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Ground states of nonlinear Schrödinger equation on star metric graphs ☆

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ABSTRACT

The existence and nonexistence of the ground state to nonlinear Schrödinger equation on several types of metric graphs are considered. In particular, for some star graphs with only one central vertex, the existence of ground state solution or positive solutions are shown. It is shown that the structure of the set of positive solution is quite different from the one for corresponding bounded n-dimensional domain. The proofs are based on variational methods, rearrangement arguments, energy estimates and phase plane analysis.

1. Introduction

The nonlinear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} + r\Delta\psi + \chi|\psi|^2\psi = 0, \qquad t > 0,\ x \in \mathbb{R}^n, \tag{1.1}$$

arises as a canonical model of physics from the studies of continuum mechanics, condensed matter, nonlinear optics, plasma physics [15,34]. A standing wave solution of (1.1) is in a form of $\psi(x,t) = \exp(\lambda it)\Psi(x)$ and Ψ satisfies a nonlinear elliptic equation:

$$r\Delta\Psi - \lambda\Psi + \chi|\Psi|^2\Psi = 0, \qquad x \in \mathbb{R}^n, \tag{1.2}$$

which has been extensively considered in the last a few decades [9,10,33]. Here r is interpreted as the normalized Plank constant, χ describes the strength of the attractive interactions and λ is the wavelength.

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Standing wave solutions of more general Schrödinger type equations have also been studied in [7,8,14,16,17,20,21,25,26,28,29,36,38].

While the standard spatial setting for the nonlinear Schrödinger equation is the Euclidean space $\mathbb{R}^n$ for $n = 1, 2, 3$, there has been recent interests on wave propagations on thin graph-like domains which can be approximated by metric graphs (or quantum graphs) [11,18,23,32]. A metric graph is a graph $G = (V, E)$ with a set V of vertices and a set E of edges, such that each edge $e \in E$ is associated with either a closed bounded interval $I_e = [0, l_e]$ of length $l_e > 0$, or a closed half-line $I_e = [0, \infty)$ with $l_e = \infty$ in this case. The notion of graph is central to this paper, and we refer the reader to [12,39] for the basic definitions in graph theory. For each edge $e \in E$ joining two vertices $v_1, v_2 \in V$, a coordinate system x_e is chosen along $I_e = [0, l_e]$, in such a way that v_1 corresponds to $x_e = 0$ and v_2 to $x_e = l_e$, or vice versa. In the case that $l_e = \infty$, we always assume that the half-line I_e is attached to the remaining part of the graph at $x_e = 0$, and the vertex corresponding to $x_e = +\infty$ is called a vertex at infinity. The subset of V consisting of all vertices at infinity is denoted by V_∞ [5].

In this paper, we investigate the existence and nonexistence of ground state solutions to a nonlinear Schrödinger (NLS) equation on a connected metric graph $G = (V, E)$:

$$\begin{cases} -u_e'' + u_e = |u_e|^{p-2}u_e, & \text{for each edge } e \in E, \\ \displaystyle\sum_{e \succ v} \frac{du_e}{dx_e}(v) = 0, & \text{for each vertex } v \in V \backslash V_\infty, \\ u_{e_i}(v) = u_{e_j}(v), & \text{if } e_i \succ v \text{ and } e_j \succ v \text{ for some } \ v \in V \backslash V_\infty, \\ u = (u_e) \in H^1(G), \end{cases} \tag{1.3}$$

where $p > 2$ and $e \succ v$ means that the edge e is incident to a vertex v. In (1.3), the sum of flux from all edges incident at the vertex v is zero, which is the Kirchhoff's circuit law; and second boundary condition that $u_{e_i}(v) = u_{e_j}(v)$ is known as the continuity condition at the vertex v. If the vertex v is an endpoint (only one edge is incident to v), then the Kirchhoff's condition becomes the Neumann boundary condition at v. If $v \in V_\infty$, there is no given boundary condition but we consider the problem in H^1 space hence we must have $\lim\limits_{x_e \to \infty} u_e(x_e) = 0$ for $u_e \in H^1(I_e)$ and $I_e = [0, +\infty)$. Here $L^p(G)$ is the space defined as the set of functions $u : G \to \mathbb{R}$ such that

$$\int_G |u|^p dx := \sum_{e \in E} \int_{I_e} |u_e|^p dx_e < \infty,$$

and $H^1(G)$ is the Sobolev space defined as the set of functions $u : G \to \mathbb{R}$ such that $u = (u_e)$ is continuous on G and $u_e \in H^1(I_e)$ for every edge $e \in E$ with the natural norm

$$\|u\|^2_{H^1(G)} = \int_G (|u'(x)|^2 + |u(x)|^2)dx = \sum_{e \in E} \int_{I_e} (|u_e'(x_e)|^2 + |u_e(x_e)|^2)dx_e.$$

The energy function corresponding to (1.3) is defined by

$$J(u, G) = \frac{1}{2} \sum_e \int_{I_e} (|u_e'(x_e)|^2 + |u_e(x_e)|^2)dx_e - \frac{1}{p} \sum_e \int_{I_e} |u_e(x_e)|^p dx_e, \quad u \in H^1(G). \tag{1.4}$$

A critical point $u \in H^1(G)$ of $J(\cdot, G)$ satisfies that for any $w = (w_e) \in H^1(G)$, we have

$$(J'(u, G), w) = \sum_e \int_{I_e} (u_e'w_e' + u_e w_e - |u_e|^{p-2}u_e w_e)\mathrm{d}x_e = 0.$$

It can be shown that u is a solution of (1.3) if and only if u is a critical point of $J(\cdot, G)$ (see Lemma 2.1), hence (1.3) is the Euler–Lagrange equation for the energy $J(\cdot, G)$. It is easy to see that $u = 0$ is always a trivial solution of (1.3), and any other solution u lies in the Nehari manifold defined as

$$N(G) = \{u \in H^1(G)\backslash\{0\} : (J'(u, G), u) = 0\}. \tag{1.5}$$

The ground state energy of $J(\cdot, G)$ is defined by

$$E(G) = \inf_{u \in N(G)} J(u, G), \tag{1.6}$$

and if $E(G)$ is attained by some $u_* \in N(G)$, that is, $J(u_*, G) = E(G)$, then u_* is called a ground state solution of (1.3).

It is known that the ground state energy is always positive for any metric graph G (see Lemma 2.4). When the metric graph G is compact (that is, $V_\infty = \emptyset$), then a positive ground state solution u_* always exists, although it is possible that $u_* = 1$ which is a trivial constant state only when G is compact. Indeed we will prove the following result for the ground state solution on a compact metric graph:

Theorem 1.1. *Suppose that G is a compact connect graph with the total length l. Then*

1. *A positive ground state solution $u_*(G)$ exists.*
2. *If the total length l is sufficiently small, then $u_*(G) \equiv 1$. Moreover $u_* = 1$ is the unique positive solution of* (1.3).

We remark that it is known that $u = 1$ is the unique positive solution of (1.2) on a bounded domain in $\mathbb{R}^n$ with Neumann boundary condition when the diffusion coefficient r is large or the domain is small [24].

For a non-compact metric graph G ($V_\infty \neq \emptyset$), the existence or nonexistence of a ground state solution depends on the topological structure of G. We are interested in the following questions:

(Q1) For what kind of non-compact metric graphs, the ground state energy can (cannot) be attained by a ground state solution u_*?

(Q2) When the ground state energy is attained, what is the ground state energy? And is the ground state solution unique and monotone?

(Q3) When the ground state energy cannot be attained, are there other nontrivial positive solutions for the non-compact metric graph G?

In general these questions are hard to answer as the graphs can have complicated topological structure. For the simplest non-compact graph $G = \mathbb{R}$ (which can be viewed as a graph with two vertices at infinity and one finite vertex), it is well-known that the ground state energy is attained by a positive solution u_0 which is symmetric with respective to some $x_0 \in \mathbb{R}$ (the center), and u_0 is strictly decreasing from the center to the infinite vertex. Without loss of generality, we assume that the center of u_0 is at the finite vertex. That is, u_0 is the positive solution of

$$u'' - u + u^{p-1} = 0,\ x \in \mathbb{R},\ u'(0) = 0,\ \lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} u'(x) = 0. \tag{1.7}$$

It is well-known that u_0 is positive, strictly decreasing for positive x, and decays exponentially at the infinity. It is also known that u_0 is the unique nontrivial solution of (1.3) in $H^1(\mathbb{R})$. We denote by $E_0 = J(u_0, \mathbb{R})$ where $J(\cdot, \mathbb{R})$ is the energy functional defined in (1.4). Indeed the explicit formula of u_0 is known [15]:

$$u_0(x) = \left(\frac{p}{2}\right)^{1/(p-2)} \operatorname{sech}^{2/(p-2)}\left(\frac{p-2}{2}x\right). \tag{1.8}$$

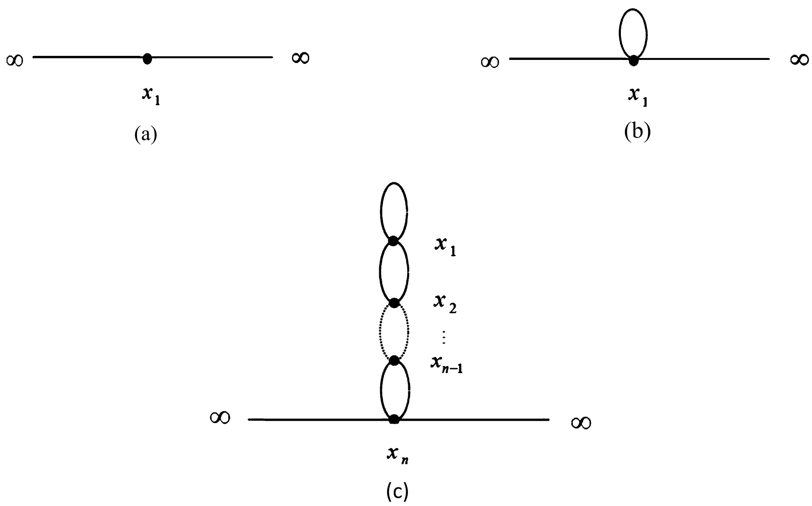

Fig. 1. Three graphs satisfying **(H)** for which a ground state exists.

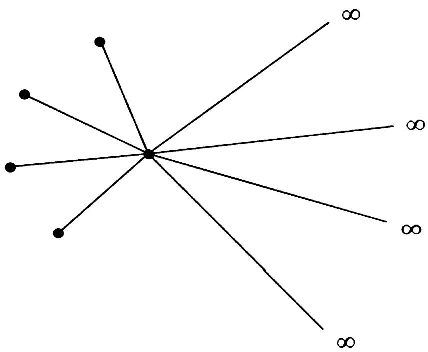

Fig. 2. The star graph with $k = 4$ and $m = 4$.

In the following we call u_0 the canonical soliton solution of (1.3) in $\mathbb{R}$, and we also denote by $u_0^+ := u_0|_{[0,\infty)}$ which is a half soliton defined on a half line.

To ensure the existence/nonexistence of ground states, a condition on a non-compact graph G was proposed in [5] as follows:

(H) After removal of any edge $e \in E$, every connected component of the graph $\tilde{G} = (V, E\backslash\{e\})$ contains at least one vertex $v \in V_\infty$.

By using similar arguments as in [5], one can prove the following result regarding the existence/nonexistence of ground states under condition **(H)** (see subsection 2.3).

Theorem 1.2. *Suppose that $G = (V, E)$ is a metric graph satisfying* **(H)**. *Then a ground state solution $u_*(G)$ of* (1.3) *exists if and only if G is one of the three graphs in* Fig. 1.

In this paper we are mainly interested in answering the questions above for star graphs. A star graph is the one with only one central vertex v_0 which is connected to any other vertex via exactly one edge, and there is no other edges. In the following we consider a star graph with k finite edges of equal length l and m infinite length edges all starting from the central vertex v_0, and we call this metric graph the (k, m)-type star graph denoted by $\mathcal{S}_{k,m}$ (see Fig. 2). Here $k, m \in \mathbb{N} \cup \{0\}$ and $k + m > 0$. Note that when $m = 0$, $\mathcal{S}_{k,0}$ is the isotropic k-star with finite length, and when $k = 0$, $\mathcal{S}_{0,m}$ is the m-infinite-star graph (a star graph

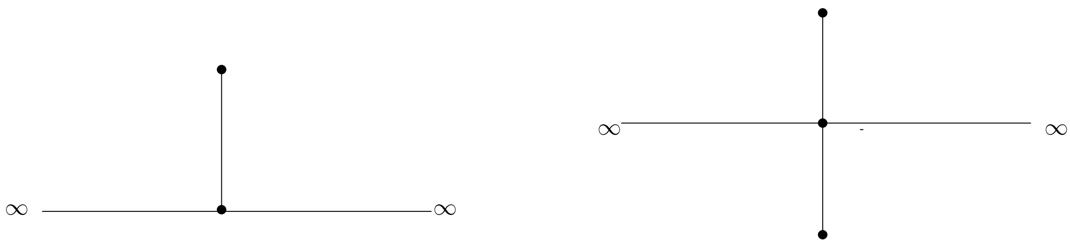

Fig. 3. Star graphs $\mathcal{S}_{k,2}$ with two infinite edges. Left: $k = 1$; Right: $k = 2$.

with m infinite length edges). Also if $\mathcal{S}_{k,m}$ satisfies the condition **(H)**, then it is necessary that $k = 0$. For the case of $k = 0$, the set of positive solutions of (1.3) can be completely classified as follows.

Theorem 1.3. *Let $\mathcal{S}_{0,m}$ be the star graph with m edges of infinite length.*

1. *A ground state solution exists for $\mathcal{S}_{0,m}$ if and only if $m = 1$ or $m = 2$;*
2. *When m is an odd integer, then (1.3) has a unique positive solution on $\mathcal{S}_{0,m}$, which equals to u_0^+ on each edge;*
3. *When $m = 2m_1$ is an even integer, and u is a positive solution of (1.3) on $\mathcal{S}_{0,m}$, then the edges of $\mathcal{S}_{0,m}$ can be labeled pairwise so that $e_{2i-1} \cup e_{2i} = \mathbb{R}$ for $1 \leqslant i \leqslant m_1$, and $u|_{e_{2i-1}\cup e_{2i}} = u_0$ so that the unique central vertex v_0 has the same coordinate $x = x_c \in \mathbb{R}$ on each $e_{2i-1} \cup e_{2i}$. Hence the positive solution of (1.3) is unique up to a translation of the maximum point of u_0 on each of $e_{2i-1} \cup e_{2i}$.*

A direct consequence is that the positive solution of (1.3) on the m-infinite-star graph has the energy level $mE_0/2$, hence it is quantized. For odd m, the maximum point of the positive solution is always at the unique central vertex v_0 of $\mathcal{S}_{0,m}$, while for even m, the maximum point of the positive solution can be translated (uniformly for each pair of (e_{2i-1}, e_{2i})) to any location on $\mathbb{R}$. We note that the latter case provides an example that a positive solution of (1.3) on a symmetric graph is not necessarily symmetric with respect to its geometric center (the central vertex here), while the celebrated result of Gidas, Ni and Nirenberg [19] showed that a positive of (1.2) is always symmetric with respect to a point $x_0 \in \mathbb{R}^n$. To be more precise, we have the following proposition.

Proposition 1.4. *A metric graph $G = (V, E)$ is defined as symmetric with respect to a vertex $v_0 \in V$, if $G\backslash\{v_0\} = \bigcup_{i\in I} G_i$, $G_i \bigcap G_j = \emptyset$ for $i \neq j$, and for any $i, j \in I$ there is an isomorphism $f_{ij} : \bar{G}_i \to \bar{G}_j$ such that $f_{ij}(v_0) = (v_0)$. A solution u of (1.3) on a symmetric metric graph $G = \{v_0\} \cup (\bigcup_{i\in I} G_i)$ is symmetric if $u|_{G_i} = u|_{f_{ij}(G_i)}$ for any $i, j \in I$. Then for the symmetric graph $\mathcal{S}_{0,2m_1}$ with $m_1 \geqslant 2$, there exists a positive solution of (1.3) which is not symmetric.*

For a star metric graph with at least one half-line and at least one finite edge, the existence of ground state solution or positive solution is much more difficult to determine. Note that such graphs ($\mathcal{S}_{k,m}$ with $k \geqslant 1$ and $m \geqslant 1$) are non-compact and they do not satisfy the condition **(H)**. We have the following results on the existence of ground state solution when the star graph has 2 half-lines and 1 or 2 (equal length) finite edges (see Fig. 3 for these graphs).

Theorem 1.5. *Let $\mathcal{S}_{k,m}$ be the (k, m)-type star graph defined as above.*

1. *If $k = 1$ or $k = 2$, and $m = 2$, then a ground state solution exists.*
2. *The ground state energy satisfies*

$$\frac{1}{2}E_0 \leqslant E(\mathcal{S}_{1,2}) < E_0, \qquad \frac{1}{2}E_0 \leqslant E(\mathcal{S}_{2,2}) \leqslant E_0. \tag{1.9}$$

Note that the ground state for $\mathcal{S}_{1,1}$ is obviously u_0 (restricted to half line). The existence of a ground state for $\mathcal{S}_{1,2}$ (two half-lines and a pendant, see Fig. 3 left) has been considered in [5] with a different setting (under a mass constraint, see below), and the result for $\mathcal{S}_{2,2}$ (two half-lines and two pendants) has not been considered previously. The existence/nonexistence of a ground state for $\mathcal{S}_{k,m}$ for other values of (k,m) is still open. On the other hand, while the existence of a ground state is not always known for star graph $\mathcal{S}_{k,m}$, we show in our last main result that a positive solution (not necessarily with ground state energy) of (1.3) always exists:

Theorem 1.6. *Let $\mathcal{S}_{k,m}$ be the (k,m)-type star graph defined as above. Then for any $k \geqslant 1$ and $m \geqslant 1$, (1.3) has a symmetric positive solution in form of $(u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m)$ satisfying $u_1 = u_2 = \cdots = u_k = u$ and $v_1 = v_2 = \cdots v_m = v$, and (u,v) satisfies*

$$\begin{cases} u'' - u + u^{p-1} = 0, & -l < x < 0, \\ v'' - v + v^{p-1} = 0, & 0 < x < \infty, \\ u'(-l) = 0, \ \lim\limits_{x\to\infty} v(x) = 0, & \\ u(0) = v(0), \ ku'(0) = mv'(0), & \end{cases} \tag{1.10}$$

and $u'(x) < 0$ for $-l < x < 0$, $v'(x) < 0$ for $x > 0$. Moreover the strictly decreasing positive solution of (1.10) is unique for any $l > 0$, $k \geqslant 1$ and $m \geqslant 1$.

It is easy to see that when $k = m$, then $(u(x), v(x)) = (u_0(x+l)|_{[-l,0]}, u_0(x+l)|_{[0,\infty)})$ is such a symmetric solution. Here in Theorem 1.6, we show that such a solution also exists when $k \neq m$. The proof of Theorem 1.6 is based on an ODE shotting method argument for the system (1.10), which is of independent interest. The symmetric solution obtained in Theorem 1.6 in general is not a ground state solution, but in some special case, the existence of such a symmetric solution implies the existence of a ground state as in the following corollary:

Corollary 1.7. *Let $\mathcal{S}_{k,m}$ be the (k,m)-type star graph defined as above. If $k = 1$ and $m \geqslant 2$, then a ground state solution exists if the length of the unique finite edge l is sufficiently large.*

Whether a ground state exists for $\mathcal{S}_{1,m}$ when the length of the unique finite edge l is small is still not known.

The ground states of the nonlinear Schrödinger equation on a metric graph have been considered in a different but related setting (see [1–6,27,30,35]). In these work, the energy functional

$$\tilde{J}(u,G) = \frac{1}{2}\|u'\|^2_{L^2(G)} - \frac{1}{p}\|u\|^p_{L^p(G)}$$

was considered with the mass constraint

$$\|u\|^2_{L^2(G)} = \mu.$$

Here $\mu > 0$ is a fixed number and it is assumed that $p \in (2,6)$. The ground state in this setting is defined as $u_* \in H^1(G)$ satisfying

$$\tilde{J}(u_*, G) = \inf\{\tilde{J}(u,G) : u \in H^1(G), \|u\|^2_{L^2(G)} = \mu\}. \tag{1.11}$$

Note that in this setting it is required that $p \in (2,6)$ so that the functional $\tilde{J}(\cdot, G)$ is bounded from below under the mass constraint $\|u\|^2_{L^1(G)} = \mu$, and when $p \geqslant 6$ the infimum in (1.11) is $-\infty$. In our results in this paper, we only need $p > 2$. Another work of nonlinear elliptic equations on a metric graph is [41], in which the connection between the stability with respect to reaction–diffusion dynamics and the graph structure is considered.

In Section 2, we will show that the existence of a solution to (1.3) corresponds to the existence of a critical point of $J(u, G)$ in $H^1(G)$. Therefore, we can apply variational methods to obtain the existence of critical points of the energy functional $J(u, G)$. Here G may be $\mathbb{R}$. The proofs of Theorems 1.1 and 1.2 are given in Section 2. The main existence results for the ground state solutions (Theorems 1.3 and 1.5) are proved in Section 3, and in Section 4, we study the symmetric solution of (1.3) using ODE techniques and we prove Theorem 1.6 and Corollary 1.7.

2. Preliminaries

In this section, we will give some basic notations and energy estimates.

2.1. Variational setting

We first show that solutions of (1.3) correspond to critical points of $J(\cdot, G)$ defined in (1.4).

Lemma 2.1. *Let G be a connected metric graph. Then u is a solution of (1.3) if and only if u is a critical point of the functional $J(\cdot, G)$ defined in (1.4).*

Proof. Suppose that $u \in H^2(G)$ is a solution of (1.3), then $u \in H^1(G)$ and $u = (u_e)$ is continuous at every vertex. Let $w = (w_e) \in H^1(G)$, then by integrating by parts and using the Kirchhoff's condition, we have

$$\int_G (-u'' + u - |u|^{p-2}u)w dx := \sum_{e\in E} \int_{I_e} (-u_e'' + u_e - |u_e|^{p-2}u_e)w_e dx_e$$
$$= \sum_{e\in E} \int_{I_e} (u_e' w_e' + u_e w_e - |u_e|^{p-2}u_e w_e)dx_e = (J'(u, G), w) = 0.$$

Therefore, u is a critical point of $J(\cdot, G)$ in $H^1(G)$. On the other hand, let $u = (u_e) \in H^1(G)$ be a critical point of $J(\cdot, G)$, then u is continuous at each finite vertex. Fixing an edge e, choosing an arbitrary $w = w_e \in C_0^\infty(I_e)$ and integrating by parts, we have

$$\int_{I_e} (-u_e'' + u_e - |u_e|^{p-2}u_e)w_e dx_e = \int_{I_e} (u_e' w_e' + u_e w_e - |u_e|^{p-2}u_e w_e)dx_e$$
$$= (J'(u, G), w) = 0,$$

and then $-u_e'' + u_e = |u_e|^{p-2}u_e$ in $H^{-1}(I_e)$ and $u_e \in H^2(I_e)$ by the elliptic regularity theory. Next we prove the Kirchhoff condition. Fixing a vertex $v \in V\backslash V_\infty$, choosing a test function $w \in H^1(G)$ which is null at every vertex of G except at v, and integrating by parts, we have

$$-\sum_{e\succ v} \frac{du_e}{dx_e}(v)w(v) = 0,$$

since only the boundary terms at v are not zero. Then we have $\sum_{e\succ v} \frac{du_e}{dx_e}(v) = 0$ since $w(v)$ is arbitrary. □

Next we recall the following classical result regarding the Mountain-Pass structure of an energy functional (see [40]).

Proposition 2.2. *Let X be a Banach space and $J \in C^1(X, \mathbb{R})$. Let Ω be a bounded open subset of X, and $e_1, e_2 \in X$ with $e_1 \in \Omega$ and $e_2 \notin \overline{\Omega}$. If $\inf_{u \in \partial\Omega} J(u) > \max\{J(e_1), J(e_2)\}$, we say that J satisfies the Mountain-Pass geometric structure. Let*

$$\Gamma = \{\gamma \in C[0,1] : \gamma(0) = e_1, \gamma(1) = e_2\},$$

and

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)).$$

Then J has a (PS)$_c$ sequence, that is, a sequence $\{u_n\} \subseteq X$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to \infty$.

Now we show that for any metric graph G, the energy functional $J(\cdot, G)$ possesses such a Mountain-Pass structure, thus it always has a (PS)$_c$ sequence.

Lemma 2.3. *Suppose that G is a connected metric graph.*

1. *There exist $r, \alpha > 0$ such that $\inf_{\|u\|_{H^1(G)} = r} J(u, G) = \alpha > 0$.*
2. *There exists $u_1 \in H^1(G)$ with $\|u_1\|_{H^1(G)} > r$ such that $J(u_1, G) < 0$.*
3. *$J(\cdot, G)$ possesses a bounded (PS)$_c$ sequence $\{u_n\} \subseteq H^1(G)$ where*

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t), G), \quad \textit{and} \;\; \Gamma = \{\gamma \in C[0,1] : \gamma(0) = 0, \gamma(1) = u_1\}. \tag{2.1}$$

Proof. 1. For $u \in H^1(G)$, since

$$J(u, G) = \frac{1}{2} \int\limits_G (|u'|^2 + |u|^2) - \frac{1}{p} \int\limits_G |u|^p = \frac{1}{2}\|u\|_{H^1(G)}^2 - \frac{1}{p}|u|_p^p \geqslant \frac{1}{2}\|u\|_{H^1(G)}^2 - C_p\|u\|_{H^1(G)}^p,$$

then we can choose a small $r > 0$ such that $J(u, G) \geqslant r^2 \left(\frac{1}{2} - C_p r^{p-2}\right) := \alpha > 0$ for any u satisfying $\|u\|_{H^1(G)} = r$.

2. Let $u \in H^1(G)$ be fixed. Since

$$J(tu, G) = \frac{1}{2}t^2\|u\|_{H^1(G)}^2 - \frac{1}{p}t^p|u|_{L^p(G)}^p,$$

then $J(tu, G) \to -\infty$ as $t \to \infty$. Therefore, we can choose $t > 0$ large enough such that $J(tu, G) < 0$ and $\|tu\|_{H^1(G)} > r$. Let $u_1 = tu$ then the conclusion holds.

3. From part 1, 2 and Proposition 2.2, $J(\cdot, G)$ possesses a (PS)$_c$ sequence $\{u_n\} \subseteq H^1(G)$ where c is given by (2.1). We prove that $\{u_n\}$ is bounded. Indeed

$$\|u_n\|_{H^1(G)} + c \geqslant J(u_n, G) - \frac{1}{p}(J'(u_n, G), u_n) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_{H^1(G)}^2, \tag{2.2}$$

which implies that $\{u_n\}$ is bounded. This proof actually shows any (PS) sequence is bounded. □

Next we prove some properties of the Nehari manifold $N(G)$, which is also similar to the ones on a region of $\mathbb{R}^n$.

Lemma 2.4. *Suppose that G is a connected metric graph.*

1. *$E(G) := \inf_{u\in N(G)} J(u,G) > 0$ where $N(G)$ is defined in (1.5).*
2. *For any $u \in H^1(G)\backslash\{0\}$, there exists a unique $t_u \in (0,\infty)$ such that $t_u u \in N(G)$ and $J(t_u u,G) = \max_{t\geqslant 0} J(tu,G)$.*

Proof. 1. First we show that if $u \in N(G)$, then $\|u\|_{H^1(G)} \geqslant C$ for some constant $C > 0$ independent of u. In fact, if $u \in N(G)$, then

$$0 = (J'(u,G),u) = \|u\|^2_{H^1(G)} - |u|^p_{L^p(G)} \geqslant \|u\|^2_{H^1(G)} - C_p\|u\|^p_{H^1(G)}.$$

Therefore, $\|u\|_{H^1(G)} \geqslant C$ for some $C > 0$. Then by using that $0 = (J'(u,G),u) = \|u\|^2_{H^1(G)} - |u|^p_{L^p(G)}$, we obtain that

$$J(u,G) = \frac{1}{2}\|u\|^2_{H^1(G)} - \frac{1}{p}|u|^p_{L^p(G)} = \left(\frac{1}{2}-\frac{1}{p}\right)\|u\|^2_{H^1(G)} \geqslant \left(\frac{1}{2}-\frac{1}{p}\right)C^2.$$

Hence $E(G) \geqslant (1/2 - 1/p)C^2 > 0$.

2. For any $u \in H^1(G)\backslash\{0\}$, define

$$g(t) := J(tu,G) = \frac{1}{2}t^2\|u\|^2_{H^1(G)} - \frac{1}{p}t^p|u|^p_{L^p(G)}.$$

Then

$$g'(t) = t\|u\|^2_{H^1(G)} - t^{p-1}|u|^p_{L^p(G)},$$

and there exists a unique $t_u \in (0,\infty)$ such that $g'(t_u) = 0$. Moreover, $g'(t) > 0$ for $t \in (0,t_u)$ and $g'(t) < 0$ for $t \in (t_u,\infty)$. Therefore, $J(t_u u,G) = \max_{t\geqslant 0} g(t) = \max_{t\geqslant 0} J(tu,G)$. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. 1. If G is a compact graph, then $J(\cdot,G)$ satisfies the (PS) condition and Proposition 2.2 implies the existence of a ground state which is also a Mountain-Pass solution. Moreover the Mountain-Pass energy level c defined in (2.1) is coincident with the ground state energy $E(G)$. This proof is similar to the one of NLS equation with subcritical nonlinearity on a bounded domain of $\mathbb{R}^N$.

2. Let u be a positive solution of (1.3) and decompose u as $u = u_G + \phi$, where

$$u_G = \frac{1}{l}\int_G u(x)dx, \qquad \int_G \phi(x)dx = 0.$$

Then from (1.3), we have

$$\begin{cases} -\phi_e'' + \phi_e = u_G^{p-1} - u_G + (p-1)(\int_0^1 |u_G + t\phi_e|^{p-2}dt)\phi_e, & \text{for each edge } e \in E,\\ \sum_{e\succ v} \frac{d\phi_e}{dx_e}(v) = 0, & \text{for each vertex } v \in V,\\ \phi_{e_i}(v) = \phi_{e_j}(v), & \text{if } e_i \succ v \text{ and } e_j \succ v \text{ for some } v \in V,\\ \phi = (\phi_e) \in H^1(G). \end{cases} \tag{2.3}$$

Multiplying both sides of (2.3) by ϕ_e and integrating over e, and using the boundary conditions, we have

$$\int_G |\phi'|^2 dx + \int_G |\phi|^2 dx = (p-1) \int_G \phi^2 (\int_0^1 |u_G + t\phi|^{p-2} dt) dx. \tag{2.4}$$

For any $v \in H^1(G)$, according to Lemma 2.1 and [13, Remark 2.1], we have

$$\left| \int_G (u'v' + uv) dx \right| = \left| \int_G |u|^{p-2} uv \right|$$
$$\leqslant \int_G |u|^{p-1} dx |v|_\infty \leqslant (l^{-1/2} + l^{1/2}) \int_G |u|^{p-1} dx \|v\|_{H^1(G)}.$$

Therefore we have

$$\|u\|_{H^1(G)} \leqslant (l^{-1/2} + l^{1/2}) \int_G |u|^{p-1} dx. \tag{2.5}$$

On the other hand, integrate (1.3), it follows from the Kirchhoff condition that

$$\int_G u dx = \int_G u^{p-1} dx. \tag{2.6}$$

The Hölder inequality implies that

$$\int_G |u|^{p-1} dx = \int_G u^{p-1} dx = \int_G u dx \leqslant \left(\int_G u^{p-1} dx \right)^{1/(p-1)} l^{(p-2)/(p-1)},$$

and hence

$$\int_G u dx = \int_G u^{p-1} dx \leqslant l. \tag{2.7}$$

Combining (2.7) with (2.5), we get

$$\|u\|_{H^1(G)} \leqslant l^{1/2} + l^{3/2}. \tag{2.8}$$

According to [13, Remark 2.1] and (2.8),

$$|u|_\infty \leqslant (l^{-1/2} + l^{1/2}) \|u\|_{H^1(G)} \leqslant (l^{1/2} + l^{3/2})(l^{-1/2} + l^{1/2}) = (1 + l)^2. \tag{2.9}$$

Using the estimate (2.9), we find that $|u_G + t\phi| = |tu + (1-t)u_G| \leqslant \max |u| \leqslant (1 + l)^2$ for $x \in G$ and $t \in [0, 1]$. Hence (2.4) is reduced to

$$\int_G (|\phi'|^2 + \phi^2) dx \leqslant (p-1)(1 + l)^{2(p-2)} \int_G \phi^2 dx. \tag{2.10}$$

Since $\int_G \phi dx = 0$, there exists a point $x_0 \in G$ such that $\phi(x_0) = 0$. For any $x \in G$, therefore [13, Remark 2.2] shows that

$$\int_G \phi^2 dx \leqslant l^2 \int_G |\phi'|^2 dx. \tag{2.11}$$

Thus, (2.10) and (2.11) show that

$$\left(l^{-2} + 1\right) \int_G \phi^2 dx \leqslant (p-1)(1+l)^{2(p-2)} \int_G \phi^2 dx. \tag{2.12}$$

Therefore, for l sufficiently small, $\phi \equiv 0$ and then $u = u_G$ is a positive constant. However, (1.3) has a unique positive constant solution 1. Hence $u \equiv 1$. □

We note that an explicit bound of l can be estimated from (2.12). For example, by using

$$\frac{1}{(l+1)^2} < l^{-2} + 1 \leqslant (p-1)(1+l)^{2(p-2)},$$

we obtain that

$$l \geqslant \left(\frac{1}{p-1}\right)^{1/(2(p-1))} - 1. \tag{2.13}$$

2.2. Rearrangement

The decreasing rearrangement u^* of a function $u \in H^1(G)$ for G being a metric graph was first used in [18]. As in the case of G being an interval, this kind of rearrangement does not increase the Dirichlet integral (see also [2,22]). Other than the decreasing rearrangement u^*, we will also need the symmetric rearrangement $\hat{u}$ which is defined below (see [5]).

Let $u \in H^1(G)$. Assume that

$$m = \inf_G |u| \geqslant 0, \quad M = \sup_G |u| > 0. \tag{2.14}$$

Let $\mu(\cdot)$ denote the distribution function of u:

$$\mu(t) = \sum_{e \in E} meas(\{x_e \in I_e : |u_e(x_e)| > t\}), \quad t \geqslant 0,$$

where the u_e is a branch of u, that is $u_e = u|_e$. Set

$$r = \sum_{e \in E} meas(I_e), \quad I^* = [0, r), \quad \hat{I} = (-r/2, r/2),$$

where $r \in [0, \infty]$ is the total length of G. One can define the following rearrangements of u:

1. the decreasing rearrangement $u^* : I^* \to \mathbb{R}$ as the function

$$u^*(x) = \inf\{t \geqslant 0 : \mu(t) \leqslant x\}, \ x \in I^*;$$

2. the symmetric decreasing rearrangement $\hat{u} : \hat{I} \to \mathbb{R}$ as the function

$$\hat{u}(x) = \inf\{t \geqslant 0 : \mu(t) \leqslant 2|x|\}, \ x \in \hat{I}.$$

If G is a noncompact graph, $I^* = [0, +\infty)$ and $\hat{I} = (-\infty, +\infty)$. Since $|u|$, u^* and $\hat{u}$ are all equimeasurable, one has that

$$\int_{I^*} |u^*(x)|^q dx = \int_{\hat{I}} |\hat{u}(x)|^q dx = \int_G |u(x)|^q dx, \quad q > 0, \tag{2.15}$$

and

$$\inf_{I^*} u^* = \inf_{\hat{I}} \hat{u} = \inf_G |u| = m, \ \ \sup_{I^*} u^* = \sup_{\hat{I}} \hat{u} = \sup_G |u| = M.$$

When G is a connected metric graph, it is known ([22]) that $u^* \in H^1(I^*)$ and $\hat{u} \in H^1(\hat{I})$ respectively. Moreover, $\widehat{su} = s\hat{u}$ for $s > 0$. In fact, we can easily see that $\mu_{su}(t) = \mu_u(t/s)$, and then

$$\begin{aligned} \widehat{su}(x) &= \inf\{t \geqslant 0 : \mu_{su}(t) \leqslant 2|x|\} = \inf\{t \geqslant 0 : \mu_u(t/s) \leqslant 2|x|\} \\ &= s \inf\{\tau \geqslant 0 : \mu_u(\tau) \leqslant 2|x|\} = s\hat{u}(x). \end{aligned} \tag{2.16}$$

Similarly we also have $(su)^* = su^*$.

Let

$$N(t) = \#\{x \in G : |u(x)| = t\}, \ \ t \in (m, M).$$

We have the following result regarding the Dirichlet integral and $N(t)$ (see Proposition 3.1 of [5]).

Proposition 2.5. *Let G be a connected metric graph, and let $u \in H^1(G)$ satisfying (2.14). Then*

$$\int_{I^*} |(u^*)'(x)|^2 dx \leqslant \int_G |u'(x)|^2 dx, \tag{2.17}$$

where equality holds only when $N(t) = 1$ for a.e. $t \in (m, M)$. Moreover, if $N(t) \geqslant 2$ for a.e. $t \in (m, M)$, then

$$\int_{\hat{I}} |(\hat{u})'(x)|^2 dx \leqslant \int_G |u'(x)|^2 dx, \tag{2.18}$$

where equality holds only when $N(t) = 2$ for a.e. $t \in (m, M)$.

Proof. By the Proposition 3.1 of [2], we only need to prove that

$$\int_G |u'|^2 dx = \int_G |(|u|)'|^2 dx.$$

In fact, let $u_+ = \max\{u, 0\}$, $u_- = \max\{-u, 0\}$, then $u = u_+ - u_-$ and $|u| = u_+ + u_-$. Moreover, $u_\pm \in H^1(G)$ and

$$\int_G |u'|^2 dx = \int_G ((u'_+)^2 + (u'_-)^2 - 2u'_+u'_-)dx = \int_G ((u'_+)^2 + (u'_-)^2)dx$$
$$= \int_G [(u_+ + u_-)']^2 dx = \int_G |(|u|)'|^2 dx. \quad \square$$

2.3. Basic estimates

First we establish the following basic energy estimate for any non-compact graph G.

Lemma 2.6. *Let E_0 be the ground state energy of (1.3) on $\mathbb{R}$. Suppose that G is a connected metric graph with at least one half-line, then the ground state energy $E(G)$ satisfies*

$$\frac{1}{2}E_0 \leqslant E(G) \leqslant E_0. \tag{2.19}$$

Proof. Let $\{u_n\} \subseteq H^1(\mathbb{R})$ be a sequence such that each u_n has a compact support and $u_n \to u_0$ strongly in $H^1(\mathbb{R})$. Since $u_n \to u_0$ also in $L^p(\mathbb{R})$ as $n \to \infty$, we see that

$$J(u_n, \mathbb{R}) \to J(u_0, \mathbb{R}) = E_0, \quad (J'(u_n, \mathbb{R}), u_n) \to (J'(u_0, \mathbb{R}), u_0) = 0, \quad n \to \infty.$$

Therefore, there exists a sequence $\{t_n\}$ satisfying $t_n u_n \in N(\mathbb{R})$ and $t_n \to 1$ as $n \to \infty$. By translation, we may assume that $t_n u_n(\cdot + x_n)$ is supported in $[0, \infty)$. Identifying this interval with one of the half-lines of G, we may consider $t_n u_n(\cdot + x_n)$ as a function in $H^1(G)$, by extending it to zero on any other edge of G. Then we obtain that

$$E(G) = \inf_{N(G)} J(u, G) \leqslant \lim_{n\to\infty} J(t_n u_n(\cdot + x_n), G) = \lim_{n\to\infty} J(t_n u_n, \mathbb{R}) = \lim_{n\to\infty} J(u_n, \mathbb{R}) = E_0.$$

On other hand, for any $u \in N(G)$, then $u^* \in H^1(\mathbb{R}^+)$ and there exists a $t > 0$ such that $tu^* \in N(\mathbb{R}^+)$. Therefore from Lemma 2.4,

$$J(u, G) \geqslant J(tu, G) \geqslant J(tu^*, \mathbb{R}^+) \geqslant \inf_{N(\mathbb{R}^+)} J(u, \mathbb{R}^+) = \frac{1}{2}E_0. \quad \square$$

Next we recall the following parametrization of a metric graph G (see [5, Lemma 5.1]).

Lemma 2.7. *Assume that G is connected and satisfies the condition* **(H)**. *Then G as a metric space satisfies the following condition as well:*

(H′) *For every point $x_0 \in G$, there exists two injective curves $\gamma_1, \gamma_2 : [0, \infty) \to G$ parameterized by arclength, with disjoint images except for finitely many points, and such that $\gamma_1(0) = \gamma_2(0) = x_0$.*

Similar to [5, Theorem 2.3], we show that under the condition **(H)**, the ground state energy $E(G)$ can be determined.

Lemma 2.8. *Assume that G is connected and satisfies condition the condition* **(H)**. *Then, $E(G) = E_0$.*

Proof. Let $u \in N(G)$. Since $J(|u|, G) = J(u, G)$, we may assume that $u \geqslant 0$. Therefore we have that $M = \max_G u > 0$ and that $m = \inf_G u = 0$, as G contains at least two half-lines. Thus, $\hat{u} \in H^1(\mathbb{R})$. We claim that

$$N(t) = \#\{x \in G : |u(x)| = u(x) = t\} \geqslant 2 \text{ for a.e. } t \in (0, M). \tag{2.20}$$

Indeed from Lemma 2.7, the condition **(H′)** holds. Let γ_1, γ_2 be as in **(H′)**, relative to a point $x_0 \in G$ where $u(x_0) = M$. We define a continuous function $v : \mathbb{R} \to \mathbb{R}$ by

$$v(y) = \begin{cases} u(\gamma_1(y)), & y \geqslant 0, \\ u(\gamma_2(-y)), & y < 0. \end{cases} \tag{2.21}$$

Clearly $v(0) = u(x_0) = M$. Moreover, as each γ_i parameterizes a half-line of G for y large enough, we have that $v(y) \to 0$ as $|y| \to \infty$. Hence v has at least two distinct preimages for every value $t \in (0, M)$. Since the images of γ_1 and γ_2 are disjoint except for finitely many points of G, then (2.20) follows.

For any $u \in N(G)$, from Lemma 2.4, there exists a $t_0 > 0$ such that $t_0\hat{u} \in N(\mathbb{R})$. Therefore from Lemma 2.4 and Proposition 2.5, we have

$$E_0 = \inf_{N(\mathbb{R})} J(u, \mathbb{R}) \leqslant J(t_0\hat{u}, \mathbb{R}) = J(\widehat{t_0 u}, \mathbb{R}) \leqslant J(t_0 u, G) \leqslant J(u, G).$$

Thus $E_0 = J(u_0, \mathbb{R}) = \inf_{N(\mathbb{R})} J(u, \mathbb{R}) \leqslant \inf_{N(G)} J(u, G) = E(G)$. Combining with Lemma 2.6, we conclude that $E(G) = E_0$. □

Now we prove Theorem 1.2.

Proof of Theorem 1.2. The proof is similar to the one for [5, Theorem 2.5]. The graph in Fig. 1 (a) is apparently isometric to $\mathbb{R}$, then u_0 can be seen as an element of $H^1(G)$, and then the infimum is achieved. For the graph G in Fig. 1 (b), suppose the length of finite loop is $2a > 0$, then u_0 can be identified as an element of $H^1(G)$ by letting $x = 0$ corresponding to the mid-point of the finite loop, and $x = \pm a$ corresponding to x_1. Again the infimum is achieved. For the graph G in Fig. 1 (c), the two edges between each x_i and x_{i+1} are of equal length, hence u_0 can also be identified as an element of $H^1(G)$ for which the two half-lines intersect at $x = \pm a_i$ for some $a_n > a_{n-1} > \cdots > a_1 > 0$, then the infimum is attained by u_0 under this correspondence.

On the other hand, assume that $u \in H^1(G)$ achieves the infimum $E(G)$. Then from Lemma 2.8, $E(G) = E_0$, (2.20) holds and we may assume that $u > 0$ and $J(u, G) = J(\hat{u}, \mathbb{R})$. Thus we must have $\hat{u} = u_0$, and $G = \Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_i is the image of the curve γ_i (defined as in Lemma 2.7). The remaining parts of the proof are same as the proof of [5, Theorem 2.5], which are omitted. □

3. Existence of ground states

In this section, we prove our main existence results (Theorems 1.3 and 1.5). First we show the following partial symmetry result for a positive solution u with energy level less than or equal to E_0.

Lemma 3.1. *Assume that G is a star graph with m infinite edges (half-lines) and k finite edges (with possibly unequal length), $m \geqslant 2$ and $m + k \geqslant 3$. If $u \in H^1(G)$ is a positive solution of (1.3) with $J(u, G) \leqslant E_0$, then u is symmetric on all m half-lines and it is strictly decreasing on $[0, \infty)$ for each half-line.*

Proof. Since u is a positive solution of (1.3), then $u \in N(G)$ and we have

$$\int_G [|u'(x)|^2 + |u(x)|^2] dx = \int_G |u(x)|^p dx. \tag{3.1}$$

Let $G = \mathcal{S}_{k,m}$ and let $u = (u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m) \in H^1(G)$ be a positive solution of (1.3), then $v_i \in H^1(\mathbb{R}_+)$ $(1 \leqslant i \leqslant m)$ is a positive solution of the equation

$$\begin{cases} -u'' + u = u^{p-1}, & x \in \mathbb{R}_+, \\ u > 0, \ u \in H^1(\mathbb{R}_+), \end{cases} \tag{3.2}$$

and $v_i(0) = v_j(0) > 0$. From the uniqueness of the positive solution of (1.7), we must have $v_i(x) = u_0(x \pm x_1)$ where $x_1 > 0$ satisfies $u_0(x_1) = v_i(0)$ for each i.

We prove that $v_i(x) = u_0(x+x_1)$ for each i. Suppose to the contrary, there exists i (without loss generality, we assume $i = 1$) such that $v_i(x) = u_0(x - x_1)$. Since $m \geqslant 2$, then $v_2(x) = u_0(x \pm x_1)$. If $v_2(x) = u_0(x+x_1)$, then from (3.1), we have

$$\begin{aligned} J(u,G) &= (\frac{1}{2} - \frac{1}{p}) \int_G [|u'(x)|^2 + |u(x)|^2] dx \\ &> (\frac{1}{2} - \frac{1}{p}) \int_0^\infty [|v_1'(x)|^2 + |v_1(x)|^2] dx + (\frac{1}{2} - \frac{1}{p}) \int_0^\infty [|v_2'(x)|^2 + |v_2(x)|^2] dx \\ &= (\frac{1}{2} - \frac{1}{p}) \int_{-x_1}^\infty [|u_0'(x)|^2 + |u_0(x)|^2] dx + (\frac{1}{2} - \frac{1}{p}) \int_{x_1}^\infty [|u_0'(x)|^2 + |u_0(x)|^2] dx = E_0, \end{aligned} \tag{3.3}$$

which contradicts with the assumption that $J(u,G) \leqslant E_0$. Note that the inequality in (3.3) is strict since $m + k \geqslant 3$. On the other hand, if $v_2(x) = u_0(x - x_1)$, then similar to (3.3), we have

$$\begin{aligned} J(u,G) &= (\frac{1}{2} - \frac{1}{p}) \int_G [|u'(x)|^2 + |u(x)|^2] dx \\ &> (\frac{1}{2} - \frac{1}{p}) \int_0^\infty [|v_1'(x)|^2 + |v_1(x)|^2] dx + (\frac{1}{2} - \frac{1}{p}) \int_0^\infty [|v_2'(x)|^2 + |v_2(x)|^2] dx \\ &= (\frac{1}{2} - \frac{1}{p}) \int_{-x_1}^\infty [|u_0'(x)|^2 + |u_0(x)|^2] dx + (\frac{1}{2} - \frac{1}{p}) \int_{-x_1}^\infty [|u_0'(x)|^2 + |u_0(x)|^2] dx > E_0, \end{aligned} \tag{3.4}$$

which again contradicts with $J(u,G) \leqslant E_0$. Hence for each $1 \leqslant i \leqslant m$, we have $v_i(x) = u_0(x + x_1)$. This implies that $v_i(x) = v_j(x)$ and are decreasing on $[0, \infty)$. □

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. 1. Suppose that u is a positive ground state solution of (1.3) when $G = \mathcal{S}_{0,m}$, the star graph with no finite edges but m infinite edges. From Lemma 2.6, $E(G) \leqslant E_0$. Suppose $m \geqslant 3$, then according to Lemma 3.1, for each $1 \leqslant i \leqslant m$, $v_i(x) = u_0(x + x_1)$ for some $x_1 > 0$. The Kirchhoff condition implies that $x_1 = 0$ hence $v_i(x) = u_0(x)$ for all $i = 1, 2, \cdots, m$. Similar to (3.3), we have

$$\begin{aligned} J(u,G) &= (\frac{1}{2} - \frac{1}{p}) \int_G [|u'(x)|^2 + |u(x)|^2] dx \\ &= (\frac{1}{2} - \frac{1}{p}) \sum_{i=1}^m \int_0^\infty [|v_i'(x)|^2 + |v_i(x)|^2] dx = \frac{m}{2} E_0 > E_0. \end{aligned} \tag{3.5}$$

That contradicts with $J(u,G) = E(G) \leqslant E_0$. Hence there is no positive ground state solution of (1.3) when $m \geqslant 3$. When $m = 1$ or $m = 2$, a ground state obviously exists: $u = u_0|_{\mathbb{R}_+}$ for $m = 1$, and $u = u_0$ for $m = 2$.

2. Suppose that u is a positive solution of (1.3) when $G = \mathcal{S}_{0,m}$ with m odd. Then similar to the proof of Lemma 3.1, $u = (v_1, \cdots, v_m)$ and each v_i is a solution of (3.2) such that $v_i(0) = v_j(0) > 0$ for all i, j. Then again $v_i(x) = u_0(x \pm x_1)$ which implies that

$$\sum_{i=1}^{m} \frac{dv_i}{dx_i}(0) = pu_0'(x_1), \tag{3.6}$$

where p is an odd integer. Then the Kirchhoff condition implies that $u_0'(x_1) = 0$ hence $x_1 = 0$. Therefore u equals to $u_0|_{\mathbb{R}_+}$ on each edge, and u is unique.

3. Suppose that u is a positive solution of (1.3) when $G = \mathcal{S}_{0,m}$ with $m = 2m_1$ even. Then same as part 2, we have $v_i(x) = u_0(x \pm x_1)$ and (3.6) holds for $p = 0$. That implies that exactly m_1 of $v_i(x) = u_0(x + x_1)$, and the other m_1 satisfying $v_i(x) = u_0(x - x_1)$. So the $2m_1$ edges can be paired into m_1 whole solitons with an arbitrary $x_1 > 0$. □

Remark 3.2. In Lemma 3.1, we show that a positive solution with energy $\leqslant E_0$ must be symmetric on all infinite edges. The results in Theorem 1.3 parts 2 and 3 show that the energy constraint is a necessary condition for the symmetry.

The next lemma is a key of proving Theorem 1.5.

Lemma 3.3. *Assume that G is a star graph with m infinite edges (half-lines) and k finite edges (with possibly unequal length), $m \geqslant 1$ and $k \geqslant 1$. If $E(G) < E_0$, then $E(G)$ can be attained by a positive solution of (1.3).*

Proof. Suppose that $G = \mathcal{S}_{k,m}$ with $m \geqslant 1$ and $k \geqslant 1$. Let $u_n \in N(G)$ be a minimizing sequence such that $J(u_n, G) \to E(G)$ as $n \to \infty$. Let $u_n = (u_n^1, u_n^2, \cdots, u_n^k, v_n^1, v_n^2, \cdots, v_n^m)$, where u_n^i is defined on the i-th finite edge for $1 \leqslant i \leqslant k$, and v_n^j is defined on the j-th half-line for $1 \leqslant j \leqslant m$.

Define

$$M(u, G) = (J'(u, G), u) = \|u\|^2_{H^1(G)} - |u|^p_{L^p(G)}.$$

Since $u_n \in N(G)$, then $M(u_n, G) = 0$ and

$$\left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|^2_{H^1(G)} = J(u_n, G) - \frac{1}{p} M(u_n, G) \to E(G), \quad n \to \infty,$$

which implies that $\{u_n\}$ is bounded in $H^1(G)$. Moreover, there exists a sequence $\lambda_n \in \mathbb{R}$ such that

$$J'(u_n, G) - \lambda_n M'(u_n, G) \to 0, \quad n \to \infty.$$

Since $u_n \in N(G)$, then by the proof of Lemma 2.4, we have

$$(M'(u_n, G), u_n) = 2\|u_n\|^2_{H^1(G)} - p|u_n|^p_{L^p(G)} = (2 - p)\|u_n\|^2_{H^1(G)} \leqslant -(p - 2)C^2.$$

Hence $\lambda_n \to 0$ and consequently $J'(u_n, G) \to 0$ as $n \to \infty$.

Therefore, subject to a subsequence, there exists a function $u \in H^1(G)$ such that

$$u_n \rightharpoonup u, \text{ in } H^1(G); \quad u_n \to u, \text{ in } L^q_{loc}(G), \ q \in [1, +\infty], \quad n \to \infty. \tag{3.7}$$

If $u \neq 0$, then $u \in N(G)$ and

$$J(u,G) = \frac{1}{2}\|u\|^2_{H^1(G)} - \frac{1}{p}|u|^p_{L^p(G)} = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2_{H^1(G)}$$
$$\leqslant \liminf_{n\to\infty}\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2_{H^1(G)} = \liminf_{n\to\infty} J(u_n,G) = E(G).$$

Hence $E(G)$ is attained by $u \neq 0$.

In the following, we consider the case of $u \equiv 0$. Then (3.7) implies that $u_n \rightharpoonup 0$ in $H^1(G)$ and $u_n \to 0$ in $L^q_{loc}(G)$ for $q \in [1,\infty]$ as $n \to \infty$. Let $\xi_R, \eta_R \in C^\infty(\mathbb{R}_+)$ be defined by

$$\xi_R(x) = \begin{cases} 1, & x > 2R, \\ 0, & x < R, \end{cases} \qquad \eta_R(x) = 1 - \xi_R(x).$$

Moreover, we may assume that $|\xi'_R|, |\eta'_R| \leqslant 2/R$, $0 \leqslant \xi_R, \eta_R \leqslant 1$ for $x \in \mathbb{R}_+$. Let l be the maximum of the lengths of all finite edges, and we fix $R > l$. Then

$$u_n\eta_R = (u^1_n, u^2_n, \cdots, u^k_n, v^1_n\eta_R, v^2_n\eta_R, \cdots, v^m_n\eta_R), \quad u_n\xi_R = (0, 0, \cdots, 0, v^1_n\xi_R, v^2_n\xi_R, \cdots, v^m_n\xi_R).$$

Clearly $u_n\eta_R, u_n\xi_R \in H^1(G)$ and $\{u_n\eta_R\}, \{u_n\xi_R\}$ are bounded in $H^1(G)$. From $J'(u_n,G) \to 0$ as $n \to \infty$, we have

$$\int_G (|u'_n|^2\eta_R + \eta'_R u'_n u_n + |u_n|^2\eta_R - |u_n|^p\eta_R)dx = (J'(u_n,G), \eta_R u_n) \to 0, \quad n \to \infty. \tag{3.8}$$

By using $u_n \to 0$ in $L^q_{loc}(G)$ $(n \to \infty)$, we have

$$\begin{aligned} \left|\int_G u'_n u_n \eta'_R dx\right| &\leqslant \frac{2}{R}\int_{x\in[0,2R]} |u'_n u_n| \leqslant \frac{2}{R}\left(\int_{x\in[0,2R]} |u_n|^2\right)^{\frac{1}{2}} \|u_n\|_{H^1(G)} \to 0, \quad \text{as } n \to \infty, \\ \left|\int_G |u_n|^2\eta_R - |u_n|^p\eta_R\right| &\leqslant \int_{x\in[0,2R]} (|u_n|^2 + |u_n|^p) \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we obtain that

$$\int_G |u'_n|^2\eta_R \to 0, \quad \text{and} \quad \int_{x\in[0,2R]} (|u'_n|^2 + |u_n|^2) \to 0 \text{ as } n \to \infty. \tag{3.10}$$

Because of $u_n \in N(G)$, we have

$$\begin{aligned} 0 = \langle J'(u_n,G), u_n\rangle &= \int_G (|u'_n|^2 + u_n^2 - |u_n|^p)dx \\ &= \int_{x>2R} (|u'_n|^2 + |u_n|^2 - |u_n|^p)dx + \int_{x\in[0,2R]} (|u'_n|^2 + |u_n|^2 - |u_n|^p)dx. \end{aligned}$$

This together with (3.10) and $u_n \to 0$ in $L^p_{\text{loc}}(G)$ implies that

$$\int_{x>2R} (|u'_n|^2 + |u_n|^2 - |u_n|^p)dx \to 0, \quad n \to \infty. \tag{3.11}$$

On the other hand, by (3.10) and $u_n \to 0$ in $L^p_{\text{loc}}(G)$, we obtain that

$$\begin{aligned}&\left|\int\limits_{x\in[0,2R]} (|u_n'|^2|\xi_R|^2 + 2u_n' u_n \xi_R \xi_R' + |u_n|^2|\xi_R'|^2 + |u_n|^2\xi_R^2 - |u_n|^p\xi_R^p)\right| \\ &\leqslant \int\limits_{x\in[0,2R]} \left(2|u_n'|^2 + (3+4R^{-2})|u_n|^2 + |u_n|^p\right) \to 0, \quad n\to\infty.\end{aligned} \tag{3.12}$$

Therefore, from (3.12) and (3.11), we obtain that

$$\begin{aligned}&(J'(u_n\xi_R, G), u_n\xi_R) \\ &= \int\limits_G (|u_n'|^2|\xi_R|^2 + 2u_n' u_n \xi_R \xi_R' + |u_n|^2|\xi_R'|^2 + |u_n|^2\xi_R^2 - |u_n|^p\xi_R^p) \\ &= \int\limits_{x>2R} (|u_n'|^2 + |u_n|^2 - |u_n|^p) \\ &\quad + \int\limits_{x\in[0,2R]} (|u_n'|^2|\xi_R|^2 + 2u_n' u_n \xi_R \xi_R' + |u_n|^2|\xi_R'|^2 + |u_n|^2\xi_R^2 - |u_n|^p\xi_R^p) \\ &= \int\limits_{x>2R} (|u_n'|^2 + |u_n|^2 - |u_n|^p) + o(1) = o(1), \quad n\to\infty.\end{aligned} \tag{3.13}$$

Similarly,

$$\begin{aligned}J(u_n\xi_R, G) &= \int\limits_G (\frac{1}{2}|u_n'|^2|\xi_R|^2 + u_n' u_n \xi_R \xi_R' + \frac{1}{2}|u_n|^2|\xi_R'|^2 + \frac{1}{2}|u_n|^2\xi_R^2 - \frac{1}{p}|u_n|^p\xi_R^p) \\ &= \int\limits_{x>2R} (\frac{1}{2}|u_n'|^2 + \frac{1}{2}|u_n|^2 - \frac{1}{p}|u_n|^p) \\ &\quad + \int\limits_{x\in[0,2R]} (\frac{1}{2}|u_n'|^2|\xi_R|^2 + u_n' u_n \xi_R \xi_R' + \frac{1}{2}|u_n|^2|\xi_R'|^2 + \frac{1}{2}|u_n|^2\xi_R^2 - \frac{1}{p}|u_n|^p\xi_R^p) \\ &= \int\limits_{x>2R} (\frac{1}{2}|u_n'|^2 + \frac{1}{2}|u_n|^2 - \frac{1}{p}|u_n|^p) + o(1), \quad n\to\infty.\end{aligned} \tag{3.14}$$

On the other hand, we have $J(u_n, G) \to E(G)$ as $n \to \infty$, and

$$\begin{aligned}J(u_n, G) &= \int\limits_G (\frac{1}{2}|u_n'|^2 + \frac{1}{2}|u_n|^2 - \frac{1}{p}|u_n|^p) \\ &= \int\limits_{x>2R} (\frac{1}{2}|u_n'|^2 + \frac{1}{2}|u_n|^2 - \frac{1}{p}|u_n|^p)dx + \int\limits_{x\in[0,2R]} (\frac{1}{2}|u_n'|^2 + \frac{1}{2}|u_n|^2 - \frac{1}{p}|u_n|^p)dx \\ &= \int\limits_{x>2R} (\frac{1}{2}|u_n'|^2 + \frac{1}{2}|u_n|^2 - \frac{1}{p}|u_n|^p)dx + o(1), \quad n\to\infty.\end{aligned} \tag{3.15}$$

Now combining (3.14), (3.15), we obtain that

$$\lim_{n\to\infty} J(u_n\xi_R, G) = E(G). \tag{3.16}$$

From $u_n\xi_R \in H^1(G)$, there exists $t_n > 0$ such that $t_n u_n\xi_R \in N(G)$. By (3.13) and (3.16), we must have $t_n \to 1$.

If $m \geqslant 2$, then $t_n u_n \xi_R \in H^1(\mathcal{S}_{0,m})$ and $t_n u_n \xi_R \in N(\mathcal{S}_{0,m})$. Denote $v_n = t_n u_n \xi_R \in N(\mathcal{S}_{0,m})$. Then $J(v_n, \mathcal{S}_{0,m}) \to E(G)$ as $n \to \infty$. Therefore, according to Lemma 2.8, we have

$$E_0 = E(\mathcal{S}_{0,m}) \leqslant J(v_n, \mathcal{S}_{0,m}) = J(v_n, G) = E(G) + o(1).$$

This is a contradiction with the assumption $E(G) < E_0$.

If $m = 1$, then $v_n \in H^1(\mathbb{R})$ and then $v_n \in N(\mathbb{R})$. Hence,

$$E_0 = E(\mathbb{R}) \leqslant J(v_n, \mathbb{R}) = J(v_n, G) = E(G) + o(1).$$

This is also a contradiction with the assumption $E(G) < E_0$. Therefore, $u \neq 0$ and the proof is completed. □

To prove the existence of ground state solution for $G = \mathcal{S}_{k,2}$ with $k = 1$, we recall the following hybrid rearrangement from [5], and here the conclusion (iv) is new.

Lemma 3.4 *(Hybrid rearrangement). Let $G = \mathcal{S}_{1,2}$ be the star graph with 1 finite edge and 2 half-lines. Assume that $u = (u_1, v_1, v_2) \in H^1(G)$ such that $u > 0$ and $meas(\{u = t\}) = 0$ for every $t > 0$. Then there exists $\tilde{u} \in H^1(G)$, $\tilde{u} = (\tilde{u}_1, \tilde{v}_1, \tilde{v}_2)$ with the following properties:*

- (i) *$\tilde{v}_1 = \tilde{v}_2$, and it is decreasing on $(0, \infty)$.*
- (ii) *$\tilde{u}_1 : [-l, 0] \to \mathbb{R}$ is decreasing so that $\min\limits_{x\in[-l,0]} \tilde{u}_1(x) = \tilde{u}_1(0) = \tilde{v}_i(0) = \max\limits_{x\in[0,\infty)} \tilde{v}_i(x)$, $i = 1, 2$.*
- (iii) *$J(\tilde{u}, G) \leqslant J(u, G)$, and the equality holds only if u_1 is decreasing on $[-l, 0]$ and $\min\limits_{x\in[-l,0]} u_1(x) = \max\limits_{x\in[0,\infty)} v_i(x)$ for $i = 1, 2$.*
- (iv) *For $t > 0$, $\widetilde{tu} = t\tilde{u}$.*

Proof. The proof of (i)–(iii) is the same as the one of [5, Lemma 6.1], and only (iv) needs to be proved. Let $\widetilde{u}$ be defined as [5, Lemma 6.1]. Then according to the definition of $\widetilde{u}$, (2.16), and the fact that $(su)^* = su^*$, the conclusion (iv) holds. □

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. 1. First we prove that $E(\mathcal{S}_{1,2}) < E_0$. Then from Lemma 3.3, a positive ground state exists. We define a function $u = (u_1, v_1, v_2) \in H^1(\mathcal{S}_{1,2})$ as follows:

$$v_1(x) = v_2(x) = u_0(x + l/2),\ x \in [0, \infty), \quad u_1(x) = u_0(x + l/2),\ x \in [-l, 0].$$

Then $u = (u_1, v_1, v_2) \in H^1(\mathcal{S}_{1,2})$ and $J(u, \mathcal{S}_{1,2}) = J(u_0, \mathbb{R}) = E_0$. We can also compute that $u \in N(\mathcal{S}_{1,2})$. Moreover, $u > 0$ and $meas(\{u = t\}) = 0$ for every $t > 0$. Then by Lemma 3.4, we have $\tilde{u} \in H^1(\mathcal{S}_{1,2})$.

From Lemma 3.4 (iii), we have $J(\tilde{u}, \mathcal{S}_{1,2}) < J(u, \mathcal{S}_{1,2}) = E_0$, and there exists a constant $t > 0$ such that $t\tilde{u} \in N(G)$. Therefore from Lemma 3.4, we have

$$E(\mathcal{S}_{1,2}) \leqslant J(t\tilde{u}, \mathcal{S}_{1,2}) < J(tu, \mathcal{S}_{1,2}) < J(u, \mathcal{S}_{1,2}) = E_0.$$

Then from Lemma 3.3, the infimum $E(\mathcal{S}_{1,2})$ is attained.

2. We prove that $E(\mathcal{S}_{2,2})$ can be attained. From Lemma 2.6, we know that $E(\mathcal{S}_{2,2}) \leqslant E_0$. If $E(\mathcal{S}_{2,2}) = E_0$, then we define

$$v_1(x) = v_2(x) = u_0(x + l),\ x \in [0, \infty), \quad u_1(x) = u_2(x) = u_0(x + l),\ x \in [-l, 0].$$

Therefore, $u = (u_1, u_2, v_1, v_2) \in H^1(G)$, u is a critical point of $J(\cdot, G)$ and $J(u, G) = J(u_0, \mathbb{R}) = E_0$. If $E(\mathcal{S}_{2,2}) < E_0$, then it follows from Lemma 3.3 that $E(\mathcal{S}_{2,2})$ is achieved by some $u \in H^1(\mathcal{S}_{2,2})$. □

4. Symmetric solutions

In this section we prove the existence of a positive solution of (1.3) for a symmetric star graph $\mathcal{S}_{k,m}$ with $k \geqslant 1$ and $m \geqslant 1$, which also implies the existence of a ground state solution for $\mathcal{S}_{1,m}$ with a sufficiently long finite edge.

Recall that u_0 is the unique positive solution of (1.7). Multiplying (1.7) by u_0' and integrating on $[0, \infty)$, we obtain that

$$0 = \int_0^\infty (u_0'' - u_0 + u_0^{p-1}) u_0' dx = -\frac{1}{2} u_0^2(0) + \frac{1}{p} u_0^p(0).$$

Hence

$$u_0(0) = (p/2)^{1/(p-2)} = \theta > 1.$$

We recall some basic facts about the solutions of the second order nonlinear ODE $u'' - u + u^{p-1} = 0$. Suppose that u is the solution of initial value problem

$$\begin{cases} u'' - u + u^{p-1} = 0, \\ u(0) = \overline{u_0},\ u'(0) = \overline{w_0}. \end{cases} \tag{4.1}$$

Let $w(x) = u'(x)$. Then $(u(x), w(x))$ is the solution of

$$\begin{cases} u' = w, \\ w' = u - u^{p-1}, \\ u(0) = \overline{u_0},\ w(0) = \overline{w_0}. \end{cases} \tag{4.2}$$

Note that (4.2) is a first order Hamiltonian ODE system with a Hamiltonian

$$H(u, w) = \frac{1}{2} w^2 - \frac{1}{2} u^2 + \frac{1}{p} u^p. \tag{4.3}$$

Hence for a solution (u, w) of (4.2),

$$\frac{d}{dx} H(u(x), w(x)) = \frac{\partial H}{\partial u} u' + \frac{\partial H}{\partial w} w' = 0.$$

In particular, $H(u(x), w(x)) \equiv H(u(0), w(0))$ for all $x > 0$.

Now we consider a solution of (4.1) or (4.2) with $\overline{w_0} = 0$. Multiplying (4.1) by u' and integrating on $[0, x]$, we obtain that

$$0 = \int_0^x [u''u' + (-u + u^{p-1})u']dy = \frac{1}{2}[u'(x)]^2 + g(u(x)) - g(u(0)), \tag{4.4}$$

where $g(u) = -\frac{1}{2}u^2 + \frac{1}{p}u^p = \int_0^u (-v + v^{p-1})dv$.

We consider a solution u of (4.1) satisfying $u'(0) = 0$, $u'(x) < 0$ for $x > 0$. Then (4.4) implies that

$$\frac{du}{dx} = u'(x) = -\sqrt{2} \cdot \sqrt{g(\overline{u_0}) - g(u(x))},\ x > 0$$

or

$$\frac{du}{\sqrt{2} \cdot \sqrt{g(\overline{u_0}) - g(u)}} = -dx,\ x > 0. \tag{4.5}$$

Integrating (4.5) for $x \in [0, x_1]$, we have

$$x_1 = \frac{1}{\sqrt{2}} \int_{u(x_1)}^{\overline{u_0}} \frac{1}{\sqrt{g(\overline{u_0}) - g(u)}} du := G(\overline{u_0}). \tag{4.6}$$

Here we assume that for $x \in [0, x_1], g(\overline{u_0}) > g(u(x))$. The quantity x_1 can be viewed as the "time" needed for a solution of (4.2) moving from $(u(0), w(0)) = (\overline{u_0}, 0)$ to $(u(x_1), w(x_1)) = (u(x_1), u'(x_1))$. Hence $G(\overline{u_0})$ is often called the time-mapping [31,37]. Note that here $(x_1, u(x_1))$ is arbitrary.

The following lemma of ODE shooting argument is a key in establishing our existence result of a symmetric solution of (1.3) on a symmetric star graph.

Lemma 4.1. *Suppose $x_2 > 0$ and u_0 is the unique positive solution of (1.7). Let $(P, Q) = (u_0(x_2), u_0'(x_2))$, and $k > 0$. Then*

1. *there exists a unique $\tilde{u} = \tilde{u}(x_2) > 0$ and $l = l(x_2) > 0$ such that*

$$\begin{cases} u' = w, \\ w' = u - u^{p-1}, \\ u(0) = \tilde{u},\ w(0) = 0, \\ u(l) = P,\ w(l) = kQ, \end{cases} \tag{4.7}$$

has a solution (u, w) such that $u(x) > 0$ and $w(x) < 0$ for $x \in (0, l)$.

2. *$\tilde{u}$ and l are continuous in $x_2 \in (0, \infty)$, and they satisfy*

$$\lim_{x_2 \to 0^+} \tilde{u}(x_2) = \theta, \quad \lim_{x_2 \to 0^+} l(x_2) = 0,$$
$$\lim_{x_2 \to \infty} \tilde{u}(x_2) = \theta, \quad \lim_{x_2 \to \infty} l(x_2) = \infty.$$

Proof. 1. Fix $x_2 > 0$ and let $(P, Q) = (u_0(x_2), u_0'(x_2))$. Then the solution orbit of (4.7) with $(u(0), w(0)) = (P, kQ)$ is on the curve

$$H(u, w) = H(P, kQ) = \frac{k^2}{2}Q^2 - \frac{1}{2}P^2 + \frac{1}{p}P^p. \tag{4.8}$$

We claim that the curve $H(u, w) = H(P, kQ)$ intersects with $w = 0$. Indeed, $g(P) = -\frac{1}{2}P^2 + \frac{1}{p}P^p < H(P, kQ)$ and $\lim_{u\to\infty} g(u) = \infty > H(P, kQ)$. From the intermediate-value Theorem, there exists $\tilde{u} \in (P, \infty)$ such that $g(\tilde{u}) = H(P, kQ)$, which implies that $H(\tilde{u}, 0) = H(P, kQ)$. From the phase portrait of (4.7), such a $\tilde{u} \in (P, \infty)$ is unique. Hence $\tilde{u}$ is uniquely determined by x_2, and the time l is also uniquely determined by x_2.

2. The continuity of $\tilde{u}(x_2)$ and $l(x_2)$ easily follows from the continuous differentiability of u_0 on x_2. When $x_2 \to 0^+$, $(P, kQ) = (u_0(x_2), ku_0'(x_2)) \to (\theta, 0)$. From $H(\tilde{u}(x_2), 0) = H(P, kQ)$, we have

$$g(\tilde{u}(x_2)) = \frac{k^2}{2}Q^2 + g(P) \to g(\theta) = 0, \quad x_2 \to 0^+.$$

Since $\tilde{u}(x_2) > P > 1$, then $\tilde{u}(x_2) \to \theta$ as $x_2 \to 0^+$. On the other hand, when $x_2 \to \infty$, $(P, kQ) = (u_0(x_2), ku_0'(x_2)) \to (0, 0)$. Hence,

$$g(\tilde{u}(x_2)) \to g(0) = 0, \quad x_2 \to \infty.$$

From the phase portrait of (4.2), $\tilde{u}(x_2) > 1$, hence $\tilde{u}(x_2) \to \theta$ as $x_2 \to \infty$. Next we determine the asymptotic behavior of $l(x_2)$. From (4.6), we have

$$l(x_2) = \frac{1}{\sqrt{2}} \int_{u_0(x_2)}^{\tilde{u}(x_2)} \frac{1}{\sqrt{g(\tilde{u}(x_2)) - g(u)}} du. \tag{4.9}$$

Since $u_0(x_2) \to \theta$ and $\tilde{u}(x_2) \to \theta$ as $x_2 \to 0^+$, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that when $0 < x_2 < \delta$, we have

$$|u_0(x_2) - \theta| < \varepsilon, \quad |\tilde{u}(x_2) - \theta| < \varepsilon.$$

From the mean-value theorem, there exists $\xi \in (u, \tilde{u}(x_2))$ such that

$$g(\tilde{u}(x_2)) - g(u) = g'(\xi)(\tilde{u}(x_2) - u).$$

Hence when $0 < x_2 < \delta$,

$$\begin{aligned} 0 < l(x_2) &\leqslant \frac{1}{\sqrt{2}} \frac{1}{\min_{|\xi-\theta|<\varepsilon} \sqrt{g'(\xi)}} \int_{u_0(x_2)}^{\tilde{u}(x_2)} \frac{1}{\sqrt{\tilde{u}(x_2) - u}} du \\ &\leqslant \sqrt{2} \frac{1}{\min_{|\xi-\theta|<\varepsilon} \sqrt{g'(\xi)}} \sqrt{\tilde{u}(x_2) - u_0(x_2)} \to 0, \text{ as } x_2 \to 0. \end{aligned}$$

On the other hand, when $x_2 \to \infty$,

$$u_0(x_2) \to 0, \quad \tilde{u}(x_2) \to \theta.$$

Thus (4.9) implies that

$$\lim_{x_2\to\infty} l(x_2) = \frac{1}{\sqrt{2}}\int_0^\theta \frac{1}{\sqrt{g(\theta)-g(u)}}du = \infty,$$

since the solution of (4.2) with $u(0) = \theta$ and $w(0) = 0$ is a homoclinic orbit. □

Now we prove the existence of a symmetric positive solution of (1.3) on the star graph $\mathcal{S}_{k,m}$ with k finite edges with equal length l and m half-lines.

Proof of Theorem 1.6. To construct a symmetric positive solution of (1.3), from the proof of Lemma 3.1, we must have $v(x) = u_0(x + x_2)$ or $v(x) = u_0(x - x_2)$ for some $x_2 > 0$. Here we take $v(x) = u_0(x + x_2)$. Then u satisfies

$$\begin{cases} u'' - u + u^{p-1} = 0, & -l < x < 0, \\ u'(-l) = 0,\ u(0) = u_0(x_2),\ u'(0) = \dfrac{m}{k}u_0'(x_2). \end{cases} \tag{4.10}$$

From Lemma 4.1, for any $l > 0$, (4.10) has a positive solution $u(x)$ such that $u'(x) < 0$ in $(-l, 0)$. Hence $(u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m)$ satisfying $u_1 = u_2 = \cdots = u_k = u$ and $v_1 = v_2 = \cdots v_m = v$ is a symmetric positive solution of (1.3), and $u'(x) < 0$ for $-l < x < 0$, $v'(x) < 0$ for $x > 0$. For the uniqueness of positive decreasing solution of (1.10), first $v(x)$ must be some $u_0(x + x_2)$ for $x_2 > 0$, thus $u(x)$ must satisfy (4.10). For fixed x_2, l, k, m, the decreasing solution of (4.10) is unique from the part 1 of Lemma 4.1. Hence (u, v) is uniquely determined by (l, k, m). □

Note that the positive solution of (1.10) in Theorem 1.6 is strictly decreasing in $(-l, \infty)$. It is possible to have solutions of (1.10) which are not decreasing in $(-l, \infty)$. Such a solution may be periodic in $(-l, 0)$ and may have a unique maximal point in $(0, \infty)$. Also a positive solution of (1.3) on $\mathcal{S}_{k,m}$ may be not symmetric. Hence the uniqueness of positive solution of (1.3) holds only in the decreasing functions. Finally we use the result of Theorem 1.6 to prove Corollary 1.7.

Proof of Corollary 1.7. Suppose that $k = 1$, $m \geqslant 2$, and let $z = (u, v)$ be the solution obtained in Theorem 1.6 for $G = \mathcal{S}_{1,m}$. From the proof of Lemma 4.1, $u(x)$ satisfies (4.10). When $l \to \infty$, then from part 2 of Lemma 4.1 the corresponding x_2 also tends to ∞. Indeed if x_2 is bounded, then $l(x_2)$ is also bounded. From part 2 of Lemma 4.1, the corresponding $\tilde{u}(x_2) = u(-l(x_2)) \to \theta$. Let $u_l(x)$ be the unique positive solution of (4.10) for a given $l > 0$. Then the above argument shows that $u_l(-l) \to \theta$ as $l \to \infty$. Define $w_l(y) = u_l(y - l)$ for $y \in [0, l]$. Since $w_l(y)$ satisfies the equation $w_l'' - w_l + w_l^{p-1} = 0$ on $y \in (0, l)$, $w_l'(0) = 0$ and $w_l(0) = u_l(-l)$. Then for any fixed $L > 0$, w_l converges to u_0 uniformly on $[0, L]$ as $l \to \infty$. In particular, $u_l(0) \to 0$ and $u_l'(0) \to 0$ as $l \to \infty$. Since $v_l(0) = u_l(0)$ and $v_l'(l) = (1/m)u'(l)$, and $v_l' < 0$ for $x > 0$, then $J(v_l, (0, \infty)) \to 0$ as $l \to \infty$. According to the construction of u in Theorem 1.6, $u = w_l$. Hence when $l > 0$ is sufficiently large, $J((u, v), G) = \frac{1}{2}E_0 + o(1) < E_0$. Now Lemma 3.3 shows that (1.3) has a ground state for $\mathcal{S}_{1,m}$ with l large. □

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