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Bifurcation of positive solutions to scalar reaction–diffusion equations with nonlinear boundary condition

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Abstract

The bifurcation of non-trivial steady state solutions of a scalar reaction–diffusion equation with nonlinear boundary conditions is considered using several new abstract bifurcation theorems. The existence and stability of positive steady state solutions are proved using a unified approach. The general results are applied to a Laplace equation with nonlinear boundary condition and bistable nonlinearity, and an elliptic equation with superlinear nonlinearity and sublinear boundary conditions.

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1. Introduction

Reaction–diffusion systems are considered as foremost important mathematical models for self-organized spatiotemporal pattern formation in morphogenesis, chemical reactions, ecology and various other scientific disciplines [4,19,31,40]. Typically one or several nonlinear parabolic equations describing the change of concentration of chemical or biological substances are defined in a spatial domain, which can be a cell or a tissue, a reactor or a Petri dish, or a habitat in the land or in the water depending on the context of the model. The spatial domain is often assumed to be an open bounded subset in $\mathbb{R}^N$ with $N = 1, 2$ or 3. For the wellposedness of the partial differential equation models, boundary conditions on the boundary of the spatial domain have to be defined so the equation is well-posed. Typical boundary conditions include Dirichlet one for which the value of state variables on the boundary are specified, or Neumann one for which the flux of state variables across the boundary are known. The homogeneous Neumann boundary condition assumes a zero flux on the boundary, and it represents insulation or a closed system. All these boundary conditions are linear equations of the functions and their normal derivatives on the boundary.

In many other situations, the chemical reactions or the biological bondings occur in a narrow layer near the boundary or on the boundary surface (cell membrane), and the nonlinear reaction on the boundary makes a nonlinear boundary condition. For example, a highly exothermic reaction can take place in a thin layer around a boundary [20], and a Bcd gradient formation are generated through a source on the boundary [15,19]. Such reaction–diffusion models are usually consisted of a linear diffusion equation in the spatial domain and a nonlinear reaction on the boundary [2,20,37], and they may represent some pattern formation mechanisms different from the classical ones derived from interior reaction and fixed boundary conditions.

Several aspects of reaction–diffusion models with nonlinear boundary conditions have been considered. The wellposedness and asymptotical behavior of solutions were studied in [1,2,8, 30,37]; the blowup of solutions were characterized in [17,21,25,48,49]; and the boundary layer solutions were constructed in [3,5,11,12]. The existence, uniqueness and stability of steady state solutions have also been studied via bifurcation method and other related methods in several special classes of problems [6,7,13,27,28,33–35,41,42,46,47]. On the other hand, a general bifurcation theorem was recently established in [39], which provides a more direct approach to the bifurcation under nonlinear boundary conditions.

In this paper, we provide a unified approach for the bifurcation of non-trivial steady state solutions of a scalar reaction–diffusion equation with nonlinear boundary conditions. More precisely we consider a scalar parabolic equation with a nonlinear boundary condition in form

$$\begin{cases} u_t = \Delta u + \lambda s(x) f(u), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} = \lambda r(x) g(u), & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_*(x), & x \in \Omega, \end{cases} \tag{1.1}$$

and the steady state solutions of (1.1) satisfy

$$\begin{cases} -\Delta u = \lambda s(x) f(u), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \lambda r(x) g(u), & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Here $\Omega \in \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with smooth boundary $\partial\Omega$, n is the unit outer normal to $\partial\Omega$ and λ is a positive parameter. Throughout the paper we always assume that f, g, r, s satisfy the following basic conditions:

(H0) $f, g \in C^{1,\gamma}(\mathbb{R}, \mathbb{R})$, $s \in C^{1,\gamma}(\overline{\Omega}, \mathbb{R})$, and $r \in C^{1,\gamma}(\partial\Omega, \mathbb{R})$ for some $0 < \gamma < 1$.

For local bifurcations, the definition and smoothness of f and g can be restricted to a neighborhood of bifurcation values, and here we assume that f, g are defined on $\mathbb{R}$ for simplicity of presentation. The equations (1.1) and (1.2) give a flexible setting for considering both the interior reaction and the boundary reaction which are controlled by a single bifurcation parameter λ, and it can also be used to consider reaction–diffusion with no-flux boundary condition by letting $r(x) \equiv 0$, or boundary reaction with diffusion in interior by letting $s(x) \equiv 0$. In our local bifurcation results, we also do not impose special algebraic forms on the nonlinearities $f(u)$ and $g(u)$. Typical assumptions on $f(u)$ are logistic growth (see [6,7,26,32,42,46]), or bistable growth (see [11,26,32]), while $g(u)$ can take a superlinear form in combustion problem [17,25], or a sublinear one in ecological application [6,42]. Our results can be applied to all these cases.

We consider the non-trivial solutions of (1.2) bifurcating from the known trivial solutions. The reaction in (1.1) is at a constant equilibrium state if

(H1) there exists $u_0 \in \mathbb{R}$ and $u_0 \geq 0$ such that $f(u_0) = g(u_0) = 0$.

In this case, (1.2) possesses a line of trivial solutions

$$\Gamma_{u_0} := \{(\lambda, u_0) : \ \lambda > 0\}. \tag{1.3}$$

On the other hand, when $\lambda = 0$, any constant $u_1 \geq 0$ is a solution to (1.2). Hence (1.2) always has another line of trivial solutions:

$$\Gamma_{u_1} := \{(0, u_1) : \ u_1 > 0, \ u_1 \in \mathbb{R}\}. \tag{1.4}$$

In Section 2, we show that non-trivial solutions of (1.2) can emerge from Γ_{u_0} at some bifurcation point $(\lambda, u) = (\lambda_*, u_0)$, or from Γ_{u_1} at some $(\lambda, u) = (0, u_1)$. In a special case, non-trivial solutions can also emerge from $(\lambda, u) = (0, u_0)$, the intersection point of Γ_{u_0} and Γ_{u_1}. To consider the solutions of (1.2) in a functional setting, we define $X = W^{2,p}(\Omega)$, $Y = L^p(\Omega) \times W^{1-\frac{1}{p},p}(\partial\Omega)$ where $p > n$, and define a nonlinear mapping $F : \mathbb{R} \times X \to Y$ by

$$F(\lambda, u) = \left(\Delta u + \lambda s(x) f(u), \frac{\partial u}{\partial n} - \lambda r(x) g(u) \right). \tag{1.5}$$

From the assumption **(H0)**, any solution (λ, u) of $F(\lambda, u) = 0$ is indeed a classical solution of class $C^{2,\gamma}(\overline{\Omega})$. For our results, we will apply abstract local bifurcation theory in [9,10], as well as the more recent ones in [22,23,38], and we also use global bifurcation theory in [36,39]. For reader's convenience, we include these results in Section 5 as reference.

To illustrate our general results, we consider two applications in Sections 3 and 4 respectively on

(a) A Laplace equation with nonlinear boundary condition and bistable nonlinearity: $f(u) = 0$ and $g(u)$ is a bistable function satisfying $g(0) = g(a) = g(1) = 0$ (see Section 3);
(b) An elliptic equation with superlinear nonlinearity and sublinear boundary conditions: $f(u) = u^q$ and $g(u) = u - u^p$ with $p, q > 1$ (see Section 4).

For the example in (a), if the nonlinearity $g(u)$ is logistic type, it has been considered in [18,27]. Such models arise from the studies of genetic evolution, see for example [26,32]. When $r(x)$ is positive, then the only nonnegative solutions of (1.2) are the constant ones $u = 0$ and $u = 1$ for all $\lambda > 0$; when $r(x)$ is sign-changing, $\int_\Omega r(x)ds < 0$ and $f''(u) \le 0$, then there exists a critical value $\lambda_1 > 0$ such that only when $\lambda > \lambda_1$, (1.2) has a unique non-constant solution u satisfying $0 \le u \le 1$, and all such non-constant solutions for $\lambda > \lambda_1$ are on a curve bifurcating from $(\lambda, u) = (\lambda_1, 0)$ (see [27]). When $r(x)$ is negative, then non-constant positive solutions bifurcate from a branch of trivial solutions at a sequence of bifurcation points, and under additional conditions on nonlinearity, the existence of a non-constant positive solution for any sufficiently large λ was shown in [18]. For example in (b), an equation in form

$$\begin{cases} -\Delta u = \lambda(m(x)u - u^2), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \lambda b(x)u^p, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

was considered in [14,45,46]. The existence of a bifurcation branch of positive solutions from trivial solutions and its asymptotic behavior and stability with respect to indefinite nonlinear boundary conditions are considered.

Our approach provide some new results of bifurcation of nontrivial solutions, and it can also be used to prove many previously known results with a simpler and unified approach. We also remark that our methods can also be adapted to more general form of equation:

$$\begin{cases} -\Delta u = f(\lambda, x, u), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = g(\lambda, x, u), & x \in \partial\Omega. \end{cases} \tag{1.7}$$

In the paper we use $||\cdot||$ as the norm of Banach space X, $\langle\cdot,\cdot\rangle$ as the duality pair of a Banach space X and its dual space X^*. For a linear operator L, we use $N(L)$ as the null space of L and $R(L)$ as the range space of L, and we use $L[w]$ to denote the image of w under the linear mapping L. For a multilinear operator L, we use $L[w_1, w_2, \cdots, w_k]$ to denote the image of $(w_1, w_2, \cdots, w_k)$ under L, and when $w_1 = w_2 = \cdots = w_k$, we use $L[w_1]^k$ instead of $L[w_1, w_1, \cdots, w_1]$. For a nonlinear operator F, we use F_u as the partial derivative of F with respect to argument u.

2. Bifurcation from trivial solutions

We say that (λ_*, u_0) is a bifurcation point on the line of trivial solutions $\Gamma_{u_0} = \{(\lambda, u_0) : \lambda > 0\}$ if there exists a sequence (λ^n, u^n) of solutions to (1.2) such that $u^n \ne u_0$, $\lambda^n \to \lambda_*$ and $u^n \to u_0$ in $C(\bar{\Omega})$ as $n \to \infty$. And a bifurcation point on the line $\Gamma_{u_1} := \{(0, u_1) : u_1 > 0, u_1 \in \mathbb{R}\}$ can be defined similarly. In this section we consider the bifurcation of non-trivial solutions of (1.2) from the two lines of trivial solutions defined in (1.3) and (1.4). We first prove the following lemma regarding the possible bifurcation points.

Lemma 2.1. *Suppose that* **(H0)** *is satisfied.*

1. *Assume that* **(H1)** *is also satisfied. Let* $\lambda_* > 0$ *so that* $(\lambda, u) = (\lambda_*, u_0)$ *is a bifurcation point of* (1.2) *with respect to the trivial branch* Γ_{u_0}*, then* λ_* *is an eigenvalue of*

$$\begin{cases} -\Delta v = \lambda f'(u_0)s(x)v, & x \in \Omega, \\ \dfrac{\partial v}{\partial n} = \lambda g'(u_0)r(x)v, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

2. *Let* $u_1 > 0$ *so that* $(\lambda, u) = (0, u_1)$ *is a bifurcation point of* (1.2) *with respect to the trivial branch* Γ_{u_1}*, then* u_1 *satisfies*
   (H2) $f(u_1)\int_\Omega s(x)dx + g(u_1)\int_{\partial\Omega} r(x)dS = 0.$

Proof. For the mapping F defined in (1.5), its partial derivative $F_u(\lambda_*, u_*) : X \to Y$ is defined as

$$F_u(\lambda_*, u_*)[v] = \left(\Delta v + \lambda_* f'(u_*)s(x)v, \frac{\partial v}{\partial n} - \lambda_* g'(u_*)r(x)v\right). \tag{2.2}$$

1. Assume that **(H1)** is also satisfied. If λ_* is not an eigenvalue of (2.1), then $F_u(\lambda_*, u_0)$ is a homeomorphism for all λ close to λ_*. The implicit function theorem implies that F has only the trivial solutions (λ, u_0) near (λ_*, u_0), then λ_* is not a bifurcation point along the line Γ_{u_0}. Hence at a bifurcation point (λ_*, u_0), the equation (2.1) must be satisfied for $\lambda = \lambda_*$.
2. Suppose that $u_1 > 0$ satisfies that $(\lambda, u) = (0, u_1)$ is a bifurcation point of (1.2) with respect to the trivial branch Γ_{u_1}. By the definition, there exists a sequence (λ^n, u^n) of solutions to (1.2) satisfying

$$0 \neq \lambda^n \to 0 \text{ and } ||u^n - u_1||_X \to 0, \text{ when } n \to \infty.$$

   By integration, we have

$$\lambda^n \int_\Omega s(x)f(u^n)dx = -\int_\Omega \Delta u^n dx = -\int_{\partial\Omega} \frac{\partial u^n}{\partial n} dS = -\lambda^n \int_{\partial\Omega} r(x)g(u^n)dS,$$

   which implies that $\int_\Omega s(x)f(u^n)dx + \int_{\partial\Omega} r(x)g(u^n)dS = 0$ since $\lambda^n \neq 0$. By taking $n \to \infty$, we obtain **(H2)**. □

We shall discuss the two bifurcation scenarios separately in the following. In subsection 2.1, we consider the bifurcations on $\Gamma_{u_0} := \{(\lambda, u_0) : \lambda > 0\}$, while in subsection 2.2, we consider the bifurcations on $\Gamma_{u_1} := \{(0, u_1) : u_1 > 0, u_1 \in \mathbb{R}\}$.

2.1. Local bifurcation from Γ_{u_0}

Lemma 2.1 shows that a bifurcation point λ_* must be an eigenvalue of the problem (2.1). We first consider the bifurcation from the principal eigenvalue, which is always a simple one. Regarding the existence of such a principal eigenvalue, we recall the following result from [44, Theorem 2.2]:

Lemma 2.2. *Consider the eigenvalue problem*

$$\begin{cases} -\Delta u = \lambda s(x)u, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \lambda r(x)u, & x \in \partial\Omega. \end{cases} \tag{2.3}$$

Assume that either $s(x) \not\leq 0$ in Ω or $r(x) \not\leq 0$ on $\partial\Omega$, the problem (2.3) has a unique positive principal eigenvalue $\lambda_1(s,r)$ if and only if

$$\int_\Omega s(x)dx + \int_{\partial\Omega} r(x)dS < 0, \tag{2.4}$$

and it is characterized by the formula

$$\lambda_1(s,r) = \inf\left\{ Q_{s,r}(u) : u \in H^1(\Omega) \text{ and } \int_\Omega s(x)u^2dx + \int_{\partial\Omega} r(x)u^2dS > 0 \right\}, \tag{2.5}$$

where

$$Q_{s,r}(u) = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega s(x)u^2dx + \int_{\partial\Omega} r(x)u^2dS}. \tag{2.6}$$

Now we can state our main result on the local bifurcation from the line of trivial solution Γ_{u_0}.

Theorem 2.3. *Suppose that f, g, r, s satisfy* **(H0)**, **(H1)**, *and also satisfy*

(H3) *Either $f'(u_0)s(x) \not\leq 0$ in Ω or $g'(u_0)r(x) \not\leq 0$ on $\partial\Omega$, and*

$$f'(u_0)\int_\Omega s(x)dx + g'(u_0)\int_{\partial\Omega} r(x)dS < 0. \tag{2.7}$$

Then the problem (2.1) has a principal eigenvalue $\lambda_1 > 0$ which is a bifurcation point of (1.2) with respect to Γ_{u_0}. Moreover the solution set of (1.2) near (λ_1, u_0) consists precisely of the curves Γ_{u_0} and

$$\Sigma_0 = \{(\lambda_0(t), u_0(t)) : \ t \in I = (-\eta, \eta) \subset \mathbb{R}\},$$

where $\lambda_0(t) = \lambda_1 + z_2(t)$*,* $u_0(t) = u_0 + t\phi_1 + tz_1(t)$ *are* C^1 *functions such that* $z_i(0) = 0$*,* $i = 1, 2$*,* ϕ_1 *is the eigenfunction corresponding to* λ_1*.*

Proof. Since f, g, r, s satisfy **(H1)**, the problem (2.1) has a principal eigenvalue λ_1 from Lemma 2.2. We apply Theorem 5.1 to the equation $F(\lambda, u) = 0$ at $(\lambda, u) = (\lambda_1, u_0)$ where F is defined as in (1.5), and we verify all the assumptions in Theorem 5.1. We prove it in several steps:

(1) $\dim N(F_u(\lambda_1, u_0)) = 1$. From Lemma 2.2, λ_1 is the principal eigenvalue of (2.1) which is simple, ϕ_1 is the eigenfunction corresponding to λ_1, it follows $N(F_u(\lambda_1, u_0)) = \text{span}\{\phi_1\}$.

(2) $\text{codim}\, R(F_u(\lambda_1, u_0)) = 1$. Let $(h_1, h_2) \in R(F_u(\lambda_1, u_0))$ and let $w \in X$ satisfies

$$\begin{cases} \Delta w + \lambda_1 f'(u_0)s(x)w = h_1, & x \in \Omega, \\ \dfrac{\partial w}{\partial n} - \lambda_1 g'(u_0)r(x)w = h_2, & x \in \partial\Omega. \end{cases} \tag{2.8}$$

Multiplying the equations in (2.1) and (2.8) by w and ϕ_1, respectively, subtracting and integrating on Ω, we have

$$\begin{aligned} \int_\Omega \phi_1 h_1 dx &= \int_\Omega (\phi_1 \Delta w - w\Delta\phi_1) dx = \int_{\partial\Omega} (\phi_1 \frac{\partial w}{\partial n} - w\frac{\partial \phi_1}{\partial n}) dS \\ &= \int_{\partial\Omega} \phi_1(\lambda_1 g'(u_0)r(x)w + h_2) - w\lambda_1 g'(u_0)r(x)\phi_1 dS = \int_{\partial\Omega} \phi_1 h_2 dS. \end{aligned}$$

This shows that $(h_1, h_2) \in R(F_u(\lambda_1, u_0))$ if and only if

$$\int_\Omega \phi_1 h_1 dx - \int_{\partial\Omega} \phi_1 h_2 dS = 0. \tag{2.9}$$

In the following we define $l \in Y^*$ by

$$\langle l, (h_1, h_2)\rangle = \int_\Omega \phi_1 h_1 dx - \int_{\partial\Omega} \phi_1 h_2 dS. \tag{2.10}$$

(3) $F_{\lambda u}(\lambda_1, u_0)[\phi_1] \notin R(F_u(\lambda_1, u_0))$. From

$$F_{\lambda u}(\lambda_1, u_0)[\phi_1] = (s(x)f'(u_0)\phi_1, -r(x)g'(u_0)\phi_1),$$

we obtain that

$$
\begin{aligned}
&\int_{\Omega} s(x) f'(u_0)\phi_1^2 dx + \int_{\partial\Omega} r(x) g'(u_0)\phi_1^2 dS \\
=&\frac{1}{\lambda_1}\int_{\Omega}(-\Delta\phi_1)\phi_1 dx + \frac{1}{\lambda_1}\int_{\partial\Omega}\frac{\partial\phi_1}{\partial n}\phi_1 dS \\
=&-\frac{1}{\lambda_1}\int_{\partial\Omega}\frac{\partial\phi_1}{\partial n}\phi_1 dS + \frac{1}{\lambda_1}\int_{\Omega}|\nabla\phi_1|^2 dx + \frac{1}{\lambda_1}\int_{\partial\Omega}\frac{\partial\phi_1}{\partial n}\phi_1 dS \\
=&\frac{1}{\lambda_1}\int_{\Omega}|\nabla\phi_1|^2 dx > 0.
\end{aligned}
\tag{2.11}
$$

Thus the theorem is proved by applying Theorem 5.1. □

Furthermore from Theorem 5.1 we can determine the bifurcation direction by calculating $\lambda_0'(0)$ via (5.1). In the following we assume that f, g are class C^2 or C^3 near $u = u_0$ as needed. Since

$$
F_{uu}(\lambda_1, u_0)[\phi_1]^2 = (\lambda_1 s(x) f''(u_0)\phi_1^2, -\lambda_1 r(x) g''(u_0)\phi_1^2),
$$

it follows that

$$
\lambda_0'(0) = -\frac{\langle l, F_{uu}(\lambda_1, u_0)[\phi_1]^2\rangle}{2\langle l, F_{\lambda u}(\lambda_1, u_0)[\phi_1]\rangle} = -\frac{\lambda_1^2 f''(u_0)\int_{\Omega} s(x)\phi_1^3 dx + \lambda_1^2 g''(u_0)\int_{\partial\Omega} r(x)\phi_1^3 dS}{2\int_{\Omega}|\nabla\phi_1|^2 dx}
\tag{2.12}
$$

from (5.1). The expression of $\lambda_0'(0)$ in (2.12) can be simplified in various cases, and we list several useful cases here:

(Case 1) Assume $s(x) \equiv 0$, $r(x) \not\equiv 0$ and $g'(u_0) \neq 0$. Then

$$
\lambda_0'(0) = -\lambda_1 \cdot \frac{g''(u_0)}{g'(u_0)} \cdot \frac{\int_{\Omega}|\nabla\phi_1|^2\phi_1 dx}{\int_{\Omega}|\nabla\phi_1|^2 dx}.
\tag{2.13}
$$

Thus a transcritical bifurcation occurs at (λ_1, u_0) if $g''(u_0) \neq 0$. If $g''(u_0) = 0$, we have $\lambda_0'(0) = 0$ and if we further assume that $g \in C^3$ near $u = u_0$ then

$$
\lambda_0''(0) = -\lambda_1 \cdot \frac{g'''(u_0)}{g'(u_0)} \cdot \frac{\int_{\Omega}|\nabla\phi_1|^2\phi_1^2 dx}{\int_{\Omega}|\nabla\phi_1|^2 dx}.
\tag{2.14}
$$

Therefore a pitchfork bifurcation occurs at (λ_1, u_0) if $g''(u_0) = 0$ and $g'''(u_0) \neq 0$.

(Case 2) Assume $r(x) \equiv 0$, $s(x) \not\equiv 0$ and $f'(u_0) \neq 0$. Then

$$\lambda_0'(0) = -\lambda_1 \cdot \frac{f''(u_0)}{f'(u_0)} \cdot \frac{\int_\Omega |\nabla\phi_1|^2 \phi_1 dx}{\int_\Omega |\nabla\phi_1|^2 dx}. \tag{2.15}$$

Hence a transcritical bifurcation occurs at (λ_1, u_0) if $f''(u_0) \neq 0$. If $f''(u_0) = 0$, then $\lambda_0'(0) = 0$ and if $f \in C^3$ near $u = u_0$, then

$$\lambda_0''(0) = -\lambda_1 \cdot \frac{f'''(u_0)}{f'(u_0)} \cdot \frac{\int_\Omega |\nabla\phi_1|^2 \phi_1^2 dx}{\int_\Omega |\nabla\phi_1|^2 dx}. \tag{2.16}$$

Thus a pitchfork bifurcation occurs at (λ_1, u_0) if $f''(u_0) = 0$ and $f'''(u_0) \neq 0$.

(Case 3) Assume $r(x) \not\equiv 0$, $s(x) \not\equiv 0$, and $f'(u_0) \neq 0$, $g'(u_0) \neq 0$. Then

$$\lambda_0'(0) = \frac{\lambda_1}{2} \cdot \left(\frac{f''(u_0)}{f'(u_0)} - \frac{g''(u_0)}{g'(u_0)} \right) \cdot \frac{\int_\Omega \phi_1^2 \Delta\phi_1 dx}{\int_\Omega |\nabla\phi_1|^2 dx} - \lambda_1 \cdot \frac{g''(u_0)}{g'(u_0)} \cdot \frac{\int_\Omega |\nabla\phi_1|^2 \phi_1 dx}{\int_\Omega |\nabla\phi_1|^2 dx}. \tag{2.17}$$

If $f'(u_0) \cdot g''(u_0) = g'(u_0) f''(u_0) \neq 0$, then (2.17) reduces to (2.13), and a transcritical bifurcation occurs. But if $f''(u_0) = g''(u_0) = 0$, then $\lambda_0'(0) = 0$ and

$$\lambda_0''(0) = \frac{\lambda_1}{3} \cdot \left(\frac{f'''(u_0)}{f'(u_0)} - \frac{g'''(u_0)}{g'(u_0)} \right) \cdot \frac{\int_\Omega \phi_1^3 \Delta\phi_1 dx}{\int_\Omega |\nabla\phi_1|^2 dx} - \lambda_1 \cdot \frac{g'''(u_0)}{g'(u_0)} \cdot \frac{\int_\Omega |\nabla\phi_1|^2 \phi_1^2 dx}{\int_\Omega |\nabla\phi_1|^2 dx}. \tag{2.18}$$

All these formulas of $\lambda_0'(0)$ and $\lambda_0''(0)$ can be calculated from (5.1), (5.2), (2.1) and the Green's formula:

$$\int_\Omega \phi_1^m \Delta\phi_1 dx = \int_{\partial\Omega} \phi_1^m \frac{\partial\phi_1}{\partial n} dS - m \int_\Omega \phi_1^{m-1} |\nabla\phi_1|^2 dx,$$

for $m = 2, 3$, so we omit the detail of calculation. It is worth pointing out that in the formulas (2.13), (2.14), (2.15) and (2.16), the sign of $\lambda_0'(0)$ or $\lambda_0''(0)$ is independent of the weight function $s(x)$ or $r(x)$, but in the more complicated cases (2.17) or (2.18), the weight function $s(x)$ or $r(x)$ may play a role.

Next we determine the stability of bifurcating solution $(\lambda_0(t), u_0(t))$ in Theorem 2.3 by using Theorem 5.4. To that end, we apply (5.5) and compute $m(\lambda)$ in (5.3). Here we define $K : W^{2,p}(\Omega) \to L^p(\Omega) \times W^{1-\frac{1}{p},p}(\partial\Omega)$ by $K(u) := (u, 0)$ in (5.3). It follows that

$$\begin{cases} \Delta z(\lambda) + \lambda s(x) f'(u_0) z(\lambda) = m(\lambda) z(\lambda), & x \in \Omega, \\ \dfrac{\partial z(\lambda)}{\partial n} = \lambda r(x) g'(u_0) z(\lambda), & x \in \partial\Omega. \end{cases} \tag{2.19}$$

Differentiating (2.19) with respect to λ and setting $\lambda = \lambda_1$, we obtain

$$\begin{cases} \Delta z'(\lambda_1) + s(x) f'(u_0)\phi_1 + \lambda_1 s(x) f'(u_0) z'(\lambda_1) = m'(\lambda_1)\phi_1, & x \in \Omega, \\ \dfrac{\partial z'(\lambda_1)}{\partial n} = r(x) g'(u_0)\phi_1 + \lambda_1 r(x) g'(u_0) z'(\lambda_1), & x \in \partial\Omega, \end{cases} \tag{2.20}$$

by using that $z(\lambda_1) = \phi_1$, $m(\lambda_1) = 0$. By integration and Green's formula, we have

$$m'(\lambda_1) = \frac{f'(u_0)\displaystyle\int_\Omega s(x)\phi_1^2 dx + g'(u_0)\int_{\partial\Omega} r(x)\phi_1^2 dS}{\displaystyle\int_\Omega \phi_1^2 dx} = \frac{\displaystyle\int_\Omega |\nabla\phi_1|^2 dx}{\lambda_1 \displaystyle\int_\Omega \phi_1^2 dx}. \tag{2.21}$$

Now from (5.5) and (2.21), we obtain that

$$sign(\mu(t)) = -sign(t\lambda_0'(t)), \quad \text{for } t \text{ small}, \tag{2.22}$$

while $\lambda_0'(t)$ can be obtained by using $\lambda_0'(0)$ and $\lambda_0''(0)$ calculated above. Note that (2.21) implies that the trivial solution (λ, u_0) is stable when $\lambda < \lambda_1$, and it becomes unstable when $\lambda > \lambda_1$. If $u_0 = 0$ and only positive solutions are physically meaningful solutions, then the transcritical bifurcation is supercritical (forward) if $\lambda_0'(0) > 0$ (or $\lambda_0''(0) > 0$ if $\lambda_0'(0) = 0$) and $\mu(t) < 0$ for $t \in (0, \eta)$, and it is subcritical (backward) if $\lambda_0'(0) < 0$ (or $\lambda_0''(0) < 0$ if $\lambda_0'(0) = 0$) and $\mu(t) > 0$ for $t \in (0, \eta)$. (See Fig. 1.)

We remark that Theorem 2.3 is based on the existence of a principal eigenvalue of (2.1), and the bifurcating solution $u(t) \approx u_0 + t\phi_1$ which is either strictly larger than or strictly smaller than u_0. When $u_0 = 0$, these solutions correspond to either positive or negative solutions, which is important in various applications. The proof of Theorem 2.3 is still valid for the bifurcation at a non-principal simple eigenvalue, but when $u_0 = 0$, one obtain sign-changing solutions in this case. The calculations of bifurcation direction and stability are also valid for the bifurcation at a non-principal simple eigenvalue.

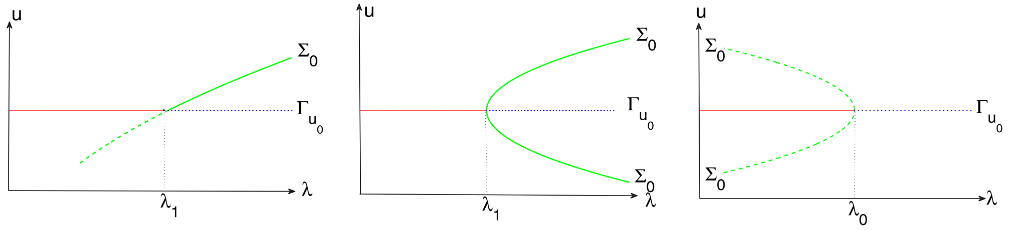

Fig. 1. Bifurcation Diagrams. (Left: transcritical bifurcation; Middle: supercritical pitchfork bifurcation; Right: subcritical pitchfork bifurcation).

2.2. *Local bifurcation at* $(0, u_1)$

In this subsection, we consider the bifurcation from the line of trivial solutions $\Gamma_{u_1} := \{(0, u_1) : \ u_1 > 0, \ u_1 \in \mathbb{R}\}$. In Lemma 2.1, we have shown that at a bifurcation point, u_1 must satisfy **(H2)**. Applying Theorem 5.5, we have the following theorem.

Theorem 2.4. *Suppose that* f, g, r, s *satisfy* **(H0)** *with* $f, g \in C^3(\mathbb{R}, \mathbb{R})$, **(H2)** *and also*

(H4) $f'(u_1) \int_\Omega s(x)dx + g'(u_1) \int_{\partial\Omega} r(x)dS \neq 0.$

Then $(0, u_1)$ *is a bifurcation point of* (1.2) *with respect to the trivial branch* Γ_{u_1}. *Moreover the solution set of* (1.2) *near* $(0, u_1)$ *consists precisely of the curves* Γ_{u_1} *and*

$$\Sigma_1 = \{(\lambda_1(t), u_1(t)) : \ t \in I = (-\eta, \eta) \subset \mathbb{R}\},$$

where $\lambda_1(t) = t + t\theta_1(t)$, $u_1(t) = u_1 + \eta_1 t + t y_1(t)$ *are* C^1 *functions such that* $\theta_1(0) = \theta_1'(0) = y_1(0) = y_1'(0) = 0$, ψ_1 *is the unique solution of*

$$\begin{cases} \Delta v + s(x) f(u_1) = 0, & x \in \Omega, \\ \dfrac{\partial v}{\partial n} = r(x) g(u_1), & x \in \partial\Omega, \\ \displaystyle\int_\Omega v(x)dx = 0, \end{cases} \tag{2.23}$$

and

$$\eta_1 = -\frac{f'(u_1) \displaystyle\int_\Omega s(x)\psi_1 dx + g'(u_1) \int_{\partial\Omega} r(x)\psi_1 dS}{f'(u_1) \displaystyle\int_\Omega s(x)dx + g'(u_1) \int_{\partial\Omega} r(x)dS}. \tag{2.24}$$

Proof. We apply Theorem 5.5 to $F(\lambda, u)$ defined as in (1.5) and we verify all the assumptions in Theorem 5.5 are satisfied. We prove it in several steps:

(1) $\dim N(F_u(0,u_1)) = \operatorname{codim} R(F_u(0,u_1)) = 1$. By (2.2), we obtain

$$F_u(0,u_1)[\phi] = \left(\Delta\phi, \frac{\partial\phi}{\partial n}\right), \tag{2.25}$$

so $N(F_u(0,u_1)) = \operatorname{span}\{1\}$ and $R(F_u(0,u_1)) = \left\{(h_1,h_2) : \int_\Omega h_1 dx - \int_{\partial\Omega} h_2 dS = 0\right\}$.

(2) $F_\lambda(0,u_1) \in R(F_u(0,u_1))$. From **(H2)**, we get

$$F_\lambda(0,u_1) = (s(x)f(u_1), -r(x)g(u_1)) \in R(F_u(0,u_1)). \tag{2.26}$$

We define Z to be $\left\{v \in X : \int_\Omega v(x)dx = 0\right\}$, and we denote by ψ_1 the unique solution of $F_\lambda(0,u_1) + F_u(0,u_1)[v] = 0$ for $v \in Z$, then ψ_1 satisfies (2.23).

(3) We show that the Hessian matrix H_0 in Theorem 5.5 is indefinite. It is easy to see that

$$H_0 = \begin{pmatrix} 2\langle l, F_{\lambda u}(0,u_1)[\psi_1]\rangle & \langle l, F_{\lambda u}(0,u_1)[1]\rangle \\ \langle l, F_{\lambda u}(0,u_1)[1]\rangle & 0 \end{pmatrix}, \tag{2.27}$$

where

$$\begin{aligned} \langle l, F_{\lambda u}(0,u_1)[\psi_1]\rangle &= f'(u_1)\int_\Omega s(x)\psi_1 dx + g'(u_1)\int_{\partial\Omega} r(x)\psi_1 dS, \\ \langle l, F_{\lambda u}(0,u_1)[1]\rangle &= f'(u_1)\int_\Omega s(x)dx + g'(u_1)\int_{\partial\Omega} r(x)dS \end{aligned} \tag{2.28}$$

by using that $F_{\lambda\lambda}(0,u_1) = 0$, $F_{uu}(0,u_1)[\psi_1]^2 = 0$, $F_{uu}(0,u_1)[1,\psi_1] = 0$, and $F_{uu}(0,u_1)[1]^2 = 0$. Hence we have $\det(H_0) = -\left[f'(u_1)\int_\Omega s(x)dx + g'(u_1)\int_{\partial\Omega} r(x)dS\right]^2 < 0$ by **(H4)**.

Now applying part 2 of Theorem 5.5 to $F(\lambda,u) = 0$ shows that $(\mu_1,\eta_1) = (1,\eta_1)$ and $(\mu_2,\eta_2) = (0,1)$. Then the solution set of (1.2) near $(0,u_1)$ consists precisely of the curves $(\lambda_1(t),u_1(t)) = (t + t\theta_1(t), u_1 + \eta_1 t + ty_1(t))$ and $(\lambda_2(t),u_2(t)) = (t\theta_2(t), u_1 + t + ty_2(t))$. Note that the solution curve $(\lambda_2(t),u_2(t))$ is identical to the trivial branch Γ_{u_1}. □

By using Lemma 5.3, we also define $K : W^{2,p}(\Omega) \to L^p(\Omega) \times W^{1-\frac{1}{p},p}(\partial\Omega)$ by $K(u) := (u,0)$. Then we have the following result for the linearized equation for the bifurcating solutions in Theorem 2.4.

Lemma 2.5. *Let X, Y, F, Z be the same as in Theorem 2.4, and let all assumptions in Theorem 2.4 on F be satisfied. Let $(\lambda_1(t),u_1(t))$ be the solution curve in Theorem 2.4. Then there exist $\varepsilon > 0$, C^2 functions $\sigma : (-\varepsilon,\varepsilon) \to \mathbb{R}$, and $v : (-\varepsilon,\varepsilon) \to X$ such that*

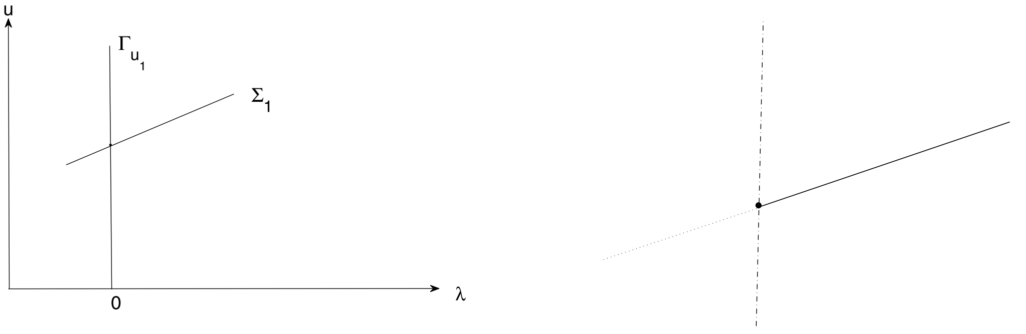

Fig. 2. Left: bifurcating solution in Theorem 2.4; Right: stability of bifurcating solution in Theorem 2.4.

$$F_u(\lambda_1(t), u_1(t))[v(t)] = \sigma(t)(v(t), 0) \quad \textit{for} \quad t \in (-\varepsilon, \varepsilon), \tag{2.29}$$

where $\sigma(0) = 0$, $v(0) = 1$, *and* $v(t) - 1 \in Z$.

Now we consider the stability of bifurcating solution $(\lambda_1(t), u_1(t))$ in Theorem 2.4 by using Theorem 5.5 and compute $\sigma(t)$ in (2.29). It follows that

$$\begin{cases} \Delta v(t) + \lambda_1(t)s(x)f'(u_1(t))v(t) = \sigma(t)v(t), & x \in \Omega, \\ \dfrac{\partial v(t)}{\partial n} = \lambda_1(t)r(x)g'(u_1(t))v(t), & x \in \partial\Omega. \end{cases} \tag{2.30}$$

Differentiating (2.19) with respect to t, we have

$$\begin{cases} \Delta v'(t) + \lambda_1(t)s(x)f'(u_1(t))v'(t) + \lambda_1'(t)s(x)f'(u_1(t))v(t) \\ +\lambda_1(t)s(x)f''(u_1(t))u_1'(t)v(t) = \sigma'(t)v(t) + \sigma(t)v'(t), & x \in \Omega, \\ \dfrac{\partial v'(t)}{\partial n} = \lambda_1(t)r(x)g'(u_1(t))v'(t) \\ +\lambda_1'(t)r(x)g'(u_1(t))v(t) + \lambda_1(t)r(x)g''(u_1(t))u_1'(t)v(t), & x \in \partial\Omega. \end{cases} \tag{2.31}$$

and setting $t = 0$, we obtain

$$\begin{cases} \Delta v'(0) + s(x)f'(u_1) = \sigma'(0), & x \in \Omega, \\ \dfrac{\partial w(0)}{\partial n} = r(x)g'(u_1), & x \in \partial\Omega, \end{cases} \tag{2.32}$$

by using that $\lambda_1(0) = 0$, $\lambda_1'(0) = 1$, $\sigma(0) = 0$ and $v(0) = 1$. By integration and Green's formula, we have

$$\sigma'(0) = \frac{1}{|\Omega|}\left[f'(u_1)\int_\Omega s(x)dx + g'(u_1)\int_{\partial\Omega} r(x)dS \right] \neq 0, \tag{2.33}$$

by **(H4)**. (See Fig. 2.)

We remark that there is no simple eigenvalue assumption needed for the results in Theorem 2.4, and it is possible that there are more than one values u_1 which satisfy **(H2)** and **(H4)**. We also notice that Theorem 2.4 can also be applied to $u_1 = u_0$ as **(H1)** implies **(H2)**, and if **(H4)** is also satisfied, then the local structure of the solution set near $(0, u_0)$ is the union of Γ_{u_0} and Γ_{u_1}.

But when the opposite of **(H4)** is satisfied at $(0, u_0)$, it is possible that (1.2) has another solution curve near $(0, u_0)$. Using Theorem 5.6 which was recently proved in [23], we prove the following result when the opposite of **(H4)** (which we call **(H4′)**) is satisfied at $(0, u_0)$:

Theorem 2.6. *Suppose that f, g, r, s satisfy* **(H0)** *with $f, g \in C^3(\mathbb{R}, \mathbb{R})$,* **(H1)***,*

(H4′) $f'(u_0)\int_\Omega s(x)dx + g'(u_0)\int_{\partial\Omega} r(x)dS = 0$; *and*

(H5) $f''(u_0)\int_\Omega s(x)dx + g''(u_0)\int_{\partial\Omega} r(x)dS \neq 0.$

Then the solution set of (1.2) near $(0, u_0)$ consists precisely of the curves Γ_{u_0}, Γ_{u_1} and

$$\Sigma_2 = \{(\lambda_2(t), u_2(t)) : \ t \in I = (-\eta, \eta) \subset \mathbb{R}\},$$

where $\lambda_2(t) = t + t\theta_1(t)$, $u_2(t) = u_0 + \eta_2 t + t\theta_2(t)$ are C^1 functions such that $\theta_i(0) = 0$, $i = 1, 2$, here

$$\eta_2 = -\frac{2\int_\Omega |\nabla\psi_2|^2 dx}{f''(u_0)\int_\Omega s(x)dx + g''(u_0)\int_{\partial\Omega} r(x)dS}, \tag{2.34}$$

where ψ_2 is the unique solution of

$$\begin{cases} \Delta v + f'(u_0)s(x) = 0, & x \in \Omega, \\ \dfrac{\partial v}{\partial n} = r(x)g'(u_0), & x \in \partial\Omega. \end{cases} \tag{2.35}$$

Proof. We apply Theorem 5.6 to $F(\lambda, u)$ defined as in (1.5). Clearly, for any $\lambda \in \mathbb{R}$, $F(\lambda, u_0) = (0, 0)$ from **(H1)**. Direct calculation shows that

$$F_u(0, u_0)[\phi] = \left(\Delta\phi, \frac{\partial\phi}{\partial n}\right), \tag{2.36}$$

then $N(F_u(0, u_0)) = \text{span}\{1\}$ and $R(F_u(0, u_0)) = \left\{(h_1, h_2) : \int_\Omega h_1 dx - \int_{\partial\Omega} h_2 dS = 0\right\}$. By **(H4′)** we see that $F_{\lambda u}(0, u_0)[1] = (f'(u_0)s(x), -r(x)g'(u_0)) \in R(F_u(0, u_0))$, and there exists a unique solution $\psi_2 \in W^{2,p}(\Omega)$ satisfying (2.35). Note that $F_{uu}(0, u_0)[1]^2 = (0, 0) \in R(F_u(0, u_0))$. It follows that conditions **(F1)**, **(F3′)** and **(F4′)** are satisfied and $v_3 = 0$ at $(0, u_0)$. By using notations in Theorem 5.6, we can calculate that

$$\begin{aligned} H_{11} &= 2\langle l, F_{\lambda u}(0, u_0)[\psi_2]\rangle = 2\left[f'(u_0)\int_\Omega s(x)\psi_2 dx + g'(u_0)\int_{\partial\Omega} r(x)\psi_2 dS \right] \\ &= 2\int_\Omega |\nabla\psi_2|^2 dx, \\ H_{12} &= \frac{1}{2}\langle l, F_{\lambda uu}(0, u_0)[1]^2\rangle = \frac{1}{2}[f''(u_0)\int_\Omega s(x)dx + g''(u_0)\int_{\partial\Omega} r(x)dS], \\ H_{22} &= \frac{1}{3}\langle l, F_{uuu}(0, u_0)[1]^3\rangle = 0. \end{aligned} \tag{2.37}$$

Therefore

$$\det H = -\frac{1}{4}\left[f''(u_0)\int_\Omega s(x)dx + g''(u_0)\int_{\partial\Omega} r(x)dS \right]^2 < 0,$$

by the assumption **(H5)**. Then all conditions in Theorem 5.6 are satisfied, and the solution set of $F(\lambda, u) = 0$ near $(0, u_0)$ is the union of curves intersecting at $(0, u_0)$ including the line of trivial solutions $\Gamma_{u_0} = \{(\lambda, u_0)\}$ and two other curves $\{(\lambda_i(t), u_i(t)) : |t| < \delta\}$ $(i = 1, 2)$ for some $\delta > 0$, with

$$\lambda_i(t) = \mu_i t + t\theta_i(t), \;\; u_i(t) = u_0 + \eta_i t w_0 + t v_i(t),$$

where (μ_1, η_1) and (μ_2, η_2) are non-zero linear independent solutions of the equation

$$H_{11}\mu^2 + 2H_{12}\mu\eta = 0. \tag{2.38}$$

Now we can choose $(\mu_1, \eta_1) = (0, 1)$, $(\mu_2, \eta_2) = (1, -H_{11}/2H_{12})$ with η_2 satisfying (2.34). Then the curve $\{(\lambda_1(t), u_1(t))\}$ corresponds to the branch Γ_{u_1} of the trivial solutions, while the curve $\Sigma_2 = \{(\lambda_2(s), u_2(s))\}$ must be distinct from Γ_{u_0} or Γ_{u_1}. The proof is complete. □

Finally, we determine the stability of bifurcating solution $(\lambda_2(t), u_2(t))$ in Theorem 2.6 by using Theorem 5.8. From (5.19), we have $\gamma'(0) = 0$ and

$$\gamma''(0) = \frac{2}{|\Omega|}\int_\Omega |\nabla\psi_2|^2 dx > 0. \tag{2.39}$$

From (5.22), we have $\sigma_2'(0) = 0$ and

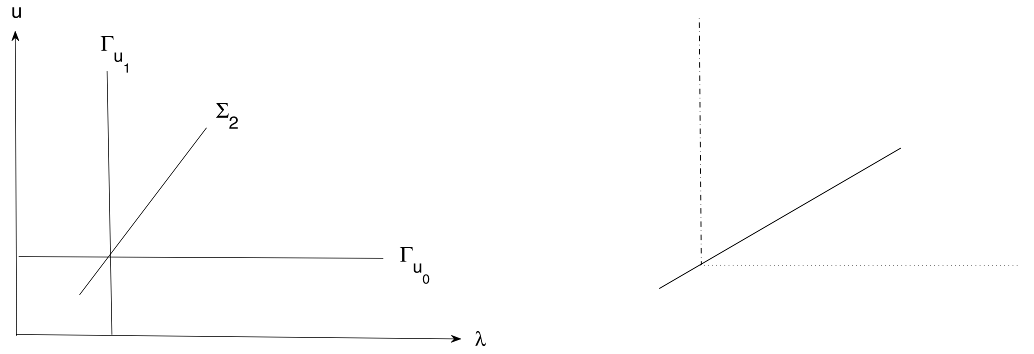

Fig. 3. Left: bifurcating solution in Theorem 2.6; Right: stability of bifurcating solution in Theorem 2.6.

$$\sigma_2''(0) = -\frac{2}{|\Omega|}\int_\Omega |\nabla\psi_2|^2 dx < 0. \tag{2.40}$$

(2.39) and (2.40) imply that the trivial solution (λ, u_0) is unstable for $\lambda \in (-\eta, \eta)$ and the non-trivial solution $(\lambda_2(t), u_2(t))$ is stable for $t \in (-\eta, \eta)$. (See Fig. 3.)

We remark that the local bifurcation analysis for (1.1) from $(0, u_1)$ with $u_1 > 0$, and from $(0, 0)$ was first carried out in [43], where f is assumed to be the logistic nonlinearity as a special case. Another type of bifurcation from $(0, u_1)$ (called double saddle-node bifurcation) was considered recently in [24,50], but the branch of nontrivial solutions is tangent to the branch of trivial solutions not crossing like here.

3. Laplace equation with nonlinear boundary condition and bistable nonlinearity

In this section, we consider a special case of (1.1) with $s(x) = 0$, that is

$$\begin{cases} u_t = \Delta u, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} = \lambda r(x) g(u), & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = u_*(x), & x \in \Omega, \end{cases} \tag{3.1}$$

where $\Omega \in \mathbb{R}^n (n \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, λ is a positive parameter, $g : \mathbb{R} \to \mathbb{R}$ is a C^3 function satisfying

(H6) $$\begin{cases} g(0) = g(a) = g(1) = 0, \\ g < 0 \text{ in } (0, a), g > 0 \text{ in } (a, 1), \\ g'(0) < 0, \ g'(a) > 0, \ g'(1) < 0, \\ \text{there exists a constant } b \in (0, 1) \text{ such that } g'' > 0 \text{ in } (0, b), g'' < 0 \text{ in } (b, 1), \end{cases}$$

and in addition to **(H0)**, we assume that

(H7) $r : \partial\Omega \to \mathbb{R}$ is a sign-changing function such that $\displaystyle\int_{\partial\Omega} r(x) dS \neq 0$.

Here we only look for steady state solutions of (3.1) satisfying $0 \le u \le 1$, which satisfy

$$\begin{cases} \Delta u = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \lambda r(x) g(u), & x \in \partial\Omega, \end{cases} \tag{3.2}$$

Define a nonlinear mapping $F : \mathbb{R} \times W^{2,p}(\Omega) \to L^p(\Omega) \times W_p^{1-\frac{1}{p}}(\partial\Omega)$, $p > n$ by

$$F(\lambda, u) = \left(\Delta u, \frac{\partial u}{\partial n} - \lambda r(x) g(u) \right). \tag{3.3}$$

The trivial branches are the curves

$$\Gamma_0 := \{(\lambda, 0) : \lambda > 0\}, \;\; \Gamma_a := \{(\lambda, a) : \lambda > 0\}, \text{ and } \Gamma_1 := \{(\lambda, 1) : \lambda > 0\}.$$

First we have the following result, which is exactly analogous to [28, Theorem 3.1], hence we omit the proof.

Proposition 3.1. *Suppose that* **(H6)** *and* **(H7)** *hold true, then for all* λ *sufficient small the only equilibria to (3.2) are constant ones.*

Secondly Lemma 2.2 implies the following proposition.

Proposition 3.2. *Suppose that* **(H6)** *and* **(H7)** *hold true.*

1. *If*

$$\int_{\partial\Omega} r(x) dS < 0, \tag{3.4}$$

then there is a bifurcation point $\lambda = \lambda_a$ *with respect to the trivial branch* Γ_a *and there is no bifurcation points with respect to* Γ_0 *and* Γ_1.

2. *If*

$$\int_{\partial\Omega} r(x) dS > 0, \tag{3.5}$$

then there are bifurcation points $\lambda = \lambda_0$ *and* $\lambda = \lambda_1$ *with respect to the trivial branch* Γ_0 *and* Γ_1 *respectively, and there is no bifurcation with respect to* Γ_a.

Proof. From Lemma 2.2, we have that $\lambda_a > 0$ is the principal eigenvalue of

$$\begin{cases} \Delta v = 0, & x \in \Omega, \\ \dfrac{\partial v}{\partial n} = \lambda g'(a) r(x) v, & x \in \partial\Omega, \end{cases} \tag{3.6}$$

since $r(x)$ is a sign-changing function satisfying (3.4) and $g'(a) > 0$. On the other hand $\lambda_0 < 0$ and $\lambda_1 < 0$ since $g'(0) < 0$ and $g'(1) < 0$. Then the conclusion in part 1 follow. Part 2 is similar. □

We remark that the critical case $\int_{\partial\Omega} r(x)dS = 0$ was considered in [29] and g is assumed to be a logistic type function. In the following we mainly consider the case (3.4). From Theorem 2.3, we have

Theorem 3.3. *Suppose that* **(H0)**, **(H6)**, **(H7)** *and* (3.4) *are satisfied. Then the principal eigenvalue λ_a of* (3.6) *is a bifurcation point of* (3.2) *with respect to Γ_a, and the solutions of* (3.2) *near (λ_a, a) consists precisely of the curves $u = a$ and*

$$S_1 = \{(\lambda(t), u(t)) : t \in I = (-\eta, \eta) \subset \mathbb{R}\},$$

where $u(t) = a + t\phi_0 + z_1(t)$, $\lambda(t) = \lambda_a + z_2(t)$ are C^2 functions such that $z_i(0) = z_i'(0) = 0$, $i = 1, 2$, and ϕ_0 is the positive principal eigenfunction of (3.6)*. Furthermore,*

1. *if $a < b$, then a transcritical bifurcation occurs at (λ_a, a) and $\lambda'(0) < 0$;*
2. *if $a = b$, then a pitchfork bifurcation occurs at (λ_a, a), $\lambda'(0) = 0$ and $\lambda''(0) > 0$;*
3. *if $a > b$, then a transcritical bifurcation occurs at (λ_a, a) and $\lambda'(0) > 0$.*

Proof. From Proposition 3.2, λ_a is a bifurcation point of (3.2) with respect to Γ_a, and the local bifurcation follows from Theorem 2.3. Now we apply (2.13) to the bifurcation curve of (3.2), then we obtain that

$$\lambda'(0) = -\lambda_a \cdot \frac{g''(a)}{g'(a)} \cdot \frac{\int_\Omega |\nabla\phi_0|^2\phi_1 dx}{\int_\Omega |\nabla\phi_0|^2 dx}. \tag{3.7}$$

From **(H6)**, we have $g'(a) > 0$, $g''(a) > 0$ if $a < b$, and $g''(a) < 0$ if $a > b$. Hence a transcritical bifurcation occurs when $a \neq b$ and we can determine the sign of $\lambda'(0)$ from (3.7). If $a = b$, then $\lambda'(0) = 0$, then we can further apply (2.14) to obtain that $\lambda''(0) > 0$ since $g''(u)$ changes from positive to negative thus $g'''(a) < 0$. □

The local branch obtained in Theorem 3.3 is indeed part of a global continuum of positive solutions of (3.2). So we have the following global bifurcation results regarding the global continuum emanating from (λ_a, a).

Theorem 3.4. *Suppose that* **(H0)**, **(H6)**, **(H7)** *and* (3.4) *are satisfied. Let Σ be the set of nontrivial solutions of* (3.2)*. Then*

1. *there exists a connected component Σ_1 of $\overline{\Sigma}$ which contains (λ_a, a) (the bifurcation point defined in Theorem 3.3).*
2. *$\Sigma_1 \backslash \{(\lambda_a, a)\} = \Sigma_1^+ \cup \Sigma_1^-$, where Σ_1^+ (Σ_1^-) is consisted with solutions (λ, u) of* (3.2) *satisfying $1 > u > a$ ($0 < u < a$), and near (λ, a), Σ_1^+ corresponds to the positive (negative)*

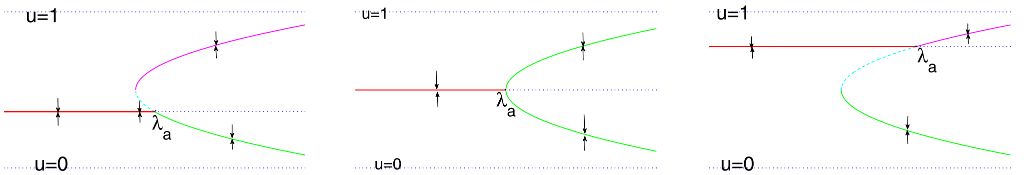

Fig. 4. Bifurcation Diagrams if (3.4) holds. (Left: $a < b$; Middle: $a = b$; Right: $a > b$).

half-branch $S_1^+ = \{(\lambda(t), u(t)) : t \in (0, \eta)\}$ ($S_1^- = \{(\lambda(t), u(t)) : t \in (-\eta, 0)\}$) *obtained in Theorem 3.3.*

3. *The projection* Σ_1^+ (Σ_1^-) *onto* λ*-axis contains the interval* (λ_a, ∞)*. Hence for any* $\lambda > \lambda_a$*, (3.2) possesses at least one solution* $u_+(\lambda)$ *such that* $u_+ > a$ *and at least one solution* $u_-(\lambda)$ *such that* $u_- < a$*.*
4. *If* $a \le b$*, then (3.2) has a unique solution* $u_-(\lambda)$ *such that* $u_- < a$ *for any* $\lambda > \lambda_a$*, and* $\Sigma_1^- = \{(\lambda, u_-(\lambda) : \lambda > \lambda_a\}$*; similarly if* $a \ge b$*, then (3.2) has a unique solution* $u_+(\lambda)$ *such that* $u_+ > a$ *for any* $\lambda > \lambda_a$*, and* $\Sigma_1^+ = \{(\lambda, u_+(\lambda) : \lambda > \lambda_a\}$ *(see Fig. 4).*

Proof. The existence of a global continuum Σ_1 follows from Theorem 5.9. From the continuity of Σ_1 and maximum principle, that for any $(\lambda, u) \in \Sigma_1$, either u is nontrivial and $0 < u < 1$, or $u \equiv a$. From [39, Theorem 4.4], each of Σ_1^+ or Σ_1^- is either unbounded, or contains another bifurcation point (λ_*, a), or contains another point $(\lambda_*, a + z)$ where z satisfies $\int_\Omega \phi_0 z = 0$. Again from the continuity of Σ_1^+ and maximum principle, any solution (λ, u) on Σ_1^+ must satisfy either (i) $a < u < 1$, or (ii) $u \equiv a$ or (iii) $u \equiv 1$. From Lemma 2.2, $\lambda = \lambda_a$ is the unique principal eigenvalue of (3.7), hence (ii) is not possible otherwise there is other principal eigenvalue of (3.7). (iii) is also not possible as from Proposition 3.2, there is no bifurcation points along Σ_1. This shows that any solution (λ, u) on Σ_1^+ must satisfy $a < u < 1$, and Σ_1^+ cannot contain another bifurcation point (λ_*, a). Finally Σ_1^+ cannot contain another point $(\lambda_*, a + z)$ where z satisfies $\int_\Omega \phi_0 z = 0$ as such an $a + z$ would violate the condition $a < u < 1$. Therefore Σ_1^+ is unbounded. Also $a < u < 1$ on Σ_1^+, then the projection of Σ_1^+ onto λ-axis is unbounded. From Proposition 3.1, the projection contains (λ_a, ∞). The proof for Σ_1^- is also similar. This completes the proof of Parts 1-3.

The proof of Part 4 is similar to the ones in [16, Lemma 10.1.5] and [27, Theorem 2.5]. We only consider the case $a \le b$. We prove that in this case, any solution (λ, u_-) on Σ_1^- is non-degenerate, *i.e.* the linearized equation

$$\begin{cases} \Delta\psi = 0, & x \in \Omega, \\ \dfrac{\partial\psi}{\partial n} = \lambda g'(u_-)r(x)\psi, & x \in \partial\Omega, \end{cases} \tag{3.8}$$

has no nonzero solution. Suppose it is not true, then there exists $\psi \not\equiv 0$ satisfying (3.8). Since $0 < u_- < a$ in $\overline{\Omega}$, then $g(u_-) < 0$ from **(H6)**. Let $\theta = \dfrac{\psi}{g(u_-)}$, then θ satisfies

$$\begin{cases} \Delta\theta + 2\dfrac{g'(u_-)}{g(u_-)}\nabla u_- \cdot \nabla\theta + \dfrac{g''(u_-)}{g(u_-)}|\nabla u_-|^2\theta = 0, & x \in \Omega, \\ \dfrac{\partial\theta}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

Here the coefficient of θ term is non-positive since $g''(u_-) > 0$ and $g(u_-) < 0$, so applying the maximum principle and Hopf's Lemma yields $\theta \equiv 0$, which is a contradiction with $\psi \not\equiv 0$. Thus (λ, u_-) is non-degenerate, and from the implicit function theorem, the local branch S_1^- can be extended to a monotone global curve Σ_1^- which is parameterized by $\lambda \in (\lambda_a, \infty)$. (3.2) cannot have other solution (λ, u) satisfying $0 < u < a$ as such a solution is also non-degenerate and the connected component which it belongs to is also a global curve can be extended as long as it satisfies $0 < u < a$. But Proposition 3.1 and Lemma 2.2 imply that such a connected component different from Σ_1^- does not exist. Hence for $\lambda > \lambda_a$, the solution of (3.2) satisfying $0 < u < a$ is unique and it is on the branch Σ_1^-. □

Remark 3.5.

1. For the case of $a < b$, (3.2) has multiple solutions satisfying $a < u < 1$ for $\lambda \in (\lambda_a - \varepsilon, \lambda_a)$ from Theorem 3.3 and Proposition 3.1, and similarly for the case of $a > b$, (3.2) has multiple solutions satisfying $0 < u < a$ for $\lambda \in (\lambda_a - \varepsilon, \lambda_a)$. When $a = b$, it is easy to show that (3.2) has only the trivial solutions when $\lambda < \lambda_a$.
2. There are other bifurcation points along Γ_a where solutions of (3.2) bifurcate from Γ_a, but there solutions satisfy that $u - a$ changes sign.

Next we consider the stability of the solutions of (3.2).

Theorem 3.6. *Suppose that* **(H0)**, **(H6)**, **(H7)** *and* (3.4) *are satisfied. Let $u_\pm$ be the nontrivial solutions of* (3.2) *obtained in Theorem 3.3.*

1. *The constant solution $u = a$ is locally asymptotically stable for $0 < \lambda < \lambda_a$, and it is unstable for $\lambda > \lambda_a$.*
2. *For all $\lambda > 0$, the solution $u = 0$ and $u = 1$ are both unstable.*
3. *If $a \le b$, then $u_-(\lambda)$ is locally asymptotically stable for $\lambda > \lambda_a$, and its basin of attraction contains any $u_0 \in \{0 \le u \le a\}$ which is not 0 or a; and if $a \ge b$, then $u_+(\lambda)$ is locally asymptotically stable for $\lambda > \lambda_a$, and its basin of attraction contains any $u_0 \in \{a \le u \le 1\}$ which is not a or 1.*
4. *If $a < b$, then $u_+(\lambda)$ is unstable for $\lambda_a - \varepsilon < \lambda < \lambda_a$; and similarly if $a > b$, then $u_-(\lambda)$ is unstable for $\lambda_a - \varepsilon < \lambda < \lambda_a$.*

Proof. The proof of Part 1 and 2 is standard, see for example [27, Theorem 3.1]. For Part 3 and 4, we use the formula (2.22) to determine the stability of the bifurcating solutions $u_\pm$. When $a \le b$, $\lambda'(t) < 0$ and $t\lambda'(t) > 0$ for $-\eta < t < 0$, hence $\mu(t) < 0$, which implies that $u(t) = u_-(\lambda)$ is locally asymptotically stable for $-\eta < t < 0$ ($\lambda_a < \lambda < \lambda_a + \varepsilon$). Similarly when $a \le b$, $u(t) = u_+(\lambda)$ is unstable for $0 < t < \eta$ ($\lambda_a - \varepsilon < \lambda < \lambda_a$). From Theorem 3.4 part 4, $u_-(\lambda)$ is unique and non-degenerate for all $\lambda > \lambda_a$, hence it is always locally asymptotically stable for all $\lambda > \lambda_a$. The global stability for all $u_0 \in \{0 \le u \le a\}$ which is not 0 or a follows from a well-known argument of Lyapunov function. □

For the case (3.5), we can similarly prove that the principal eigenvalue λ_0 (λ_1) is a bifurcation point with respect to Γ_0 (Γ_1), and a local (also global) bifurcation occurs at $(\lambda_0, 0)$ or $(\lambda_1, 1)$. There is a connected component of solutions of (3.2) satisfying $0 < u < 1$ emanating from $(\lambda_0, 0)$ or $(\lambda_1, 1)$, and the direction and stability of bifurcating solutions can also be determined. But no other global information of the solution branches is known, which still awaits further investigation.

4. An elliptic equation with superlinear nonlinearity and sublinear boundary conditions

In this section we consider the following elliptic equation with nonlinear boundary condition:

$$\begin{cases} -\Delta u = \lambda s(x) u^q, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \lambda r(x)(u - u^p), & x \in \partial\Omega. \end{cases} \tag{4.1}$$

Here $q > 1$ and $p > 1$. We have the following propositions by applying Theorems 2.3, 2.4 and 2.6. First a bifurcation occurs from the line of trivial solutions $\Gamma_0 = \{(\lambda, 0) : \lambda > 0\}$:

Proposition 4.1. *Suppose that* **(H0)**, **(H7)** *and* (3.4) *hold, then the problem*

$$\begin{cases} -\Delta u = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \lambda r(x) u, & x \in \partial\Omega, \end{cases} \tag{4.2}$$

has a principal eigenvalue $\lambda_1 > 0$ which is a bifurcation point of (4.1) with respect to Γ_0. Moreover the solution set of (4.1) near $(\lambda_1, 0)$ consists precisely of the curves $u = 0$ and

$$\Sigma_0 = \{(\lambda_0(t), u_0(t)) : \ t \in I = (-\eta, \eta) \subset \mathbb{R}\},$$

where $\lambda_0(t) = \lambda_1 + z_2(t)$, $u_0(t) = t\phi_1 + t z_1(t)$ are C^1 functions such that $z_i(0) = 0$, $i = 1, 2$, ϕ_1 is the eigenfunction corresponding to λ_1.

Proof. (4.1) is a special case of (1.2) with $f(u) = u^q$ and $g(u) = u - u^p$. From Lemma 2.2, the principal eigenvalue $\lambda_1 > 0$ exists and it is a bifurcation point. It is easy to verify the conditions **(H0)**, **(H1)** and **(H3)**. Therefore Theorem 2.3 can be applied to obtain desired result. □

Secondly we have the result of bifurcation from the line $\Gamma_{u_1} = \{(0, u_1) : u_1 > 0\}$.

Proposition 4.2. *Suppose that* **(H0)** *and* (3.4) *hold, u_1 is a positive root of*

$$u_1^{q-1} \int_\Omega s(x)dx + (1 - u_1^{p-1}) \int_{\partial\Omega} r(x)dS = 0, \tag{4.3}$$

and one of following three conditions holds:

1. *$p > q > 1$; or*
2. *$p = q > 1$ and $\int_{\partial\Omega} r(x)dS < \int_\Omega s(x)dx$; or*
3. *$q > p > 1$, $\int_\Omega s(x)dx > 0$ and $u_1 \neq (\frac{q-1}{q-p})^{\frac{1}{p-1}}$,*

then $(0,u_1)$ *is a bifurcation point of* (4.1) *with respect to the trivial branch* Γ_{u_1}. *Moreover the solution set of* (4.1) *near* $(0,u_1)$ *consists precisely of the curves* Γ_{u_1} *and*

$$\Sigma_1=\{(\lambda_1(t),u_1(t)):\ \ t\in I=(-\eta,\eta)\subset\mathbb{R}\},$$

where $\lambda_1(t)=t+t\theta_1(t)$, $u_1(t)=u_1+\eta_1 t+ty_1(t)$ *are* C^1 *functions such that* $\theta_1(0)=\theta_1'(0)=y_1(0)=y_1'(0)=0$, ψ_1 *is the unique solution of*

$$\begin{cases}\Delta v+s(x)u_1^q=0, & x\in\Omega,\\ \dfrac{\partial v}{\partial n}=r(x)(u_1-u_1^p), & x\in\partial\Omega,\\ \displaystyle\int_\Omega v(x)dx=0,\end{cases}\tag{4.4}$$

and

$$\eta_1=-\frac{qu_1^{q-1}\displaystyle\int_\Omega s(x)\psi_1dx+(1-pu_1^{p-1})\int_{\partial\Omega}r(x)\psi_1dS}{qu_1^{q-1}\displaystyle\int_\Omega s(x)dx+(1-pu_1^{p-1})\int_{\partial\Omega}r(x)dS}.\tag{4.5}$$

Proof. For $f(u)=u^q$ and $g(u)=u-u^p$, it is easy to verify that (4.3) implies **(H2)**. We denote

$$\begin{aligned}Q(u_1):&=u_1^{q-1}\int_\Omega s(x)dx+(1-u_1^{p-1})\int_{\partial\Omega}r(x)dS,\\ &=u_1^{q-1}(\int_\Omega s(x)dx-u_1^{p-q}\int_{\partial\Omega}r(x)dS)+\int_{\partial\Omega}r(x)dS.\end{aligned}$$

It is clear that $Q(0)=\int_{\partial\Omega}r(x)dS<0$ and $\lim_{u_1\to\infty}Q(u_1)=\infty$. As a result, we can find a positive u_1 satisfying (4.3) and **(H4)**. Therefore Theorem 2.4 can be applied to obtain results here. □

Finally a bifurcation from $(\lambda,u)=(0,0)$ is also possible as shown in the following result.

Proposition 4.3. *Consider* (4.1) *with* $q=2$ *and* $p\ge 2$. *If*

$$\int_{\partial\Omega}r(x)dS=0\ \ \text{and}\ \ \int_\Omega s(x)dx<0.\tag{4.6}$$

Then the solution set of (4.1) *near* $(0,0)$ *consists precisely of the curves* $\lambda=0$, $u=0$ *and*

$$\Sigma_2=\{(\lambda_2(t),u_2(t)):\ \ t\in I=(-\eta,\eta)\subset\mathbb{R}\},$$

where $\lambda_2(t) = t + t\theta_1(t)$, $u_2(t) = \eta_2 t + t\theta_2(t)$ *are* C^1 *functions such that* $\theta_i(0) = 0$, $i = 1, 2$, *here*

$$\eta_2 = -\frac{\int_\Omega |\nabla\psi_2|^2 dx}{\int_\Omega s(x)dx}, \tag{4.7}$$

where ψ_2 *is the unique solution of*

$$\begin{cases} \Delta v = 0, & x \in \Omega, \\ \dfrac{\partial v}{\partial n} = r(x), & x \in \partial\Omega. \end{cases} \tag{4.8}$$

Proof. It is easy to verify the conditions **(H4′)** and **(H5)**, then Theorem 2.6 can be applied. □

Note that the bifurcation at $(0, u_1)$ for $u_1 > 0$ and the one from $(0, 0)$ cannot occur simultaneously from (3.4) and (4.6). Here only the local bifurcations for (4.1) are considered here to illustrate the applications of our main results in Section 3. It is possible to adapt techniques in [14,45,46] to consider the global bifurcations of positive solutions, which will be done in a future work.

5. Appendix

In this section we recall some known abstract bifurcation theorems which are used in this paper.

5.1. Local bifurcation theorems in Banach spaces

A nonlinear problem can often be formulated as an abstract equation

$$F(\lambda, u) = 0,$$

where $F : \mathbb{R} \times X \to Y$ is a nonlinear differentiable mapping, and X, Y are Banach spaces. We say that 0 is a simple eigenvalue of $F_u(\lambda_0, u_0)$, if the following assumption is satisfied:

(F1) $\dim N(F_u(\lambda_0, u_0)) = \operatorname{codim} R(F_u(\lambda_0, u_0)) = 1$, and $N(F_u(\lambda_0, u_0)) = \operatorname{span}\{w_0\}$.

Crandall and Rabinowitz [9] prove the celebrated “bifurcation from a simple eigenvalue” theorem (see [9, Theorem 1.7]), and here is an expanded version for our purpose:

Theorem 5.1. *Let* U *be a neighborhood of* (λ_0, u_0) *in* $\mathbb{R} \times X$, *and let* $F : U \to Y$ *be a twice continuously differentiable mapping. Assume that* $F(\lambda, u_0) = 0$ *for* $(\lambda, u_0) \in U$. *At* (λ_0, u_0), F *satisfies* **(F1)** *and*

(F3) $F_{\lambda u}(\lambda_0, u_0)[w_0] \notin R(F_u(\lambda_0, u_0))$.

Let Z be any complement of $\operatorname{span}\{w_0\}$ *in X. Then the solution set of $F(\lambda, u) = 0$ near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $\{(\lambda(s), u(s)) : s \in I = (-\epsilon, \epsilon)\}$, where $\lambda : I \to \mathbb{R}$, $z : I \to Z$ are C^1 functions such that $u(s) = u_0 + sw_0 + sz(s)$, $\lambda(0) = \lambda_0$, $z(0) = 0$, and*

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0]^2\rangle}{2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle}, \tag{5.1}$$

where $l \in Y^*$ satisfying $N(l) = R(F_u(\lambda_0, u_0))$. If F also satisfies

(F4) $F_{uu}(\lambda_0, u_0)[w_0]^2 \notin R(F_u(\lambda_0, u_0))$,

then we have $\lambda'(0) \neq 0$, and it is called a transcritical bifurcation; If F satisfies

(F4′) $F_{uu}(\lambda_0, u_0)[w_0]^2 \in R(F_u(\lambda_0, u_0))$,

and in addition $F \in C^3$, then $\lambda'(0) = 0$ and

$$\lambda''(0) = -\frac{\langle l, F_{uuu}(\lambda_0, u_0)[w_0]^3\rangle + 3\langle l, F_{uu}(\lambda_0, u_0)[w_0, \theta]\rangle}{3\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle}, \tag{5.2}$$

where θ satisfies $F_{uu}(\lambda_0, u_0)[w_0]^2 + F_u(\lambda_0, u_0)[\theta] = 0$. If $\lambda'(0) = 0$ and $\lambda''(0) \neq 0$, then it is called a pitchfork bifurcation.

For the bifurcation from simple eigenvalue described in Theorem 5.1, an "exchange of stability" occurs at the bifurcation point between the branches of known trivial solutions and bifurcating non-trivial solutions. To introduce the principle of exchange of stability, we first recall the definition of K-simple eigenvalue and a fundamental result due to Crandall and Rabinowitz [10]. Here we denote by $B(X, Y)$ the set of bounded linear maps from X into Y.

Definition 5.2. ([10, Definition 1.2]) Let $T, K \in B(X, Y)$. We say that $\mu \in \mathbb{R}$ is a K-simple eigenvalue of T, if

$$\dim N(T - \mu K) = \operatorname{codim} R(T - \mu K) = 1, \quad N(T - \mu K) = \operatorname{span}\{w_0\},$$

and

$$K[w_0] \notin R(T - \mu K).$$

The following basic perturbation result for the eigenvalues can be proved:

Lemma 5.3. ([10, Lemma 1.3]) *Suppose that $T_0, K \in B(X, Y)$ and μ_0 is a K-simple eigenvalue of T_0. Then there exists $\delta > 0$ such that if $T \in B(X, Y)$ and $\|T - T_0\| < \delta$, then there exists a unique $\mu(T) \in \mathbb{R}$ satisfying $\|\mu(T) - \mu_0\| < \delta$ such that $N(T - \mu(T)K) \neq \emptyset$ and $\mu(T)$ is a K-simple eigenvalue of T. Moreover if $N(T_0 - \mu_0 K) = \operatorname{span}\{w_0\}$ and Z is a complement of* $\operatorname{span}\{w_0\}$ *in X, then there exists a unique $w(T) \in X$ such that $N(T - \mu(T)K) = \operatorname{span}\{w(T)\}$, $w(T) - w_0 \in Z$ and the map $T \mapsto (\mu(T), w(T))$ is analytic.*

As in [10, Corollary 1.13], by using Lemma 5.3, we have the following result for the linearized equation for the bifurcating solutions in Theorem 5.1.

Theorem 5.4. *Let X, Y, U, F, Z, λ_0, and w_0 be the same as in Theorem 5.1, and let all assumptions in Theorem 5.1 on F be satisfied. Let $(\lambda(t), u(t))$ be the solution curves in Theorem 5.1, there exist C^2 functions $m(\lambda) : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to \mathbb{R}$, $z : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to X$, $\mu : (-\delta, \delta) \to \mathbb{R}$, and $\phi : (-\delta, \delta) \to X$ such that*

$$F_u(\lambda, u_0)z(\lambda) = m(\lambda)K(z(\lambda)), \quad \text{for} \quad \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon), \tag{5.3}$$

$$F_u(\lambda(t), u(t))\phi(t) = \mu(t)K(\phi(t)), \quad \text{for} \quad t \in (-\delta, \delta). \tag{5.4}$$

where $m(\lambda_0) = \mu(0) = 0$, $z(\lambda_0) = \phi(0) = w_0$. Moreover, near $t = 0$ the functions $\mu(t)$ and $-t\lambda'(t)m'(\lambda_0)$ have the same zeroes and, whenever $\mu(t) \neq 0$ the same sign and satisfy

$$\lim_{t \to 0} \frac{-t\lambda'(t)m'(\lambda_0)}{\mu(t)} = 1. \tag{5.5}$$

Theorem 5.1 describes a bifurcation of non-trivial solutions from a known branch. However sometimes the bifurcation is not from a known branch or the known branch is not in a form of $\{(\lambda, u_0) : \lambda \in \mathbb{R}\}$. In such situation, the following crossing curve bifurcation theorem is more useful.

Theorem 5.5. ([22, Theorem 2.1]) *Let $F : \mathbb{R} \times X \to Y$ be a C^2 mapping. Suppose that $F(\lambda_0, u_0) = 0$, F satisfies* **(F1)** *and* **(F2′)**. *Let $X = N(F_u(\lambda_0, u_0)) \oplus Z$ be a fixed splitting of X, let $v_1 \in Z$ be the unique solution of*

$$F_\lambda(\lambda_0, u_0) + F_u(\lambda_0, u_0)[v] = 0 \tag{5.6}$$

and let $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v\rangle = 0\}$. We assume that the matrix (all derivatives are evaluated at (λ_0, u_0))

$$H_0 = H_0(\lambda_0, u_0) \equiv \begin{pmatrix} \langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1]^2\rangle & \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle \\ \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle & \langle l, F_{uu}[w_0]^2\rangle \end{pmatrix} \tag{5.7}$$

is non-degenerate, i.e., $\det(H_0) \neq 0$.

1. *If H_0 is definite, i.e.* $\det(H_0) > 0$, *then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is $\{(\lambda_0, u_0)\}$.*
2. *If H_0 is indefinite, i.e.* $\det(H_0) < 0$, *then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is the union of two intersecting C^1 curves, and the two curves are in form of $(\lambda_i(s), u_i(s)) = (\lambda_0 + \mu_i s + s\theta_i(s), u_0 + \eta_i s w_0 + s y_i(s))$, $i = 1, 2$, where $s \in (-\delta, \delta)$ for some $\delta > 0$, (μ_1, η_1) and (μ_2, η_2) are non-zero linear independent solutions of the equation*

$$\begin{aligned} &\langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1]^2\rangle\mu^2 + 2\langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle\eta\mu \\ &+ \langle l, F_{uu}[w_0]^2\rangle\eta^2 = 0, \end{aligned} \tag{5.8}$$

where $\theta_i(s)$, $y_i(s)$ are some functions defined on $s \in (-\delta, \delta)$ which satisfy $\theta_i(0) = \theta_i'(0) = 0$, $y_i(s) \in Z$, and $y_i(0) = y_i'(0) = 0$, $i = 1, 2$.

Finally we recall a degenerate version of Theorem 5.1, which can be used to obtain more than two intersecting solution curves near the bifurcation point.

Theorem 5.6. ([23, Theorem 3.1]) *Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F \in C^3(U, Y)$. Assume that $F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$ and at (λ_0, u_0), F satisfies* **(F1)**,

(F3′) $F_{\lambda u}(\lambda_0, u_0)[w_0] \in R(F_u(\lambda_0, u_0))$; *and*

(F4′) $F_{uu}(\lambda_0, u_0)[w_0]^2 \in R(F_u(\lambda_0, u_0))$.

Let $X = N(F_u(\lambda_0, u_0)) \oplus Z$ be a fixed splitting of X, and let $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v\rangle = 0\}$. Denote by $v_2 \in Z$ the unique solution of

$$F_{\lambda u}(\lambda_0, u_0)[w_0] + F_u(\lambda_0, u_0)[v] = 0, \tag{5.9}$$

and $v_3 \in Z$ the unique solution of

$$F_{uu}(\lambda_0, u_0)[w_0]^2 + F_u(\lambda_0, u_0)[v] = 0. \tag{5.10}$$

We assume that the matrix (all derivatives are evaluated at (λ_0, u_0))

$$H = H(\lambda_0, u_0) = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} \tag{5.11}$$

is non-degenerate, i.e., $\det(H) \neq 0$, *where H_{ij} is given by*

$$H_{11} = \langle l, F_{\lambda\lambda u}[w_0] + 2F_{\lambda u}[v_2]\rangle, \tag{5.12}$$

$$H_{12} = \frac{1}{2}\langle l, F_{\lambda uu}[w_0]^2 + F_{\lambda u}[v_3] + 2F_{uu}[w_0, v_2]\rangle, \tag{5.13}$$

$$H_{22} = \frac{1}{3}\langle l, F_{uuu}[w_0]^3 + 3F_{uu}[w_0, v_3]\rangle. \tag{5.14}$$

1. *If H is definite, i.e.* $\det(H) > 0$, *then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is the line $\{(\lambda, u_0)\}$.*
2. *If H is indefinite, i.e.* $\det(H) < 0$, *then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is the union of C^1 curves intersecting at (λ_0, u_0), including the line of trivial solutions $\Gamma_0 = \{(\lambda, u_0)\}$ and two other curves $\Gamma_i = \{(\lambda_i(s), u_i(s)) : |s| < \delta\}$ ($i = 1, 2$) for some $\delta > 0$, with*

$$\lambda_i(s) = \lambda_0 + \mu_i s + s\theta_i(s), \quad u_i(s) = u_0 + \eta_i s w_0 + s v_i(s),$$

where (μ_1, η_1) and (μ_2, η_2) are non-zero linear independent solutions of the equation

$$H_{11}\mu^2 + 2H_{12}\mu\eta + H_{22}\eta^2 = 0, \tag{5.15}$$

$\theta_i(0) = \theta_i'(0) = 0$, $v_i(s) \in Z$, *and* $v_i(0) = v_i'(0) = 0$, $i = 1, 2$.

We have the following result for the linearized equation for the bifurcating solutions in Theorem 5.6.

Proposition 5.7. ([23, Proposition 3.3]) *Let X, Y, U, F, Z, λ_0, w_0, v_1 and v_2 be the same as in Theorem 5.6, and let all assumptions in Theorem 5.6 on F be satisfied. In addition we assume that $X \subset Y$, and the inclusion mapping $i : X \to Y$ is continuous. Let $(\lambda_i(s), u_i(s))$ $(i = 1, 2)$ be the solution curves in Theorem 5.6. Then there exist $\varepsilon > 0$, C^2 functions $\gamma : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to \mathbb{R}$, $\sigma_i : (-\varepsilon, \varepsilon) \to \mathbb{R}$, $v : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to X$, $w_i : (-\varepsilon, \varepsilon) \to X$ such that*

$$F_u(\lambda, u_0)[v(\lambda)] = \gamma(\lambda)v(\lambda) \quad \text{for} \quad \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon), \tag{5.16}$$

$$F_u(\lambda_i(s), u_i(s))[w_i(s)] = \sigma_i(s)w_i(s) \quad \text{for} \quad s \in (-\varepsilon, \varepsilon), \tag{5.17}$$

where $\gamma(\lambda_0) = \sigma_i(0) = 0$, $v(\lambda_0) = w_i(0) = w_0$, and $v(\lambda) - w_0 \in X$, $w_i(s) - w_0 \in Z$.

The signs of $\gamma(\lambda)$ and $\sigma_i(s)$ determine the stability of the bifurcating solutions. Here we consider the stability of bifurcating solutions obtained in Theorem 5.6.

Theorem 5.8. ([23, Theorem 3.4]) *Let the assumptions of Proposition 5.7 hold and let γ, σ_i be the functions provided by Proposition 5.7. In addition, we assume that*

$$w_0 \notin R(F_u(\lambda_0, u_0)), \quad \text{where } w_0(\neq 0) \in N(F_u(\lambda_0, u_0)). \tag{5.18}$$

Then

1. $\gamma'(\lambda_0) = 0$, *and*

$$\gamma''(\lambda_0) = \frac{H_{11}}{\langle l, w_0 \rangle}. \tag{5.19}$$

If $H_{11} = 0$ and we assume that $F \in C^4$ near (λ_0, u_0), then we have $\gamma''(\lambda_0) = 0$ and

$$\gamma'''(\lambda_0) = -\frac{\langle l, F_{\lambda\lambda\lambda u}(\lambda_0, u_0)[w_0] + 3F_{\lambda\lambda u}(\lambda_0, u_0)[v_1] + 3F_{\lambda u}(\lambda_0, u_0)[v_3] \rangle}{\langle l, w_0 \rangle}, \tag{5.20}$$

where $v_3 \in Z$ is the unique solution of

$$F_{\lambda\lambda u}(\lambda_0, u_0)[w_0] + 2F_{\lambda u}(\lambda_0, u_0)[v_1] + F_u(\lambda_0, u_0)[v_3] = 0. \tag{5.21}$$

2. $\sigma_i'(0) = 0$ *and*

$$\sigma_i''(0) = \frac{H_{22}\eta_i^2 - H_{11}\mu_i^2}{\langle l, w_0 \rangle}, \tag{5.22}$$

where H_{11}, H_{22}, μ_i, η_i $(i = 1, 2)$ are defined in Theorem 5.6.

5.2. Global bifurcation theorems

We recall the following global bifurcation theorem due to [39] which basically is based on almost the same conditions of Theorem 5.1, and it is also an generalization of the classical Rabinowitz global bifurcation theorem [36].

Theorem 5.9. *Let V be an open connected subset of $\mathbb{R} \times X$ and $(\lambda_0, u_0) \in V$, and let F be a continuously differentiable mapping from V into Y. Suppose that*

1. $F(\lambda, u_0) = 0$ *for* $(\lambda, u_0) \in V$,
2. *The partial derivative* $F_{\lambda u}(\lambda, u)$ *exists and is continuous in* (λ, u) *near* (λ_0, u_0),
3. $F_u(\lambda_0, u_0)$ *is a Fredholm operator with index* 0*, and dim* $N(F_u(\lambda_0, u_0)) = 1$,
4. $F_{\lambda u}(\lambda_0, u_0)[w_0] \notin R(F_u(\lambda_0, u_0))$*, where* $w_0 \in X$ *spans* $N(F_u(\lambda_0, u_0))$.

Let Z be any complement of $\text{span}\{w_0\}$ *in X. Then there exist an open interval $I_1 = (-\epsilon, \epsilon)$ and continuous functions $\lambda : I_1 \to \mathbb{R}$, $\psi : I_1 \to Z$, such that $\lambda(0) = \lambda_0$, $\psi(0) = 0$, and, if $u(s) = u_0 + sw_0 + s\psi(s)$ for $s \in I_1$, then $F(\lambda(s), u(s)) = 0$. Moreover, $F^{-1}(\{0\})$ near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $\Gamma = \{(\lambda(s), u(s)) : s \in I_1\}$. If in addition, $F_u(\lambda, u)$ is a Fredholm operator for all $(\lambda, u) \in V$, then the curve Γ is contained in $\mathcal{C}$, which is a connected component of $\overline{S}$ where $S = \{(\lambda, u) \in V : F(\lambda, u) = 0, u \neq u_0\}$; and either $\mathcal{C}$ is not compact in V, or $\mathcal{C}$ contains a point (λ_*, u_0) with $\lambda_* \neq \lambda_0$.*

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