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Global stability and pattern formation in a nonlocal diffusive Lotka–Volterra competition model ☆

Wenjie Ni [a], Junping Shi [b,*], Mingxin Wang [a]

[a] Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, China

[b] Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795, USA

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Abstract

A diffusive Lotka–Volterra competition model with nonlocal intraspecific and interspecific competition between species is formulated and analyzed. The nonlocal competition strength is assumed to be determined by a diffusion kernel function to model the movement pattern of the biological species. It is shown that when there is no nonlocal intraspecific competition, the dynamics properties of nonlocal diffusive competition problem are similar to those of classical diffusive Lotka–Volterra competition model regardless of the strength of nonlocal interspecific competition. Global stability of nonnegative constant equilibria are proved using Lyapunov or upper–lower solution methods. On the other hand, strong nonlocal intraspecific competition increases the system spatiotemporal dynamic complexity. For the weak competition case, the nonlocal diffusive competition model may possess nonconstant positive equilibria for some suitably large nonlocal intraspecific competition coefficients.

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Keywords: Diffusive Lotka–Volterra competition model; Nonlocal interaction; Global stability; Non-constant equilibrium solution

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* Corresponding author.
E-mail address: jxshix@wm.edu (J. Shi).

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1. Introduction

Mathematical dynamic models have been established to describe the competition of living space and resource between biological individuals in the same or different species [31,55,56], and such models are used to predict the outcome of competition. Various empirical studies and mathematical models support the principle of competition exclusion: if two similar species compete for the same limiting resource, then they cannot coexist [27]. On the other hand, coexistence of species are extensively observed in natural world, and spatial heterogeneity is suggested as one of reasons for the coexistence [36,46,53].

Reaction–diffusion models have been used to describe the competition of two population distributed in a spatial region, and the individuals in the population disperse following random movement [10,32,49]. The classical diffusive Lotka–Volterra competition system takes the following form:

$$\begin{cases} u_t = d_1 \Delta u + u(\alpha - a_{11}u - a_{12}v), & x \in \Omega,\ t > 0, \\ v_t = d_2 \Delta v + v(\beta - a_{22}v - a_{21}u), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ are the population density of two biological species, and $d_1 > 0$ and $d_2 > 0$ are the diffusion coefficients corresponding to two species u and v, respectively. The positive constants α and β are the maximum intrinsic growth rates of the two species; the negative feedbacks $-a_{11}u^2$ and $-a_{22}v^2$ represent the crowding effect which is caused by the *intraspecific* competition with peers of the same species, while the other two negative feedback terms $-a_{12}uv$ and $-a_{21}vu$ represent the *interspecific* competition between the two species. Here $a_{ij} > 0$ for $i, j = 1, 2$ are constants, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward unit normal vector over $\partial\Omega$, and the homogeneous Neumann boundary condition indicates that this system is self-contained with zero population flux across the boundary. The initial data u_0, v_0 are continuous non-negative functions. It is well known that the positive equilibrium of (1.1) is globally asymptotic stable for the weak competition case ($\frac{a_{21}}{a_{11}} < \frac{\beta}{\alpha} < \frac{a_{22}}{a_{12}}$), and it is unstable for the strong competition case ($\frac{a_{22}}{a_{12}} < \frac{\beta}{\alpha} < \frac{a_{21}}{a_{11}}$). On the other hand, when $\max\left\{\frac{a_{22}}{a_{12}}, \frac{a_{21}}{a_{11}}\right\} < \frac{\beta}{\alpha}$ or $\min\left\{\frac{a_{22}}{a_{12}}, \frac{a_{21}}{a_{11}}\right\} > \frac{\beta}{\alpha}$, there is no positive equilibrium for (1.1), and one of boundary equilibria (in which only one species is present) is globally asymptotically stable.

When the space is heterogeneous, for example $\alpha = \beta = m(x)$ representing that the two species compete for the identical heterogeneous resource, it is shown in [23] that if the two species are also almost identical ($a_{11} = a_{22}$ and $a_{12} = a_{21}$) but have different diffusion coefficients (say, $d_2 < d_1$), then the slower species v will prevail so that the corresponding boundary equilibrium is globally asymptotically stable. Note that such "slower disperser wins" scenario only occurs when $m(x)$ is not a constant function. When the species are not identical but in the weak competition regime ($a_{11}a_{22} > a_{21}a_{12}$), it has been shown that there is always a globally asymptotically stable non-negative equilibrium, and the dynamics can be completely determined according to the values of a_{ij} and heterogeneous $m(x)$ [30,35,41]. The dynamics of diffusive Lotka–Volterra systems with advective effect has been studied recently in [43,44,61–63], and a technical analytic approach to directly exclude the existence of any co-existence steady state was developed.

In reality, taking the competition for common resource into consideration, individuals may compete for resource with peers of the same species not only in their immediate neighborhood but also in the entire spatial domain. Similarly the competition between individuals of two species is also not necessarily occurring between individuals at the same location but also occurring between individuals at different locations. Hence nonlocal competition effect has been incorporated into biological interaction models [25,26,54]. We propose the following generalized diffusive Lotka–Volterra competition model with local and nonlocal intraspecific and interspecific competition:

$$\begin{cases} u_t = d_1\Delta u + u\left(\alpha - a_{11}u - c_{11}\displaystyle\int_\Omega K_1(x,y)u(y,t)dy\right) \\ \qquad - u\left(a_{12}v + c_{12}\displaystyle\int_\Omega K_2(x,y)v(y,t)dy\right), & x\in\Omega,\ t>0, \\ v_t = d_2\Delta v + v\left(\beta - a_{22}v - c_{22}\displaystyle\int_\Omega K_2(x,y)v(y,t)dy\right) \\ \qquad - v\left(a_{21}u + c_{21}\displaystyle\int_\Omega K_1(x,y)u(y,t)dy\right), & x\in\Omega,\ t>0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0(x)\ge 0,\quad v(x,0)=v_0(x)\ge 0, & x\in\Omega. \end{cases} \tag{1.2}$$

Here the variables $u(x,t)$, $v(x,t)$ and parameters $\alpha, \beta, d_i, a_{ij}$ are the same as in (1.1), and note that (1.2) is reduced to (1.1) when $c_{ij}=0$ for all $i, j = 1, 2$. In the modified model (1.2), $c_{11} \ge 0$ and $c_{22} \ge 0$ are the strength of nonlocal intraspecific competition, and $c_{12} \ge 0$ and $c_{21} \ge 0$ are the strength of nonlocal interspecific competition. In the first equation of (1.2), the nonlocal term $\int_\Omega K_1(x,y)u(y,t)dy$ representing the intraspecific competitive effect at location x depends on a weighted average population density of u in its neighborhood considering the depletion of resource in their neighborhood (see [8,9]). For the same reason as for the intraspecific competition, the interspecific competitive effect of v on u relies on a weighted spatial average of v, that is $\int_\Omega K_2(x,y)v(y,t)dy$. The nonlocal terms in the second equation of (1.2) are similarly defined. While more general conditions can be imposed on $K_i(x,y)$, here we assume that $K_i(x,y)$ $(i=1,2)$ is the Green function of the operator $-\hat{d}_i\Delta + I$ with Neumann boundary condition and $\hat{d}_i$ are positive constants. This is in consistence with the diffusive movement of u and v assumed in (1.2), but we do not need to assume $\hat{d}_i = d_i$ though that could be the case. For a bounded domain Ω, the Green's function can be derived from eigenfunctions of a related Fredholm integral operator (see [14, Remark 4.6]). An explicit expression of the Green's function for $\Omega = \mathbb{R}^n$ is known as the Yukawa potential (see [38, pp. 163–164]), and for the one-dimensional case $\Omega = \mathbb{R}$, the kernels $K_i(x,y)$ take the form $1/(2\sqrt{\hat{d}_i})e^{-|x-y|/\sqrt{\hat{d}_i}}$ which is a monotonically decreasing function of $|x-y|$ and is consistent with the actual that individuals are in stronger competition with others nearby than with those further away (see [8]). More biological interpretations of this choice of kernel function are referred to [28,34]. Traveling wave solutions of reaction diffusion model with such nonlocal integral terms have been investigated in [7,28,29], and a nonlocal Lotka–Volterra competition model with similar nonlocal terms on unbounded domain has been studied in [8].

With a change of variables, (1.2) can be simplified to

$$\begin{cases} u_t = d_1\Delta u + u\left(\alpha - u - c_{11}\int_\Omega K_1(x,y)u(y,t)dy\right) \\ \qquad - u\left(a_1 v + c_{12}\int_\Omega K_2(x,y)v(y,t)dy\right), & x\in\Omega,\ t>0, \\ v_t = d_2\Delta v + v\left(\beta - v - c_{22}\int_\Omega K_2(x,y)v(y,t)dy\right) \\ \qquad - v\left(a_2 u + c_{21}\int_\Omega K_1(x,y)u(y,t)dy\right), & x\in\Omega,\ t>0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0(x)\ge 0, \quad v(x,0)=v_0(x)\ge 0, & x\in\Omega. \end{cases} \tag{1.3}$$

In the remaining part of this paper, we consider the equilibrium solutions and associated dynamics of (1.3). If the nonlocal competition coefficients $c_{ij}=0$ in (1.3), then (1.3) is reduced to (1.1) with $a_{11}=a_{22}=1$ and $a_{12}=a_1$, $a_{21}=a_2$. Similar to (1.1), the problem (1.3) has a trivial equilibrium $(0,0)$, and semi-trivial equilibria $\hat{\mathbf{u}}_1=(\alpha/(1+c_{11}),\ 0)$ and $\hat{\mathbf{u}}_2=(0,\ \beta/(1+c_{22}))$ for which only one species persists. Also (1.3) has a unique positive constant equilibrium $\hat{\mathbf{u}}_3$ (see (3.3) for precise expression).

Our main mathematical results concern the effect of the nonlocal competition on the dynamics of (1.3) in the weak competition regime and strong competition regime. Define

$$A_{11}=1+c_{11},\quad A_{12}=a_1+c_{12},\quad A_{21}=a_2+c_{21},\quad A_{22}=1+c_{22}$$

to be the combined strength of local and nonlocal competition between species 1 and species 2. Then the weak competition regime is when $\frac{A_{21}}{A_{11}}<\frac{A_{22}}{A_{12}}$ is satisfied, and the strong competition regime is when $\frac{A_{21}}{A_{11}}>\frac{A_{22}}{A_{12}}$. We further define

$$E_1=(a_1+c_{12})\frac{\beta}{\alpha}-1,\quad E_2=(a_2+c_{21})\frac{\alpha}{\beta}-1,\quad E_3=\frac{c_{12}}{2a_1+c_{12}},\quad E_4=\frac{c_{21}}{2a_2+c_{21}}.$$

Then our main results for the weak competition case of (1.3) are summarized as follows:

1. Suppose that $c_{11}=c_{22}=0$ and $\frac{A_{21}}{A_{11}}>\frac{A_{22}}{A_{12}}$ (Fig. 1, left panel), then for any $d_1,\ d_2>0$ we have (Proposition 3.1 and Theorems 3.3–3.5):
   (a) for $\frac{\beta}{\alpha}<\frac{A_{22}}{A_{12}}$, $\hat{\mathbf{u}}_1$ is globally asymptotically stable;
   (b) for $\frac{\beta}{\alpha}>\frac{A_{21}}{A_{11}}$, $\hat{\mathbf{u}}_2$ is globally asymptotically stable;
   (c) for $\frac{A_{22}}{A_{12}}<\frac{\beta}{\alpha}<\frac{A_{21}}{A_{11}}$, both $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ are locally asymptotically stable.

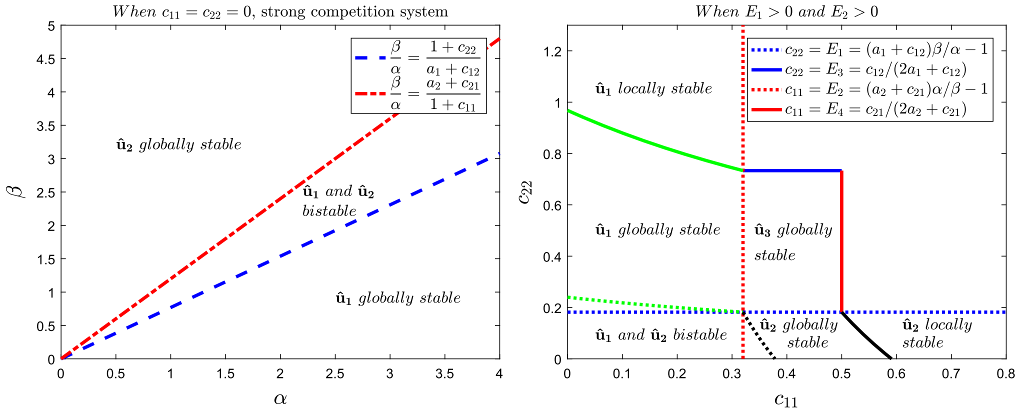

Fig. 1. Dynamics of problem (1.3). Here parameters $a_1 = 0.2$, $a_2 = 0.4$, $c_{12} = 1.1$, $c_{21} = 0.8$. And $c_{11} = c_{22} = 0$ for the left panel, $\alpha = 2.2$ and $\beta = 2$ for the right panel. (For interpretation of the colors in this figure, the reader is referred to the web version of this article.)

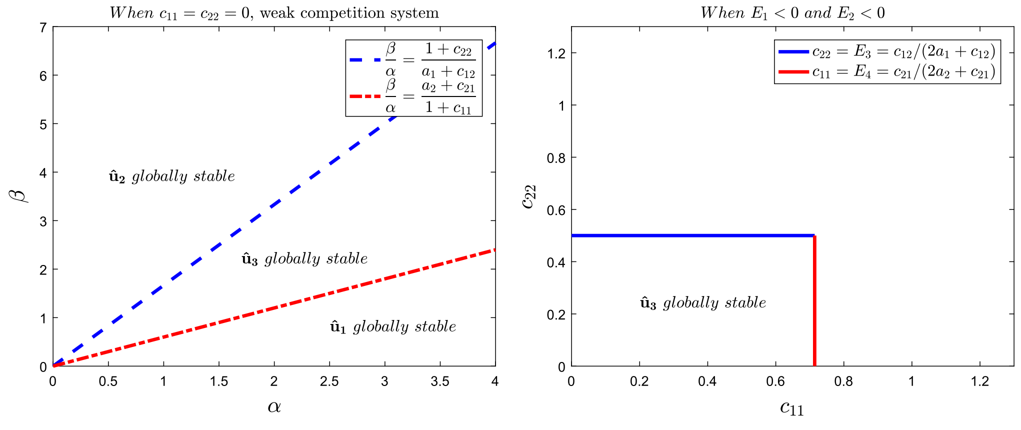

Fig. 2. Dynamics of problem (1.3). Here parameters $a_1 = 0.2$, $a_2 = 0.1$, $c_{12} = 0.4$, $c_{21} = 0.5$. And $c_{11} = c_{22} = 0$ for the left panel, $\alpha = 2.2$ and $\beta = 2$ for the right panel. (For interpretation of the colors in this figure, the reader is referred to the web version of this article.)

2. Suppose $c_{11} = c_{22} = 0$ and $\frac{A_{21}}{A_{11}} < \frac{A_{22}}{A_{12}}$ (Fig. 2, left panel), then for any $d_1, d_2 > 0$ we have (Theorems 3.3–3.5):
   (a) for $\frac{\beta}{\alpha} < \frac{A_{21}}{A_{11}}$, $\hat{\mathbf{u}}_1$ is globally asymptotically stable;
   (b) for $\frac{\beta}{\alpha} > \frac{A_{22}}{A_{12}}$, $\hat{\mathbf{u}}_2$ is globally asymptotically stable;
   (c) for $\frac{A_{21}}{A_{11}} < \frac{\beta}{\alpha} < \frac{A_{22}}{A_{12}}$, $\hat{\mathbf{u}}_3$ is globally asymptotically stable.
3. Suppose $c_{11} > 0$, $c_{22} > 0$ and $E_1 > 0$, $E_2 > 0$ (Fig. 1, right panel), then for any $d_1, d_2 > 0$ we have (Proposition 3.1 and Theorems 3.3–3.5):
   (a) for $0 < c_{11} < E_2$ and $E_1 \frac{A_{21}}{A_{11}} \frac{\alpha}{\beta} < c_{22} \le E_3 \frac{A_{21}}{A_{11}} \frac{\alpha}{\beta}$, $\hat{\mathbf{u}}_1$ is globally asymptotically stable;
   (b) for $E_2 \frac{A_{12}}{A_{22}} \frac{\beta}{\alpha} < c_{11} \le E_4 \frac{A_{12}}{A_{22}} \frac{\beta}{\alpha}$ and $0 < c_{22} < E_1$, $\hat{\mathbf{u}}_2$ is globally asymptotically stable;
   (c) for $0 < c_{11} < E_2$, $\hat{\mathbf{u}}_1$ is locally asymptotically stable;
   (d) for $0 < c_{22} < E_1$, $\hat{\mathbf{u}}_2$ is locally asymptotically stable;
   (e) for $E_2 < c_{11} \le E_4$ and $E_1 < c_{22} \le E_3$, $\hat{\mathbf{u}}_3$ is globally asymptotically stable.

4. Suppose $c_{11} > 0$, $c_{22} > 0$ and $E_1 < 0$, $E_2 < 0$ (Fig. 2, right panel), then for any $d_1, d_2 > 0$ we have (Proposition 3.1 and Theorems 3.3–3.5):
   (a) for $0 < c_{11} \le E_4$ and $0 < c_{22} \le E_3$, $\hat{\mathbf{u}}_3$ is globally asymptotically stable;
   (b) for $c_{11} > 0$, $c_{22} > 0$, both $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ are unstable.
5. If $c_{11}c_{22} - c_{12}c_{21} > (c_{12} + c_{21}) - (c_{11} + c_{22}) > 0$, $a_1 = a_2 = a > 0$ satisfies $0 < 1 - a^2 \ll 1$, and $\frac{A_{21}}{A_{11}} < \frac{\beta}{\alpha} < \frac{A_{22}}{A_{12}}$, then (1.3) has a positive non-constant coexistence equilibrium solution while the constant coexistence equilibrium $\hat{\mathbf{u}}_3$ is unstable (Theorem 4.11, Corollary 4.12 and Remark 4.13).

The results in Part 1 and 2 show that when there is no nonlocal intraspecific competition, the dynamics of (1.3) is similar to that of local counterpart (1.1) and it can be completely classified according to the ratio β/α with the threshold values shifted because of nonlocal interspecific competition. In all cases except the bistable one, a non-negative constant equilibrium is reached as the asymptotic state. On the other hand, when there is nonlocal intraspecific competition in addition to the nonlocal interspecific one, results in Part 3 and Part 4 provide the ranges of (c_{11}, c_{22}) so that the semi-trivial or coexistence state is globally asymptotically stable. In Part 3, the region $\{0 \le c_{11} < E_2,\ 0 \le c_{22} < E_1\}$ is the strong competition regime, while the region $\{c_{11} > E_2,\ c_{22} > E_1\}$ is the weak competition one; and in Part 4, all of $\{c_{11} \ge 0,\ c_{22} \ge 0\}$ is in the weak competition regime. The dynamics of (1.3) when $c_{11} > 0$ and $c_{22} > 0$ (as in Part 3 and 4) is not completely classified as there are several regions in (c_{11}, c_{22}) plane for which the global dynamical behavior is not known yet from our results. But our result in Part 5 shows that the positive constant coexistence state could be unstable even in the weak competition regime, and non-constant positive equilibria exist and appear to be locally asymptotically stable ones, which is supported by numerical simulations (see Section 4). The parameter diagrams in Fig. 1 right panel and Fig. 2 right panel suggest that the non-constant positive equilibria can only exist when c_{11} and c_{22} are properly large in the weak competition case.

The existence of non-constant positive equilibria (or spatial patterns) suggests that the constant coexistence state can be destabilized by the nonlocal competition effect in the weak competition case. More precisely, our result in Part 2 above shows that when $c_{11} = c_{22} = 0$ and $c_{12} > 0$, $c_{21} > 0$ (no nonlocal intraspecific competition, and nonlocal interspecific competition only), then no spatial patterns are possible. But when $c_{11} > 0$ and $c_{22} > 0$ also hold (with nonlocal intraspecific and interspecific competitions), stationary spatial pattern can be generated (see Part 5 above). This shows that the nonlocal intraspecific competition not the nonlocal interspecific competition is the main driving force for the pattern formation.

The numerical simulations (see Section 4) also indicate that the two species concentrate in different areas of the habitat, which is known as the spatial segregation in competition model (see [16,22]). Spatial heterogeneity has been thought as one of main reasons that similar species can coexist in the environment. It is known that the classical diffusive Lotka–Volterra competition model (1.1) cannot have stable non-constant equilibria if the domain Ω is convex [33]. Note that this excludes any one-dimensional domain. Various mechanisms have been suggested as possible causes of stable coexistence with spatial segregation (hence non-constant): cross-diffusion in one-dimensional domain [46], nonlinear diffusion in one-dimensional domain [47], dumbbell-shaped higher-dimensional domain [45], advection in one-dimensional or higher-dimensional domain [11,15]. The result here can be regarded as another mechanism to achieve spatial segregated coexistence for spatial Lotka–Volterra competition models. Some other recent studies of diffusive competition models can be found in [24,43,48].

It is notable that the classical diffusive Lotka–Volterra competition model (1.1) generates a monotone dynamical system. However the nonlocal diffusive competition problem (1.3) is not a monotone dynamical system, and the maximum principle is not applicable. Indeed our main strategy of this paper is to use the diffusion kernel to convert (1.3) into a parabolic–elliptic system of four equations (see Section 3), and the new system is not a competition system any more. Such conversion removes the nonlocal terms in the system so classical PDE theory can be applied, while the number of equations in the system increases which brings additional difficulty of analysis.

Our analysis of the competition system (1.3) depends on a detailed analysis of the semi-trivial equilibrium solution of (1.3), which is a positive equilibrium solution of the scalar equation:

$$\begin{cases} u_t = d\Delta u + u\left(a - u - \int\limits_{\Omega} K(x,y)u(y,t)dy\right), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{1.4}$$

In this paper we consider a more general version of (1.4), and we show that when $K(x, y)$ is the Green's function of the operator $-d\Delta + I$ with Neumann boundary condition, then (1.4) has a unique positive equilibrium solution which is constant and globally asymptotically stable with respect to the dynamics of (1.4). Our global stability results are proved using both a modified Lyapunov functional method and also an upper–lower solution method with different conditions on nonlinearities. Related Lyapunov functional methods have also been used in [51,59]. The diffusive logistic equation with nonlocal carrying capacity as (1.4) has been considered in many other work recently, but the kernel function $K(x, y)$ in previous studies may take other forms. In [1,13,54], the steady state solutions with constant kernel function K were considered; in [17,60], the kernel function $K(x, y)$ is assumed to be separable, *i.e.* $K(x, y) = f(x)g(y)$; and in [2,19,54], the kernel function satisfies $K(x, y) = K(|x - y|)$. The steady state solutions for more general kernel function have also been considered in [3,54,60], and some related nonlocal eigenvalue problems have been investigated in [5,6,18,37,52].

This paper is organized as follows. In Section 2, we study global stability of positive equilibrium for the nonlocal problem of one species (1.4). Section 3 is devoted to analyze the global dynamical properties of the nonlocal diffusive competition problem (1.3). Section 4 concerns with the existence of nonconstant positive equilibrium solutions in the weak competition case.

2. The nonlocal problem of one species

In the section, we study the dynamics of the scalar parabolic equation with nonlocal interaction:

$$\begin{cases} u_t = d\Delta u + u\left(a - f(u) - \int\limits_{\Omega} K(x,y)g(u(y,t))dy\right), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{2.1}$$

where $K(x, y)$ is the Green function of the operator $-d_3\Delta + I$ with Neumann boundary condition, and $f(s)$, $g(s)$ satisfy the assumption

($\mathbf{F_1}$) $f, g : [0, \infty) \to [0, \infty)$ are continuously differentiable, $f(0) \geq 0$, $g(0) \geq 0$, $f(0) + g(0) < a$, $f'(s) > 0$, $g'(s) > 0$ for $s > 0$ and $\lim_{s\to\infty} f(s) > a > 0$.

Set $w(x, t) = \int_\Omega K(x, y)g(u(y, t))dy$, then the problem (2.1) can be converted to the following problem with $\tau = 0$:

$$\begin{cases} u_t = d\Delta u + u(a - f(u) - w), & x \in \Omega,\ t > 0, \\ \tau w_t = d_3\Delta w - w + g(u), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x) \geq 0,\ \tau w(x, 0) = \tau w_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{2.2}$$

Throughout this section we assume that $u_0 \in C(\bar{\Omega})$, and $w_0 \in C(\bar{\Omega})$ when $\tau > 0$.

In the following, we will investigate asymptotic behavior of the solutions of the problem (2.2) with $\tau \geq 0$. Clearly $\tilde{\mathbf{u}}_0 = (0, g(0))$ and $\tilde{\mathbf{u}}_1 = (\tilde{u}, \tilde{w})$ are nonnegative constant equilibria of problem (2.2), where $\tilde{u}$ is the unique positive solution of $f(u) + g(u) = a$ and $\tilde{w} = g(\tilde{u})$.

2.1. Local stability

The equilibrium solutions of (2.2) satisfy a system of elliptic equations:

$$\begin{cases} d\Delta u + u(a - f(u) - w) = 0, & x \in \Omega, \\ d_3\Delta w - w + g(u) = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{2.3}$$

Denote $\mathbf{u} = (u, w)$ and

$$H(\mathbf{u}) = \begin{bmatrix} u(a - f(u) - w) \\ -w + g(u) \end{bmatrix}, \quad H_{\mathbf{u}}(\mathbf{u}) = \begin{bmatrix} a - f(u) - w - uf'(u) & -u \\ g'(u) & -1 \end{bmatrix},$$

where $H_{\mathbf{u}}(\mathbf{u})$ is the linearization of $H(\mathbf{u})$ at $\mathbf{u}$. The linearization of (2.3) at the equilibrium $\tilde{\mathbf{u}}_i$ can be written as

$$\begin{cases} -D\Delta\mathbf{u} = H_{\mathbf{u}}(\tilde{\mathbf{u}}_i)\mathbf{u} - \xi\mathbf{u}, & x \in \Omega, \\ \dfrac{\partial \mathbf{u}}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{2.4}$$

where $D = \text{diag}\{d, d_3\}$ and $i \in \{0, 1\}$. The local stabilities of the equilibria $\tilde{\mathbf{u}}_0$ and $\tilde{\mathbf{u}}_1$ with respect to (2.3) are determined by the eigenvalues of problem (2.4).

Let $0 = \mu_0 < \mu_1 < \cdots < \mu_i < \cdots$ be the complete set of eigenvalues of the operator $-\Delta$ in Ω with homogeneous Neumann boundary condition, and let $E(\mu_i)$ be the subspace generated by the eigenfunctions corresponding to μ_i. Let m_i be the algebraic multiplicity of μ_i, *i.e.*, $m_i = \dim E(\mu_i)$, and let $\{\phi_{ij}\}_{j=1}^{m_i}$ be a basis of $E(\mu_i)$, *i.e.*, $\{\phi_{ij}\}_{j=1}^{m_i}$ constitute a complete set of linearly independent eigenfunctions corresponding to μ_i. Define for $n \in \mathbb{N}$,

$$\begin{cases} \mathbf{X} = \left\{ u \in C^1(\bar{\Omega}) : \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}, \quad \mathbf{X}^n = \overbrace{\mathbf{X} \times \ldots \times \mathbf{X}}^{n}, \\ \mathbf{X}^n_{ij} = \{c\,\phi_{ij} : c \in \mathbb{R}^n\}, \quad \mathbf{X}^n_i = \bigoplus_{j=1}^{m_i} \mathbf{X}^n_{ij}. \end{cases} \tag{2.5}$$

Then

$$\mathbf{X}^n = \bigoplus_{i=0}^{\infty} \mathbf{X}^n_i.$$

We have the following result regarding the local stability of $\tilde{\mathbf{u}}_0$ and $\tilde{\mathbf{u}}_1$.

Proposition 2.1. *Suppose that d, d_3 and a are positive.*

1. *The semitrivial equilibrium $\tilde{\mathbf{u}}_0 = (0, g(0))$ is unstable with respect to* (2.2) *when $\tau > 0$.*
2. *The positive equilibrium $\tilde{\mathbf{u}}_1 = (\tilde{u}, \tilde{w})$ is locally asymptotically stable with respect to* (2.2) *when $\tau > 0$.*

Proof. For the equilibrium $\tilde{\mathbf{u}}_0 = (0, g(0))$, a direct computation yields

$$H_{\mathbf{u}}(\tilde{\mathbf{u}}_0) = \begin{bmatrix} a - f(0) - g(0) & 0 \\ g(0) & -1 \end{bmatrix}.$$

Clearly $\lambda = a - f(0) - g(0) > 0$ and $\lambda = -1$ are eigenvalues of $H_{\mathbf{u}}(\tilde{\mathbf{u}}_0)$. Therefore the equilibrium $(0, 0)$ is unstable.

For the equilibrium $\tilde{\mathbf{u}}_1 = (\tilde{u}, \tilde{w})$, denote $\mathcal{L} = D\Delta + H_{\mathbf{u}}(\tilde{\mathbf{u}}_1)$. Then for each $j \in \mathbb{N} \cup \{0\}$, $\mathbf{X}^2_j$ is invariant under the operator $\mathcal{L}$, and ξ is an eigenvalue of $\mathcal{L}$ on $\mathbf{X}^2_j$ if and only if ξ is an eigenvalue of the matrix

$$M_j = -\mu_j D + H_{\mathbf{u}}(\tilde{\mathbf{u}}_1) = \begin{bmatrix} -\mu_j d - \tilde{u} f'(\tilde{u}) & -\tilde{u} \\ g'(\tilde{u}) & -\mu_j d_3 - 1 \end{bmatrix}.$$

The direct calculation gives

$$\begin{aligned} \operatorname{Tr} M_j &= -1 - \tilde{u} f'(\tilde{u}) - (d + d_3)\mu_j < 0, \\ \det M_j &= (\mu_j d + \tilde{u} f'(\tilde{u}))(\mu_j d_3 + 1) + \tilde{u} g'(\tilde{u}) > 0, \end{aligned}$$

which means that the real part of the two eigenvalues of M_j are negative. Hence $(\tilde{u}, \tilde{w})$ is locally asymptotically stable and the proof is complete. □

2.2. Global stability

In this subsection, we prove the global stability of positive equilibrium of the problem (2.2) with $\tau \geq 0$. First we have the following results regarding the global existence and boundedness of solutions to (2.1) and (2.2).

Theorem 2.2. *Suppose that f and g satisfy the assumption* ($\mathbf{F_1}$) *and $u_0(x) \geq 0, \not\equiv 0$ and $w_0(x) \geq 0, \not\equiv 0$ when $\tau > 0$.*

1. *Let $p > \max\{1, N/2\}$. Then* (2.1) *has a unique solution $u(x,t) > 0$ for $x \in \bar{\Omega}$, $t > 0$ and*

$$u \in C([0,\infty); L^p(\Omega)) \cap C((0,\infty); W_p^2\Omega)) \cap C^1((0,\infty); L^p(\Omega)).$$

Consequently, when $\tau = 0$ the problem (2.2) *has a unique solution $(u(x,t), w(x,t))$ with $u(x,t), w(x,t) > 0$ for $x \in \bar{\Omega}$, $t > 0$ and $u \in C^{2+\gamma,1+\gamma/2}(\bar{\Omega} \times (0,\infty))$, $w(\cdot,t) \in C^{2+\gamma}(\bar{\Omega})$ for $t > 0$ where $0 < \gamma < 1$. Moreover, there exists a constant $M_1 > 0$ such that*

$$\|u(\cdot,t)\|_{C^2(\bar{\Omega})},\ \ \|w(\cdot,t)\|_{C^2(\bar{\Omega})} \leq M_1, \quad \forall\, t \geq 1. \tag{2.6}$$

2. *When $\tau > 0$, the problem* (2.2) *has a unique solution $(u(x,t), w(x,t))$ with $u(x,t)$, $w(x,t) > 0$ for $x \in \bar{\Omega}$, $t > 0$ and $u, w \in C^{2+\gamma,1+\gamma/2}(\bar{\Omega} \times (0,\infty))$ where $0 < \gamma < 1$. Moreover, there exists a constant $M_2 > 0$ such that*

$$\|u(\cdot,t)\|_{C^2(\bar{\Omega})},\ \ \|w(\cdot,t)\|_{C^2(\bar{\Omega})} \leq M_2, \quad \forall\, t \geq 1. \tag{2.7}$$

The proof of Theorem 2.2 for (2.1) or equivalently (2.2) with $\tau = 0$ can be done using semi-group theory and Banach's fixed point theorem similar to the one for [60, Theorem 2.1]; and for (2.2) with $\tau > 0$, the upper and lower solutions method can be used to show the global existence of solutions. And making use of the boundedness of solutions of problem (2.2) for $\tau \geq 0$, one can prove the estimates (2.6) and (2.7) by similar methods as the ones for [58, Theorem 2.1], and the details are omitted here.

To prove the global stability of the positive equilibrium $\tilde{\mathbf{u}}_1 = (\tilde{u}, \tilde{w})$, we will use the following well known elementary lemma.

Lemma 2.3. (Barbălat's Lemma [4].) *Suppose that $h : [0,\infty) \to \mathbb{R}$ is uniformly continuous and that $\lim_{t\to\infty} \int_0^t h(s)ds$ exists. Then $\lim_{t\to\infty} h(t) = 0$ holds.*

Now we are ready to prove the following global stability results for (2.2).

Theorem 2.4. *Let $(u(x,t), w(x,t))$ be the positive solution of* (2.2) *with $\tau \geq 0$. If f and g satisfy* ($\mathbf{F_1}$) *and*

($\mathbf{F_2}$) *$g(s) \leq sg'(s)$ for $0 < s \leq a_* = f^{-1}(a)$.*

Then $(u(x,t), w(x,t))$ converges to the positive constant equilibrium $\tilde{\mathbf{u}}_1$ uniformly in $\bar{\Omega}$ when $t \to \infty$.

Proof. Step 1. Firstly, from (2.2) we see that u satisfies

$$\begin{cases} u_t \le d\Delta u + u(a - f(u)), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \ge 0, & x \in \Omega. \end{cases}$$

Clearly, $\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} u(x,t) \le a_*$. In view of the conditions $(\mathbf{F_1})$ *and* $(\mathbf{F_2})$ *and the fact that* $g(\tilde{u}) > 0$, there exists a constant $T > 0$ such that

$$ug'(u) - g(u) + g(\tilde{u}) > 0, \quad \forall\, t \ge T. \tag{2.8}$$

Step 2. Define a function $Q : [0,\infty) \to \mathbb{R}$ by

$$Q(t) = \int_\Omega \int_{\tilde{u}}^{u(x,t)} \frac{g(s) - g(\tilde{u})}{s} ds dx + \frac{\tau}{2} \int_\Omega (w(x,t) - \tilde{w})^2 dx.$$

Then $Q(t) \ge 0$ by the condition $(\mathbf{F_1})$. Now we study the properties of $Q(t)$.

If $\tau > 0$, using (2.8) we obtain that, for $t > T$,

$$\begin{aligned}
\frac{dQ}{dt} &= \int_\Omega \frac{g(u) - g(\tilde{u})}{u} u_t dx + \tau \int_\Omega (w - \tilde{w}) w_t dx \\
&= -\int_\Omega \left(d\frac{ug'(u) - (g(u) - g(\tilde{u}))}{u^2} |\nabla u|^2 + d_3 |\nabla w|^2 \right) dx \\
&\quad - \int_\Omega [(g(u) - g(\tilde{u}))(f(u) - f(\tilde{u})) + (g(u) - g(\tilde{u}))(w - \tilde{w})] dx \\
&\quad + \int_\Omega [(g(u) - g(\tilde{u}))(w - \tilde{w}) - (w - \tilde{w})^2] dx \\
&= -\int_\Omega \left(d\frac{ug'(u) - (g(u) - g(\tilde{u}))}{u^2} |\nabla u|^2 + d_3 |\nabla w|^2 \right) dx \\
&\quad - \int_\Omega [(g(u) - g(\tilde{u}))(f(u) - f(\tilde{u})) + (w - \tilde{w})^2] dx \\
&\le -\int_\Omega [(g(u) - g(\tilde{u}))(f(u) - f(\tilde{u})) + (w - \tilde{w})^2] dx \le 0.
\end{aligned} \tag{2.9}$$

With $\tau = 0$, multiplying the second equation of (2.2) by $w - \tilde{w}$ and integrating over Ω, we have

$$\int_\Omega (g(u)-g(\tilde u))(w-\tilde w)dx=\int_\Omega [d_3|\nabla w|^2+(w-\tilde w)^2]dx. \tag{2.10}$$

Combining (2.10) with (2.8), similarly we obtain that, for $t>T$,

$$\begin{aligned}\frac{dQ}{dt}&=\int_\Omega \frac{g(u)-g(\tilde u)}{u}u_t dx\\ &=-\int_\Omega d\frac{[ug'(u)-(g(u)-g(\tilde u))]|\nabla u|^2}{u^2}dx\\ &\quad-\int_\Omega (g(u)-g(\tilde u))[f(u)-f(\tilde u)+(w-\tilde w)]dx\\ &=-\int_\Omega \left(d\frac{[ug'(u)-(g(u)-g(\tilde u))]|\nabla u|^2}{u^2}+d_3|\nabla w|^2\right)dx\\ &\quad-\int_\Omega [(g(u)-g(\tilde u))(f(u)-f(\tilde u))+(w-\tilde w)^2]dx\\ &\le-\int_\Omega [(g(u)-g(\tilde u))(f(u)-f(\tilde u)]dx\le 0.\end{aligned} \tag{2.11}$$

It follows from (2.9), (2.11) and $Q(t)\ge 0$ for $t\ge 0$ that $\lim\limits_{t\to\infty} Q(t)$ exists for $\tau\ge 0$.

Step 3. When $\tau>0$, we denote

$$S(t)=-\int_\Omega [(g(u)-g(\tilde u))(f(u)-f(\tilde u))+(w-\tilde w)^2]dx.$$

Then by (2.9),

$$\overline{\lim_{t\to\infty}}\int_T^t |S(s)|ds\le\overline{\lim_{t\to\infty}}\int_T^t |Q'(s)|ds=-\lim_{t\to\infty}\int_T^t Q'(s)ds=Q(T)-\lim_{t\to\infty}Q(t)<\infty,$$

which means that $\lim_{t\to\infty}\int_0^t S(s)ds$ exists. Moreover, from (2.7), it can be shown that $S'(t)$ is uniformly bounded for $t\ge 1$. Consequently $S(t)$ is uniformly continuous for $t\ge 1$. Then taking advantage of Lemma 2.3, we obtain that $\lim\limits_{t\to\infty} S(t)=0$.

From the boundedness property in (2.7), the sets $\{u(\cdot,t):t\ge 1\}$ and $\{w(\cdot,t):t\ge 1\}$ are relatively compact in $C(\bar\Omega)$. Assume that

$$\|u(x,t_k)-u_\infty(x)\|_{C(\bar\Omega)}\to 0,\quad \|w(x,t_k)-w_\infty(x)\|_{C(\bar\Omega)}\to 0\quad \text{as } t_k\to\infty$$

for some $u_\infty(x), w_\infty(x)\in C(\bar\Omega)$. Combining this result with the convergence of $S(t)$, we have $u_\infty(x)\equiv\tilde u$ and $w_\infty(x)\equiv\tilde w$. This implies that $(u(\cdot,t),w(\cdot,t))$ converges to $(\tilde u,\tilde w)$ uniformly in $\bar\Omega$ as $t\to\infty$.

Step 4. When $\tau = 0$, we set

$$S_0(t) = -\int_\Omega (g(u) - g(\tilde{u}))(f(u) - f(\tilde{u}))dx.$$

From (2.6) and (2.11), similarly we can deduce that $\lim_{t\to\infty} S_0(t) = 0$ and $\lim_{t\to\infty} u(x,t) = \tilde{u}$ uniformly in $\bar{\Omega}$. Then combining this with (2.10), we obtain $\lim_{t\to\infty} \int_\Omega (w - \tilde{w})^2 dx = 0$. Making use of (2.6) again, we can show that $(u(\cdot,t), w(\cdot,t))$ converges to $(\tilde{u}, \tilde{w})$ uniformly in $\bar{\Omega}$ as $t \to \infty$. The proof is complete. □

Remark 2.5. Assume that the assumption $(\mathbf{F_1})$ holds. Then the proof of Theorem 2.4 also shows that $\lim_{t\to\infty} u(t) = \tilde{u}$ and $\lim_{t\to\infty} w(t) = \tilde{w}$ for $\tau > 0$ if $(u(t), w(t))$ is the solution of the following ordinary differential equations:

$$\begin{cases} u' = u\,(a - f(u) - w)\,, \\ \tau w' = -w + g(u), \\ u(0) > 0,\ w(0) > 0. \end{cases} \tag{2.12}$$

Next we prove the global stability of the positive equilibrium $\tilde{\mathbf{u}}_1$ of (2.2) by the upper and lower solutions method for $\tau > 0$ and different conditions about f, g. Let $(\bar{u}(t), \underline{u}(t), \bar{w}(t), \underline{w}(t))$ be the solution of

$$\begin{cases} \bar{u}' = \bar{u}\left(a - f(\bar{u}) - \underline{w}\right), \\ \underline{u}' = \underline{u}\left(a - f(\underline{u}) - \bar{w}\right), \\ \tau \bar{w}' = -\bar{w} + g(\bar{u}), \\ \tau \underline{w}' = -\underline{w} + g(\underline{u}), \\ \bar{u}(0) = \max_{x\in\bar{\Omega}} u_0(x),\ \underline{u}(0) = \min_{x\in\bar{\Omega}} u_0(x) > 0, \\ \bar{w}(0) = \max_{x\in\bar{\Omega}} w_0(x),\ \underline{w}(0) = \min_{x\in\bar{\Omega}} w_0(x) > 0, \end{cases} \tag{2.13}$$

where $u_0(x)$ and $w_0(x)$ are the initial conditions defined in (2.2). Then it can be shown that $(\bar{u}, \bar{w})$ and $(\underline{u}, \underline{w})$ are a pair of coupled ordered upper and lower solutions of problem (2.2) with $\tau > 0$, and

$$0 < \underline{u}(t) \le u(x,t) \le \bar{u}(t),\quad 0 < \underline{w}(t) \le w(x,t) \le \bar{w}(t),\quad \forall\, x \in \bar{\Omega},\ t > 0. \tag{2.14}$$

Moreover, if $\bar{u}(0) - \underline{u}(0) + \bar{w}(0) - \underline{w}(0) > 0$, we get

$$\underline{u}(t) < \bar{u}(t),\quad \underline{w}(t) < \bar{w}(t),\quad \forall\, t > 0. \tag{2.15}$$

Now we show the following global stability of the positive equilibrium $\tilde{\mathbf{u}}_1$ when $\tau > 0$ with some different conditions on f, g:

Theorem 2.6. *Let $(u(x,t), w(x,t))$ be the positive solution of (2.2) with $\tau > 0$. Suppose that f and g satisfy $(\mathbf{F_1})$ and*

(F$_3$) $g'(s) < f'(s)$ *for* $0 \le s \le a_* = f^{-1}(a)$.

Then $(u(x,t), w(x,t))$ *converges to* $(\tilde{u}, \tilde{w})$ *uniformly in* $\bar{\Omega}$ *as* $t \to \infty$.

Proof. Let $(\bar{u}, \bar{w})$ and $(\underline{u}, \underline{w})$ be defined as above. If $\bar{u}(0) = \underline{u}(0) > 0$ and $\bar{w}(0) = \underline{w}(0) > 0$, then $\bar{u} = \underline{u}$ and $\bar{w} = \underline{w}$, so $(\bar{u}(t), \bar{w}(t))$ is the solution of (2.12). It follows from Remark 2.5 that $\lim_{t\to\infty} \bar{u}(t) = \lim_{t\to\infty} \underline{u}(t) = \tilde{u}$ and $\lim_{t\to\infty} \bar{w}(t) = \lim_{t\to\infty} \underline{w}(t) = \tilde{w}$. Combining this formula with (2.14), we obtain $\lim_{t\to\infty} u(x,t) = \tilde{u}$ and $\lim_{t\to\infty} w(x,t) = \tilde{w}$.

In the following, we consider the case of $\bar{u}(0) - \underline{u}(0) + \bar{w}(0) - \underline{w}(0) > 0$. By the assumption (**F$_3$**), there exists $\varepsilon > 0$ such that

$$(1+\varepsilon)g'(s) < f'(s), \quad \forall\, 0 \le s \le a_* + \varepsilon. \tag{2.16}$$

For the problem (2.13), it is well known that $\limsup_{t\to\infty} \bar{u}(t) \le a_*$. Recalling (2.15), there exists $T > 0$ such that, for $t > T$,

$$\underline{u}(t) < \bar{u}(t) < a_* + \varepsilon. \tag{2.17}$$

Define

$$Q_1(t) = \ln \frac{\bar{u}(t)}{\underline{u}(t)} + \tau(1+\varepsilon)(\bar{w}(t) - \underline{w}(t)), \quad t \ge 0.$$

Then $Q_1(t)$ is well defined for $t \in [0, \infty)$ and $Q_1(t) > 0$ for $t > 0$ by (2.15). Making use of (2.15), (2.16) and (2.17) we have that, for $t > T$,

$$\begin{aligned} \frac{dQ_1(t)}{dt} &= \frac{\bar{u}'}{\bar{u}} - \frac{\underline{u}'}{\underline{u}} + \tau(1+\varepsilon)(\bar{w}' - \underline{w}') \\ &= -(f(\bar{u}) - f(\underline{u})) - \varepsilon(\bar{w}(t) - \underline{w}(t)) + (1+\varepsilon)(g(\bar{u}) - g(\underline{u})) \\ &= -[(f(\bar{u}) - f(\underline{u})) - (1+\varepsilon)(g(\bar{u}) - g(\underline{u}))] - \varepsilon(\bar{w}(t) - \underline{w}(t)) \\ &< 0, \end{aligned} \tag{2.18}$$

which means that $Q_1(t)$ decreases for $t > T$. The inequality (2.18) also implies that

$$\liminf_{t\to\infty} \{[(f(\bar{u}) - f(\underline{u})) - (1+\varepsilon)(g(\bar{u}) - g(\underline{u}))] + \varepsilon(\bar{w} - \underline{w})\} = 0.$$

Together with (2.15), we conclude that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{t_n\to\infty} [(f(\bar{u}) - f(\underline{u})) - (1+\varepsilon)(g(\bar{u}) - g(\underline{u}))] = 0, \quad \lim_{t_n\to\infty} (\bar{w}(t_n) - \underline{w}(t_n)) = 0,$$

and so $\lim_{t_n\to\infty} (\bar{u}(t_n) - \underline{u}(t_n)) = 0$ by (2.16). Since $\limsup_{t\to\infty} \bar{u}(t) > 0$ by Remark 2.5, it follows from the definition of $Q_1(t)$ that $\lim_{t_n\to\infty} Q_1(t_n) = 0$. Then we obtain $\lim_{t\to\infty} Q_1(t) = 0$ since $Q_1(t)$ decreases for $t > T$ and

$$\lim_{t\to\infty}(\bar{u}(t)-\underline{u}(t))=0, \quad \lim_{t\to\infty}(\bar{w}(t)-\underline{w}(t))=0. \tag{2.19}$$

Let $(u_1(t), w_1(t))$ be the solution of (2.12) with initial conditions $u_1(0) = (\bar{u}(0) + \underline{u}(0))/2$ and $w_1(0) = (\bar{w}(0) + \underline{w}(0))/2$. Recall that $(\bar{u}(t), \underline{u}(t))$ and $(\bar{w}(t), \underline{w}(t))$ are a pair of coupled ordered upper and lower solutions of (2.12), then

$$\underline{u}(t) \le u_1(t) \le \bar{u}(t), \quad \underline{w}(t) \le w_1(t) \le \bar{w}(t), \quad \forall\, t > 0. \tag{2.20}$$

On the other hand, using Remark 2.5 we have

$$\lim_{t\to\infty} u_1(t) = \tilde{u}, \quad \lim_{t\to\infty} w_1(t) = \tilde{w}. \tag{2.21}$$

It follows from (2.19)–(2.21) that $\lim_{t\to\infty}\bar{u}(t) = \lim_{t\to\infty}\underline{u}(t) = \tilde{u}$, $\lim_{t\to\infty}\bar{w}(t) = \lim_{t\to\infty}\underline{w}(t) = \tilde{w}$. Combining these convergences with (2.14) we have $\lim_{t\to\infty} u(x,t) = \tilde{u}$, $\lim_{t\to\infty} w(x,t) = \tilde{w}$. $\square$

Finally we show the global stability of the positive equilibrium $\tilde{\mathbf{u}}_1$ when $\tau = 0$ by using the upper and lower solutions method similar to the one in [12]. Let $(\bar{u}(t), \underline{u}(t))$ be the solution of the following system of ordinary differential equations:

$$\begin{cases} \bar{u}' = \bar{u}(a - f(\bar{u}) - g(\underline{u})), \\ \underline{u}' = \underline{u}(a - f(\underline{u}) - g(\bar{u})), \\ \bar{u}(0) = \max_{x\in\bar{\Omega}} u_0(x), \ \underline{u}(0) = \min_{x\in\bar{\Omega}} u_0(x) > 0, \end{cases} \tag{2.22}$$

where $u_0(x)$ is the initial condition defined in (2.1). It is easy to see that $(\tilde{u}, \tilde{u})$ is the unique positive equilibrium of problem (2.22) when $f'(s) \ne g'(s)$ for $0 \le s \le a_*$, where $\tilde{u}$ is the unique positive root of $f(u) + g(u) = a$.

We first prove the following lemma regarding the system of ordinary differential equations (2.22).

Lemma 2.7. *Let $(\bar{u}(t), \underline{u}(t))$ be the unique solution of (2.22). If f and g satisfy* **(F1)** *and* **(F3)**, *then $\underline{u}(t) \le \bar{u}(t)$ for $t \ge 0$ and the positive equilibrium $(\tilde{u}, \tilde{u})$ is globally asymptotically stable with respect to (2.22).*

Proof. Let $h(t; h_0)$ be the solution of the ordinary differential equation:

$$h' = h(a - f(h) - g(h)), \quad h(0) = h_0 > 0.$$

Then we have $\lim_{t\to\infty} h(t) = \tilde{u}$ and $h(t; h_1) < h(t; h_2)$ for $0 < h_1 < h_2$. If $\bar{u}(0) = \underline{u}(0)$, then we get that $\bar{u}(t) = \underline{u}(t) = h(t; \bar{u}(0))$ and $\lim_{t\to\infty}\bar{u}(t) = \lim_{t\to\infty}\underline{u}(t) = \tilde{u}$.

In the following we consider the case of $\underline{u}(0) < \bar{u}(0)$. Set $T = \sup\{t : \underline{u}(t) < \bar{u}(t)\}$. If $T < \infty$, it then follows that $\underline{u}(t) < \bar{u}(t)$ for $0 \le t < T$ and $\bar{u}(T) = \underline{u}(T)$. It follows from (2.22) that, for $0 \le t \le T$,

$$\begin{cases} \bar{u}' \geq \bar{u}(a - f(\bar{u}) - g(\bar{u})), \\ \underline{u}' \leq \underline{u}(a - f(\underline{u}) - g(\underline{u})), \\ \bar{u}(0) > \underline{u}(0), \end{cases}$$

which means that $\underline{u}(t) \leq h(t; \underline{u}(0)) < h(t; \bar{u}(0)) \leq \bar{u}(t)$ for $0 \leq t \leq T$. This contradicts to $\bar{u}(T) = \underline{u}(T)$. Therefore $T = \infty$ and

$$\underline{u}(t) \leq h(t; \underline{u}(0)) < h(t; \bar{u}(0)) \leq \bar{u}(t), \quad \forall\, t \geq 0. \tag{2.23}$$

Thanks to the assumption $(\mathbf{F_3})$, there exists $\varepsilon > 0$ such that

$$g'(s) < f'(s), \quad \forall\, 0 \leq s \leq a_* + \varepsilon. \tag{2.24}$$

For the problem (2.22), clearly $\limsup_{t\to\infty} \bar{u}(t) \leq a$. Hence there exists $T > 0$ such that

$$\underline{u}(t) < \bar{u}(t) < a + \varepsilon, \ \ \forall\, t > T. \tag{2.25}$$

Denote $Q_2(t) = \ln(\bar{u}(t)/\underline{u}(t))$. Then $Q_2(t)$ is well defined for $t \geq 0$ and $Q_2(t) > 0$ for $t > 0$ by (2.23). In view of (2.24) and (2.25) it deduces that, for $t > T$,

$$\frac{dQ_2(t)}{dt} = \frac{\bar{u}'}{\bar{u}} - \frac{\underline{u}'}{\underline{u}} = -(f(\bar{u}) - f(\underline{u})) + (g(\bar{u}) - g(\underline{u})) \leq 0,$$

which means that $Q_2(t)$ decreases for $t > T$. Similar to the proof of Theorem 2.6, we can obtain $\lim_{t\to\infty}(\bar{u}(t) - \underline{u}(t)) = 0$. Then it follows from (2.23) and $\lim_{t\to\infty} h(t; \bar{u}(0)) = \tilde{u}$ that $\lim_{t\to\infty} \bar{u}(t) = \lim_{t\to\infty} \underline{u}(t) = \tilde{u}$. □

Now we prove the global stability of the positive equilibrium $\tilde{\mathbf{u}}_1$ when $\tau = 0$ under $(\mathbf{F_1})$ and $(\mathbf{F_3})$.

Theorem 2.8. *Let $(u(x,t), w(x,t))$ be the positive solution of (2.2) with $\tau = 0$. If f and g satisfy $(\mathbf{F_1})$ and $(\mathbf{F_3})$, then $\lim_{t\to\infty} u(x,t) = \tilde{u}$ and $\lim_{t\to\infty} w(x,t) = \tilde{w}$ uniformly for $x \in \bar{\Omega}$.*

Proof. For any fixed $0 < T < \infty$ we define a set

$$A_T = \{\phi \in C(\bar{\Omega} \times [0,T]) \ \text{ and } \ \underline{u}(t) \leq \phi(x,t) \leq \bar{u}(t) \ \text{ in } \Omega \times [0,T]\}.$$

Then A_T is a bounded and closed convex subset of $C(\bar{\Omega} \times [0,T])$. For the given $\phi \in A_T$, it is easy to see that the problem

$$\begin{cases} d_3 \Delta w_1 - w_1 + g(\phi) = 0, & x \in \Omega,\ t > 0, \\ \dfrac{\partial w_1}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{2.26}$$

admits a unique solution w_1 and $w_1(\cdot,t) \in C^{2+\gamma}(\bar{\Omega})$ for $0 \leq t \leq T$. Moreover, the maximum principle yields

$$\min_{x\in\Omega} g(\phi(x,t)) \le \min_{x\in\Omega} w_1(x,t) \le w_1(x,t) \le \max_{x\in\Omega} w_1(x,t) \le \max_{x\in\Omega} g(\phi(x,t)), \quad \forall\, 0<t\le T.$$

Making use of the definition of ϕ and the properties of g we get

$$g(\underline{u}(t)) \le w_1(x,t) \le g(\bar{u}(t)), \quad \forall\, x\in\Omega,\ 0<t\le T. \tag{2.27}$$

And so the problem

$$\begin{cases} (u_1)_t = d\Delta u_1 + u_1(a - f(u_1) - w_1), & x\in\Omega,\ 0<t\le T,\\ \dfrac{\partial u_1}{\partial \nu} = 0, & x\in\partial\Omega,\ 0<t\le T,\\ u_1(x,0) = u_0(x) > 0. & x\in\Omega, \end{cases} \tag{2.28}$$

admits a unique solution $u_1 \in W_p^{2,1}(\Omega\times(0,T))$ for any given $p>1$, where $u_0(x)$ is given by (2.1). We define $J(\phi) = u_1$.

We first prove that the operator J has a fixed point in the set A_T. It follows from (2.27) that

$$u_1(a - f(u_1) - g(\bar{u})) \le (u_1)_t - d\Delta u_1 = u_1(a - f(u_1) - w_1) \le u_1(a - f(u_1) - g(\underline{u})),$$

which means that $\bar{u}$ and $\underline{u}$ are the upper and lower solutions of (2.28). Therefore

$$\underline{u}(t) \le u_1(x,t) \le \bar{u}(t), \quad \forall\, x\in\bar{\Omega},\ 0\le t\le T. \tag{2.29}$$

Noticing that

$$g(0) \le g(\underline{u}(t)) \le g(\phi(x,t)) \le g(\bar{u}(t)) \le g\left(\max_{0\le t\le T} \bar{u}(t)\right)$$

for any given $\phi\in A_T$. The elliptic L^p theory shows that $\|w_1(\cdot,t)\|_{W_p^2(\Omega)} < C(\bar{u},p)$ for any $p\ge 1$ and all $0\le t\le T$, $\phi\in A_T$. This combined with (2.28) allows us to derive that $u_1 = J(\phi)$ is uniformly bounded in $W_p^{2,1}(\Omega\times(0,T))$ with respect to $\phi\in A_T$ for any fixed $p>N$. Since $W_p^{2,1}(\Omega\times(0,T)) \hookrightarrow C(\bar{\Omega}\times[0,T])$ is a compact embedding, we obtain that $J(A_T)$ is a relatively compact subset of $C(\bar{\Omega}\times[0,T])$ and $J(A_T)\subset A_T$ by (2.29). Applying the Schauder fixed point theorem we obtain that J has a fixed point $\hat{u}$ in the set A_T. Combining this with Theorem 2.2 we see that $\hat{u} = u$ is the unique solution of the problem (2.2) in $[0,T]$ with $\tau = 0$. Thus $\underline{u}(t) \le u(x,t) \le \bar{u}(t)$ in $\bar{\Omega}\times[0,T]$, and then $\underline{u}(t) \le u(x,t) \le \bar{u}(t)$ in $\bar{\Omega}\times[0,\infty)$ by the arbitrariness of T. Now using Lemma 2.7 we have $\lim_{t\to\infty} u(x,t) = \tilde{u}$ and $\lim_{t\to\infty} w(x,t) = g(\tilde{u}) = \tilde{w}$ uniformly on $\bar{\Omega}$. The proof is complete. □

The global stability results for (2.2) with $\tau = 0$ in Theorems 2.4 and 2.8 naturally imply the global stability of constant equilibrium for the nonlocal problem (2.1):

Corollary 2.9. *Let $u(x,t)$ be the solution of problem (2.1) with $u_0(x)\ge 0, \not\equiv 0$. If f and g satisfy* (**F₁**) *and* (**F₂**) *or* (**F₃**)*, then $\lim_{t\to\infty} u(x,t) = \tilde{u}$ uniformly for $x\in\bar{\Omega}$.*

Remark 2.10. We have proved the global stability under two different conditions ($\mathbf{F_2}$) and ($\mathbf{F_3}$). Some f, g satisfy both conditions: for example $f(u) = bu$ and $g(u) = cu$ with $b > c$. But neither of ($\mathbf{F_2}$) or ($\mathbf{F_3}$) covers the other. Set $a = 1$, $f(u) = u$ and $g(u) = u/(1+u)$. Then f, g satisfy ($\mathbf{F_1}$) and ($\mathbf{F_3}$), but they do not satisfy ($\mathbf{F_2}$). On the other hand, set $a = 1$, $f(u) = u$ and $g(u) = u^p$ with $p > 1$, then f, g satisfy ($\mathbf{F_1}$) and ($\mathbf{F_2}$) but not ($\mathbf{F_3}$).

3. Nonlocal diffusive competition problem

In this section, we consider the global stability of the equilibria of diffusive Lotka–Volterra system with nonlocal interaction.

3.1. Equilibria

First we convert the nonlocal problem (1.2) into an equivalent parabolic–elliptic system. Denote $w(x,t) = \int_\Omega K_1(x,y)u(y,t)dy$ and $z(x,t) = \int_\Omega K_2(x,y)v(y,t)dy$. After doing some scaling, the problem (1.2) becomes

$$\begin{cases} u_t = d_1\Delta u + u\left(\alpha - u - c_{11}w - c_{12}z - a_1 v\right), & x\in\Omega,\ t>0,\\ v_t = d_2\Delta v + v\left(\beta - v - c_{21}w - c_{22}z - a_2 u\right), & x\in\Omega,\ t>0,\\ 0 = d_3\Delta w - w + u, & x\in\Omega,\ t>0,\\ 0 = d_4\Delta z - z + v, & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = \dfrac{\partial w}{\partial\nu} = \dfrac{\partial z}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x)\ge 0,\ v(x,0) = v_0(x)\ge 0, & x\in\Omega, \end{cases} \tag{3.1}$$

which is equivalent to the problem (1.3). For the simplicity of notations, we denote $\varrho = (\alpha, \beta, a_1, a_2, c_{11}, c_{12}, c_{21}, c_{22})$ to be the vector of parameters.

For system (3.1), $\tilde{\mathbf{u}}_\mathbf{0} = (0, 0, 0, 0)$ is the trivial equilibrium, and there are two semi-trivial equilibria

$$\begin{cases} \tilde{\mathbf{u}}_\mathbf{1} = (\tilde u_1, \tilde v_1, \tilde w_1, \tilde z_1) = \left(\dfrac{\alpha}{1+c_{11}},\ 0,\ \dfrac{\alpha}{1+c_{11}},\ 0\right),\\ \tilde{\mathbf{u}}_\mathbf{2} = (\tilde u_2, \tilde v_2, \tilde w_2, \tilde z_2) = \left(0,\ \dfrac{\beta}{1+c_{22}},\ 0,\ \dfrac{\beta}{1+c_{22}}\right). \end{cases} \tag{3.2}$$

Moreover system (3.1) has a unique positive equilibrium $\tilde{\mathbf{u}}_\mathbf{3} = (\tilde u_3, \tilde v_3, \tilde w_3, \tilde z_3)$ with

$$\begin{cases} \tilde u_3 = \dfrac{\alpha(1+c_{22}) - \beta(a_1+c_{12})}{(1+c_{11})(1+c_{22}) - (a_1+c_{12})(a_2+c_{21})}, & \tilde w_3 = \tilde u_3,\\ \tilde v_3 = \dfrac{\beta(1+c_{11}) - \alpha(a_2+c_{21})}{(1+c_{11})(1+c_{22}) - (a_1+c_{12})(a_2+c_{21})}, & \tilde z_3 = \tilde v_3, \end{cases} \tag{3.3}$$

provided that the parameters $\varrho = (\alpha,\ \beta,\ a_1,\ a_2,\ c_{11},\ c_{12},\ c_{21},\ c_{22})$ satisfy one of the following:

($\mathbf{G_1}$) $\dfrac{a_2+c_{21}}{1+c_{11}} < \dfrac{\beta}{\alpha} < \dfrac{1+c_{22}}{a_1+c_{12}}$, or

$(\mathbf{G_2})$ $\dfrac{1+c_{22}}{a_1+c_{12}} < \dfrac{\beta}{\alpha} < \dfrac{a_2+c_{21}}{1+c_{11}}$.

Similar to the classic competition system without nonlocal term (1.1), we say the system (3.1) is a weak competition system if ϱ satisfies $(\mathbf{G_1})$; and it is a strong competition system if ϱ satisfies $(\mathbf{G_2})$.

In the following discussion, we focus on the weak competition case of (3.1). When $c_{ij} = 0$ for $i = 1, 2$ and $j = 1, 2$, the problem (3.1) and (1.1) are essentially the same. However, when $c_{ij} > 0$, some results different from problem (1.1) will be obtained.

3.2. Local stability

In this subsection, we study the local stability of constant equilibria of problem (3.1). Let $\mathbf{u} = (u, v, w, z)$ be a steady state of problem (3.1) and denote by L_1 the corresponding linearized operator. Then L_1 can be expressed as

$$L_1 = D\Delta + H_{\mathbf{u}}(\mathbf{u}),$$

where $D = \mathrm{diag}(d_1, d_2, d_3, d_4)$ and

$$H_{\mathbf{u}}(\mathbf{u}) = \begin{bmatrix} \alpha - 2u - c_{11}w - c_{12}z - a_1 v & -a_1 u & -c_{11}u & -c_{12}u \\ -a_2 v & \beta - 2v - c_{21}w - c_{22}z - a_2 u & -c_{21}v & -c_{22}v \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

From (2.5), we obtain

$$\begin{cases} \mathbf{X}^4 = \left\{(u, v, w, z) \in [C^1(\bar{\Omega})]^4 : \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0 \text{ on } \partial\Omega\right\}, \\ \mathbf{X}^4_{ij} = \{c\phi_{ij} : c \in \mathbb{R}^4\}, \quad \mathbf{X}^4_i = \bigoplus_{j=1}^{m_i} \mathbf{X}^4_{ij}, \quad \mathbf{X}^4 = \bigoplus_{i=0}^{\infty} \mathbf{X}^4_i. \end{cases} \tag{3.4}$$

We have the following local stability results for the semi-trivial equilibria.

Proposition 3.1. *Suppose the parameters in ϱ and d_i for $i \in \{1, 2, 3, 4\}$ are positive.*

1. *The equilibrium $\tilde{\mathbf{u}}_1 = (\alpha/(1 + c_{11}), 0, \alpha/(1 + c_{11}), 0)$ is locally asymptotically stable if $\frac{\beta}{\alpha} < \frac{a_2+c_{21}}{1+c_{11}}$, and is unstable if $\frac{\beta}{\alpha} > \frac{a_2+c_{21}}{1+c_{11}}$.*
2. *The equilibrium $\tilde{\mathbf{u}}_2 = (0, \beta/(1 + c_{22}), 0, \beta/(1 + c_{22}))$ is locally asymptotically stable if $\frac{\beta}{\alpha} > \frac{1+c_{22}}{a_1+c_{12}}$, and is unstable if $\frac{\beta}{\alpha} < \frac{1+c_{22}}{a_1+c_{12}}$.*

Proof. We only prove it for $\tilde{\mathbf{u}}_1$ as the proof for $\tilde{\mathbf{u}}_2$ is the same. From the definition of L_1, we obtain that for each $j \in \mathbb{N} \cup \{0\}$, $\mathbf{X}^4_j$ is invariant under the operator L_1, and ξ is an eigenvalue of L_1 on $\mathbf{X}^4_j$ if and only if ξ is an eigenvalue of the matrix $M_j = -\mu_j D + H_{\mathbf{u}}(\tilde{\mathbf{u}}_1)$. To find eigenvalues of M_j, we get

$$\lambda I - M_j = \lambda I + \begin{bmatrix} \tilde{u}_1 + d_1\mu_j & a_1\tilde{u}_1 & c_{11}\tilde{u}_1 & c_{12}\tilde{u}_1 \\ 0 & A_1 + d_2\mu_j & 0 & 0 \\ -1 & 0 & 1 + d_3\mu_j & 0 \\ 0 & -1 & 0 & 1 + d_4\mu_j \end{bmatrix},$$

where $A_1 = \frac{\alpha(a_2+c_{21})}{1+c_{11}} - \beta$, and the characteristic polynomial of M_j is

$$\det(\lambda I - M_j) = (\lambda + A_1 + d_2\mu_j)(\lambda + 1 + d_4\mu_j)[(\lambda + \tilde{u}_1 + d_1\mu_j)(\lambda + 1 + d_3\mu_j) + c_{11}\tilde{u}_1].$$

The four eigenvalues of M_j satisfy $\lambda_1^{(j)} = -A_1 - d_2\mu_j$, $\lambda_2^{(j)} = -1 - d_4\mu_j < 0$ and $\mathrm{Re}\lambda_3^{(j)}$, $\mathrm{Re}\lambda_4^{(j)} < 0$ since

$$\lambda_3^{(j)} + \lambda_4^{(j)} = -1 - \tilde{u}_1 - (d_1 + d_2)\mu_j < 0,\ \lambda_3^{(j)}\lambda_4^{(j)} = (\tilde{u}_1 + \mu_j d_1)(1 + \mu_j d_3) + c_{11}\tilde{u}_1 > 0.$$

The equilibrium $\tilde{\mathbf{u}}_1$ is locally asymptotically stable if $A_1 > 0$, and is unstable if $A_1 < 0$. Notice $A_1 > 0\,(< 0)$ is equivalent to $\frac{\beta}{\alpha} < (>) \frac{a_2+c_{21}}{1+c_{11}}$, the proof is complete. □

The local stability of the positive equilibrium $\tilde{\mathbf{u}}_3$ is more complicated to compute and we will show in Section 4 that even in the weak competition case, $\tilde{\mathbf{u}}_3$ may be unstable.

3.3. Global stability

In this subsection, we investigate the global stability of equilibria of (3.1) by using Lyapunov functional method and upper–lower solution methods. First we state the following global existence and boundedness results for the solutions of (3.1) and (1.3). The proof is similar to that of Theorem 2.2 and is omitted here.

Theorem 3.2. *Let $u_0, v_0 \in C(\bar{\Omega})$ with $u_0, v_0 \geq 0, \not\equiv 0$ and $p > \max\{1, N/2\}$. Then (1.3) has a unique solution (u, v) with $u(x,t), v(x,t) > 0$ for $x \in \bar{\Omega}$, $t > 0$ and $u, v \in C([0,\infty); L^p(\Omega)) \cap C((0,\infty); W_p^2(\Omega)) \cap C^1((0,\infty); L^p(\Omega))$. Consequently, the problem (3.1) has a unique solution (u, v, w, z) with $u(x,t), w(x,t), w(x,t), z(x,t) > 0$ for $x \in \bar{\Omega}, t > 0$ and $u, v \in C^{2+\gamma,1+\gamma/2}(\bar{\Omega} \times (0,\infty))$, $w, z \in C((0,\infty); C^{2+\gamma}(\bar{\Omega}))$ where $0 < \gamma < 1$. Moreover, there exists a constant $M_3 > 0$ such that*

$$\|u(\cdot,t)\|_{C^2(\bar{\Omega})},\ \|v(\cdot,t)\|_{C^2(\bar{\Omega})},\ \|w(\cdot,t)\|_{C^2(\bar{\Omega})},\ \|z(\cdot,t)\|_{C^2(\bar{\Omega})} \leq M_3,\quad \forall\, t \geq 1.$$

We first show the global stability of the positive equilibrium $\tilde{\mathbf{u}}_3$.

Theorem 3.3. *If the parameters in ϱ satisfy* $(\mathbf{G_1})$ *and*

$$(\mathbf{G_3})\quad c_{11} \leq \frac{c_{21}}{2a_2 + c_{21}},\quad c_{22} \leq \frac{c_{12}}{2a_1 + c_{12}},$$

then for any $d_i > 0$, $i \in \{1,2,3,4\}$, the constant coexistence equilibrium $\tilde{\mathbf{u}}_3 = (\tilde{u}_3, \tilde{v}_3, \tilde{w}_3, \tilde{z}_3)$ is globally asymptotically stable with respect to (3.1).

Proof. To simplify the proof, we introduce some notations. Define

$$\varepsilon = a_2 + c_{21}, \quad \eta = a_1 + c_{12}, \quad \rho_1 = \frac{\varepsilon a_1 + \eta a_2}{2\varepsilon\eta}, \quad \rho_2 = \frac{c_{12}}{2\eta}, \quad \rho_3 = \frac{c_{21}}{2\varepsilon}, \tag{3.5}$$

and

$$\langle u, u\rangle_i = \int_\Omega (u - \tilde{u}_i)(u - \tilde{u}_i)dx, \quad \langle u, v\rangle_i = \int_\Omega (u - \tilde{u}_i)(v - \tilde{v}_i)dx,$$

where $\tilde{u}_i$ and $\tilde{v}_i$ for $i \in \{1, 2, 3\}$ are given by (3.2) and (3.3). Similarly we can define $\langle u, w\rangle_i$, $\langle u, z\rangle_i$, $\langle v, w\rangle_i$, $\langle v, z\rangle_i$, $\langle w, z\rangle_i$ for $i \in \{1, 2, 3\}$. A direct computation shows that

$$\rho_1 + \rho_2 + \rho_3 = 1. \tag{3.6}$$

Multiplying the third equation of (3.1) by $w - \tilde{w}_i$, multiplying the fourth equation of (3.1) by $z - \tilde{z}_i$ and integrating over Ω, we have that, for $i \in \{1, 2, 3\}$,

$$\begin{cases} \langle u, w\rangle_i &= \int_\Omega (u - \tilde{u}_i)(w - \tilde{w}_i)dx = \int_\Omega [d_3|\nabla w|^2 + (w - \tilde{w}_i)^2]dx \\ &= d_3 \int_\Omega |\nabla w|^2 dx + \langle w, w\rangle_i, \\ \langle v, z\rangle_i &= \int_\Omega (v - \tilde{v}_i)(z - \tilde{z}_i)dx = \int_\Omega [d_3|\nabla z|^2 + (z - \tilde{z}_i)^2]dx \\ &= \int_\Omega d_3|\nabla z|^2 dx + \langle z, z\rangle_i. \end{cases} \tag{3.7}$$

And also

$$\begin{cases} -\langle u, u\rangle_i &= -\langle u, u\rangle_i + 2\langle u, w\rangle_i - 2\langle u, w\rangle_i \\ &= -\langle u, u\rangle_i + 2\langle u, w\rangle_i - 2\langle w, w\rangle_i - 2d_3 \int_\Omega |\nabla w|^2 dx \\ &\le -\langle w, w\rangle_i, \\ -\langle v, v\rangle_i &= -\langle v, v\rangle_i + 2\langle v, z\rangle_i - 2\langle v, z\rangle_i \\ &= -\langle v, v\rangle_i + 2\langle v, z\rangle_i - 2\langle z, z\rangle_i - 2d_3 \int_\Omega |\nabla z|^2 dx \\ &\le -\langle z, z\rangle_i. \end{cases} \tag{3.8}$$

By $(\mathbf{G_1})$, there exists a positive constant ζ such that for any $x, y \in \mathbb{R}$,

$$\begin{aligned} 2\varepsilon\eta xy &= 2xy\sqrt{\varepsilon^2\eta^2} = 2xy\sqrt{\varepsilon\eta(a_1 + c_{12})(a_2 + c_{21})} \\ &\le 2|xy|\sqrt{\varepsilon\eta(1 + c_{11} - \zeta/\varepsilon)(1 + c_{22} - \zeta/\eta)} \\ &\le [\varepsilon(1 + c_{11}) - \zeta]x^2 + [\eta(c_{22} + 1) - \zeta]y^2, \end{aligned} \tag{3.9}$$

and applying (3.9), we obtain

$$\begin{cases} (\varepsilon a_1+\eta a_2)\langle u,v\rangle_3 = 2\rho_1\varepsilon\eta\langle u,v\rangle_3 \le \rho_1[(\varepsilon(1+c_{11})-\zeta)\langle u,u\rangle_3+(\eta(1+c_{22})-\zeta)\langle v,v\rangle_3], \\ \varepsilon c_{12}\langle u,z\rangle_3 = 2\rho_2\varepsilon\eta\langle u,z\rangle_3 \le \rho_2[(\varepsilon(1+c_{11})-\zeta)\langle u,u\rangle_3+(\eta(1+c_{22})-\zeta)\langle z,z\rangle_3], \\ \eta c_{21}\langle v,w\rangle_3 = 2\rho_3\varepsilon\eta\langle v,w\rangle_3 \le \rho_3[(\varepsilon(1+c_{11})-\zeta)\langle w,w\rangle_3+(\eta(1+c_{22})-\zeta)\langle v,v\rangle_3], \end{cases} \tag{3.10}$$

where ε, η and ρ_i for $i\in\{1,2,3\}$ are given by (3.5). Adding the three inequalities in (3.10), we get

$$\begin{aligned} &(\varepsilon a_1+\eta a_2)\langle u,v\rangle_3+\varepsilon c_{12}\langle u,z\rangle_3+\eta c_{21}\langle v,w\rangle_3 \\ \le\,&\varepsilon(\rho_1+\rho_2)(1+c_{11})\langle u,u\rangle_3+\eta(\rho_1+\rho_3)(1+c_{22})\langle v,v\rangle_3 \\ &+\varepsilon\rho_3(1+c_{11})\langle w,w\rangle_3+\eta\rho_2(1+c_{22})\langle z,z\rangle_3-\zeta E_3, \end{aligned} \tag{3.11}$$

where $E_3=(\rho_1+\rho_2)\langle u,u\rangle_3+(\rho_1+\rho_3)\langle v,v\rangle_3+\rho_3\langle w,w\rangle_3+\rho_2\langle z,z\rangle_3>0$.

Define $Q_3:[0,\infty)\to\mathbb{R}$ by

$$Q_3(t)=\varepsilon\int_\Omega\int_{\tilde u_3}^{u(x,t)}\frac{s-\tilde u_3}{s}dsdx+\eta\int_\Omega\int_{\tilde v_3}^{v(x,t)}\frac{s-\tilde v_3}{s}dsdx,$$

where $(u(x,t),v(x,t))$ is the positive solution of (1.3) or (3.1) and the positive constants ε, η are given by (3.5). Then by (3.7) and (3.11) we get

$$\begin{aligned} \frac{dQ_3(t)}{dt}=&-\int_\Omega\left(\varepsilon d_1\tilde u_3\frac{|\nabla u|^2}{u^2}+\eta d_2\tilde v_3\frac{|\nabla v|^2}{v^2}\right)dx+\varepsilon\int_\Omega(u-\tilde u_3)(\alpha-u-c_{11}w)dx \\ &-\varepsilon\int_\Omega(u-\tilde u_3)(c_{12}z+a_1v)dx+\eta\int_\Omega(v-\tilde v_3)(\beta-v-c_{21}w-c_{22}z-a_2u)dx \\ =&-\int_\Omega\left(\varepsilon d_1\tilde u_3\frac{|\nabla u|^2}{u^2}+\eta d_2\tilde v_3\frac{|\nabla v|^2}{v^2}\right)dx-\varepsilon\langle u,u\rangle_3-\varepsilon c_{11}\langle u,w\rangle_3-\varepsilon c_{12}\langle u,z\rangle_3 \\ &-\varepsilon a_1\langle u,v\rangle_3-\eta\langle v,v\rangle_3-\eta c_{21}\langle v,w\rangle_3-\eta c_{22}\langle v,z\rangle_3-\eta a_2\langle u,v\rangle_3 \\ =&-\int_\Omega\left(\varepsilon d_1\tilde u_3\frac{|\nabla u|^2}{u^2}+\eta d_2\tilde v_3\frac{|\nabla v|^2}{v^2}+\varepsilon c_{11}d_3|\nabla w|^2+\eta c_{22}d_4|\nabla z|^2\right)dx \\ &-\varepsilon\langle u,u\rangle_3-\varepsilon c_{11}\langle w,w\rangle_3-\varepsilon c_{12}\langle u,z\rangle_3-(\varepsilon a_1+\eta a_2)\langle u,v\rangle_3 \\ &-\eta\langle v,v\rangle_3-\eta c_{21}\langle v,w\rangle_3-\eta c_{22}\langle z,z\rangle_3 \\ :=&B_3-\varepsilon\langle u,u\rangle_3-\varepsilon c_{11}\langle w,w\rangle_3-\varepsilon c_{12}\langle u,z\rangle_3-(\varepsilon a_1+\eta a_2)\langle u,v\rangle_3 \\ &-\eta\langle v,v\rangle_3-\eta c_{21}\langle v,w\rangle_3-\eta c_{22}\langle z,z\rangle_3 \\ \le&B_3-\varepsilon\langle u,u\rangle_3-\varepsilon c_{11}\langle w,w\rangle_3-\eta\langle v,v\rangle_3-\eta c_{22}\langle z,z\rangle_3 \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon(\rho_1+\rho_2)(1+c_{11})\langle u,u\rangle_3+\eta(\rho_1+\rho_3)(1+c_{22})\langle v,v\rangle_3\\
&+\varepsilon\rho_3(1+c_{11})\langle w,w\rangle_3+\eta\rho_2(1+c_{22})\langle z,z\rangle_3-\zeta E_3\\
=&B_3-\varepsilon[1-(\rho_1+\rho_2)(1+c_{11})]\langle u,u\rangle_3+\varepsilon[\rho_3(1+c_{11})-c_{11}]\langle w,w\rangle_3\\
&-\eta[1-(\rho_1+\rho_3)(1+c_{22})]\langle v,v\rangle_3+\eta[\rho_2(1+c_{22})-c_{22}]\langle z,z\rangle_3-\zeta E_3.
\end{aligned}
$$

Taking advantage of ($\mathbf{G_3}$), we get

$$
(\rho_1+\rho_2)(1+c_{11})\le 1,\quad (\rho_1+\rho_3)(1+c_{22})\le 1. \tag{3.12}
$$

Then it follows from (3.6), (3.8) and (3.12) that

$$
\begin{aligned}
\frac{dQ_3(t)}{dt}\le&B_3-\varepsilon[1-(\rho_1+\rho_2)(1+c_{11})]\langle w,w\rangle_3+\varepsilon[\rho_3(1+c_{11})-c_{11}]\langle w,w\rangle_3\\
&-\eta[1-(\rho_1+\rho_3)(1+c_{22})]\langle z,z\rangle_3+\eta[\rho_2(1+c_{22})-c_{22}]\langle v,v\rangle_3-\zeta E_3\\
=&B_3-\zeta E_3\le 0.
\end{aligned}
$$

Similar to the proof of Theorem 2.4, we can show that for any $d_i>0$, $i\in\{1,2,3,4\}$, the solution (u,v,w,z) converges to the constant coexistence equilibrium $\tilde{\mathbf{u}}_3$ as $t\to\infty$. □

Next we show the global stability of the semi-trivial equilibrium $\tilde{\mathbf{u}}_1$.

Theorem 3.4. *If the parameters in* ϱ *satisfy*

$$
\frac{\beta}{\alpha}<\min\left\{\frac{a_2+c_{21}}{1+c_{11}},\ \frac{a_2+c_{21}}{(a_1+c_{12})(a_2+c_{21})-(1+c_{11})c_{22}}\right\}, \tag{3.13}
$$

and

$$
c_{11}\le\frac{c_{21}}{2a_2+c_{21}},\quad c_{22}\le\frac{c_{12}}{2a_1+c_{12}}\frac{a_2+c_{21}}{1+c_{11}}\frac{\alpha}{\beta}, \tag{3.14}
$$

then for any $d_i>0$, $i\in\{1,2,3,4\}$, *the equilibrium* $\tilde{\mathbf{u}}_1=(\alpha/(1+c_{11}),0,\alpha/(1+c_{11}),0)$ *is globally asymptotically stable with respect to* (3.1).

Proof. As the parameters α,β,a_1,a_2 and $c_{11},c_{12},c_{21},c_{22}$ satisfy (3.13), we get

$$
(a_1+c_{12})(a_2+c_{21})<(1+c_{11})\left(c_{22}+\frac{a_2+c_{21}}{1+c_{11}}\frac{\alpha}{\beta}\right).
$$

Similar to the proof of Theorem 3.3, we have

$$
\begin{aligned}
&(\varepsilon a_1+\eta a_2)\langle u,v\rangle_1+\varepsilon c_{12}\langle u,z\rangle_1+\eta c_{21}\langle v,w\rangle_1\\
\le&\,\varepsilon(\rho_1+\rho_2)(1+c_{11})\langle u,u\rangle_1+\eta\left[(\rho_1+\rho_3)\left(c_{22}+\frac{a_2+c_{21}}{1+c_{11}}\right)-\theta\right]\langle v,v\rangle_1\\
&+\varepsilon\rho_3(1+c_{11})\langle w,w\rangle_1+\eta\rho_2\left(c_{22}+\frac{a_2+c_{21}}{1+c_{11}}\right)\langle z,z\rangle_1-\theta E_1,
\end{aligned} \tag{3.15}
$$

where $\theta > 0$ is a small constant, the positive constants ε, η and ρ_i for $i \in \{1, 2, 3\}$ are given by (3.5) and $E_1 = (\rho_1 + \rho_2)\langle u, u\rangle_1 + (\rho_1 + \rho_3)\langle v, v\rangle_1 + \rho_3\langle w, w\rangle_1 + \rho_2\langle z, z\rangle_1 > 0$. From the second equation of (3.1), we can see that $\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} v(x, t) \le \beta$. Together with the inequality $\frac{a_2+c_{21}}{1+c_{11}}\alpha - \beta - \beta\theta > 0$ (cf. (3.13)), there exists $T > 0$ such that for $t > T$ and $x \in \bar{\Omega}$,

$$-\left(\frac{a_2 + c_{21}}{1 + c_{11}}\alpha - \beta\right) < -v(x, t)\left(\frac{a_2 + c_{21}}{1 + c_{11}}\frac{\alpha}{\beta} - 1 - \theta\right) < 0. \tag{3.16}$$

Define $Q_4 : [0, \infty) \to \mathbb{R}$ by

$$Q_4(t) = \varepsilon \int_{\Omega} \int_{\tilde{u}_1}^{u(x,t)} \frac{s - \tilde{u}_1}{s} ds dx + \eta \int_{\Omega} v(x, t) dx,$$

where $(u(x, t), v(x, t))$ is the positive solution of (1.3) or (3.1). Then by (3.7), (3.15) and (3.16), we get for $t > T$,

$$\begin{aligned}
\frac{dQ_4(t)}{dt} =& - \varepsilon d_1\tilde{v}_1 \int_{\Omega} \frac{|\nabla u|^2}{u^2} dx + \varepsilon \int_{\Omega} (u - \tilde{u}_1)(\alpha - u - c_{11}w) dx \\
& - \varepsilon \int_{\Omega} (u - \tilde{u}_1)(c_{12}z + a_1 v) dx + \eta \int_{\Omega} v(\beta - v - c_{21}w - c_{22}z - a_2 u) dx \\
=& - \int_{\Omega} \left[\varepsilon d_1\tilde{v}_1 \frac{|\nabla u|^2}{u^2} + \eta v\left(\frac{a_2 + c_{21}}{1 + c_{11}}\alpha - \beta\right)\right] dx - \varepsilon\langle u, u\rangle_1 - \varepsilon c_{11}\langle u, w\rangle_1 \\
& - \varepsilon c_{12}\langle u, z\rangle_1 - \varepsilon a_1\langle u, v\rangle_1 - \eta\langle v, v\rangle_1 - \eta c_{21}\langle v, w\rangle_1 - \eta c_{22}\langle v, z\rangle_1 - \eta a_2\langle u, v\rangle_1 \\
\le& - \int_{\Omega} \left(\varepsilon d_1\tilde{v}_1 \frac{|\nabla u|^2}{u^2} + \varepsilon d_3 c_{11}|\nabla w|^2 + \eta d_4 c_{22}|\nabla z|^2\right) dx \\
& - \varepsilon\langle u, u\rangle_1 - \varepsilon c_{11}\langle w, w\rangle_1 - \varepsilon c_{12}\langle u, z\rangle_1 - (\varepsilon a_1 + \eta a_2)\langle u, v\rangle_1 \\
& - \eta\langle v, v\rangle_1 - \eta c_{21}\langle v, w\rangle_1 - \eta c_{22}\langle z, z\rangle_1 - \eta\langle v, v\rangle_1 \left(\frac{a_2 + c_{21}}{1 + c_{11}}\frac{\alpha}{\beta} - 1 - \theta\right) \\
:=& B_1 - \varepsilon\langle u, u\rangle_1 - \varepsilon c_{11}\langle w, w\rangle_1 - \varepsilon c_{12}\langle u, z\rangle_1 - (\varepsilon a_1 + \eta a_2)\langle u, v\rangle_1 \\
& - \eta\left(\frac{a_2 + c_{21}}{1 + c_{11}}\frac{\alpha}{\beta} - \theta\right)\langle v, v\rangle_1 - \eta c_{21}\langle v, w\rangle_1 - \eta c_{22}\langle z, z\rangle_1 \\
\le& B_1 - \varepsilon\langle u, u\rangle_1 - \varepsilon c_{11}\langle w, w\rangle_1 - \eta\left(\frac{a_2 + c_{21}}{1 + c_{11}}\frac{\alpha}{\beta} - \theta\right)\langle v, v\rangle_1 - \eta c_{22}\langle z, z\rangle_1 \\
& + \varepsilon(\rho_1 + \rho_2)(1 + c_{11})\langle u, u\rangle_1 + \eta\left[(\rho_1 + \rho_3)\left(c_{22} + \frac{a_2 + c_{21}}{1 + c_{11}}\frac{\alpha}{\beta}\right) - \theta\right]\langle v, v\rangle_1 \\
& + \varepsilon\rho_3(1 + c_{11})\langle w, w\rangle_1 + \eta\rho_2\left(c_{22} + \frac{a_2 + c_{21}}{1 + c_{11}}\frac{\alpha}{\beta}\right)\langle z, z\rangle_1 - \theta E_1
\end{aligned}$$

$$
\begin{aligned}
&= B_1 - \varepsilon[1 - (\rho_1 + \rho_2)(1 + c_{11})]\langle u, u\rangle_1 + \varepsilon[\rho_3(1 + c_{11}) - c_{11}]\langle w, w\rangle_1 \\
&\quad - \eta\left[\frac{a_2 + c_{21}}{1 + c_{11}}\frac{\alpha}{\beta} - (\rho_1 + \rho_3)\left(c_{22} + \frac{a_2 + c_{21}}{1 + c_{11}}\frac{\alpha}{\beta}\right)\right]\langle v, v\rangle_1 \\
&\quad + \eta\left[\rho_2\left(c_{22} + \frac{a_2 + c_{21}}{1 + c_{11}}\frac{\alpha}{\beta}\right) - c_{22}\right]\langle z, z\rangle_1 - \theta E_1.
\end{aligned}
$$

Taking advantage of (3.14),

$$
(\rho_1 + \rho_2)(1 + c_{11}) \le 1, \quad (\rho_1 + \rho_3)\left(c_{22} + \frac{a_2 + c_{21}}{1 + c_{11}}\frac{\alpha}{\beta}\right) \le \frac{a_2 + c_{21}}{1 + c_{11}}\frac{\alpha}{\beta}. \tag{3.17}
$$

Then it follows from (3.6), (3.8) and (3.17) that, for $t > T$,

$$
\frac{dQ_4(t)}{dt} \le B_1 - \theta E_1 \le 0.
$$

Similar to the proof of Theorem 2.4, we obtain that for any $d_i > 0$, $i \in \{1, 2, 3, 4\}$, the solution (u, v, w, z) converges to the equilibrium $\tilde{\mathbf{u}}_1$ as $t \to \infty$. □

Parallel to Theorem 3.4 we also have

Theorem 3.5. *If the parameters in ϱ satisfy*

$$
\frac{\beta}{\alpha} > \max\left\{\frac{1 + c_{22}}{a_1 + c_{12}},\ a_2 + c_{21} - \frac{1 + c_{22}}{a_1 + c_{12}}c_{11}\right\}, \tag{3.18}
$$

and

$$
c_{11} \le \frac{c_{21}}{2a_2 + c_{21}}\frac{a_1 + c_{12}}{1 + c_{22}}\frac{\beta}{\alpha}, \quad c_{22} \le \frac{c_{12}}{2a_1 + c_{12}}, \tag{3.19}
$$

then for any $d_i > 0$, $i \in \{1, 2, 3, 4\}$, the equilibrium $\tilde{\mathbf{u}}_2 = (0, \beta/(1 + c_{22}), 0, \beta/(1 + c_{22}))$ is globally asymptotically stable with respect to (3.1)*.*

Remark 3.6. When $c_{11} = c_{22} = 0$, the inequalities in $(\mathbf{G_3})$, (3.14) and (3.19) are automatically satisfied, and the inequalities (3.13), (3.18) are similar to the conditions of global stability of semi-travel equilibria for problem (1.1). Hence the Theorems 3.3–3.5 imply that, when $c_{11} = c_{22} = 0$, the dynamics of problem (3.1) is similar to that of problem (1.1) regardless of the strength of nonlocal interspecific competition c_{12} and c_{21}. And when c_{11}, c_{22} are small, for the weak competition case we obtain similar results to the problem (1.1). However the dynamical properties of problem (3.1) may be different from those of problem (1.1) when c_{11}, c_{22} are suitably large.

Finally we point out that the global stability of the positive equilibrium $\tilde{\mathbf{u}}_3$ can also be proved by using methods similar to the proof of Theorem 2.8. As the proof is very similar, we omit the details but only sketch the key points. Let $(\bar{u}(t), \underline{u}(t), \bar{v}(t), \underline{v}(t))$ be the solution of the following system of ordinary differential equations:

$$\begin{cases} \bar{u}' = \bar{u}(\alpha - \bar{u} - c_{11}\underline{u} - c_{12}\underline{v} - a_1\underline{v}), \\ \underline{u}' = \underline{u}(\alpha - \underline{u} - c_{11}\bar{u} - c_{12}\bar{v} - a_1\bar{v}), \\ \bar{v}' = \bar{v}(\beta - \bar{v} - c_{21}\underline{u} - c_{22}\underline{v} - a_2\underline{u}), \\ \underline{v}' = \underline{v}(\beta - \underline{v} - c_{21}\bar{u} - c_{22}\bar{v} - a_2\bar{u}), \\ \bar{u}(0) \geq \underline{u}(0) > 0, \ \ \bar{v}(0) \geq \underline{v}(0) > 0, \\ \bar{u}(0) = \max_{x\in\Omega} u_0(x), \ \underline{u}(0) = \min_{x\in\Omega} u_0(x) > 0, \\ \bar{v}(0) = \max_{x\in\Omega} v_0(x), \ \underline{v}(0) = \min_{x\in\Omega} v_0(x) > 0. \end{cases} \tag{3.20}$$

We can see that $(\tilde{u}_3, \tilde{u}_3, \tilde{v}_3, \tilde{v}_3)$ is the unique positive equilibrium of (3.20) when $(c_{11} - 1)(c_{22} - 1) \neq (c_{12} + a_1)(c_{21} + a_2)$ and one of $(\mathbf{G_1})$ or $(\mathbf{G_2})$ holds, where $\tilde{u}_3$ and $\tilde{v}_3$ are defined in (3.3). Then by using the approach used in the proof of Lemma 2.7 and Theorem 2.8, we can prove the following global stability result for $\tilde{\mathbf{u}}_3$.

Theorem 3.7. *Suppose that the parameters in ϱ satisfy* $(\mathbf{G_1})$ *and*

$(\mathbf{G_4})$ $a_1 + c_{12} + c_{22} < 1, a_2 + c_{21} + c_{11} < 1.$

1. *Suppose that $(\bar{u}(t), \underline{u}(t), \bar{v}(t), \underline{v}(t))$ is the unique solution of problem* (3.20)*. Then $\bar{u}(t) \geq \underline{u}(t)$ and $\bar{v}(t) \geq \underline{v}(t)$ for $t \geq 0$, and $(\tilde{u}, \tilde{u}, \tilde{v}, \tilde{v})$ is globally asymptotically stable with respect to* (3.20)*.*
2. *Let $(u(x,t), w(x,t), w(x,t), z(x,t))$ be the unique solution of problem* (3.1) *with initial data $u_0(x), v_0(x) \geq 0$, $\not\equiv 0$ on $\bar{\Omega}$. Then for any $d_i > 0$, $i \in \{1, 2, 3, 4\}$, $\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} w(x,t) = \tilde{u}$ and $\lim_{t\to\infty} v(x,t) = \lim_{t\to\infty} z(x,t) = \tilde{v}$ uniformly for $x \in \bar{\Omega}$.*

Remark 3.8. Compared with $(\mathbf{G_3})$, the assumption $(\mathbf{G_4})$ may be weaker when c_{12}, c_{21} are small. For example, set $\alpha = \beta = 1$, $a_1 = a_2 = 0.2$, $c_{12} < 0.1$, $c_{21} < 0.1$ and $c_{11} = c_{22} = 0.5$, then α, β, a_1, a_2 and $c_{11}, c_{12}, c_{21}, c_{22}$ satisfy $(\mathbf{G_1})$ and $(\mathbf{G_4})$ but not $(\mathbf{G_3})$.

4. Nonconstant stationary patterns

In Section 3, we have obtained some conditions which guarantee the global stability of the constant equilibria $\tilde{\mathbf{u}}_i$ of (3.1) for $i \in \{1, 2, 3\}$ respectively when the nonlocal intraspecific competition coefficients c_{11} and c_{22} are small. In this section we shall study the existence and non-existence of nonconstant positive equilibrium of (3.1), and we consider the following elliptic equations:

$$\begin{cases} d_1\Delta u + u\,(\alpha - u - c_{11}w - c_{12}z - a_1 v) = 0, & x \in \Omega, \\ d_2\Delta v + v\,(\beta - v - c_{21}w - c_{22}z - a_2 u) = 0, & x \in \Omega, \\ d_3\Delta w - w + u = 0, & x \in \Omega, \\ d_4\Delta z - z + v = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

Our results show that the equilibrium problem (4.1) may possess nonconstant positive solutions for the weak competition case when c_{11} and c_{22} are suitably large.

4.1. A priori estimates

To discuss the existence and nonexistence of nonconstant positive solutions of (4.1), in this subsection we shall give some *a priori* upper and lower bounds for the positive solutions of (4.1). We first recall the following well known results.

Proposition 4.1. (Harnack inequality [39].) *Let $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a positive solution of $\Delta w(x) + c(x)w(x) = 0$, where $c \in C(\Omega) \cap L^\infty(\Omega)$, satisfying the homogeneous Neumann boundary condition. Then there exists a positive constant $\tilde{C}$ which depends only on M where $\|c\|_\infty \le M$ such that $\max_{\bar{\Omega}} w \le \tilde{C} \min_{\bar{\Omega}} w$.*

Proposition 4.2. (Maximum principle [42].) *Let $g \in C(\bar{\Omega})$, and $b_j \in C(\bar{\Omega})$, $1 \le j \le N$. Assume that $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ and satisfies*

$$\begin{cases} \Delta u + \sum_{j=1}^{N} b_j(x) u_{x_j} + g(x) \ge (\le) 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} \le (\ge) 0, & x \in \partial\Omega. \end{cases}$$

If $u(x_0) = \max_{\bar{\Omega}} u \left(\min_{\bar{\Omega}} u \right)$, then $g(x_0) \ge (\le) 0$.

Making use of Proposition 4.2, we can easily derive the following rough bound.

Proposition 4.3. *Let (u, v, w, z) be any positive solution of (4.1). Then*

$$0 < \min_{\bar{\Omega}} u \le \min_{\bar{\Omega}} w \le \max_{\bar{\Omega}} w \le \max_{\bar{\Omega}} u \le \alpha,$$
$$0 < \min_{\bar{\Omega}} v \le \min_{\bar{\Omega}} z \le \max_{\bar{\Omega}} z \le \max_{\bar{\Omega}} v \le \beta.$$

The following result can be proved by the standard Schauder theory for elliptic equations, and its proof will be omitted here.

Proposition 4.4. *Let $d_0 > 0$ be a fixed constant. Then there exists a positive constant $\bar{C} = \bar{C}(d_0, \varrho, \Omega, N)$ such that, when $d_1, d_2, d_3, d_4 \ge d_0$, any positive solution of (4.1) satisfies $(u, v, w, z) \in [C^{2+\gamma}(\bar{\Omega})]^2$ and $\max\{\|u\|_{2+\gamma}, \|v\|_{2+\gamma}, \|w\|_{2+\gamma}, \|z\|_{2+\gamma}\} \le \bar{C}$, where $0 < \gamma < 1$ and ϱ is the vector of parameters.*

We can also prove the following lemma in the same way as that of [51, Lemma 2], and the details are omitted here.

Lemma 4.5. *Suppose that* $(\mathbf{G_1})$ *holds. Let* $d_{ij} \in (0, \infty)$ *with* $i \in \{1, 2, 3, 4\}$, $j \in \mathbb{N}$, *and* $(u^{(j)}, v^{(j)})$ *be the positive solutions of* (4.1) *with* $d_i = d_{ij}$. *Assume that* $d_{ij} \to \hat{d}_i \in [0, \infty]$ *as* $j \to \infty$ *for* $i \in \{1, 2, 3, 4\}$, *and*

$$\lim_{j\to\infty} (u^{(j)}, v^{(j)}, w^{(j)}, z^{(j)}) = (u^*, v^*, w^*, z^*) \text{ uniformly on } \bar{\Omega}.$$

If u^*, v^*, w^* *and* z^* *are nonnegative constants, then* $(u^*, v^*, w^*, z^*) = \tilde{\mathbf{u}}_3$ *where the positive equilibrium* $\tilde{\mathbf{u}}_3$ *is the unique positive equilibrium of* (3.1) *defined in* (3.3).

Now we can show the following result on positive lower bound of positive solutions of (4.1).

Theorem 4.6. *Let* $d_0 > 0$ *be a fixed constant. Then there is a constant* $\underline{C} = \underline{C}(d_0, \varrho) > 0$ *such that, when* $d_i \geq d_0$, $i \in \{1, 2, 3, 4\}$, *every possible positive solution* (u, v, w, z) *of* (4.1) *satisfies*

$$\min_{x\in\bar{\Omega}} u \geq \underline{C}, \quad \min_{x\in\bar{\Omega}} v \geq \underline{C}, \quad \min_{x\in\bar{\Omega}} w \geq \underline{C}, \quad \min_{x\in\bar{\Omega}} z \geq \underline{C}. \tag{4.2}$$

Proof. By Proposition 4.4, the solution (u, v, w, z) of (4.1) is bounded in $[C^{2+\gamma}(\bar{\Omega})]^4$. Then it follows from Proposition 4.1 that there exists a positive constant $\widetilde{C}$ such that

$$\max_{\bar{\Omega}} u \leq \widetilde{C} \min_{\bar{\Omega}} u, \quad \max_{\bar{\Omega}} v \leq \widetilde{C} \min_{\bar{\Omega}} v, \quad \max_{\bar{\Omega}} w \leq \widetilde{C} \min_{\bar{\Omega}} w, \quad \max_{\bar{\Omega}} z \leq \widetilde{C} \min_{\bar{\Omega}} z. \tag{4.3}$$

Assume on the contrary that (4.2) does not hold. Then there is a sequence $\{(u_j, v_j, w_j, z_j)\}_{j=1}^{\infty}$, which are positive solutions of (4.1) with the corresponding diffusion coefficients $d_i = d_{ij} \geq d_0$ for $i \in \{1, 2, 3, 4\}$ and $j \in \mathbb{N}$, such that

$$\begin{aligned} & d_{ij} \to \hat{d}_i \in [d_0, \infty], \quad i \in \{1, 2, 3, 4\}, \\ & \min_{\bar{\Omega}} u_j \to 0, \ or \ \min_{\bar{\Omega}} v_j \to 0, \ or \ \min_{\bar{\Omega}} w_j \to 0, \ or \ \min_{\bar{\Omega}} z_j \to 0 \end{aligned}$$

as $j \to \infty$. Together with (4.3), we get

$$\max_{\bar{\Omega}} u_j \to 0, \ or \ \max_{\bar{\Omega}} v_j \to 0, \ or \ \max_{\bar{\Omega}} w_j \to 0, \ or \ \max_{\bar{\Omega}} z_j \to 0, \tag{4.4}$$

as $j \to \infty$. By Proposition 4.4, the set $\{(u_j, v_j, w_j, z_j)\}_{j=1}^{\infty}$ is bounded in $[C^{2+\gamma}(\bar{\Omega})]^4$, which is compactly embedded into $[C^2(\bar{\Omega})]^4$. Hence there exist a subsequence of $\{(u_j, v_j, w_j, z_j)\}_{j=1}^{\infty}$, without loss of generality we still denote it by $\{(u_j, v_j, w_j, z_j)\}_{j=1}^{\infty}$, and four nonnegative functions $u_\infty, v_\infty, w_\infty, z_\infty \in C^2(\bar{\Omega})$ such that

$$\|u_j - u_\infty\|_{C^2(\bar{\Omega})}, \ \|v_j - v_\infty\|_{C^2(\bar{\Omega})}, \ \|w_j - w_\infty\|_{C^2(\bar{\Omega})}, \ \|z_j - z_\infty\|_{C^2(\bar{\Omega})} \to 0 \tag{4.5}$$

as $j \to \infty$. According to (4.4), at least one of $u_\infty, v_\infty, w_\infty$ and z_∞ is zero. Integrating over Ω of the third and fourth equations of (4.1), we have

$$\frac{1}{|\Omega|}\int_\Omega u_j(x)dx = \frac{1}{|\Omega|}\int_\Omega w_j(x)dx, \quad \frac{1}{|\Omega|}\int_\Omega v_j(x)dx = \frac{1}{|\Omega|}\int_\Omega z_j(x)dx.$$

Combining this with (4.5) we get

$$\frac{1}{|\Omega|}\int_\Omega u_\infty(x)dx = \frac{1}{|\Omega|}\int_\Omega w_\infty(x)dx, \quad \frac{1}{|\Omega|}\int_\Omega v_\infty(x)dx = \frac{1}{|\Omega|}\int_\Omega z_\infty(x)dx, \tag{4.6}$$

and then

$$u_\infty = 0 \Longleftrightarrow w_\infty = 0, \quad v_\infty = 0 \Longleftrightarrow z_\infty = 0. \tag{4.7}$$

In the following, the proof will be divided into two cases.

Case 1: $\hat{d}_i < \infty$ for $i \in \{1,2,3,4\}$. In view of (4.5), we can see that $(u_\infty, v_\infty, w_\infty, z_\infty)$ is the solution of (4.1) with the corresponding diffusion coefficients $d_i = \hat{d}_i$ for $i \in \{1,2,3,4\}$. By (4.4) and (4.7), we just need to consider the two subcases: $u_\infty = w_\infty = 0$ and $v_\infty = z_\infty = 0$.

If $u_\infty = w_\infty = 0$, then (v_∞, z_∞) satisfies

$$\begin{cases} -d_2\Delta v = v\left(\beta - v - c_{22}z\right), & x \in \Omega, \\ -d_4\Delta z = -z + v, & x \in \Omega, \\ \dfrac{\partial v}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.8}$$

Notice that the functions $f(s) = s$ and $g(s) = c_{22}s$ satisfies the assumption $(\mathbf{F_2})$, then from Theorem 2.4, the problem (4.8) has a unique positive solution $(\beta/(1+c_{22}), \beta/(1+c_{22}))$. Thus $(u_\infty, v_\infty, w_\infty, z_\infty) = (0, \beta/(1+c_{22}), 0, \beta/(1+c_{22}))$. This is a contradiction with Lemma 4.5. If $v_\infty = z_\infty = 0$, we will get a similar contradiction.

Case 2: $\hat{d}_i = \infty$ for some $i \in \{1,2,3,4\}$. In the following we consider the cases of $\hat{d}_1 = \infty$ and $\hat{d}_3 = \infty$ respectively. Similarly we can prove the desired conclusion for the cases of $\hat{d}_2 = \infty$ and $\hat{d}_4 = \infty$.

If $\hat{d}_1 = \infty$, clearly $u_\infty = C \geq 0$ for some constant C. Now we claim that $w_\infty = C$. If $\hat{d}_3 = \infty$, then w_∞ is also a constant, and by (4.6) we get $w_\infty = C$. On the other hand, if $\hat{d}_3 < \infty$, then w_∞ satisfies

$$\begin{cases} -d_3\Delta w = -w + C, & x \in \Omega, \\ \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

which means that $w_\infty = C$. Thus $w_\infty = C$ always holds. In the following we will show that v_∞ and z_∞ are also constants. If $C \neq 0$, it follows from (4.4) and (4.7) that $v_\infty = z_\infty = 0$. Thus $(u_\infty, v_\infty, w_\infty, z_\infty) = (C, 0, C, 0)$, which is a contradiction with Lemma 4.6. If $C = 0$, then there are four cases for $\hat{d}_2$ and $\hat{d}_4$:

$$(1)\ \hat{d}_2 < \infty,\ \hat{d}_4 < \infty, \quad (2)\ \hat{d}_2 = \infty,\ \hat{d}_4 = \infty$$
$$(3)\ \hat{d}_2 < \infty,\ \hat{d}_4 = \infty, \quad (4)\ \hat{d}_2 = \infty,\ \hat{d}_4 < \infty.$$

If $\hat{d}_2, \hat{d}_4 < \infty$, it follows from (4.5) that (v_∞, z_∞) satisfies (4.8). Similarly to Case 1, a contradiction can be derived. Obviously $v_\infty = z_\infty = C_1 \geq 0$ is a constant if $\hat{d}_2 = \hat{d}_4 = \infty$. Thus $(u_\infty, v_\infty, w_\infty, z_\infty) = (0, C_1, 0, C_1)$, which contradicts with Lemma 4.5. For the case $\hat{d}_2 < \infty$, $\hat{d}_4 = \infty$, it can be shown that $z_\infty = C_2 \geq 0$ is a constant and v_∞ satisfies

$$\begin{cases} -d_2\Delta v = v(\beta - v - c_{22}C_2), & x \in \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

which means that $v_\infty \geq 0$ is a constant. And making use of (4.6), we get $v_\infty = z_\infty = C_2$. Hence $(u_\infty, v_\infty, w_\infty, z_\infty) = (0, C_2, 0, C_2)$, which contradicts with Lemma 4.5. If $\hat{d}_2 = \infty$, $\hat{d}_4 < \infty$, we obtain that $v_\infty = C_3 \geq 0$ is a constant and z_∞ satisfies

$$\begin{cases} -d_4\Delta z = -z + C_3, & x \in \Omega, \\ \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

which means that $z_\infty = C_3$. Hence $(u_\infty, v_\infty, w_\infty, z_\infty) = (0, C_3, 0, C_3)$, which contradicts to Lemma 4.5.

If $\hat{d}_1 < \infty$ and $\hat{d}_3 = \infty$, we obtain that $w_\infty = C_4 \geq 0$ for some constant C_4 and u_∞ satisfies

$$\begin{cases} -d_1\Delta u = u(\alpha - u - c_{11}C_4), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

which means that $u_\infty \geq 0$ is a constant. Then it follows from (4.6) that $u_\infty = w_\infty = C_4$. In the following we will show that v_∞ and z_∞ are also constants. If $C_4 = 0$, then $u_\infty = w_\infty = 0$. And there are four cases for $\hat{d}_2$ and $\hat{d}_4$ which are the same as the above. Similarly we can obtain a contradiction. If $C_4 \neq 0$. Together with (4.4) and (4.7), we have $v_\infty = z_\infty = 0$. Therefore $(u_\infty, v_\infty, w_\infty, z_\infty) = (C_4, 0, C_4, 0)$, which contradicts to Lemma 4.5. □

4.2. Nonexistence of nonconstant equilibria

In this subsection, we shall give some conditions to guarantee the nonexistence of nonconstant positive solutions of (4.1). First the following corollary is a direct consequence of the global stability proved in Section 3.

Corollary 4.7. *If the parameters in ϱ satisfy the conditions in one of the Theorems 3.3–3.5 and Theorem 3.7, then for any $d_i > 0$, $i \in \{1, 2, 3, 4\}$, the problem (4.1) has no nonconstant positive solution.*

Next we show that there is no nonconstant positive solution of (4.1) when diffusion coefficients d_1, d_2 are sufficiently large.

Theorem 4.8. *Suppose that all parameters in ϱ are all positive. Let $d_0 > 0$ be a fixed constant. Then there exists a positive constant $d_5 = d_5(d_0)$ such that the problem (4.1) has no nonconstant positive solution provided d_1, $d_2 \geq d_5$ and $d_3, d_4 \geq d_0$.*

Proof. Let (u, v, w, z) be a positive solution of (4.1), and denote

$$\bar{u} = \frac{1}{|\Omega|}\int_\Omega u(x)\mathrm{d}x, \quad \bar{v} = \frac{1}{|\Omega|}\int_\Omega v(x)\mathrm{d}x, \quad \bar{w} = \frac{1}{|\Omega|}\int_\Omega w(x)\mathrm{d}x, \quad \bar{z} = \frac{1}{|\Omega|}\int_\Omega z(x)\mathrm{d}x.$$

Multiplying the third equation of (4.1) by $w - \bar{w}$, multiplying the fourth equation by $z - \bar{z}$ and integrating them over Ω we have

$$\begin{cases} \int_\Omega (u-\bar{u})(w-\bar{w})dx = \int_\Omega d_3|\nabla w|^2 + (w-\bar{w})^2 dx, \\ \int_\Omega (v-\bar{v})(z-\bar{z})dx = \int_\Omega d_3|\nabla z|^2 + (z-\bar{z})^2 dx. \end{cases} \tag{4.9}$$

Then multiplying the equation of u in (4.1) by $u - \bar{u}$ and integrating the result over Ω we have

$$\begin{aligned} d_1\int_\Omega |\nabla(u-\bar{u})|^2\mathrm{d}x &= \int_\Omega (u-\bar{u})u(\alpha - u - c_{11}w - c_{12}z - a_1 v)\mathrm{d}x \\ &= \int_\Omega (u-\bar{u})(u-\bar{u})(\alpha - u - c_{11}w - c_{12}z - a_1 v)\mathrm{d}x \\ &\quad + \int_\Omega (u-\bar{u})\bar{u}(\alpha - u - c_{11}w - c_{12}z - a_1 v)\mathrm{d}x \\ &\le \int_\Omega \alpha(u-\bar{u})^2\mathrm{d}x + \int_\Omega (u-\bar{u})\bar{u}(-u - c_{11}w - c_{12}z - a_1 v)\mathrm{d}x \\ &= \int_\Omega \alpha(u-\bar{u})^2\mathrm{d}x - \bar{u}\int_\Omega [(u-\bar{u})^2 + c_{11}(u-\bar{u})(w-\bar{w})]\mathrm{d}x \\ &\quad - \bar{u}\int_\Omega [c_{12}(u-\bar{u})(z-\bar{z}) + a_1(u-\bar{u})(v-\bar{v})]\mathrm{d}x. \end{aligned}$$

Taking advantage of Proposition 4.3, Theorem 4.6 and (4.9) we obtain

$$\begin{aligned} d_1\int_\Omega |\nabla(u-\bar{u})|^2\mathrm{d}x &\le \int_\Omega [(\alpha - \underline{C})(u-\bar{u})^2\mathrm{d}x - c_{11}\underline{C}(w-\bar{w})^2]\mathrm{d}x \\ &\quad - \bar{u}\int_\Omega [c_{12}(u-\bar{u})(z-\bar{z}) + a_1(u-\bar{u})(v-\bar{v})]\mathrm{d}x \\ &\le \int_\Omega \left[(\alpha - \underline{C})(u-\bar{u})^2\mathrm{d}x - c_{11}\underline{C}(w-\bar{w})^2 + \frac{\alpha^2 c_{12}^2}{4c_{22}\underline{C}}(u-\bar{u})^2\right]\mathrm{d}x \\ &\quad + \int_\Omega \left[c_{22}\underline{C}(z-\bar{z})^2 + \frac{\alpha a_1}{2}(u-\bar{u})^2 + \frac{\alpha a_1}{2}(v-\bar{v})^2\right]\mathrm{d}x. \end{aligned}$$

Similarly, by Proposition 4.3, Theorem 4.6 and (4.9), one has

$$
\begin{aligned}
d_2\int_\Omega |\nabla(v-\bar v)|^2\mathrm{d}x &= \int_\Omega (v-\bar v)v(\beta - v - c_{21}w - c_{22}z - a_2u)\mathrm{d}x\\
&= \int_\Omega (v-\bar v)^2(\beta - v - c_{21}w - c_{22}z - a_2u)\mathrm{d}x\\
&\quad + \int_\Omega (v-\bar v)\bar v(\beta - v - c_{21}w - c_{22}z - a_2u)\mathrm{d}x\\
&\le \int_\Omega \beta(v-\bar v)^2\mathrm{d}x + \int_\Omega [(v-\bar v)\bar v(-v - c_{21}w - c_{22}z - a_2u)]\mathrm{d}x\\
&\le \int_\Omega [(\beta-\underline{C})(v-\bar v)^2 - c_{22}\underline{C}(z-\bar z)^2]\mathrm{d}x\\
&\quad - \bar v\int_\Omega [c_{21}(v-\bar v)(w-\bar w) + a_2(u-\bar u)(v-\bar v)]\,\mathrm{d}x\\
&\le \int_\Omega \left[(\beta-\underline{C})(v-\bar v)^2 - c_{22}\underline{C}(z-\bar z)^2 + \frac{4\beta^2c_{21}^2}{4c_{11}\underline{C}}(v-\bar v)^2\right]\mathrm{d}x\\
&\quad + \int_\Omega \left[c_{11}\underline{C}(w-\bar w)^2 + \frac{\beta a_2}{2}(u-\bar u)^2 + \frac{\beta a_2}{2}(v-\bar v)^2\right]\mathrm{d}x.
\end{aligned}
$$

Summing up the above estimates we have

$$
d_1\int_\Omega |\nabla(u-\bar u)|^2\mathrm{d}x + d_2\int_\Omega |\nabla(v-\bar v)|^2\mathrm{d}x \le K\int_\Omega \left[(u-\bar u)^2 + (v-\bar v)^2\right]\mathrm{d}x
$$

for some positive constant $K > 0$. Then, by the Poincaré inequality, there exists a constant $C > 0$ such that

$$
d_1\int_\Omega |\nabla(u-\bar u)|^2\mathrm{d}x + d_2\int_\Omega |\nabla(v-\bar v)|^2\mathrm{d}x \le C\int_\Omega \left(|\nabla(u-\bar u)|^2 + |\nabla(v-\bar v)|^2\right)\mathrm{d}x.
$$

It follows that, when $d_1, d_2 \gg 1$, $\nabla(u-\bar u) = \nabla(v-\bar v) = 0$, *i.e.*, $u \equiv \bar u$, $v \equiv \bar v$. Together with (4.9), we obtain $w \equiv \bar w$, $z \equiv \bar z$. $\square$

4.3. Existence of nonconstant equilibria

From Corollary 4.7, for the weak competition case, there is no nonconstant positive solutions of problem (4.1) when c_{11} and c_{22} are small. In the following we concern about the nonconstant equilibria for large c_{11} and c_{22}. We will use the Leray–Schauder degree theory (see

[42,50,51,57]) to show that the problem (4.1) may have nonconstant positive solutions when c_{11} and c_{22} are suitably large and $(\mathbf{G_1})$ holds.

Throughout this subsection, μ_j is given by Section 2, and $\mathbf{X}_j^4$, $\mathbf{X}^4$ are defined in (3.4). Let $\mathbf{u} = (u, v, w, z)$ be any positive solution of problem (4.1). Then the problem (4.1) can be rewritten as

$$F(d_1, d_2; \mathbf{u}) := \mathbf{u} - (I - \Delta)^{-1}\{D^{-1}H(\mathbf{u}) + \mathbf{u}\} = 0 \ \text{ in } \mathbf{X}^4, \tag{4.10}$$

where $D = \operatorname{diag}(d_1, d_2, d_3, d_4)$, D^{-1} is the inverse matrix of D, $(I - \Delta)^{-1}$ is the inverse of $I - \Delta$ with homogeneous Neumann boundary condition and

$$H(\mathbf{u}) = \begin{bmatrix} u\,(\alpha - u - c_{11}w - c_{12}z - a_1 v) \\ v\,(\beta - v - c_{21}w - c_{22}z - a_2 u) \\ -w + u \\ -z + v \end{bmatrix}. \tag{4.11}$$

By direct computation, we have

$$F_{\mathbf{u}}(d_1, d_2; \tilde{\mathbf{u}}_3) = I - (I - \Delta)^{-1}\{D^{-1}H_{\mathbf{u}}(\tilde{\mathbf{u}}_3) + I\},$$

where $F_{\mathbf{u}}$ and $H_{\mathbf{u}}$ are their Jacobian matrices respectively. We note that for each $\mathbf{X}_j^4$, ξ is an eigenvalue of $F_{\mathbf{u}}(d_1, d_2; \tilde{\mathbf{u}}_3)$ on $\mathbf{X}_j^4$ if and only if $\xi(1 + \mu_j)$ is an eigenvalue of the matrix

$$M(\mu_j) = \mu_j I - D^{-1}H_{\mathbf{u}}(\tilde{\mathbf{u}}_3) = \mu_j I + \begin{bmatrix} \tilde{u}_3^* & a_1\tilde{u}_3^* & c_{11}\tilde{u}_3^* & c_{12}\tilde{u}_3^* \\ a_2\tilde{v}_3^* & \tilde{v}_3^* & c_{21}\tilde{v}_3^* & c_{22}\tilde{v}_3^* \\ -d_3^* & 0 & d_3^* & 0 \\ 0 & -d_4^* & 0 & d_4^* \end{bmatrix},$$

where $\tilde{u}_3^* = \tilde{u}_3/d_1,\ \tilde{v}_3^* = \tilde{v}_3/d_2,\ d_3^* = 1/d_3,\ d_4^* = 1/d_4$. We define

$$G(d_1, d_2; \lambda) := \det M(\lambda) = \lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4, \tag{4.12}$$

where

$$\begin{aligned}
A_1 =& d_3^* + d_4^* + \tilde{u}_3^* + \tilde{v}_3^*, \\
A_2 =& (d_3^* + d_4^*)(\tilde{u}_3^* + \tilde{v}_3^*) + d_3^*d_4^* + c_{11}d_3^*\tilde{u}_3^* + c_{22}d_4^*\tilde{v}_3^* + \tilde{u}_3^*\tilde{v}_3^* - a_1a_2\tilde{u}_3^*\tilde{v}_3^*, \\
A_3 =& d_3^*d_4^*(\tilde{u}_3^* + \tilde{v}_3^*) + c_{11}d_3^*d_4^*\tilde{u}_3^* + c_{22}d_3^*d_4^*\tilde{v}_3^* + \tilde{u}_3^*\tilde{v}_3^*[d_3^* + d_4^* + c_{11}d_3^* + c_{22}d_4^*] \\
& - \tilde{u}_3^*\tilde{v}_3^*[a_1a_2(d_3^* + d_4^*) + a_1c_{21}d_3^* + a_2c_{12}d_4^*], \\
A_4 =& d_3^*d_4^*\tilde{u}_3^*\tilde{v}_3^*[(1 + c_{11})(1 + c_{22}) - (a_1 + c_{12})(a_2 + c_{21})].
\end{aligned}$$

To apply Leray–Schauder degree theory to show the existence of nonconstant positive solutions of (4.1), we recall the following well-known lemma (see for example [50, Lemma 5.1]) which provides the calculation formula of the fixed point index of a fixed point of F.

Lemma 4.9. *Let F and G be defined as in (4.10) and (4.12) respectively. Define*

$$S = \{\mu_i : i \in \mathbb{N} \cup \{0\}\}, \quad E(d_1, d_2) = \{\lambda \geq 0 : G(d_1, d_2, \lambda) < 0\},$$

and let $m(\mu_j)$ be the algebraic multiplicity of μ_j. Suppose that $G(d_1, d_2; \mu_j) \neq 0$ for all $\mu_j \in S$. Then

$$\text{index}(F(d_1, d_2; \cdot), \tilde{\mathbf{u}}_3) = (-1)^\gamma,$$

where

$$\gamma = \begin{cases} \displaystyle\sum_{\mu_j \in E(d_1,d_2)\cap S} m(\mu_j) & \text{if } E(d_1, d_2) \cap S \neq \emptyset, \\ 0 & \text{if } E(d_1, d_2) \cap S = \emptyset. \end{cases}$$

In particular, if $G(d_1, d_2; \lambda) > 0$ for all $\lambda \geq 0$, then $\gamma = 0$.

From Lemma 4.9, we consider the properties of the function $G(d_1, d_2; \lambda)$ to gain information on stability/instability of $\tilde{\mathbf{u}}_3$, which would imply the existence of other positive equilibria. For that purpose we have the following algebraic properties.

Lemma 4.10. *Let G be defined as in (4.12), and assume that $(\mathbf{G_1})$ is satisfied.*

1. *For any $d_1, d_2 > 0$, $G(d_1, d_2; \lambda) = 0$ has at most two positive roots $0 < \lambda_3 \leq \lambda_4$.*
2. *If either d_1 or d_2 is sufficiently large, then $G(d_1, d_2; \lambda) > 0$ for $\lambda \geq 0$.*
3. *Denote*

$$\begin{cases} \tilde{A}_3 = d_3^* + d_4^* + c_{11} d_3^* + c_{22} d_4^* - a_1 a_2 (d_3^* + d_4^*) - a_1 c_{21} d_3^* - a_2 c_{12} d_4^*, \\ \tilde{A}_4 = d_3^* d_4^* [(1 + c_{11})(1 + c_{22}) - (a_1 + c_{12})(a_2 + c_{21})], \end{cases} \tag{4.13}$$

and we assume that

$$a_1 a_2 < 1, \quad \tilde{A}_3 < 0 \ \textit{and} \ (\tilde{A}_3)^2 - 4(1 - a_1 a_2)\tilde{A}_4 > 0. \tag{4.14}$$

Then when both of d_1 and d_2 are sufficiently small, $G(d_1, d_2; \lambda) = 0$ has exactly two positive roots $\lambda_3 = \lambda_3(d_1, d_2)$ and $\lambda_4 = \lambda_4(d_1, d_2)$ such that $G(d_1, d_2; \lambda) < 0$ for $\lambda_3 < \lambda < \lambda_4$, and

$$\begin{cases} \displaystyle\lim_{d_1, d_2 \to 0^+} \lambda_3(d_1, d_2) = \tilde{\lambda}_3(d_3^*, d_4^*) := \frac{-\tilde{A}_3 - \sqrt{\tilde{A}_3^2 - 4(1 - a_1 a_2)\tilde{A}_4}}{2(1 - a_1 a_2)}, \\ \displaystyle\lim_{d_1, d_2 \to 0^+} \lambda_4(d_1, d_2) = \tilde{\lambda}_4(d_3^*, d_4^*) := \frac{-\tilde{A}_3 + \sqrt{\tilde{A}_3^2 - 4(1 - a_1 a_2)\tilde{A}_4}}{2(1 - a_1 a_2)}. \end{cases} \tag{4.15}$$

Proof. 1. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the roots of $G(d_1, d_2; \lambda) = 0$ with $\mathrm{Re}\lambda_1 \le \mathrm{Re}\lambda_2 \le \mathrm{Re}\lambda_3 \le \mathrm{Re}\lambda_4$. Note that $G(d_1, d_2; 0) = A_4 > 0$ from $(\mathbf{G_1})$ and $\lim_{\lambda\to\infty} G(d_1, d_2; \lambda) = \infty$, which implies that $G(d_1, d_2; \lambda) = 0$ has 0 or 2 or 4 zeros (counting multiplicity) for $\lambda \in (0, \infty)$. But $A_1 > 0$ since $d_3^*, d_4^*, \tilde{u}_3^*, \tilde{v}_3^* > 0$, thus $G = 0$ cannot have 4 positive roots. Therefore $G(d_1, d_2; \lambda) = 0$ has at most two positive roots counting multiplicity.

2. When $d_1 \to \infty$, we have $\tilde{u}_3^* = \tilde{u}_3/d_1 \to 0$, $A_4 \to 0$ and

$$G(d_1, d_2; \lambda) = \lambda^4 + [B_1 + \epsilon_1(d_1)]\lambda^3 + [B_2 + \epsilon_2(d_1)]\lambda^2 + [B_3 + \epsilon_3(d_1)]\lambda + A_4,$$

where

$$B_1 = d_3^* + d_4^* + \tilde{v}_3^* > 0,\quad B_2 = (d_3^* + d_4^* + c_{22}d_4^*)\tilde{v}_3^* + d_3^* d_4^* > 0,$$
$$B_3 = (1 + c_{22})d_3^* d_4^* \tilde{v}_3^* > 0,\quad \lim_{d_1\to\infty} \epsilon_i(d_1) = \lim_{d_1\to\infty} (A_i - B_i) = 0,\ \ i = 1, 2, 3.$$

Hence there exists $d_6 > 0$ such that when $d_1 > d_6$,

$$G(d_1, d_2; \lambda) \ge \lambda^4 + \frac{B_1}{2}\lambda^3 + \frac{B_2}{2}\lambda^2 + \frac{B_3}{2}\lambda + A_4 > 0,\ \ \lambda \in [0, \infty).$$

Similarly we can prove the conclusion when d_2 is sufficiently large.

3. From the definition of G, we observe that

$$d_1 d_2 G(d_1, d_2; \lambda) = G_1(d_1, d_2; \lambda) + G_2(\lambda),$$

where

$$G_1(d_1, d_2; \lambda) = d_1 d_2 \lambda^4 + \delta_1(d_1, d_2)\lambda^3 + \delta_2(d_1, d_2)\lambda^2 + \delta_3(d_1, d_2)\lambda,$$
$$G_2(\lambda) = \tilde{u}_3 \tilde{v}_3[(1 - a_1 a_2)\lambda^2 + \tilde{A}_3\lambda + \tilde{A}_4],$$

with

$$\delta_1(d_1, d_2) =(d_3^* + d_4^*)d_1 d_2 + \tilde{u}_3 d_2 + \tilde{v}_3 d_1,$$
$$\delta_2(d_1, d_2) =(d_3^* + d_4^*)(\tilde{u}_3 d_2 + \tilde{v}_3 d_1) + d_3^* d_4^* d_1 d_2 + c_{11} d_3^* \tilde{u}_3 d_2 + c_{22} d_4^* \tilde{v}_3 d_1,$$
$$\delta_3(d_1, d_2) =d_3^* d_4^*[\tilde{u}_3 d_2 + c_{11} u_3 d_2 + \tilde{v}_3 d_1 + c_{22} v_3 d_1],$$

and $\tilde{A}_3$ and $\tilde{A}_4$ are defined as in (4.13).

From the assumption $(\mathbf{G_1})$ and (4.14), it follows that $G_2(\lambda) = 0$ has two positive roots: $0 < \tilde{\lambda}_3 < \tilde{\lambda}_4$ as defined in (4.15), and both roots are non-degenerate in the sense that $G_2'(\tilde{\lambda}_i) \neq 0$ for $i = 3, 4$. Choosing $I = [\tilde{\lambda}_3/2, 2\tilde{\lambda}_4]$, we have $G_1(d_1, d_2; \lambda) \to 0$ uniformly for $\lambda \in I$ and $d_1, d_2 \to 0$. It follows from implicit function theorem that for $d_1, d_2 > 0$ sufficiently small, $d_1 d_2 G(d_1, d_2; \lambda) = 0$ also have exactly two positive roots $\lambda_3(d_1, d_2)$ and $\lambda_4(d_1, d_2)$ in I near $\tilde{\lambda}_3$ and $\tilde{\lambda}_4$ respectively. On the other hand, for $\lambda \in [0, \infty)\backslash I$, $G_1(d_1, d_2; \lambda) > 0$ as $\delta_i(d_1, d_2) > 0$ for $i = 1, 2, 3$, and $G_2(\lambda) > 0$ from its quadratic form. Hence $G(d_1, d_2; \lambda) > 0$ for $\lambda \in [0, \infty)\backslash I$. This shows that when both of d_1 and d_2 are sufficiently small, $G(d_1, d_2; \lambda) = 0$ has exactly two positive roots $\lambda_3 = \lambda_3(d_1, d_2)$ and $\lambda_4 = \lambda_4(d_1, d_2)$ such that $G(d_1, d_2; \lambda) < 0$ for $\lambda_3 < \lambda < \lambda_4$ and (4.15) holds. □

Now we prove the following result on the existence of non-constant solution of (4.1).

Theorem 4.11. *Suppose that parameters in* ϱ, d_3 *and* d_4 *satisfy* $(\mathbf{G_1})$ *and* (4.14), *and suppose that* $\tilde{\lambda}_3, \tilde{\lambda}_4$ *are defined as in* (4.15). *Assume that* $\tilde{\lambda}_3 \in (\mu_k, \mu_{k+1})$ *and* $\tilde{\lambda}_4 \in (\mu_q, \mu_{q+1})$ *for some* $k, q \in \mathbb{N} \cup \{0\}$ *such that* $0 \le k < q$. *If* $\sigma_{k,q} = \sum_{i=k+1}^{q} m_i$ *is odd where* m_i *is the algebraic multiplicity of the eigenvalue* μ_i, *then there is a positive constant* d_*, *such that for any* $0 < d_1, d_2 \le d_*$, *the problem* (4.1) *has at least one nonconstant positive solution, and the constant positive solution* $\tilde{\mathbf{u}}_3$ *is unstable.*

Proof. Thanks to $\tilde{\lambda}_3 \in (\mu_k, \mu_{k+1})$ and $\tilde{\lambda}_4 \in (\mu_q, \mu_{q+1})$ and (4.15), there exists $d_* > 0$ such that, for all $0 < d_1, d_2 \le d_*$,

$$\lambda_3(d_1, d_2) \in (\mu_k, \mu_{k+1}), \quad \lambda_4(d_1, d_2) \in (\mu_q, \mu_{q+1}). \tag{4.16}$$

Make use of Theorem 4.8, there exists $d_5 > d_*$ such that (4.1) has no nonconstant positive solution for all $d_1, d_2 \ge d_5$. Moreover, by part 2 of Lemma 4.10, there exists $d_6 > d_5$ such that $G(d_1, d_2; \lambda) > 0$ for $\lambda \ge 0$ when $d_1 \ge d_6$ and $d_2 = d_5$. Hence

$$E(d_1, d_2) \cap S = \emptyset, \quad \forall d_1 \ge d_6,\ d_2 = d_5. \tag{4.17}$$

We will prove that, for any $d_1, d_2 \le d_*$, the problem (4.1) has at least one nonconstant positive solution. Suppose, to the contrary that, for some $\tilde{d}_1$, $\tilde{d}_2 \le d_*$, the problem (4.1) has no nonconstant positive solution. For these fixed parameters $\tilde{d}_1$, $\tilde{d}_2 \le d_*$ and $d_3, d_4, d_5, d_6 > 0$, we define

$$D(t) = \begin{bmatrix} t\tilde{d}_1 + (1-t)d_6 & 0 & 0 & 0 \\ 0 & t\tilde{d}_2 + (1-t)d_5 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}, \quad 0 \le t \le 1,$$

and consider the problem

$$\begin{cases} -\Delta \mathbf{u} = D^{-1}(t) H(\mathbf{u}), & x \in \Omega, \\ \dfrac{\partial \mathbf{u}}{\partial \nu} = 0, & x \in \Omega, \end{cases} \tag{4.18}$$

where $H(\mathbf{u})$ is given by (4.11). Noted that $\mathbf{u}$ is a positive solution of (4.1) if and only if it is a solution of (4.18) for $t = 1$. Obviously, $\tilde{\mathbf{u}}_3$ is the unique positive solution of (4.18) when $t = 0$. And for any $0 \le t \le 1$, $\mathbf{u}$ is a nonconstant solution of (4.18) if and only if it is a solution of the problem

$$\Phi(\mathbf{u}; t) = \mathbf{u} - (I - \Delta)^{-1}\{D^{-1}(t) H(\mathbf{u}) + \mathbf{u}\} = 0 \quad \text{on } \mathbf{X}^4.$$

From our assumptions, both equations $\Phi(\mathbf{u}; 1) = 0$ and $\Phi(\mathbf{u}; 0) = 0$ have no nonconstant positive solution. It follows from (4.16) and (4.17) that

$$E(d_6, d_5) \cap S = \emptyset, \quad E(\tilde{d}_1, \tilde{d}_2) \cap S = \{\mu_i : k + 1 \le i \le q\}.$$

Since $\sigma_{k,q}$ is odd, we have

$$\begin{cases} \text{index}(\Phi(\cdot;0),\tilde{\mathbf{u}}_3) = \text{index}(F(d_6,d_5;\cdot),\tilde{\mathbf{u}}_3) = 1, \\ \text{index}(\Phi(\cdot;1),\tilde{\mathbf{u}}_3) = \text{index}(F(\tilde{d}_1,\tilde{d}_2;\cdot),\tilde{\mathbf{u}}_3) = (-1)^{\sigma_{k,q}} = -1. \end{cases} \tag{4.19}$$

In view of Proposition 4.3 and Theorem 4.6, there exists a positive constant $\underline{C} = \underline{C}(d_7, \varrho)$, where $d_7 = \{\tilde{d}_1/2, \tilde{d}_2/2, d_3, d_4\}$, such that for all $0 \le t \le 1$ the positive solution $\mathbf{u} = (u, v, w, z)$ of (4.18) satisfies $\underline{C} < u(x), w(x) \le \alpha$ and $\underline{C} < v(x), z(x) \le \beta$ on $\bar{\Omega}$. Set

$$\Sigma = \{\mathbf{u} \in \mathbf{X}^4 : \ \underline{C} < u(x), w(x) < \alpha + 1, \ \ \underline{C} < v(x), z(x) < \beta + 1 \text{ on } \bar{\Omega}\}.$$

Then $\Phi(\mathbf{u};t) \ne 0$ for all $\mathbf{u} \in \partial\Sigma$ and $0 \le t \le 1$, and by the homotopy invariance of the Leray–Schauder degree,

$$\deg(\Phi(\cdot;0), \Sigma, 0)) = \deg(\Phi(\cdot;1), \Sigma, 0)). \tag{4.20}$$

Since both equations $\Phi(\mathbf{u};0) = 0$ and $\Phi(\mathbf{u};1) = 0$ have only one positive constant solution $\tilde{\mathbf{u}}_3$ in Σ, by (4.19), we have

$$\begin{cases} \deg(\Phi(\cdot;0), \Sigma, 0) = \text{index}(\Phi(\cdot;0);\tilde{\mathbf{u}}_3) = 1, \\ \deg(\Phi(\cdot;1), \Sigma, 0) = \text{index}(\Phi(\cdot;1);\tilde{\mathbf{u}}_3) = -1. \end{cases} \tag{4.21}$$

Now (4.21) apparently contradicts with (4.20). This completes the proof of existence of nonconstant positive solution of (4.1).

Finally we prove that $\tilde{\mathbf{u}}_3$ is unstable. Similar to the proof of Proposition 3.1, the local stability of the equilibrium $\tilde{\mathbf{u}}_3$ with respect to (3.1) is determined by the eigenvalues of the matrices $M_3(\mu_j) := -\mu_j D + H_{\mathbf{u}}(\tilde{\mathbf{u}}_3)$ for $j \in \mathbb{N} \cup \{0\}$. And it follows from the definition of $G(d_1, d_2; \lambda)$ that $\det(-M_3(\mu_j)) = d_1 d_2 d_3 d_4 G(d_1, d_2; \mu_j)$. Then when both of d_1 and d_2 are sufficiently small, we obtain that $\det(-M_3(\mu_{j_0})) < 0$ for $j_0 \in E(d_1, d_2) \cap S$ if parameters in ϱ, d_3 and d_4 satisfy $(\mathbf{G_1})$, (4.14) and $E(d_1, d_2) \cap S \ne \emptyset$, which means that one of eigenvalues of the matrix $M_3(\mu_{j0})$ is positive and the equilibrium $\tilde{\mathbf{u}}_3$ is unstable. □

The conditions on the parameters ϱ in Theorem 4.11 can be satisfied. Indeed we can have more explicit conditions on the parameters if we assume that $a_1 = a_2$ and $d_3 = d_4$.

Corollary 4.12. *Suppose that $a_1 = a_2 = a \in (0, 1)$ and $d_3 = d_4 = d > 0$. Then (4.14) holds if a and $c_{ij} \ge 0$ $(i, j = 1, 2)$ satisfy*

$$1 - a^2 + \det(C) > a(c_{21} + c_{12}) - (c_{11} + c_{22}) > \max\{2(1 - a^2), 2\sqrt{(1 - a^2)\det(C)}\}, \tag{4.22}$$

where $\det(C) = c_{11}c_{22} - c_{12}c_{21}$. *Moreover when (4.22) holds, one can choose α, β so that* $(\mathbf{G_1})$ *is satisfied.*

Table 1
Parameter values used for numerical simulations.

Parameter	c_{11}	c_{22}	c_{12}	c_{21}	α	β	a_1	a_2	d_1	d_2	d_3	d_4
Value	4	4	15	0.5	70	20	0.8	0.8	0.2	0.2	10	10

Proof. It is easy to calculate that when $a_1 = a_2 = a \in (0, 1)$ and $d_3 = d_4 = d > 0$, we have

$$\tilde{A}_3 = d^{-1}[2(1-a^2) - a(c_{21} + c_{12}) + (c_{11} + c_{22})],$$
$$\tilde{A}_4 = d^{-2}[(1-a^2) - a(c_{21} + c_{12}) + (c_{11} + c_{22}) + \det(C)],$$
$$(\tilde{A}_3)^2 - 4(1-a_1a_2)\tilde{A}_4 = [a(c_{21} + c_{12}) - (c_{11} + c_{22})]^2 - 4(1-a^2)\det(C).$$

Then from $\tilde{A}_3 < 0$, $\tilde{A}_4 > 0$ and $(\tilde{A}_3)^2 - 4(1-a_1a_2)\tilde{A}_4 > 0$, we obtain (4.22). Moreover, when (4.22) holds, we have $\tilde{A}_4 > 0$ hence one can choose α, β so that $(\mathbf{G_1})$ is satisfied. □

Remark 4.13.

1. The condition (4.22) holds if c_{ij} satisfies

$$\det(C) > (c_{21} + c_{12}) - (c_{11} + c_{22}) > 0, \tag{4.23}$$

   and $0 < 1 - a^2 \ll 1$. To make (4.23) satisfied, one can fix $c_{11}, c_{22} > 1$, and choose $c_{12} \ll 1 \ll c_{21}$ or $c_{21} \ll 1 \ll c_{12}$.
2. From the definition of $\tilde{\lambda}_3$ and $\tilde{\lambda}_4$, it is easy to see that for any $k \in \mathbb{R}^+$,

$$\tilde{\lambda}_3(kd_3^*, kd_4^*) = k\tilde{\lambda}_3(d_3^*, d_4^*), \quad \tilde{\lambda}_4(kd_3^*, kd_4^*) = k\tilde{\lambda}_4(d_3^*, d_4^*).$$

   Hence we can scale $d_3 = 1/d_3^*$ and $d_4 = 1/d_4^*$ properly so that $\sigma_{k,q}$ is odd as in Theorem 4.11. Generically for n-dimensional domain Ω and always for 1-dimensional domain $\Omega = (0, l)$, the eigenvalue μ_i is simple so $\sigma_{k,k+1} = 1$ which is odd. In that case if $q = k + 1$ then $\sigma_{k,q}$ is odd. Similarly if the parameter ϱ is fixed, one can scale the domain by $\Omega_k = \{kx : x \in \Omega\}$ for $k \in \mathbb{R}^+$ so that $\sigma_{k,q}$ is odd.

We use a numerical example to illustrate the existence of non-constant equilibrium solutions for the weak competition case for problem (3.1). We choose the parameters as in Table 1 and take $\Omega = (0, 5)$. Then the condition (4.22) is satisfied, $\lambda_3 \approx 0.1525$, $\lambda_4 \approx 0.5564$ and $E(d_1, d_2) \cap S = \{\pi^2/25\}$ which implies that $\sigma_{0,1}$ is odd. Hence all conditions in Theorem 4.11 and Corollary 4.12 are satisfied.

When $d_1 = d_2 = 0.2$, we observe a solution of (3.1) (or equivalently (1.3)) converges to a non-constant equilibrium solution (Fig. 3), and the steady state pattern corresponds to the one of $\cos(\pi x/5)$ which is monotone in $(0, 5)$ (see Fig. 4). More numerical experiments show that such patterns exist whenever $d_1 = d_2 \leq 0.4$. Indeed Fig. 4 shows two distinct steady state solutions which are symmetric under the reflection $x \mapsto 5 - x$, and the convergence to these two distinct steady state solutions for different initial conditions suggests that there exist two locally asymptotically stable positive steady state solutions for the parameter values given in Table 1.

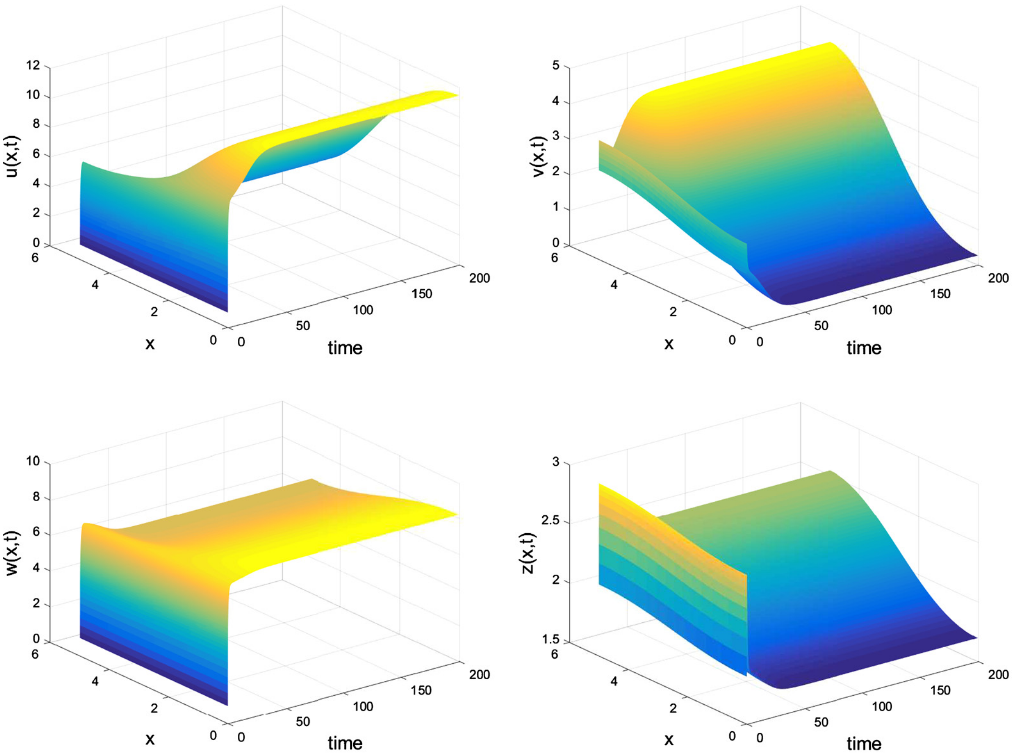

Fig. 3. Convergence to non-constant equilibrium solution for (3.1) or (1.3). Here parameters are given in Table 1, $\Omega = (0, 5)$ and the initial conditions are $u(x, 0) = 1$ and $v(x, 0) = 2 - 0.5\cos(\pi x/5)$.

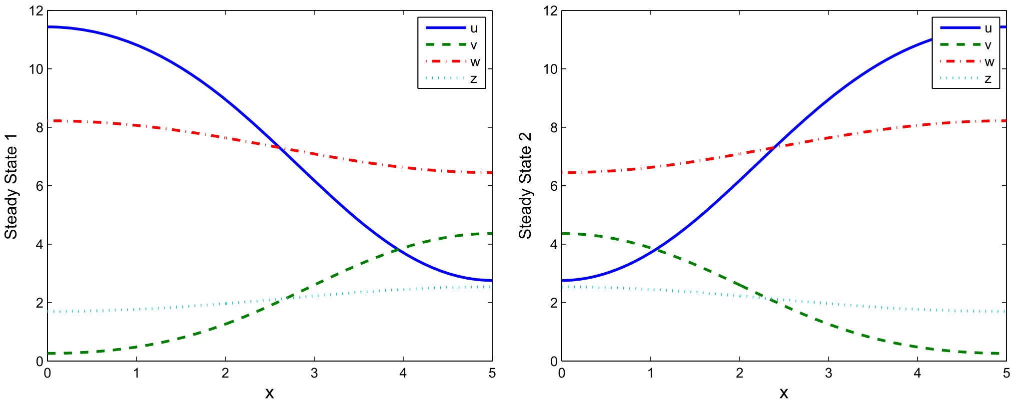

Fig. 4. Multiple positive equilibria for (3.1). Here parameters are given in Table 1, $\Omega = (0, 5)$; the initial conditions for left panel is: $u(x, 0) = 1$ and $v(x, 0) = 2 - 0.5\cos(\pi x/5)$; and the initial conditions for right panel is: $u(x, 0) = 4$ and $v(x, 0) = 2 + 0.6\cos(\pi x/5)$.

Bifurcation method can be used to prove that a pitchfork bifurcation occurs using parameter $d = d_1 = d_2$ [20,40], and the stability of bifurcating solution can be proved to be stable [21].

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