \frac{e^{4s}}{2}\int_{\mathbb{R}^2} |\nabla v|^2 \mathrm{d}x + \frac{e^{2s}}{2}\int_{\mathbb{R}^2} v^2 \mathrm{d}x + \frac{e^{4s}}{4}[I_1(v) - I_2(v)] \\ &\quad - \frac{se^{4s}}{8\pi}\left(\int_{\mathbb{R}^2} v^2 \mathrm{d}x\right)^2 - \frac{1}{e^{2s}}\int_{\mathbb{R}^2} F(e^{2s}v)\mathrm{d}x.\end{aligned}$$

It is easy to see that $\Psi \in \mathcal{C}^1(\tilde{E}, \mathbb{R})$. As in the proof of [16, (3.3),(3.4)], we have

$$\partial_s\Psi(s,v) = J(h(s,v)), \quad \partial_v\Psi(s,v)w = \Phi'(h(s,v))h(s,w) \ \text{ for } s \in \mathbb{R} \text{ and } v, w \in E. \tag{2.21}$$

Now, we define a minimax value $\tilde{c}$ for Ψ by

$$\tilde{c} = \inf_{\tilde{\gamma}\in\tilde{\Gamma}} \max_{t\in[0,1]} \Psi(\tilde{\gamma}(t)),$$

where

$$\tilde{\Gamma} = \left\{\tilde{\gamma} \in \mathcal{C}([0,1], \tilde{E}) : \tilde{\gamma}(0) = (0,0),\ \Psi(\tilde{\gamma}(1)) < 0\right\}.$$

Since $\Gamma = \left\{h \circ \tilde{\gamma} : \tilde{\gamma} \in \tilde{\Gamma}\right\}$, the minimax value of Φ and Ψ coincide, i.e., $c = \tilde{c}$. By the definition of c, for every $n \in \mathbb{N}$, there exists $\gamma_n \in \Gamma$ such that

$$\max_{t\in[0,1]} \Psi(0, \gamma_n(t)) = \max_{t\in[0,1]} \Phi(\gamma_n(t)) \leq c + \frac{1}{n^2}.$$

Next, we apply Lemma 2.1 to Ψ, $M = [0,1]$, $M_0 = \{0,1\}$ and $\tilde{E}$, $\tilde{\Gamma}$ in place of X, Γ. Let $\varepsilon_n = 1/n^2$, $\delta_n = 1/n$ and $\tilde{\gamma}_n(t) = (0, \gamma_n(t))$. Since (2.20) implies that

$\varepsilon_n = 1/n^2 \in (0, c/2)$ for large $n \in \mathbb{N}$, Lemma 2.1 yields the existence of $(s_n, v_n) \in \tilde{E}$ such that, as $n \to \infty$,

$$\Psi(s_n, v_n) \to c, \tag{2.22}$$

$$\|\Psi'(s_n, v_n)\|_{\tilde{E}^*} (1 + \|(s_n, v_n)\|_{\tilde{E}}) \to 0, \tag{2.23}$$

$$\operatorname{dist}((s_n, v_n), \{0\} \times \gamma_n([0,1])) \to 0. \tag{2.24}$$

Moreover, (2.24) implies that $s_n \to 0$. Note that

$$\langle \Psi'(s_n, v_n), (\tau, w) \rangle = \langle \Phi'(h(s_n, v_n)), h(s_n, w) \rangle + J(h(s_n, v_n))\tau, \quad \forall\ (\tau, w) \in \tilde{E}. \tag{2.25}$$

Let $u_n := h(s_n, v_n)$. By using the same way as [16, Lemma 3.2], we deduce that $\{u_n\}$ satisfies (2.12). □

Next we show the boundedness of the Cerami sequence obtained in Lemma 2.2 under the assumption (F4) or (F5).

Lemma 2.3. *Assume that (F1)-(F4) hold. Let $\{u_n\} \subset E$ be a sequence satisfying (2.12). Then $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$.*

Proof. By (F4), (2.9), (2.11) and (2.12), one has

$$\begin{aligned} c + o(1) &= \Phi(u_n) - \frac{1}{4}J(u_n) \\ &= \frac{1}{4}\|u_n\|_2^2 + \frac{1}{32\pi}\|u_n\|_2^4 + \frac{1}{2}\int_{\mathbb{R}^2} [f(u_n)u_n - 3F(u_n)]\, \mathrm{d}x \quad (2.26) \\ &= \left(\frac{1}{4} - \frac{\alpha_0}{2}\right)\|u_n\|_2^2 + \frac{1}{32\pi}\|u_n\|_2^4 \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2} \left[f(u_n)u_n - 3F(u_n) + \alpha_0 u_n^2\right] \mathrm{d}x \\ &\geq \frac{1}{32\pi}\|u_n\|_2^4 - \frac{\alpha_0}{2}\|u_n\|_2^2, \quad (2.27) \end{aligned}$$

which implies

$$\|u_n\|_2 \leq C_4, \qquad \int_{\mathbb{R}^2} \left[f(u_n)u_n - 3F(u_n) + \alpha_0 u_n^2\right] \mathrm{d}x \leq C_5 \tag{2.28}$$

for some constants $C_4, C_5 > 0$. Now, we prove that $\{\|u_n\|\}$ is also bounded. Arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n := \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$, and $\|v_n\|_2 \to 0$ due to (2.28). Set $\kappa' = \kappa/(\kappa - 1)$. By the Gagliardo-Nirenberg inequality, one has

$$\|v_n\|_{2\kappa'}^{2\kappa'} \leq C_6 \|v_n\|_2^2 \|\nabla v_n\|_2^{2\kappa' - 2} = o(1). \tag{2.29}$$

Set

$$\Omega_n := \left\{ x \in \mathbb{R}^2 : \left|\frac{f(u_n)}{u_n}\right| \leq \beta_0 \right\}.$$

Then, we have

$$\int_{\Omega_n} \left|\frac{f(u_n)}{u_n}\right| v_n^2 \mathrm{d}x \leq \beta_0 \|v_n\|_2^2 = o(1). \tag{2.30}$$

Moreover, by (F4), (2.28), (2.29) and the Hölder inequality, we have

$$\begin{aligned}
& \int_{\mathbb{R}^2\setminus\Omega_n} \left|\frac{f(u_n)}{u_n}\right| v_n^2 \mathrm{d}x \\
\leq & \left(\int_{\mathbb{R}^2\setminus\Omega_n} \left|\frac{f(u_n)}{u_n}\right|^\kappa \mathrm{d}x\right)^{1/\kappa} \left(\int_{\mathbb{R}^2\setminus\Omega_n} |v_n|^{2\kappa'} \mathrm{d}x\right)^{1/\kappa'} \\
\leq & c_0^{1/\kappa} \left(\int_{\mathbb{R}^2\setminus\Omega_n} \left[f(u_n)u_n - 3F(u_n) + \alpha_0 u_n^2\right] \mathrm{d}x\right)^{1/\kappa} \|v_n\|_{2\kappa'}^2 \\
= & o(1). \qquad (2.31)
\end{aligned}$$

From (2.5), (2.28) and the Gagliardo-Nirenberg inequality, we have

$$I_2(u_n) \leq \mathcal{C}_1 \|u_n\|_{8/3}^4 \leq C_7 \|u_n\|_2^3 \|\nabla u_n\|_2 \leq C_8 \|\nabla u_n\|_2. \qquad (2.32)$$

Thus, it follows from (2.9), (2.12), (2.30), (2.31) and (2.32) that

$$\begin{aligned}
1 + o(1) &= \frac{\|u_n\|^2 - \langle \Phi'(u_n), u_n\rangle}{\|u_n\|^2} \\
&= \frac{-I_1(u_n) + I_2(u_n) + \int_{\mathbb{R}^2} f(u_n)u_n \mathrm{d}x}{\|u_n\|^2} \\
&\leq \frac{C_8}{\|u_n\|} + \int_{\Omega_n} \left|\frac{f(u_n)}{u_n}\right| v_n^2 \mathrm{d}x + \int_{\mathbb{R}^2\setminus\Omega_n} \left|\frac{f(u_n)}{u_n}\right| v_n^2 \mathrm{d}x \\
&= o(1), \qquad (2.33)
\end{aligned}$$

which is a contradiction. Hence, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. □

Lemma 2.4. *Assume that (F1)-(F3) and (F5) hold. Let $\{u_n\} \subset E$ be a sequence satisfying (2.12). Then $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$.*

Proof. First, we prove that $\{\|\nabla u_n\|_2\}$ is bounded. Arguing by contradiction, suppose that $\|\nabla u_n\|_2 \to \infty$. Inspired by [16, Proposition 3.3], we let $t_n = \|\nabla u_n\|_2^{-1/2}$ and $v_n = t_n^2 (u_n)_{t_n}$. Then $t_n \to 0$ and

$$\|\nabla v_n\|_2 = 1, \qquad \|v_n\|_q^q = t_n^{2q-2}\|u_n\|_q^q, \qquad \forall\ 2 \leq q < \infty. \qquad (2.34)$$

By (2.34) and the Gagliardo-Nirenberg inequality, one has

$$\|v_n\|_s^s \leq C_9 \|v_n\|_2^2 \|\nabla v_n\|_2^{s-2} = C_{10}\|v_n\|_2^2, \qquad \text{for } \ s = p_0, p_1, p_2. \qquad (2.35)$$

From (F5), (2.26), (2.34) and (2.35), we deduce

$$\begin{aligned}
c + o(1) &= \Phi(u_n) - \frac{1}{4}J(u_n) \\
&\geq \frac{1}{32\pi}\|u_n\|_2^4 - \frac{\alpha_1}{2}\left(\|u_n\|_{p_1}^{p_1} + \|u_n\|_{p_2}^{p_2}\right) \qquad (2.36) \\
&= \frac{1}{32\pi} t_n^{-4}\|v_n\|_2^4 - \frac{\alpha_1}{2}\left(t_n^{-2p_1+2}\|v_n\|_{p_1}^{p_1} + t_n^{-2p_2+2}\|v_n\|_{p_2}^{p_2}\right) \\
&\geq \frac{1}{32\pi} t_n^{-4}\|v_n\|_2^4 - \frac{\alpha_1 C_{10}}{2}\left(t_n^{-2p_1+2} + t_n^{-2p_2+2}\right)\|v_n\|_2^2. \qquad (2.37)
\end{aligned}$$

Multiplying the above inequalities by t_n^4 and $t_n^4|\ln t_n|$, respectively, we have

$$\begin{aligned}\frac{1}{32\pi}\|v_n\|_2^4 &\leq ct_n^4+\frac{\alpha_1 C_{10}}{2}\left[t_n^{2(3-p_1)}+t_n^{2(3-p_2)}\right]\|v_n\|_2^2+o(t_n^4)\\ &\leq ct_n^4+8\pi\alpha_1^2C_{10}^2\left[t_n^{4(3-p_1)}+t_n^{4(3-p_2)}\right]+\frac{1}{64\pi}\|v_n\|_2^4+o(t_n^4) \quad (2.38)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{32\pi}|\ln t_n|\|v_n\|_2^4 &\leq ct_n^4|\ln t_n|+\frac{\alpha_1 C_{10}}{2}\left[t_n^{2(3-p_1)}+t_n^{2(3-p_2)}\right]|\ln t_n|\|v_n\|_2^2\\ &\quad +o(t_n^4|\ln t_n|)\\ &\leq ct_n^4|\ln t_n|+8\pi\alpha_1^2C_{10}^2\left[t_n^{4(3-p_1)}|\ln t_n|+t_n^{4(3-p_2)}|\ln t_n|\right]\\ &\quad +\frac{1}{64\pi}|\ln t_n|\|v_n\|_2^4+o(t_n^4|\ln t_n|). \quad (2.39)\end{aligned}$$

Let $\tau>0$ and $p_0-3\leq\tau<3-\max\{p_1,p_2\}$. Since $t_n\to 0$ and $|t_n|^q|\ln t_n|\to 0$ for $q>0$, it follows from (2.38), (2.39) and (F5) that

$$\|v_n\|_2\to 0,\qquad t_n^{-4\tau}\|v_n\|_2^4\to 0,\qquad |\ln t_n|\|v_n\|_2^4\to 0. \quad (2.40)$$

Thus, by (2.9), (2.11), (2.14) and (2.26), one has

$$\begin{aligned}&c-\Phi(v_n)+o(1)=\Phi(u_n)-\Phi\left(t_n^2(u_n)_{t_n}\right)\\ =&\frac{1-t_n^4}{2}\|\nabla u_n\|_2^2+\frac{1-t_n^2}{2}\|u_n\|_2^2+\frac{1-t_n^4}{4}I_0(u_n)+\frac{t_n^4\ln t_n}{8\pi}\|u_n\|_2^4\\ &+\int_{\mathbb{R}^2}\left[\frac{1}{t_n^2}F\left(t_n^2u_n\right)-F(u_n)\right]\mathrm{d}x\\ =&\frac{1-t_n^4}{4}J(u_n)+\frac{t_n^4-2t_n^2+1}{4}\|u_n\|_2^2+\frac{1}{32\pi}\|u_n\|_2^4\\ &+\frac{1}{2}\int_{\mathbb{R}^2}[f(u_n)u_n-3F(u_n)]\,\mathrm{d}x+\frac{t_n^4(4\ln t_n-1)}{32\pi}\|u_n\|_2^4\\ &+\int_{\mathbb{R}^2}\left[\frac{t_n^4}{2}F(u_n)+\frac{1}{t_n^2}F\left(t_n^2u_n\right)-\frac{t_n^4}{2}f(u_n)u_n\right]\mathrm{d}x\\ =&c+o(1)+\frac{t_n^4-2t_n^2}{4}\|u_n\|_2^2+\frac{t_n^4(4\ln t_n-1)}{32\pi}\|u_n\|_2^4\\ &+\int_{\mathbb{R}^2}\left[\frac{t_n^4}{2}F(u_n)+\frac{1}{t_n^2}F\left(t_n^2u_n\right)-\frac{t_n^4}{2}f(u_n)u_n\right]\mathrm{d}x. \quad (2.41)\end{aligned}$$

It follows from (2.17) with $p=p_0$ that

$$\begin{aligned}&\left|\frac{t_n^4}{2}F(u_n)+\frac{1}{t_n^2}F\left(t_n^2u_n\right)-\frac{t_n^4}{2}f(u_n)u_n\right|\\ &\leq\left(t_n^4+t_n^2\right)|u_n|^2+C_{11}\left(t_n^4+t_n^{2p_0-2}\right)|u_n|^{p_0}.\end{aligned} \quad (2.42)$$

From (2.34),(2.35), (2.40), (2.41) and (2.42), one has

$$\begin{aligned}|\Phi(v_n)| =& \left|\frac{t_n^4-2t_n^2}{4}\|u_n\|_2^2+\frac{t_n^4(4\ln t_n-1)}{32\pi}\|u_n\|_2^4\right.\\ &\left.+\int_{\mathbb{R}^2}\left[\frac{t_n^4}{2}F(u_n)+\frac{1}{t_n^2}F(t_n^2u_n)-\frac{t_n^4}{2}f(u_n)u_n\right]\mathrm{d}x\right|+o(1)\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{t_n^2+2t_n^4}{4}\|u_n\|_2^2 + \frac{t_n^4(4|\ln t_n|+1)}{32\pi}\|u_n\|_2^4 \\
&\quad + \left(t_n^4+t_n^2\right)\|u_n\|_2^2 + C_{11}\left(t_n^4+t_n^{2p_0-2}\right)\|u_n\|_{p_0}^{p_0} + o(1) \\
&= \frac{1+2t_n^2}{4}\|v_n\|_2^2 + \frac{4|\ln t_n|+1}{32\pi}\|v_n\|_2^4 \\
&\quad + \left(t_n^2+1\right)\|v_n\|_2^2 + C_{11}\left(t_n^{2(3-p_0)}+1\right)\|v_n\|_{p_0}^{p_0} + o(1) \\
&\leq C_{11}t_n^{-2\tau}\|v_n\|_2^2 + o(1) \\
&= o(1).
\end{aligned} \tag{2.43}
$$

Moreover, by (2.5), (2.17), (2.34), (2.40) and the Gagliardo-Nirenberg inequality, one has

$$
0 \leq I_2(v_n) \leq \mathcal{C}_1\|v_n\|_{8/3}^4 \leq C_{12}\|v_n\|_2^3 \cdot \|\nabla v_n\|_2 = o(1) \tag{2.44}
$$

and

$$
\left|\int_{\mathbb{R}^2} F(v_n)\mathrm{d}x\right| \leq \|v_n\|_2^2 + C_{13}\|v_n\|_{p_0}^{p_0} \leq \|v_n\|_2^2 + C_{14}\|v_n\|_2^2 = o(1). \tag{2.45}
$$

Thus, it follows from (2.9), (2.34), (2.43), (2.44) and (2.45) that

$$
\begin{aligned}
o(1) &= \Phi(v_n) \\
&= \frac{1}{2}\|\nabla v_n\|_2^2 + \frac{1}{2}\|v_n\|_2^2 + \frac{1}{4}[I_1(v_n) - I_2(v_n)] - \int_{\mathbb{R}^2} F(v_n)\mathrm{d}x \\
&\geq \frac{1}{2} + o(1).
\end{aligned} \tag{2.46}
$$

This contradiction shows that $\{\|\nabla u_n\|_2\}$ is bounded. Next, we prove $\{\|u_n\|_2\}$ is also bounded. By (2.36) and the Gagliardo-Nirenberg inequality, we have

$$
\begin{aligned}
\frac{1}{32\pi}\|u_n\|_2^4 &\leq c + \frac{\alpha_1}{2}\left(\|u_n\|_{p_1}^{p_1} + \|u_n\|_{p_2}^{p_2}\right) + o(1) \\
&\leq c + \frac{\alpha_1 C_{15}}{2}\left(\|u_n\|_2^2\|\nabla u_n\|_2^{p_1-2} + \|u_n\|_2^2\|\nabla u_n\|_2^{p_2-2}\right) + o(1) \\
&\leq c + C_{16}\|u_n\|_2^2 + o(1),
\end{aligned} \tag{2.47}
$$

which implies that $\{\|u_n\|_2\}$ is also bounded. Hence, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. □

To find nontrivial critical points of Φ, we present the following important lemma.

Lemma 2.5. *[13, Lemma 2.1] Let $\{u_n\}$ be be a sequence in $L^2(\mathbb{R}^2)$ such that $u_n \to u \in L^2(\mathbb{R}^2) \setminus \{0\}$ a.e. on $\mathbb{R}^2$. If $\{v_n\}$ be a bounded sequence in $L^2(\mathbb{R}^2)$ such that*

$$
\sup_{n\in\mathbb{N}} A_1(u_n^2, v_n^2) < \infty,
$$

then $\{\|v_n\|_*\}$ is bounded. If, moreover,

$$
A_1(u_n^2, v_n^2) \to 0 \quad \text{and} \quad \|v_n\|_2 \to 0 \quad \text{as } n \to \infty,
$$

then $\|v_n\|_* \to 0$ as $n \to \infty$.

3. **Least energy solutions.** In this section, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. In view of Lemmas 2.2-2.4, there exists a sequence $\{u_n\} \subset E$ satisfying (2.12) and $\|u_n\|^2 \leq M_1$ for some constant $M_1 > 0$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_2(y)} |u_n|^2 \mathrm{d}x = 0,$$

then by Lions' concentration compactness principle [25] or [40, Lemma 1.21], $u_n \to 0$ as $n \to \infty$ in $L^s(\mathbb{R}^2)$ for $s > 2$. This, together with (2.5), implies that $I_2(u_n) \to 0$ as $n \to \infty$. By (2.17), for $\varepsilon = c/(3M_1)$, there exists $C_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^2} \left| \frac{1}{2} f(u_n)u_n - F(u_n) \right| \mathrm{d}x \leq \frac{3}{2}\varepsilon \|u_n\|_2^2 + C_\varepsilon \|u_n\|_p^p \leq \frac{c}{2} + o(1). \tag{3.1}$$

Thus, it follows from (2.9), (2.10), (2.12) and (3.1) that

$$\begin{aligned} c + o(1) &= \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle \\ &= -\frac{1}{4} I_1(u_n) + \frac{1}{4} I_2(u_n) + \int_{\mathbb{R}^2} \left[\frac{1}{2} f(u_n)u_n - F(u_n)\right] \mathrm{d}x \\ &\leq \frac{c}{2} + o(1). \end{aligned} \tag{3.2}$$

This contradiction shows $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $y_n \in \mathbb{R}^2$ such that

$$\int_{B_1(y_n)} |u_n|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{3.3}$$

Let $\tilde{u}_n(x) = u_n(x + y_n)$. Then

$$\int_{B_1(0)} |\tilde{u}_n|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{3.4}$$

Note that

$$\|\tilde{u}_n\|_*^2 = \int_{\mathbb{R}^2} \ln(1 + |x - y_n|) u_n^2 \mathrm{d}x \leq \|u_n\|_*^2 + \ln(1 + |y_n|)\|u_n\|_2^2. \tag{3.5}$$

Thus $\tilde{u}_n \in E$ for every $n \in \mathbb{N}$. Since $\|\tilde{u}_n\| = \|u_n\|$ and $I_i(\tilde{u}_n) = I_i(u_n)$ for $i = 0, 1, 2$, then (2.12) implies

$$\Phi(\tilde{u}_n) \to c > 0, \quad \langle \Phi'(\tilde{u}_n), \tilde{u}_n\rangle \to 0, \quad n \to \infty. \tag{3.6}$$

Passing to a subsequence, we have $\tilde{u}_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^2)$, $\tilde{u}_n \to u_0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^2)$ for $s \in [2, \infty)$ and $\tilde{u}_n \to u_0$ a.e. on $\mathbb{R}^2$ as $n \to \infty$. Thus, (3.4) implies that $u_0 \neq 0$. By (2.5), (2.10), (2.17), (3.6) and Sobolev embedding inequality that

$$\begin{aligned} \|\tilde{u}_n\|^2 + I_1(\tilde{u}_n) + o(1) &= I_2(\tilde{u}_n) + \int_{\mathbb{R}^2} f(\tilde{u}_n)\tilde{u}_n \mathrm{d}x \\ &\leq \mathcal{C}_1 \|\tilde{u}_n\|_{8/3}^4 + \|\tilde{u}_n\|_2^2 + C_1 \|\tilde{u}_n\|_p^p \\ &\leq C_2 \|\tilde{u}_n\|^4 + \|\tilde{u}_n\|^2 + C_3 \|\tilde{u}_n\|^p, \end{aligned} \tag{3.7}$$

which implies that $\sup_{n\in\mathbb{N}} I_1(\tilde{u}_n) = \sup_{n\in\mathbb{N}} A_1(\tilde{u}_n^2, \tilde{u}_n^2) < \infty$ due to the boundedness of $\{\|\tilde{u}_n\|\}$. Applying Lemma 2.5, $\{\|\tilde{u}_n\|_*\}$ is bounded. Hence, $\{\tilde{u}_n\}$ is bounded in E. We may thus assume, passing to a subsequence again if necessary, that as $n \to \infty$

$$\tilde{u}_n \rightharpoonup u_0 \ \text{ in } E, \quad \tilde{u}_n \to u_0 \ \text{ in } L^s(\mathbb{R}^2) \text{ for } s \in [2, \infty), \quad \tilde{u}_n \to u_0 \text{ a.e. on } \mathbb{R}^2. \tag{3.8}$$

Now, we prove that $\Phi'(u_0)=0$. To this end, we claim that

$$\langle \Phi'(u_0), w\rangle = \lim_{n\to\infty}\langle \Phi'(\tilde{u}_n), w\rangle = \lim_{n\to\infty}\langle \Phi'(u_n), w(\cdot - y_n)\rangle = 0, \quad \forall\, w\in E. \tag{3.9}$$

In fact, it is easy to see that

$$\|w(\cdot - y_n)\|_E^2 = \|w\|^2 + \int_{\mathbb{R}^2}\ln(1+|x+y_n|)w^2\mathrm{d}x \le \|w\|_E^2 + \ln(1+|y_n|)\|w\|_2^2, \quad \forall\, w\in E. \tag{3.10}$$

Moreover, by (3.4), we have

$$\begin{aligned}\|u_n\|_*^2 &= \int_{\mathbb{R}^2}\ln(1+|x+y_n|)\tilde{u}_n^2\mathrm{d}x\\ &\ge \int_{B_1(0)}\ln(1+|x+y_n|)\tilde{u}_n^2\mathrm{d}x\\ &\ge \frac{\delta\ln|y_n|}{2} \ge \frac{\delta\ln(1+|y_n|)}{4}, \qquad \forall\, |y_n|\ge 2.\end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we deduce

$$\|w(\cdot - y_n)\|_E^2 \le \|w\|_E^2 + \left(\frac{4\|u_n\|_*^2}{\delta} + \ln 3\right)\|w\|_2^2, \quad \forall\, w\in E,\ n\in\mathbb{N}. \tag{3.12}$$

Thus, it follows from (2.10), (2.12) and (3.12) that

$$\begin{aligned}|\langle \Phi'(\tilde{u}_n), w\rangle| &= |\langle \Phi'(u_n), w(\cdot - y_n)\rangle|\\ &\le \|\Phi'(u_n)\|_{E^*}\left[\|w\|_E^2 + \left(\frac{4\|u_n\|_*^2}{\delta}+\ln 3\right)\|w\|_2^2\right]^{1/2}\\ &= o(1), \quad \forall\, w\in E.\end{aligned} \tag{3.13}$$

Then it follows from (3.13) that

$$\langle \Phi'(\tilde{u}_n), u_0\rangle = o(1). \tag{3.14}$$

By (2.5) and (3.8), one has

$$\left|A_2\left(\tilde{u}_n^2, \tilde{u}_n(\tilde{u}_n - u_0)\right)\right| \le \mathcal{C}_1\|\tilde{u}_n\|_{8/3}^3\|\tilde{u}_n - u_0\|_{8/3} = o(1). \tag{3.15}$$

Using (F1), (F2), (3.8) and Lebesgue's dominated convergence theorem, a standard argument shows that

$$\int_{\mathbb{R}^2} f(\tilde{u}_n)(\tilde{u}_n - u_0)\mathrm{d}x = o(1). \tag{3.16}$$

According to [13, Lemma 2.6], we have

$$A_1\left(\tilde{u}_n^2, (\tilde{u}_n - u_0)w\right) = o(1), \quad \forall\, w\in E. \tag{3.17}$$

Thus, it follows from (3.6), (3.8), (3.14), (3.15), (3.16) and (3.17) that

$$\begin{aligned} o(1) &= \langle \Phi'(\tilde{u}_n), \tilde{u}_n - u_0\rangle\\ &= \|\tilde{u}_n\|^2 - \|u_0\|^2 + A_1\left(\tilde{u}_n^2, (\tilde{u}_n - u_0)^2\right) + A_1\left(\tilde{u}_n^2, (\tilde{u}_n - u_0)u_0\right)\\ &\quad - A_2\left(\tilde{u}_n^2, \tilde{u}_n(\tilde{u}_n - u_0)\right) - \int_{\mathbb{R}^2} f(\tilde{u}_n)(\tilde{u}_n - u_0)\mathrm{d}x\\ &= \|\tilde{u}_n\|^2 - \|u_0\|^2 + A_1\left(\tilde{u}_n^2, (\tilde{u}_n - u_0)^2\right) + o(1),\end{aligned} \tag{3.18}$$

which, together with $\tilde{u}_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^2)$, yields

$$\|\tilde{u}_n - u_0\| \to 0, \quad A_1\left(\tilde{u}_n^2, (\tilde{u}_n - u_0)^2\right) \to 0. \tag{3.19}$$

Applying Lemma 2.5, we have $\|\tilde{u}_n - u_0\|_* \to 0$. Hence, $\|\tilde{u}_n - u_0\|_E \to 0$. From this and (2.7), we have

$$\begin{aligned} |A_1(\tilde{u}_n^2 - u_0^2, u_0 w)| &\le \|\tilde{u}_n - u_0\|_* \|\tilde{u}_n + u_0\|_* \|u_0\|_2 \|w\|_2 \\ &\quad + \|\tilde{u}_n - u_0\|_2 \|\tilde{u}_n + u_0\|_2 \|u_0\|_* \|w\|_* \\ &= o(1), \quad \forall\, w \in E. \end{aligned} \tag{3.20}$$

Similar to (3.15) and (3.16), we can get

$$A_2\left(\tilde{u}_n^2, (\tilde{u}_n - u_0)w\right) = o(1), \quad A_2\left(\tilde{u}_n^2 - u_0^2, u_0 w\right) = o(1) \tag{3.21}$$

and

$$\int_{\mathbb{R}^2} [f(\tilde{u}_n) - f(u_0)]\, w \mathrm{d}x = o(1), \quad \forall\, w \in E. \tag{3.22}$$

Then, from (2.10), (3.8), (3.17), (3.20), (3.21) and (3.22), we deduce that

$$\begin{aligned} & \langle \Phi'(\tilde{u}_n) - \Phi'(u_0), w \rangle \\ = {} & (\tilde{u}_n - u_0, w) + A_1\left(\tilde{u}_n^2, (\tilde{u}_n - u_0)w\right) + A_1\left(\tilde{u}_n^2 - u_0^2, u_0 w\right) \\ & - A_2\left(\tilde{u}_n^2, (\tilde{u}_n - u_0)w\right) - A_2\left(\tilde{u}_n^2 - u_0^2, u_0 w\right) - \int_{\mathbb{R}^2} [f(\tilde{u}_n) - f(u_0)]\, w \mathrm{d}x \\ = {} & o(1), \quad \forall\, w \in E. \end{aligned} \tag{3.23}$$

Therefore, (3.9) follows from (3.13) and (3.23). This shows that $u_0 \in E$ is a nontrivial solution of (1.1), and $\Phi(u_0) = c > 0$.

Set

$$\mathcal{K} := \{u \in E \setminus \{0\} : \Phi'(u) = 0\}.$$

Since $u_0 \in \mathcal{K}$, we have $\mathcal{K} \neq \emptyset$. By (F1) and (F2), there exist $C_4 > 0$ and $q_1 \ge 4$ such that

$$|f(u)u| \le \frac{1}{2} u^2 + C_4 |u|^{q_1}, \quad \forall\, u \in \mathbb{R}. \tag{3.24}$$

Since $\langle \Phi'(u), u \rangle = 0$, $\forall u \in \mathcal{K}$, by (2.10), (3.24) and Sobolev embedding inequality, one has

$$\begin{aligned} \|u\|^2 &\le \|u\|^2 + I_1(u) = I_2(u) + \int_{\mathbb{R}^2} f(u) u \mathrm{d}x \\ &\le C_5 \|u\|^4 + \frac{1}{2}\|u\|^2 + C_6 \|u\|^{q_1}, \quad \forall\, u \in \mathcal{K}, \end{aligned} \tag{3.25}$$

which implies

$$\|u\| \ge \varrho_0 := \min\left\{1, 2^{-1/2}(C_5 + C_6)^{-1/2}\right\} > 0, \quad \forall\, u \in \mathcal{K}. \tag{3.26}$$

It is easy to see that $\inf_{\mathcal{K}} \Phi > -\infty$. Next, we let $\{u_n\} \subset \mathcal{K}$ and $\Phi(u_n) \to \inf_{\mathcal{K}} \Phi$. Then, the sequence $\{u_n\}$ satisfies (2.12). Under assumptions of Theorem 1.1, we conclude from Lemmas 2.3 and 2.4 that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. We claim that $\{u_n\}$ does not vanish. Otherwise, if $\{u_n\}$ is a vanishing sequence, then by Lions' concentration compactness principle [25] or [40, Lemma 1.21], we have $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for $2 < s < \infty$. Thus, using (2.5) and (2.17), it is easy to see that

$$I_2(u_n) = o(1), \quad \int_{\mathbb{R}^2} f(u_n) u_n \mathrm{d}x = o(1).$$

From this, (3.25) and (3.26), we deduce a contradiction. By the same argument as above, there exists $\hat{u} \in \mathcal{K}$ such that $\Phi(\hat{u}) = \inf_{\mathcal{K}} \Phi > -\infty$. This shows that $\hat{u} \in E$ is a least energy solution of (1.1). □

4. **Ground state solutions.** In this section, we give the proof of Theorem 1.2. Inspired by [34–36], first, we establish some new inequalities to find ground state solutions for (1.1).

Lemma 4.1. *Assume that (F1), (F2) and (F6) hold. Then*

$$g(t,u) := \frac{1-t^4}{2}f(u)u+\frac{t^4-3}{2}F(u)+\frac{1}{t^2}F(t^2u)+\frac{(1-t^2)^2}{4}u^2 \geq 0, \quad \forall\, t>0,\ u\in\mathbb{R}. \tag{4.1}$$

Proof. Using (F1) and (F2), it is easy to see that (4.1) holds for $u=0$. For any fixed $u\neq 0$, by (F6), one has

$$\begin{aligned} \frac{\mathrm{d}(g(t,u))}{\mathrm{d}t} &= 2t^3|u|^3\left[\frac{f(t^2u)t^2u-F(t^2u)}{t^6|u|^3}-\frac{f(u)u-F(u)}{|u|^3}-\frac{(1-t^2)}{2t^2|u|}\right] \\ & \begin{cases} \geq 0, & t\geq 1, \\ \leq 0, & 0<t<1, \end{cases} \end{aligned}$$

which implies that $g(t,u)\geq g(1,u)=0$ for $t>0$. □

Lemma 4.2. *Assume that (F1), (F2) and (F6) hold. Then*

$$\Phi(u) \geq \Phi(t^2u_t)+\frac{1-t^4}{4}J(u), \qquad \forall\, u\in E, \quad t>0, \tag{4.2}$$

$$\Phi(u) \geq \frac{1}{4}J(u)+\frac{1}{32\pi}\|u\|_2^4, \qquad \forall\, u\in E. \tag{4.3}$$

Proof. By an elementary computation, one has

$$1-t^4+4t^4\ln t\geq 0, \qquad \forall\, t>0. \tag{4.4}$$

From (2.11), (2.14), (4.1) and (4.4), we deduce that

$$\begin{aligned} \Phi(u)-\Phi(t^2u_t) &= \frac{1}{2}\int_{\mathbb{R}^2}\left[(1-t^4)|\nabla u|^2+(1-t^2)u^2\right]\mathrm{d}x+\frac{1-t^4}{4}I_0(u) \\ &\quad +\frac{t^4\ln t}{8\pi}\|u\|_2^4+\int_{\mathbb{R}^2}\left[\frac{1}{t^2}F(t^2u)-F(u)\right]\mathrm{d}x \\ &= \frac{1-t^4}{4}J(u)+\frac{(1-t^2)^2}{4}\int_{\mathbb{R}^2}u^2\mathrm{d}x+\frac{1-t^4+4t^4\ln t}{32\pi}\|u\|_2^4 \\ &\quad +\int_{\mathbb{R}^2}\left[\frac{1-t^4}{2}f(u)u+\frac{t^4-3}{2}F(u)+\frac{1}{t^2}F(t^2u)\right]\mathrm{d}x \\ &\geq \frac{1-t^4}{4}J(u), \qquad \forall\, u\in E, \quad t>0. \end{aligned}$$

This shows that (4.2) holds. Note that (F1), (F2) and (4.1) imply

$$\lim_{t\to 0}g(t,u)=\frac{1}{2}f(u)u-\frac{3}{2}F(u)+\frac{1}{4}u^2\geq 0, \qquad u\in\mathbb{R}. \tag{4.5}$$

Thus, it follows from (2.9), (2.11) and (4.5) that

$$\begin{aligned} \Phi(u)-\frac{1}{4}J(u) &= \frac{1}{32\pi}\|u\|_2^4+\int_{\mathbb{R}^2}\left[\frac{1}{2}f(u)u-\frac{3}{2}F(u)+\frac{1}{4}u^2\right]\mathrm{d}x \\ &\geq \frac{1}{32\pi}\|u\|_2^4, \qquad \forall\, u\in E. \end{aligned}$$

This shows that (4.3) holds. □

From Lemma 4.2, we have the following corollary immediately.

Corollary 4.3. *Assume that (F1), (F2) and (F6) hold. Then*

$$\Phi(u) = \max_{t>0} \Phi(t^2 u_t), \quad \forall\, u \in \mathcal{M}. \tag{4.6}$$

Lemma 4.4. *Assume that (F1)-(F3) and (F6) hold. Then for any $u \in E \setminus \{0\}$, there exists a constant $t(u) > 0$ such that $t(u)^2 u_{t(u)} \in \mathcal{M}$.*

Proof. Let $u \in E \setminus \{0\}$ be fixed and define a function $\zeta(t) := \Phi(t^2 u_t)$ on $(0, \infty)$. Clearly, by (2.11) and (2.14), we have

$$\begin{aligned} \zeta'(t) = 0 \quad \Leftrightarrow \quad & \int_{\mathbb{R}^2} \left(2t^3 |\nabla u|^2 + t u^2\right) \mathrm{d}x + t^3 I_0(u) - \frac{4t^3 \ln t + t^3}{8\pi} \|u\|_2^4 \\ & \qquad + \frac{2}{t^3} \int_{\mathbb{R}^2} F(t^2 u) \mathrm{d}x - \frac{2}{t} \int_{\mathbb{R}^2} f(t^2 u) u \mathrm{d}x = 0 \\ \Leftrightarrow \quad & J(t^2 u_t) = 0 \quad \Leftrightarrow \quad t^2 u_t \in \mathcal{M}, \qquad \forall\, t > 0. \end{aligned}$$

It is easy to verify, using (F1)-(F3), that $\lim_{t\to 0} \zeta(t) = 0$, $\zeta(t) > 0$ for $t > 0$ small and $\zeta(t) < 0$ for t large. Therefore $\max_{t\in(0,\infty)} \zeta(t)$ is achieved at $t_0 = t(u) > 0$ so that $\zeta'(t_0) = 0$ and $t_0^2 u_{t_0} \in \mathcal{M}$. □

Lemma 4.5. *Assume that (F1)-(F3) and (F6) hold. Then*

$$\inf_{u\in\mathcal{M}} \Phi(u) := m = \inf_{u\in E\setminus\{0\}} \max_{t>0} \Phi(t^2 u_t). \tag{4.7}$$

Proof. Combining Corollary 4.3 and Lemma 4.4, we obtain the conclusion stated here. □

Lemma 4.6. *Assume that (F1)-(F3) and (F6) hold. Then*

(i) there exists $\varrho > 0$ such that $\|u\| \geq \varrho$, $\forall\, u \in \mathcal{M}$;

(ii) $m = \inf_{u\in\mathcal{M}} \Phi(u) > 0$.

Proof. (i) By (F1) and (F2), there exist $C_1 > 0$ and $q_2 \geq 4$ such that

$$|f(u)u| + |F(u)| \leq \frac{1}{4} u^2 + C_1 |u|^{q_2}, \quad \forall\, u \in \mathbb{R}. \tag{4.8}$$

Since $J(u) = 0$, $\forall u \in \mathcal{M}$, by (2.11), (4.8) and Sobolev embedding inequality, one has

$$\begin{aligned} \|u\|^2 \quad & \leq \quad 2\|\nabla u\|_2^2 + \|u\|_2^2 + I_1(u) = I_2(u) + \frac{1}{8\pi} \|u\|_2^4 + 2 \int_{\mathbb{R}^2} [f(u)u - F(u)] \mathrm{d}x \\ & \leq \quad C_2 \|u\|^4 + \frac{1}{2} \|u\|^2 + C_3 \|u\|^{q_2}, \qquad \forall\, u \in \mathcal{M}, \end{aligned} \tag{4.9}$$

which implies

$$\|u\| \geq \varrho := \min\left\{1, 2^{-1/2}(C_2 + C_3)^{-1/2}\right\}, \quad \forall\, u \in \mathcal{M}. \tag{4.10}$$

(ii) Let $\{u_n\} \subset \mathcal{M}$ be such that $\Phi(u_n) \to m$. There are two possible cases: 1) $\inf_{n\in\mathbb{N}} \|u_n\|_2 > 0$ or 2) $\inf_{n\in\mathbb{N}} \|u_n\|_2 = 0$.

Case 1). $\inf_{n\in\mathbb{N}} \|u_n\|_2 := \varrho_1 > 0$. In this case, from (4.3), one has

$$m + o(1) = \Phi(u_n) \geq \frac{1}{32\pi} \|u_n\|_2^4 \geq \frac{1}{32\pi} \varrho_1^4. \tag{4.11}$$

Case 2). $\inf_{n\in\mathbb{N}} \|u_n\|_2 = 0$. In view of (4.10), passing to a subsequence, we have

$$\|u_n\|_2 \to 0, \quad \|\nabla u_n\|_2 \geq \varrho. \tag{4.12}$$

By (2.5) and the Gagliardo-Nirenberg inequality, one has

$$0 \le I_2(u_n) \le \mathcal{C}_1\|u_n\|_{8/3}^4 \le C_4\|u_n\|_2^3\|\nabla u_n\|_2, \quad \|u_n\|_p^p \le C_5\|u_n\|_2^2\|\nabla u_n\|_2^{p-2}. \tag{4.13}$$

Moreover, using (4.12), it is easy to see that

$$\frac{|\ln(\|\nabla u_n\|_2)|}{\|\nabla u_n\|_2^2} \le C_6. \tag{4.14}$$

Let $t_n = \|\nabla u_n\|_2^{-1/2}$. Since $J(u_n) = 0$, it follows from (2.5), (2.14), (2.17), (4.6), (4.12), (4.13) and (4.14) that

$$\begin{aligned}
m + o(1) &= \Phi(u_n) \ge \Phi(t_n^2(u_n)_{t_n}) \\
&= \frac{t_n^4}{2}\|\nabla u_n\|_2^2 + \frac{t_n^2}{2}\|u_n\|_2^2 + \frac{t_n^4}{4}[I_1(u_n) - I_2(u_n)] - \frac{t_n^4\ln t_n}{8\pi}\|u_n\|_2^4 \\
&\quad -\frac{1}{t_n^2}\int_{\mathbb{R}^2} F(t_n^2 u_n)\mathrm{d}x \\
&\ge \frac{t_n^4}{2}\|\nabla u_n\|_2^2 - \frac{t_n^4}{4}I_2(u_n) - \frac{t_n^4\ln t_n}{8\pi}\|u_n\|_2^4 \\
&\quad -\frac{1}{t_n^2}\int_{\mathbb{R}^2}\left[|t_n^2u_n|^2 + C_7|t_n^2u_n|^p\right]\mathrm{d}x \\
&\ge \frac{t_n^4}{2}\|\nabla u_n\|_2^2 - \frac{C_4}{4}t_n^4\|u_n\|_2^3\|\nabla u_n\|_2 - \frac{t_n^4\ln t_n}{8\pi}\|u_n\|_2^4 \\
&\quad -t_n^2\|u_n\|_2^2 - C_8t_n^{2p-2}\|u_n\|_2^2\|\nabla u_n\|_2^{p-2} \\
&= \frac{1}{2} - \frac{C_4\|u_n\|_2^3}{4\|\nabla u_n\|_2} + \frac{\ln(\|\nabla u_n\|_2)}{16\pi\|\nabla u_n\|_2^2}\|u_n\|_2^4 - \frac{\|u_n\|_2^2}{\|\nabla u_n\|_2} - \frac{C_8\|u_n\|_2^2}{\|\nabla u_n\|_2} \\
&= \frac{1}{2} + o(1).
\end{aligned}$$

Cases 1) and 2) show that $m = \inf_{u\in\mathcal{M}}\Phi(u) > 0$. □

In the following, we will show that the Cerami sequence $\{u_n\}$ obtained in Lemma 2.2 is a minimizing sequence for Φ. This idea goes back to Tang [32,33].

Lemma 4.7. *Assume that (F1)-(F3) and (F6) hold. Then there exists a sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to c \in (0, m], \quad \|\Phi'(u_n)\|_{E^*}(1 + \|u_n\|_E) \to 0, \quad J(u_n) \to 0. \tag{4.15}$$

Proof. In view of Lemmas 4.5 and 4.6, we choose $v_k \in \mathcal{M}$ such that

$$0 < m \le \Phi(v_k) < m + \frac{1}{k}, \qquad k \in \mathbb{N}. \tag{4.16}$$

Applying Lemma 2.2, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$ satisfying (2.12). Now we choose $T_k > 0$ such that $\Phi(T_k^2(v_k)_{T_k}) < 0$. Let $\gamma_k(t) = (tT_k)^2(v_k)_{tT_k}$ for $t \in [0,1]$. Then $\gamma_k \in \Gamma$. Moreover, using (2.19), it is easy to see that $c \in [a_0, \sup_{t>0}\Phi(t^2(v_k)_t)]$. By virtue of Corollary 4.3, one has $\Phi(v_k) = \sup_{t>0}\Phi(t^2 (v_k)_t)$. Hence, by (4.16), one has

$$c \le \sup_{t>0}\Phi(t^2(v_k)_t) < m + \frac{1}{k}, \qquad k \in \mathbb{N}. \tag{4.17}$$

Let $k \to \infty$. Then we deduce the conclusion by Lemma 2.2. □

Proof of Theorem 1.2. In view of Lemma 4.7, there exists a sequence $\{u_n\} \subset E$ satisfying (4.15). Then, it follows from (4.3) and (4.15) that

$$c + o(1) = \Phi(u_n) - \frac{1}{4}J(u_n) \geq \frac{1}{32\pi}\|u_n\|_2^4. \tag{4.18}$$

This shows that $\{\|u_n\|_2\}$ is bounded. Now, we prove that $\{\|\nabla u_n\|_2\}$ is also bounded. Arguing by contradiction, suppose that $\|\nabla u_n\|_2 \to \infty$. By the Gagliardo-Nirenberg inequality, one has

$$\|u_n\|_p^p \leq C_1\|u_n\|_2^2\|\nabla u_n\|_2^{p-2}, \quad 0 \leq I_2(u_n) \leq \mathcal{C}_1\|u_n\|_{8/3}^4 \leq C_2\|u_n\|_2^3\|\nabla u_n\|_2. \tag{4.19}$$

Let $t_n = (2\sqrt{m}/\|\nabla u_n\|_2)^{1/2}$. Since $t_n \to 0$, we have $t_n^4 \ln t_n \to 0$. Thus, it follows from (2.5), (2.14), (2.17), (4.2), (4.15), (4.18) and (4.19) that

$$\begin{aligned}
m + o(1) \quad &\geq \quad c + o(1) = \Phi(u_n) \\
&\geq \quad \Phi(t_n^2(u_n)_{t_n}) + \frac{1-t_n^4}{4}J(u_n) \\
&= \quad \frac{t_n^4}{2}\|\nabla u_n\|_2^2 + \frac{t_n^2}{2}\|u_n\|_2^2 + \frac{t_n^4}{4}[I_1(u_n) - I_2(u_n)] - \frac{t_n^4\ln t_n}{8\pi}\|u_n\|_2^4 \\
&\quad\quad -\frac{1}{t_n^2}\int_{\mathbb{R}^2} F(t_n^2u_n)\mathrm{d}x \\
&\geq \quad \frac{t_n^4}{2}\|\nabla u_n\|_2^2 - \frac{t_n^4}{4}I_2(u_n) - t_n^2\|u_n\|_2^2 - C_3t_n^{2p-2}\|u_n\|_p^p + o(1) \\
&\geq \quad \frac{t_n^4}{2}\|\nabla u_n\|_2^2 - C_2t_n^4\|u_n\|_2^3\|\nabla u_n\|_2 - t_n^2\|u_n\|_2^2 \\
&\quad\quad -C_4t_n^{2p-2}\|u_n\|_2^2\|\nabla u_n\|_2^{p-2} + o(1) \\
&= \quad 2m - \frac{4C_2m}{\|\nabla u_n\|_2}\|u_n\|_2^3 - \frac{2\sqrt{m}}{\|\nabla u_n\|_2}\|u_n\|_2^2 - \frac{C_4(2\sqrt{m})^{p-1}}{\|\nabla u_n\|_2}\|u_n\|_2^2 + o(1) \\
&= \quad 2m + o(1).
\end{aligned} \tag{4.20}$$

This contradiction implies that $\{\|\nabla u_n\|_2\}$ is also bounded, and so $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. By the same argument as in the first part of the proof of Theorem 1.1, we conclude that there exists $\bar{u} \in E\setminus\{0\}$ such that $\Phi'(\bar{u}) = 0$ and $\Phi(\bar{u}) = c \in (0, m]$. Moreover, since $\bar{u} \in \mathcal{M}$, we have $\Phi(\bar{u}) \geq m$. This shows that $\bar{u} \in E$ is a ground state solution for (1.1) with $\Phi(\bar{u}) = m = \inf_{\mathcal{M}} \Phi > 0$. □

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