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Existence and concentration of nontrivial nonnegative ground state solutions to Kirchhoff-type system with Hartree-type nonlinearity

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Abstract. A Kirchhoff-type fractional elliptic system with Hartree-type nonlinearity is proposed to provide a unified framework for well-known nonlinear Schrödinger equations, Kirchhoff equations and Schrödinger–Poisson systems. The existence of nontrivial nonnegative ground state solutions to the system is proved when the coefficient of the potential function is larger than a threshold value, and a precise estimate of the threshold value is given for a prototypical example. It is also shown that the ground state solution concentrates on the zero set of the potential function when the coefficient tends to infinity.

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1. Introduction and main results

The nonlinear Schrödinger equation is one of the canonical mathematical equations describing various important physical phenomena such as the evolution of a free nonrelativistic quantum particle, Bose–Einstein condensation, propagation of light in nonlinear optical material. The standing waves of nonlinear Schrödinger equation satisfy the equation

$$-\Delta u + \lambda V(x)u = f(u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $\lambda > 0$ is a parameter, V is a continuous potential function, and f is a proper nonlinear function. Many results have been obtained regarding the existence of solutions of (1.1) under various assumptions on V and f; see for examples [3,7,9,13,14,38].

There are several physical equations which can be regarded as generalizations of (1.1) such as the Kirchhoff equation and the Schrödinger–Poisson system. The Kirchhoff equation takes the form of

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + \lambda V(x)u = f(u), \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $a, b, \lambda > 0$ and V, f are similar to the ones in (1.1). The solutions of (1.2) are the steady states of a hyperbolic Kirchhoff equation which was proposed by Kirchhoff in 1876 [22] as an extension of the classical d'Alembert wave equation for free vibrations of elastic strings. Kirchhoff's model considers the changes in the length of the string produced by transverse vibrations. There have been numerous studies of the solutions of Kirchhoff equation in the recent years; see for examples [2,15,18,21,25–28,42,47].

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Table 1. *Comparison of three well-known systems and our new system*

b	l	Equation
$\geqslant 0$	$\geqslant 0$	Kirchhoff–Schrödinger–Poisson system (KH)
$= 0$	$= 0$	Schrödinger equation (1.1)
> 0	$= 0$	Kirchhoff equation (1.2)
$= 0$	> 0	Schrödinger–Poisson system ($\alpha = p = 2$) (1.3)

Moreover, the concentration behavior of solutions to (1.2) in the semiclassical form has been studied by [11,12,16,17,19,43].

On the other hand, the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \lambda V(x)u + \phi u = f(u), & \text{in} \quad \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{1.3}$$

describes a charged wave interacting with its own electrostatic field [8]. The existence of positive solutions, ground state solutions and multiple solutions has been established; see for examples [4,5,10,20,23,24,37, 39–41,44,45,49–51]. The three elliptic equations/systems (1.1) (Schrödinger), (1.2) (Kirchhoff) and (1.3) (Schrödinger–Poisson) come from different physical background, but they share a similar mathematical structure. Often the mathematical methods of studying (1.2) or (1.3) originate from the ones of treating (1.1), but yet some delicate modifications are needed.

Inspired by the works mentioned above, in this paper, we propose a new generalized Kirchhoff-Schrödinger–Poisson system

$$\begin{cases} -\left(a + b\int\limits_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + \lambda V(x)u + l\phi|u|^{p-2}u = f(u), & \text{in} \quad \mathbb{R}^3, \\ (-\Delta)^{\alpha/2}\phi = l|u|^p, & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{KH}$$

where $a > 0$, $b, l \geqslant 0$ are constants, $\lambda > 0$ is a parameter, the potential function $V \in C(\mathbb{R}^3, \mathbb{R}_+)$ where $\mathbb{R}_+ = [0, \infty)$, $\alpha \in (0, 3)$ and $p \in [2, 3 + \alpha)$. The system (KH) provides a unified framework of studying the Schrödinger equation (1.1) ($b = l = 0$), the Kirchhoff equation (1.2) ($b > 0$ and $l = 0$), and the Schrödinger–Poisson system (1.3) ($b = 0$, $l > 0$ and $\alpha = p = 2$) (see Table 1). All these three well-known systems are special cases of (KH). On the other hand, in full generality ($b > 0$, $l > 0$), the new Kirchhoff-Schrödinger–Poisson system (KH) is not covered by any of (1.1), (1.2) or (1.3); hence, it offers new mathematical insight and challenge.

We assume that V and f satisfy the following conditions, respectively.

- (V_0) $V \in C(\mathbb{R}^3, \mathbb{R}_+)$ and there is $M_0 > 0$ such that the set $\Lambda := \{x \in \mathbb{R}^3 : V(x) < M_0\}$ has finite Lebesgue measure;
- (V_1) $\Omega := \operatorname{int} V^{-1}(0)$ is nonempty with a smooth boundary and $\overline{\Omega} = V^{-1}(0)$;
- (f_0) $f \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and there exist $q \in (2, 6)$ and $C > 0$ such that $|f'(t)| \leqslant C(1 + |t|^{q-2})$ for all $t \in \mathbb{R}_+$;
- (f_1) $f(t)/t^{2p-1}$ is increasing on $(0, \infty)$ and $\lim\limits_{t\to\infty} f(t)/t^{2p-1} = \infty$.

It is known that for the second equation of (KH), that is, the fractional differential equation $(-\Delta)^{\alpha/2}\phi = |u|^p$, the unique solution is $\phi = I * |u|^p$, where I is the Riesz potential defined by

$$I(x) = \frac{\Gamma((3-\alpha)/2)}{\Gamma(\alpha/2)\pi^{3/2}2^{\alpha}} \frac{1}{|x|^{3-\alpha}}, \quad x \in \mathbb{R}^3 \setminus \{0\}$$

and $*$ is the convolution of two functions in $\mathbb{R}^3$. Hence, (KH) can also be rewritten as a Hartree-type equation

$$-\left(a+b\int\limits_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+\lambda V(x)u+l(I*|u|^p)|u|^{p-2}u=f(u), \text{ in } \mathbb{R}^3. \tag{1.4}$$

The Hartree-type nonlinearity $(I*|u|^p)|u|^{p-2}u$ appears in many physical applications such as quantum theory of large systems of nonrelativistic bosonic atoms and molecules, physics of multiple-particle systems (see for instance [31]). If u is a solution of (1.4), then the pair $(u, I*u)$ is a solution of (KH). For the sake of simplicity, in many cases we just say u, instead of $(u, I*u)$, is a solution of (KH). Under the assumptions above, we have the following results on the existence of ground state solutions and their concentration behaviors for the system (KH):

Theorem 1.1. *Assume the constants $a, \lambda > 0$, $b \geqslant 0$, $l \geqslant 0$, $\alpha \in (0,3)$ and $p \in [2, 3+\alpha)$, and the conditions (V_0), (V_1), (f_0) and (f_1) hold. Then*

(i) There exists $\lambda_0 > 0$ such that the system (KH) has a nontrivial nonnegative ground state solution $u_\lambda \in H^1(\mathbb{R}^3)$ when $\lambda > \lambda_0$.

(ii) Let $\{u_{\lambda_n}\}$ be any nontrivial nonnegative ground state solution of (KH) with $\lambda = \lambda_n > 0$ and $\lambda_n \to 0$ as $n \to \infty$. Then, passing to a subsequence, one has that u_{λ_n} converges to u_0 strongly in $H^1(\mathbb{R}^3)$ as $n \to \infty$, where $u_0 \in H^1(\mathbb{R}^3)$ satisfies $u_0(x) = 0$ almost everywhere for $x \in \mathbb{R}^3 \setminus \overline{\Omega}$, and u_0 is a nontrivial nonnegative solution of

$$\begin{cases} -\left(a+b\int\limits_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+l\phi|u|^{p-2}u=f(u), & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\alpha/2}\phi=l|u|^p, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.5}$$

From the construction in our proof, the threshold value λ_0 for the existence of ground state can be explicitly estimated for some given f, V and parameters. For example, when $f(t) = t^{9/2}$, $V(x) = \max\{|x|-1, 0\}$ for $|x| \leqslant 4$ and $V(x) = 3$ for $|x| > 4$, $p = 5/2$, $\alpha = 2$, $a = 3/(4\pi)$, $b = 171/(4\pi^2)$ and $l = 238.178$, we can obtain a precisely estimated value of $\lambda_0 > 0.858$ (see Sect. 3 for more details). Here, the chosen f clearly satisfies (f_0) and (f_1) with $p = 5/2$ and $q = 11/2$, and V satisfies the conditions (V_0) and (V_1).

It is worth mentioning that the system (KH) has two nonlocal terms, which implies that the system (KH) is no longer a point-wise identity. This brings some mathematical difficulties. Besides, we do not impose any compactness condition, but prove the convergence of a bounded Palais–Smale sequence under the given conditions here. We also remark that when the condition (V_0) is replaced by (V_0'): $V \in C(\mathbb{R}^3, \mathbb{R}_+)$ and $\liminf\limits_{|x|\to\infty} V(x) = V_\infty > 0$, the conclusions of Theorem 1.1 still hold. Our result also shows that only when a delicate balance between the exponents p, q, α is satisfied, the existence of nontrivial nonnegative ground state solutions of the system (KH) can be proved. First the conditions (f_0) and (f_1) naturally imply that $q > 2p$. Secondly, although our result hold for all $\alpha \in (0,3)$ and $p \in [2, 3+\alpha)$, the convergence for the Hartree-type nonlinearities needs to be proved in two separate cases: (i) $\alpha \in (0,3)$ and $p \in [\max\{2, (5+\alpha)/3\}, 3+\alpha)$, and (ii) $\alpha \in (1,3)$ and $p \in [2, (5+\alpha)/3)$. See Lemmas 2.4 (i) and 3.2 (ii) for more details.

Since the system (KH) unifies the Schrödinger equation, the Kirchhoff equation, and the Schrödinger–Poisson system, and these systems are indeed special cases of (KH), the results in Theorem 1.1 and the strong maximum principle naturally imply the following three corollaries (when the convergence is in the sense of subsequence).

Corollary 1.2. *Assume that (V_0), (V_1), (f_0) and (f_1) hold. Then, there exists $\lambda_0 > 0$ such that (1.1) has a positive ground state solution $u_\lambda \in H^1(\mathbb{R}^3)$ when $\lambda > \lambda_0$. Furthermore, $u_\lambda \to u_0$ in $H^1(\mathbb{R}^3)$ as $\lambda \to \infty$*

and $u_0 \in H_0^1(\Omega)$ *is a positive solution of the limit system*

$$\begin{cases} -\Delta u = f(u), & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

Corollary 1.3. *Assume that* (V_0), (V_1), (f_0) *and* (f_1) *hold. Then there exists* $\lambda_0 > 0$ *such that* (1.2) *has a positive ground state solution* $u_\lambda \in H^1(\mathbb{R}^3)$ *when* $\lambda > \lambda_0$. *Furthermore* $u_\lambda \to u_0$ *in* $H^1(\mathbb{R}^3)$ *as* $\lambda \to \infty$ *and* $u_0 \in H_0^1(\Omega)$ *is a positive solution of the limit system*

$$\begin{cases} -\left(a + b\int\limits_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u = f(u), & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

Corollary 1.4. *Assume that* (V_0), (V_1), (f_0) *and* (f_1) *hold. Then there exists* $\lambda_0 > 0$ *such that the equation* (1.3) *has a positive ground state solution* u_λ *when* $\lambda > \lambda_0$. *Furthermore,* $u_\lambda \to u_0$ *in* $H^1(\mathbb{R}^3)$ *as* $\lambda \to \infty$, $u_0 \in H_0^1(\Omega)$, $u_0 = 0$ *outside of* Ω, $u_0(x) > 0$ *for* $x \in \Omega$ *and satisfies*

$$\begin{cases} -\Delta u + \phi u = f(u), & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in} \quad \mathbb{R}^3. \end{cases}$$

The result in Corollary 1.4 is similar to the one in [48], and the ones in Corollaries 1.2 and 1.3 appear to be new. The advantage of our result Theorem 1.1 is to obtain a unified existence and concentration result for the general system (KH) which include all these special cases. Other than the results mentioned above, the existence and uniqueness of positive solution to the Hartree-type equation (or Choquard equation) without potential function (corresponding to (1.4) with $\lambda = 0$ and $b = 0$) has been considered in [29,32,35,36] and many others. The existence and concentration of positive solution to a Kirchhoff-type equation with Hartree-type nonlinearity has been recently considered in [34], but the system (KH) considered here has both local and Hartree-type nonlinearity. Our result for (1.2) also improves the one in [48] as we weaken several assumptions.

The paper is organized as follows. In Sect. 2, we introduce some notations, establish mountain pass geometry structure and give some preliminary results. In Sect. 3, we use the mountain pass theorem to prove the existence of nontrivial nonnegative ground state solutions to the system (KH) and give the specific numerical estimate of λ_0. In Sect. 4, the concentration of the obtained solutions above is proved.

2. Preliminaries

In this section, we mainly establish some preliminaries. Let us first introduce a linear function space

$$H_V^1(\mathbb{R}^3) = \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \int\limits_{\mathbb{R}^3} Vu^2 < \infty \right\}$$

with the inner product and the corresponding norm defined as

$$(u,v) = \int\limits_{\mathbb{R}^3} [\nabla u \cdot \nabla v + Vuv], \quad \|u\|_\lambda = (u,u)^{1/2}, \quad u,v \in H_V^1(\mathbb{R}^3).$$

Here, $\mathcal{D}^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$ is the Hilbert space with the inner product $(u,v)_{\mathcal{D}^{1,2}} = \int\limits_{\mathbb{R}^3} \nabla u \cdot \nabla v$ and the corresponding norm $\|u\|_{\mathcal{D}^{1,2}} = \left(\int\limits_{\mathbb{R}^3} |\nabla u|^2\right)^{1/2}$. It can be proved that $H_V^1(\mathbb{R}^3)$ is a Hilbert space under the condition (V_0) and that there is a continuous embedding $H_V^1(\mathbb{R}^3) \hookrightarrow H^1(\mathbb{R}^3)$.

It is well known that the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ is continuous for $s \in [2,6]$. Hence, for every $s \in [2,6]$, there exists $\gamma_s > 0$ such that

$$|u|_s \leqslant \gamma_s \|u\|, \quad u \in H^1_V(\mathbb{R}^3), \tag{2.1}$$

where $|\cdot|_s$ denotes the usual $L^s(\mathbb{R}^3)$ norm.

For convenience, for each $\lambda > 0$, we also define an equivalent inner product and norm on $H^1_V(\mathbb{R}^3)$ by

$$(u,v)_\lambda = \int_{\mathbb{R}^3} [a\nabla u \cdot \nabla v + \lambda V uv], \quad \|u\| = (u,u)_\lambda^{1/2}, \quad u,v \in H^1_V(\mathbb{R}^3).$$

It then follows from (2.1) that for $\lambda \geqslant a$,

$$|u|_s \leqslant \frac{\gamma_s}{a^{1/2}} \|u\|_\lambda, \quad u \in H^1_V(\mathbb{R}^3). \tag{2.2}$$

In this paper, we make use of the following notations:

- Letters C_i ($i \in \mathbb{N} := \{0,1,2,\ldots\}$) are used to denote various positive constants;
- $p' = p/(p-1)$ is the conjugate exponent of p, that is $1/p + 1/p' = 1$;
- $B_r = \{x \in \mathbb{R}^3 : |x| < r\}$ denotes an open ball of $\mathbb{R}^3$ with center at the origin and radius $r > 0$.

We recall the celebrated Hardy–Littlewood–Sobolev inequality [30, p. 106] which will be used in the paper.

Lemma 2.1. (Hardy–Littlewood–Sobolev inequality) *Let $r, s \in (1,\infty)$ and $\mu \in (0,N)$ with $1/r + \mu/N + 1/s = 2$. Then, there exists a sharp constant $C(r,N,\mu,s)$ such that for all $f \in L^r(\mathbb{R}^N)$ and $g \in L^s(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \leqslant C(r,N,\mu,s)|f|_r|g|_s.$$

The sharp constant satisfies

$$C(r,N,\mu,s) \leqslant \frac{N}{N-\mu}\frac{1}{rs}\alpha(N)^{\mu/N}\left[\left(\frac{\mu/N}{1-1/r}\right)^{\mu/N} + \left(\frac{\mu/N}{1-1/s}\right)^{\mu/N}\right],$$

where $\alpha(N)$ is the volume of unit sphere in $\mathbb{R}^N$. If $r = s = 2N/(2N-\mu)$, then

$$C(N,\mu) = \pi^{\mu/2}\frac{\Gamma(N/2-\mu/2)}{\Gamma(N-\mu/2)}\left[\frac{\Gamma(N/2)}{\Gamma(N)}\right]^{-1+\mu/N}.$$

Since we are only concerned with the existence of nontrivial nonnegative ground state solutions to (KH), it may be assumed that $f(t) = 0$ for all $t \in (-\infty, 0]$. We define a functional

$$J_\lambda(u) = \frac{1}{2}\|u\|_\lambda^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2 + \frac{l}{2p}\int_{\mathbb{R}^3}(I * |u|^p)|u|^p - \int_{\mathbb{R}^3} F(u), \quad u \in H^1_V(\mathbb{R}^3), \tag{2.3}$$

where $F(t) = \int_0^t f(s)\mathrm{d}s$ for all $t \in \mathbb{R}$. It can be proved that $J_\lambda \in C^1(H^1_V(\mathbb{R}^3), \mathbb{R})$ and

$$\langle J'_\lambda(u), v\rangle = (u,v)_\lambda + b\int_{\mathbb{R}^3}|\nabla u|^2\int_{\mathbb{R}^3}\nabla u\cdot\nabla v + l\int_{\mathbb{R}^3}(I*|u|^p)|u|^{p-2}uv - \int_{\mathbb{R}^3} f(u)v \tag{2.4}$$

for all $u, v \in H^1_V(\mathbb{R}^3)$. It is standard to show that finding a weak solution of the system (KH) is equivalent to finding a critical point of the functional J_λ. We also define

$$J(u) = \frac{1}{2}\int_\Omega a|\nabla u|^2 + \frac{b}{4}\left(\int_\Omega|\nabla u|^2\right)^2 + \frac{l}{2p}\int_\Omega(I*|u|^p)|u|^p - \int_\Omega F(u), \quad u \in H^1_0(\Omega), \tag{2.5}$$

and

$$\langle J'(u),v\rangle = \int_\Omega a\nabla u\cdot\nabla v + b\int_\Omega |\nabla u|^2\int_\Omega \nabla u\cdot\nabla v + l\int_\Omega (I*|u|^p)|u|^{p-2}uv - \int_\Omega f(u)v, \tag{2.6}$$

for all $u,v\in H_0^1(\Omega)$.

We show that the energy functional J_λ possesses a mountain pass structure.

Lemma 2.2. *Suppose that (V_0), (V_1), (f_0) and (f_1) hold.*

(i) There exist $\rho,\beta>0$ such that $J_\lambda(u)>0$ and $\langle J_\lambda'(u),u\rangle>0$ for all $u\in H_V^1(\mathbb{R}^3)$ with $\|u\|_\lambda\in(0,\rho]$, and $\inf_{\|u\|_\lambda=\rho} J_\lambda(u)=\beta$, where $\beta>0$ is independent of λ.

(ii) For any given $u\in H_V^1(\mathbb{R}^3)$ with $u^+\neq 0$, we have $\lim_{t\to\infty} J_\lambda(tu)=-\infty$ and there exists a unique $t_u>0$ such that $\langle J_\lambda'(t_uu),u\rangle=0$ and $J_\lambda(t_uu)=\max_{t\in\mathbb{R}_+} J_\lambda(tu)$.

Proof. (i) According to conditions (f_0), (f_1), we have $\lim_{t\to 0^+} f(t)/t=0$. So for any given $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$|f(t)|\leqslant \varepsilon|t|+C_\varepsilon|t|^{q-1},\quad t\in\mathbb{R}. \tag{2.7}$$

Particularly, for $\varepsilon=1$, there exists $C_0>0$ such that

$$|f(t)|\leqslant |t|+C_0|t|^{q-1},\quad t\in\mathbb{R}. \tag{2.8}$$

It follows from (2.7) that there exists a $C_1>0$ such that

$$F(t)\leqslant \frac{a}{4\gamma_2^2}|t|^2+C_1|t|^q,\quad t\in\mathbb{R},$$

where $\gamma_2>0$ is from (2.1). By (2.2), we then have

$$J_\lambda(u)\geqslant \frac{1}{2}\|u\|_\lambda^2-\frac{a}{4\gamma_2^2}|u|_2^2-C_1|u|_q^q\geqslant \frac{1}{4}\|u\|_\lambda^2-C_2\|u\|_\lambda^q,\quad u\in H_V^1(\mathbb{R}^3).$$

Therefore, there exist $\rho,\beta>0$ such that the conclusion (i) for J_λ holds. Since C_2 is independent of λ, the same result holds for β. Similarly, we can prove the conclusion for $\langle J_\lambda'(u),u\rangle$.

(ii) Let $u\in H_V^1(\mathbb{R}^3)$ with $u^+\neq 0$. Then for $t\in\mathbb{R}_+$,

$$J_\lambda(tu)=t^{2p}\left[\frac{\|u\|_\lambda^2}{2t^{2(p-1)}}+\frac{b}{4t^{2(p-2)}}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2+\frac{l}{2p}\int_{\mathbb{R}^3}(I*|u|^p)|u|^p-\int_{\mathbb{R}^3}\frac{F(tu)}{t^{2p}}\right]. \tag{2.9}$$

Thus, the condition (f_1) implies that $J_\lambda(tu)\to-\infty$ as $t\to\infty$. Therefore, it follows from (i) that there exists $t_u>0$ such that $J_\lambda(t_uu)=\max_{t\in\mathbb{R}_+}J_\lambda(tu)$ and $\langle J_\lambda'(t_uu),u\rangle=0$. Furthermore, t_u is also unique from the condition (f_1). □

Next we define the Nehari manifold for (2.3) and (2.5), respectively

$$N_\lambda=\{u\in H_V^1(\mathbb{R}^3)\setminus\{0\}:\langle J_\lambda'(u),u\rangle=0\},$$
$$N=\{u\in H_0^1(\Omega)\setminus\{0\}:\langle J'(u),u\rangle=0\},$$

and

$$m_\lambda=\inf_{u\in N_\lambda}J_\lambda(u),\quad m_0=\inf_{u\in N}J(u). \tag{2.10}$$

We also define

$$c_\lambda=\inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]}J_\lambda(\gamma(t)),\quad c_0=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}J(\gamma(t)), \tag{2.11}$$

where

$$\Gamma_\lambda = \{\gamma \in C([0,1], H^1_V(\mathbb{R}^3)) : \gamma(0) = 0, J_\lambda(\gamma(1)) < 0\},$$
$$\Gamma = \{\gamma \in C([0,1], H^1_0(\Omega)) : \gamma(0) = 0, J(\gamma(1)) < 0\}.$$

Lemma 2.3. *Let m_λ, m_0, c_λ and c_0 be defined as in (2.10) and (2.11). Then*
(i) For $\lambda > 0$, $m_\lambda = c_\lambda$.
(ii) $m_0 = c_0$.
(iii) For $\lambda > 0$, $c_\lambda \leqslant c_0$.

Proof. (i) Let $u \in N_\lambda$. Then $u^+ \neq 0$. Otherwise, $\int\limits_{\mathbb{R}^3} f(u)u = 0$ and then $\langle J'_\lambda(u), u\rangle > 0$, which contradicts to $\langle J'_\lambda(u), u\rangle = 0$. It follows from Lemma 2.2 (ii) that there exists a $t_0 > 1$ such that $J_\lambda(t_0u) < 0$. Thus, $J_\lambda(u) = \max\limits_{t\in\mathbb{R}_+} J_\lambda(tu) \geqslant \max\limits_{t\in[0,1]} J_\lambda(tt_0u) \geqslant c_\lambda$. This implies that $m_\lambda \geqslant c_\lambda$.

According to the condition (f$_1$), we obtain that $f(t)t - 2pF(t) \geqslant 0$ for $t \in \mathbb{R}$. It follows from (2.3) that

$$J_\lambda(u) - \frac{1}{2p}\langle J'_\lambda(u), u\rangle \geqslant \frac{1}{2p'}\|u\|^2_\lambda \geqslant 0, \quad u \in H^1_V(\mathbb{R}^3). \tag{2.12}$$

Let $\gamma \in \Gamma_\lambda$. Then, it follows from (2.12) that $\langle J'_\lambda(\gamma(1)), \gamma(1)\rangle \leqslant 2pJ_\lambda(\gamma(1)) < 0$. Let us define $t_1 = \inf\{t \in [0,1) : \langle J'_\lambda(\gamma(s)), \gamma(s)\rangle < 0, s \in (t,1]\}$. Then $\langle J'_\lambda(\gamma(t_1)), \gamma(t_1)\rangle = 0$ and $\gamma(s) \neq 0$ for all $s \in (t_1, 1]$. We now claim that $\gamma(t_1) \neq 0$. Otherwise, $\gamma(t_1) = 0$ and then Lemma 2.2 (i) implies that $\langle J'_\lambda(\gamma(s)), \gamma(s)\rangle > 0$ as $s \to t_1^+$. This contradicts with the definition of t_1. Thus, we have that $\gamma(t_1) \in N_\lambda$ and $c_\lambda \geqslant m_\lambda$. This completes the proof of (i). The proof of (ii) is similar to (i), we have $m_0 = c_0$.

To prove (iii), for any given $\eta \in N$, by (2.9) and $t \in \mathbb{R}_+$, we have

$$\begin{aligned} J_\lambda(t\eta) &= t^{2p}\left[\frac{a}{2t^{2(p-1)}}\int\limits_\Omega |\nabla\eta|^2 + \frac{b}{4t^{2(p-2)}}\left(\int\limits_\Omega |\nabla\eta|^2\right)^2 + \frac{l}{2p}\int\limits_\Omega (I * |\eta|^p)|\eta|^p - \int\limits_\Omega \frac{F(t\eta)}{t^{2p}}\right] \\ &= J(t\eta). \end{aligned}$$

By Lemma 2.2 (ii), there exists a unique $t_\eta > 0$ such that $t_\eta\eta \in N_\lambda$. Thus

$$c_\lambda \leqslant J_\lambda(t_\eta\eta) = \max_{t\in\mathbb{R}_+} J_\lambda(t\eta) = \max_{t\in\mathbb{R}_+} J(t\eta) = J(\eta).$$

Since $\eta \in N$ is arbitrary, we have $c_\lambda \leqslant c_0$, where c_0 is independent of $\lambda \in (0,\infty)$. The proof is complete. □

Next we prove the following convergence of integrals.

Lemma 2.4. *Assume that (f$_0$) and (f$_1$) hold. Suppose that $\{u_n\}$ is a sequence in $H^1_V(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $H^1_V(\mathbb{R}^3)$ and $u_n \to u$, a.e. on $\mathbb{R}^3$ as $n \to \infty$.*
(i) $\lim\limits_{n\to\infty}\int\limits_{\mathbb{R}^3} (I * |u_n|^p)|u_n|^{p-2}u_nv = \int\limits_{\mathbb{R}^3} (I * |u|^p)|u|^{p-2}uv$ for any $v \in H^1_V(\mathbb{R}^3)$;
(ii) $\lim\limits_{n\to\infty}\int\limits_{\mathbb{R}^3} f(u_n)v = \int\limits_{\mathbb{R}^3} f(u)v$ for any $v \in H^1_V(\mathbb{R}^3)$.

Proof. (i) In the following proof, we always assume the convergence is for $n \to \infty$. From that $u_n \rightharpoonup u$ in $H^1_V(\mathbb{R}^3)$ and $6p/(3+\alpha) \in [2,6]$, we have that $u_n \rightharpoonup u$ in $L^{6p/(3+\alpha)}(\mathbb{R}^3)$. It follows from $u_n(x) \to u(x)$ a.e. on $\mathbb{R}^3$ that $|u_n|^p \rightharpoonup |u|^p$ in $L^{6/(3+\alpha)}(\mathbb{R}^3)$. By Lemma 2.1, we have that $I * (|u|^{p-2}uv) \in L^{6/(3-\alpha)}(\mathbb{R}^3)$ and then

$$\int\limits_{\mathbb{R}^3} \left[I * (|u|^{p-2}uv)\right](|u_n|^p - |u|^p) \to 0. \tag{2.13}$$

If $\alpha \in (0,3)$ and $p \in [\max\{2,(5+\alpha)/3\}, 3+\alpha)$, then it follows from $u_n(x) \to u(x)$ a.e. on $\mathbb{R}^3$ that $\left||u_n|^{p-2}u_n - |u|^{p-2}u\right|^{6/(3+\alpha)} \rightharpoonup 0$ in $L^{(3+\alpha)/(2+\alpha)}(\mathbb{R}^3)$. Thus, for $v \in H_V^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} \left||u_n|^{p-2}u_n - |u|^{p-2}u\right|^{6/(3+\alpha)} |v|^{6/(3+\alpha)} \to 0. \tag{2.14}$$

For the case $\alpha \in (1,3)$ and $p \in [2,(5+\alpha)/3)$, in a similar way, it follows from $u_n \to u$, a.e. on $\mathbb{R}^3$ that $\left||u_n|^{p-2}u_n - |u|^{p-2}u\right|^{6/(3+\alpha)} \rightharpoonup 0$ in $L^{(3+\alpha)/3}(\mathbb{R}^3)$. Thus, for $v \in H_V^1(\mathbb{R}^3) \hookrightarrow L^{6/\alpha}(\mathbb{R}^3)$, (2.14) also holds.

It follows from the Hölder inequality, (2.13) and (2.14) that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} (I * |u_n|^p)|u_n|^{p-2}u_n v - \int_{\mathbb{R}^3} (I * |u|^p)|u|^{p-2}uv\right| \\
&\leqslant \int_{\mathbb{R}^3} \left|(I * |u_n|^p)\left[|u_n|^{p-2}u_n v - |u|^{p-2}uv\right]\right| + \left|\int_{\mathbb{R}^3} \left[I * |u_n|^p - I * |u|^p\right]|u|^{p-2}uv\right| \\
&\leqslant |I * |u_n|^p|_{6/(3-\alpha)} \left|\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)v\right|_{6/(3+\alpha)} + \left|\int_{\mathbb{R}^3} \left[I * (|u|^{p-2}uv)\right](|u_n|^p - |u|^p)\right| \\
&\to 0.
\end{aligned}$$

Thus, the conclusion (i) holds.

(ii) For any given $\varepsilon > 0$, by the local compactness of Sobolev embedding, (2.7) and the Hölder inequality, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} f(u_n)v - \int_{\mathbb{R}^3} f(u)v\right| \leqslant \int_{B_r} |f(u_n) - f(u)||v| + \int_{B_r^c} |f(u_n) - f(u)||v| \\
&\leqslant |f(u_n) - f(u)|_{L^{q/(q-1)}(B_r)}|v|_q + \varepsilon(|u_n|_2 + |u|_2)|v|_{L^2(B_r^c)} + C_\varepsilon\left(|u_n|_q^{q-1} + |u|_q^{q-1}\right)|v|_{L^q(B_r^c)} \\
&\leqslant o(1) + C_3\varepsilon|v|_{L^2(B_r^c)} + C_4|v|_{L^q(B_r^c)}.
\end{aligned}$$

Since $|v|_{L^2(B_r^c)} \to 0$ and $|v|_{L^q(B_r^c)} \to 0$ as $r \to \infty$, the conclusion holds. □

3. Existence of nontrivial nonnegative ground state solutions

In this section, we use the mountain pass theorem to show the existence of nontrivial nonnegative ground state solutions of the system (KH) and give the proof of existence part in Theorem 1.1. We shall show that the energy functional J_λ satisfies the Palais–Smale (PS) condition at the c_λ, which is a key in the existence of nontrivial nonnegative ground states. We establish this result in the following three lemmas. Throughout this section, we always assume the conditions of Theorem 1.1 hold.

Lemma 3.1. *If $\{u_n\}$ is a $(PS)_{c_\lambda}$ sequence of J_λ, then there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in H_V^1(\mathbb{R}^3)$ such that $u_{n_k} \rightharpoonup u$ in $H_V^1(\mathbb{R}^3), u_{n_k} \to u$, a.e. on $\mathbb{R}^3$, and $J'_\lambda(u) = 0$.*

Proof. Since $\{u_n\}$ is a $(\mathrm{PS})_{c_\lambda}$ sequence of J_λ, it follows from (2.12) that for n large enough,

$$c_\lambda + 1 + \|u_n\|_\lambda \geqslant J_\lambda(u_n) - \frac{1}{2p}\langle J'_\lambda(u_n), u_n\rangle \geqslant \frac{1}{2p'}\|u_n\|_\lambda^2.$$

Thus, $\{u_n\}$ is bounded in $H_V^1(\mathbb{R}^3)$. We may assume that, up to a subsequence, $u_n \rightharpoonup u$ in $H_V^1(\mathbb{R}^3)$ and $u_n \to u$, a.e. on $\mathbb{R}^3$. If $u = 0$, it is easy to see from (2.4) that $J'_\lambda(u) = 0$. If $u \neq 0$, then by the weakly lower semi-continuity of the norm of $\mathcal{D}^{1,2}(\mathbb{R}^3)$, it follows that $\int_{\mathbb{R}^3} |\nabla u|^2 \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 := A^2$. Without loss of generality, we may assume that $\lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 = A^2$. Now we prove that $b\int_{\mathbb{R}^3} |\nabla u|^2 = bA^2$. In fact, suppose, by contradiction, that $b\int_{\mathbb{R}^3} |\nabla u|^2 < bA^2$. Since $J'_\lambda(u_n) \to 0$, by Lemma 2.4, we have

$$(u,v)_\lambda + bA^2 \int_{\mathbb{R}^3} \nabla u \cdot \nabla v + l \int_{\mathbb{R}^3} (I * |u|^p)|u|^{p-2}uv - \int_{\mathbb{R}^3} f(u)v = 0, \quad v \in H^1_V(\mathbb{R}^3). \tag{3.1}$$

Taking $v = u$, we get that $\langle J'_\lambda(u), u\rangle < 0$. This yields that $u^+ \neq 0$. Thus, it follows from Lemma 2.2 (*ii*) that there exists a unique $t_u \in (0,1)$ such that $\langle J'_\lambda(t_u u), t_u u\rangle = 0$. Moreover, according to the condition (f_1), we can obtain that $f(t)t - 2pF(t)$ is increasing on $\mathbb{R}_+$. Let $l_1 = (1-2/p)/4$, then

$$\begin{aligned}
c_\lambda &\leqslant J_\lambda(t_u u) - \frac{1}{2p}\langle J'_\lambda(t_u u), t_u u\rangle \\
&= \frac{1}{2p'} t_u^2 \|u\|_\lambda^2 + l_1 b t_u^4 \|u\|^4_{\mathcal{D}^{1,2}} + \frac{1}{2p}\int_{\mathbb{R}^3} [f(t_u u)t_u u - 2pF(t_u u)] \\
&< \frac{1}{2p'} \|u\|_\lambda^2 + l_1 b \|u\|^4_{\mathcal{D}^{1,2}} + \frac{1}{2p}\int_{\mathbb{R}^3} [f(u)u - 2pF(u)] \\
&\leqslant \liminf_{n\to\infty} \left(\frac{1}{2p'} \|u_n\|_\lambda^2 + l_1 b \|u_n\|^4_{\mathcal{D}^{1,2}} + \frac{1}{2p}\int_{\mathbb{R}^3} [f(u_n)u_n - 2pF(u_n)] \right) \\
&= \liminf_{n\to\infty} \left(J_\lambda(u_n) - \frac{1}{2p}\langle J'_\lambda(u_n), u_n\rangle \right) = c_\lambda,
\end{aligned}$$

which is apparently a contradiction. Thus, $b\int_{\mathbb{R}^3} |\nabla u|^2 = bA^2$ and then (3.1) implies that $J'_\lambda(u) = 0$. The proof is complete. □

Lemma 3.2. *Let $\{u_n\}$ be a $(PS)_{c_\lambda}$ sequence of J_λ and let $v_k = u_{n_k} - u$ for $k \in \mathbb{N}$, where $\{u_{n_k}\}$ and u are defined in Lemma 3.1. Then*

(i) $\lim_{k\to\infty} \left[\int_{\mathbb{R}^3} (I * |u_{n_k}|^p)|u_{n_k}|^p - \int_{\mathbb{R}^3} (I * |v_k|^p)|v_k|^p - \int_{\mathbb{R}^3} (I * |u|^p)|u|^p \right] = 0.$

(ii) $\lim_{k\to\infty} \left[(I * |u_{n_k}|^p)|u_{n_k}|^{p-2}u_{n_k} - (I * |v_k|^p)|v_k|^{p-2}v_k - (I * |u|^p)|u|^{p-2}u \right] = 0$ *in* $H_V^{-1}(\mathbb{R}^3)$, *where* $H_V^{-1}(\mathbb{R}^3)$ *is the dual space of* $H^1_V(\mathbb{R}^3)$.

(iii) $\lim_{k\to\infty} [f(u_{n_k}) - f(v_k) - f(u)] = 0$ *in* $H_V^{-1}(\mathbb{R}^3)$.

(iv) $J'_\lambda(v_k) \to 0$ *as* $k \to \infty$.

Proof. For (i), the proof can be found in [36, Lemma 2.4]. For (ii), we divide the proof into the following two cases. We can see in the proof that the Hartree-type nonlinearities convergence is crucial.

Case 1. $\alpha \in (0,3)$ and $p \in [\max\{2, (5+\alpha)/3\}, 3+\alpha)$.

From $u_{n_k} \rightharpoonup u$ in $H^1_V(\mathbb{R}^3)$ and $6p/(3+\alpha) \in [2,6]$, we have that $u_{n_k} \rightharpoonup u$ in $L^{6p/(3+\alpha)}(\mathbb{R}^3)$. Noting that $u_{n_k} \to u$, a.e. on $\mathbb{R}^3$, it follows from [23, Lemma 2.2] that

$$\lim_{k\to\infty} (|u_{n_k}|^p - |v_k|^p - |u|^p) = 0, \quad \text{in } L^{6/(3+\alpha)}(\mathbb{R}^3).$$

Thus, from Lemma 2.1, we obtain that

$$\lim_{k\to\infty}\left[I*(|u_{n_k}|^p-|v_k|^p-|u|^p)\right]=0,\quad \text{in}\ \ L^{6/(3-\alpha)}(\mathbb{R}^3). \tag{3.2}$$

Moreover, since $u_{n_k}\rightharpoonup u$ in $H^1_V(\mathbb{R}^3)$ and $6(p-1)/(2+\alpha)\in[2,6]$, we have that $u_{n_k}\rightharpoonup u$ in $L^{6(p-1)/(2+\alpha)}(\mathbb{R}^3)$. Noting that $u_{n_k}\to u$, a.e. on $\mathbb{R}^3$, it follows from [23, Lemma 2.2] that

$$\lim_{k\to\infty}\left[|u_{n_k}|^{p-2}u_{n_k}-|v_k|^{p-2}v_k-|u|^{p-2}u\right]=0,\quad \text{in}\ \ L^{6/(2+\alpha)}(\mathbb{R}^3). \tag{3.3}$$

For any $v\in H^1_V(\mathbb{R}^3)$, using simple calculation, we have

$$\begin{aligned}
B_k &:= \left|\int_{\mathbb{R}^3}(I*|u_{n_k}|^p)|u_{n_k}|^{p-2}u_{n_k}v-\int_{\mathbb{R}^3}(I*|v_k|^p)|v_k|^{p-2}v_kv-\int_{\mathbb{R}^3}(I*|u|^p)|u|^{p-2}uv\right|\\
&=\left|\int_{\mathbb{R}^3}[I*(|u_{n_k}|^p-|v_k|^p-|u|^p)]|u_{n_k}|^{p-2}u_{n_k}v\right.\\
&\quad+\int_{\mathbb{R}^3}(I*|v_k|^p)\left[|u_{n_k}|^{p-2}u_{n_k}v-|v_k|^{p-2}v_kv-|u|^{p-2}uv\right]\\
&\quad+\int_{\mathbb{R}^3}(I*|u|^p)[|u_{n_k}|^{p-2}u_{n_k}v-|v_k|^{p-2}v_kv-|u|^{p-2}uv]\\
&\quad\left.+\int_{\mathbb{R}^3}(I*|v_k|^p)|u|^{p-2}uv+\int_{\mathbb{R}^3}(I*|u|^p)|v_k|^{p-2}v_kv\right|.
\end{aligned} \tag{3.4}$$

By (3.2) and (3.3), we get

$$\begin{aligned}
B_k &\leqslant |I*(|u_{n_k}|^p-|v_k|^p-|u|^p)|_{6/(3-\alpha)}\left||u_{n_k}|^{p-1}\right|_{6/(2+\alpha)}|v|_6\\
&\quad+|I*|v_k|^p|_{6/(3-\alpha)}\left||u_{n_k}|^{p-2}u_{n_k}-|v_k|^{p-2}v_k-|u|^{p-2}u\right|_{6/(2+\alpha)}|v|_6\\
&\quad+|I*|u|^p|_{6/(3-\alpha)}\left||u_{n_k}|^{p-2}u_{n_k}-|v_k|^{p-2}v_k-|u|^{p-2}u\right|_{6/(2+\alpha)}|v|_6\\
&\quad+\int_{\mathbb{R}^3}|(I*|v_k|^p)|u|^{p-1}v|+\int_{\mathbb{R}^3}|(I*|u|^p)|v_k|^{p-1}v|\\
&=o(1)|v|_6+\int_{\mathbb{R}^3}|(I*|v_k|^p)|u|^{p-1}v|+\int_{\mathbb{R}^3}|(I*|u|^p)|v_k|^{p-1}v|.
\end{aligned} \tag{3.5}$$

Since

$$|v_k|^p\rightharpoonup 0,\quad \text{in}\ \ L^{6/(3+\alpha)}(\mathbb{R}^3),\ \ \text{as}\ \ k\to\infty, \tag{3.6}$$

and $I*|v_k|^p\geqslant 0$, in view of the Hölder inequality, Lemma 2.1 and (3.6), we have

$$\begin{aligned}
\int_{\mathbb{R}^3}|(I*|v_k|^p)|u|^{p-1}v| &\leqslant \left(\int_{\mathbb{R}^3}(I*|v_k|^p)|u|^p\right)^{1/p'}\left(\int_{\mathbb{R}^3}(I*|v_k|^p)|v|^p\right)^{1/p}\\
&\leqslant C_5\left(\int_{\mathbb{R}^3}(I*|u|^p)|v_k|^p\right)^{1/p'}|v_k|_{6p/(3+\alpha)}|v|_{6p/(3+\alpha)}=o(1)|v|_{6p/(3+\alpha)}.
\end{aligned} \tag{3.7}$$

On the other hand, since $|v_k|^{6(p-1)/5} \rightharpoonup 0$ in $L^{5/(2+\alpha)}(\mathbb{R}^3)$ as $k \to \infty$ and $(I * |u|^p)^{6/5} \in L^{5/(3-\alpha)}(\mathbb{R}^3)$, by the Hölder inequality, we have

$$\int\limits_{\mathbb{R}^3} |(I * |u|^p)|v_k|^{p-1}v| \leqslant \left(\int\limits_{\mathbb{R}^3} \left|(I * |u|^p)|v_k|^{p-1}\right|^{6/5} \right)^{5/6} |v|_6 = o(1)|v|_6. \tag{3.8}$$

Combining (3.5), (3.7) and (3.8), we find that the conclusion (ii) holds in this case.
Case 2. $\alpha \in (1, 3)$ and $p \in [2, (5 + \alpha)/3)$.

Similar to Case 1, from $u_{n_k} \rightharpoonup u$ in $H^1_V(\mathbb{R}^3)$ as $k \to \infty$ and $2(p-1) \in [2, 6]$, we have that $u_{n_k} \rightharpoonup u$ in $L^{2(p-1)}(\mathbb{R}^3)$ as $k \to \infty$. Noting that $u_{n_k}(x) \to u(x)$ a.e. on $\mathbb{R}^3$ as $k \to \infty$, it follows from [23, Lemma 2.2] that (3.3) holds in $L^2(\mathbb{R}^3)$. For any $v \in H^1_V(\mathbb{R}^3)$, by (3.2) and (3.3) in $L^2(\mathbb{R}^3)$, we get from (3.4) that

$$\begin{aligned} B_k &\leqslant |I * (|u_{n_k}|^p - |v_k|^p - |u|^p)|_{6/(3-\alpha)} \left||u_{n_k}|^{p-1}\right|_2 |v|_{6/\alpha} \\ &\quad + |I * |v_k|^p|_{6/(3-\alpha)} \left||u_{n_k}|^{p-2}u_{n_k} - |v_k|^{p-2}v_k - |u|^{p-2}u\right|_2 |v|_{6/\alpha} \\ &\quad + |I * |u|^p|_{6/(3-\alpha)} \left||u_{n_k}|^{p-2}u_{n_k} - |v_k|^{p-2}v_k - |u|^{p-2}u\right|_2 |v|_{6/\alpha} \\ &\quad + \int\limits_{\mathbb{R}^3} |(I * |v_k|^p)|u|^{p-1}v| + \int\limits_{\mathbb{R}^3} |(I * |u|^p)|v_k|^{p-1}v| \\ &= o(1)|v|_{6/\alpha} + \int\limits_{\mathbb{R}^3} |(I * |v_k|^p)|u|^{p-1}v| + \int\limits_{\mathbb{R}^3} |(I * |u|^p)|v_k|^{p-1}v|. \end{aligned}$$

Since $|v_k|^{6(p-1)/(6-\alpha)} \rightharpoonup 0$ in $L^{(6-\alpha)/3}(\mathbb{R}^3)$ as $k \to \infty$ and $(I * |u|^p)^{6/(6-\alpha)} \in L^{(6-\alpha)/(3-\alpha)}(\mathbb{R}^3)$, by the Hölder inequality, we have

$$\int\limits_{\mathbb{R}^3} |(I * |u|^p)|v_k|^{p-1}v| \leqslant \left(\int\limits_{\mathbb{R}^3} \left|(I * |u|^p)|v_k|^{p-1}\right|^{6/(6-\alpha)} \right)^{(6-\alpha)/6} |v|_{6/\alpha} = o(1)|v|_{6/\alpha}.$$

Thus combining with (3.7), the conclusion (ii) also holds in this case.

(iii) To prove this conclusion, we recall that $f \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ from (f_0). Thus, we have

$$\begin{aligned} |f(v_k + u) - f(v_k)| &= |f'(v_k + \theta u)u| \\ &\leqslant C\left[1 + (|v_k| + |u|)^{q-2}\right]|u| \leqslant C|u| + C_6\left[|v_k|^{q-2}|u| + |u|^{q-1}\right], \end{aligned}$$

where $\theta \in (0, 1)$. For any given $\varepsilon > 0$, by the Young inequality and (2.7), there is $C_\varepsilon > 0$ such that

$$|f(v_k + u) - f(v_k) - f(u)| \leqslant \varepsilon|v_k|^{q-1} + C_\varepsilon|u|^{q-1} + C_\varepsilon|u|.$$

Defining a function sequence

$$h_k = \max\{|f(v_k + u) - f(v_k) - f(u)| - \varepsilon|v_k|^{q-1}, 0\}, \quad k \in \mathbb{N},$$

we have that $h_k \to 0$, a.e. on $\mathbb{R}^3$ and $0 \leqslant h_k \leqslant C_\varepsilon|u|^{q-1} + C_\varepsilon|u|$ for all $k \in \mathbb{N}$. We choose $\chi \in C_0^\infty(\mathbb{R}, [0, 1])$ satisfying $\chi(t) = 1$ for all $t \in [-1, 1]$, and $\chi(t) = 0$ for all $|t| \geqslant 2$. Let $g_k = \chi(u)h_k$ and $\varphi_k = (1 - \chi(u))h_k$. Then, there exist $C_7, C_8 > 0$ such that $|g_k| \leqslant C_7|u|, |\varphi_k| \leqslant C_8|u|^{q-1}$ for all $k \in \mathbb{N}$. Therefore, by the dominated convergence theorem, we have

$$\int\limits_{\mathbb{R}^3} g_k^2 \to 0, \quad \int\limits_{\mathbb{R}^3} |\varphi_k|^{q/(q-1)} \to 0, \quad k \to \infty.$$

From the definition of $\{h_k\}$, it follows that

$$|f(v_k + u) - f(v_k) - f(u)| \leqslant g_k + \varphi_k + \varepsilon|v_k|^{q-1}.$$

Therefore, for any $v \in H^1_V(\mathbb{R}^3)$, we obtain that

$$\limsup_{k\to\infty}\left|\int_{\mathbb{R}^3}[f(v_k+u)-f(v_k)-f(u)]v\right| \leqslant \limsup_{k\to\infty}\int_{\mathbb{R}^3}|f(v_k+u)-f(v_k)-f(u)||v|$$
$$\leqslant \limsup_{k\to\infty}\left[|g_k|_2|v|_2+|\varphi_k|_{q/(q-1)}|v|_q+\varepsilon|v_k|_q^{q-1}|v|_q\right] \leqslant \varepsilon C_9\|v\|.$$

Thus, the conclusion (iii) holds.

Finally for (iv), by using the fact of $\lim_{k\to\infty} b\int_{\mathbb{R}^3}|\nabla u_{n_k}|^2 = b\int_{\mathbb{R}^3}|\nabla u|^2$ in Lemma 3.1 and part (i), (ii) and (iii), we have

$$\langle J'_\lambda(u_{n_k}),v\rangle - \langle J'_\lambda(v_k),v\rangle - \langle J'_\lambda(u),v\rangle$$
$$= b\left[\int_{\mathbb{R}^3}|\nabla u_{n_k}|^2\int_{\mathbb{R}^3}\nabla u_{n_k}\cdot\nabla v - \int_{\mathbb{R}^3}|\nabla v_k|^2\int_{\mathbb{R}^3}\nabla v_k\cdot\nabla v - \int_{\mathbb{R}^3}|\nabla u|^2\int_{\mathbb{R}^3}\nabla u\cdot\nabla v\right]$$
$$+ l\int_{\mathbb{R}^3}\left[(I*|u_{n_k}|^p)|u_{n_k}|^{p-2}u_{n_k} - (I*|v_k|^p)|v_k|^{p-2}v_k - (I*|u|^p)|u|^{p-2}u\right]v$$
$$- \int_{\mathbb{R}^3}[f(u_{n_k})-f(v_k)-f(u)]v = o(1)\|v\|.$$

From $J'_\lambda(u_{n_k})\to 0$ and $J'_\lambda(u)=0$, we have $J'_\lambda(v_k)\to 0$. □

Now we are ready to show that the (PS) condition is satisfied at the c_λ level.

Lemma 3.3. *Let $\{u_n\}$ be a $(PS)_{c_\lambda}$ sequence of J_λ and let $\{u_{n_k}\}$ and u be defined as in Lemma 3.1. Then there exists $\lambda_0>0$ such that $u_{n_k}\to u$ in $H^1_V(\mathbb{R}^3)$ as $k\to\infty$ for $\lambda>\lambda_0$.*

Proof. We choose $\sigma=(2p'c_0)^{1/2}$ and $\lambda_0=\max\{a,\lambda_*\}$, where λ_* is the unique positive solution satisfying $h_1(\lambda_*)=0$ where

$$h_1(\lambda) := 1-\frac{1}{\lambda M_0}-\frac{C_0\gamma_6^{3(q-2)/2}(2\sigma)^{q-2}}{a^{3(q-2)/4}(\lambda M_0)^{(6-q)/4}}. \tag{3.9}$$

Here, M_0 is defined in (V_0), C_0 is defined in (2.8), and γ_6 is defined in (2.1). Then, $h_1(\lambda_0)\geqslant 0$ and for each $\lambda>\lambda_0$, there exists $\varepsilon_0>0$ such that

$$h_1(\lambda) = 1-\frac{1}{\lambda M_0}-\frac{C_0\gamma_6^{3(q-2)/2}[2(\sigma+\varepsilon_0)]^{q-2}}{a^{3(q-2)/4}(\lambda M_0)^{(6-q)/4}} := \delta_{\varepsilon_0}>0.$$

Since

$$\lim_{\nu,\tau\to 0^+}\left[\tau p' + \left(\tau^2{p'}^2+2p'(c_0+\nu)\right)^{1/2}\right] = \sigma,$$

there are $\nu_0,\tau_0>0$ such that $\tau_0 p' + \left(\tau_0^2{p'}^2+2p'(c_0+\nu_0)\right)^{1/2} < \sigma+\varepsilon_0$.

For each given $\lambda>\lambda_0$, let $\{u_{n_k}\},\{v_k\}$ and u be defined as in Lemmas 3.1 and 3.2. By (2.12) and Lemma 2.3, there is an $n_\lambda\in\mathbb{N}$ such that for all $k\geqslant n_\lambda$,

$$\frac{1}{2p'}\|u_{n_k}\|_\lambda^2 \leqslant J_\lambda(u_{n_k}) - \frac{1}{2p}\langle J'_\lambda(u_{n_k}),u_{n_k}\rangle \leqslant c_\lambda+\nu_0+\tau_0\|u_{n_k}\|_\lambda \leqslant c_0+\nu_0+\tau_0\|u_{n_k}\|_\lambda.$$

This implies that $\|u_{n_k}\|_\lambda \leqslant \tau_0 p' + \left(\tau_0^2 {p'}^2 + 2p'(c_0+\nu_0)\right)^{1/2} < \sigma + \varepsilon_0$. From Lemma 3.1, $u_{n_k} \rightharpoonup u$ in $H_V^1(\mathbb{R}^3)$ as $k \to \infty$, and by the weakly lower semi-continuity of the norm of $H_V^1(\mathbb{R}^3)$, we have $\|u\|_\lambda \leqslant \sigma + \varepsilon_0$. Therefore,

$$\|v_k\|_\lambda \leqslant \|u_{n_k}\|_\lambda + \|u\|_\lambda < 2(\sigma+\varepsilon_0) \tag{3.10}$$

for all $k \geqslant n_\lambda$.

From (V_0), $\Lambda = \{x \in \mathbb{R}^3 : V(x) < M_0\}$ is a set of finite measure, then we obtain that

$$|v_k|_2^2 = \int_\Lambda v_k^2 + \int_{\Lambda^c} v_k^2 \leqslant o(1) + \frac{1}{\lambda M_0}\|v_k\|_\lambda^2. \tag{3.11}$$

For each given $\lambda > \lambda_0$, in order to prove $\lim_{k\to\infty} \|v_k\|_\lambda \to 0$, it is sufficient to show $\limsup_{k\to\infty} \|v_k\|_\lambda = 0$. We prove by contradiction. Assume that to the contrary, we have $\limsup_{k\to\infty} \|v_k\|_\lambda > 0$. Then passing to a subsequence, we may assume $\|v_k\|_\lambda \to \mu > 0$ for all $k \in \mathbb{N}$. Letting $k \to \infty$ in (3.11), we have

$$\limsup_{k\to\infty} |v_k|_2^2 \leqslant \frac{\mu^2}{\lambda M_0}. \tag{3.12}$$

It follows from $J_\lambda'(v_k) \to 0$ (Lemma 3.2 (iv)), (2.8), (2.2) and (3.10) that for all $k \geqslant n_\lambda$,

$$\begin{aligned} o_\lambda(1) &= \|v_k\|_\lambda^2 + b\left(\int_{\mathbb{R}^3} |\nabla v_k|^2\right)^2 + l\int_{\mathbb{R}^3} (I * |v_k|^p)|v_k|^p - \int_{\mathbb{R}^3} f(v_k)v_k \\ &\geqslant \|v_k\|_\lambda^2 - |v_k|_2^2 - C_0|v_k|_q^q \\ &\geqslant \|v_k\|_\lambda^2 - |v_k|_2^2 - C_0|v_k|_2^{(6-q)/2}|v_k|_6^{3(q-2)/2} \\ &\geqslant \|v_k\|_\lambda^2 - |v_k|_2^2 - C_0\gamma_6^{3(q-2)/2}a^{-3(q-2)/4}[2(\sigma+\varepsilon_0)]^{q-2}|v_k|_2^{(6-q)/2}\|v_k\|_\lambda^{(q-2)/2}. \end{aligned} \tag{3.13}$$

Letting $k \to \infty$ and taking the lower limit on both sides of above inequality, and by (3.12), we have

$$0 \geqslant \left[1 - \frac{1}{\lambda M_0} - \frac{C_0\gamma_6^{3(q-2)/2}[2(\sigma+\varepsilon_0)]^{q-2}}{a^{3(q-2)/4}(\lambda M_0)^{(6-q)/4}}\right]\mu^2 = \delta_{\varepsilon_0}\mu^2,$$

which is a contradiction. Thus, $v_k \to 0$ in $H_V^1(\mathbb{R}^3)$ as $k \to \infty$ and then we obtain that $u_{n_k} \to u$ in $H_V^1(\mathbb{R}^3)$ as $k \to \infty$. The proof is complete. □

Now we prove the existence part of Theorem 1.1.

Proof of Theorem 1.1 (i). From Lemma 2.2, the energy functional J_λ possesses a mountain pass geometry. From Lemmas 3.1 and 3.3, J_λ satisfies the $(\mathrm{PS})_{c_\lambda}$ condition. Thus, from the mountain pass theorem, there exists a nontrivial critical point u_λ of J_λ when $\lambda > \lambda_0$, and the energy $J_\lambda(u_\lambda)$ is in the interval $[\beta, c_0]$. From the assumptions on f and standard arguments, we can see that the nontrivial solution u_λ is a nontrivial nonnegative ground state solution. □

We end this section by the following example in which we estimate the threshold value λ_0 for a set of specific functions f, V and parameters.

Example 3.4. We set

$$p = \frac{5}{2},\ q = \frac{11}{2},\ f(t) = t^{9/2},\ \ t \geqslant 0,\ \ V(x) = \begin{cases} \max\{|x|-1, 0\}, & |x| \leqslant 4, \\ 3, & |x| > 4. \end{cases} \tag{3.14}$$

It is easy to verify that conditions (V_0), (V_1), (f_0) and (f_1) are satisfied for the functions and parameters defined in (3.14). Similarly to the proof of Lemma 3.3, we have when $\lambda > \tilde{\lambda}_0$, which is defined as

$$\tilde{\lambda}_0 := \max\left\{\frac{\gamma_6^{42}(2\sigma)^{28}}{a^{21}M_0}, a\right\}, \tag{3.15}$$

and $k \geqslant n_\lambda$, then

$$\begin{aligned} o_\lambda(1) &= \|v_k\|_\lambda^2 + b\left(\int_{\mathbb{R}^3}|\nabla v_k|^2\right)^2 + l\int_{\mathbb{R}^3}(I*|v_k|^{5/2})|v_k|^{5/2} - \int_{\mathbb{R}^3} f(v_k)v_k \\ &\geqslant \|v_k\|_\lambda^2 - \gamma_6^{21/4}a^{-21/8}[2(\sigma+\varepsilon_0)]^{7/2}|v_k|_2^{1/4}\|v_k\|_\lambda^{7/4}. \end{aligned}$$

Letting $k \to \infty$ and taking the lower limit on both sides of above inequality, we have

$$0 \geqslant \left[1 - \frac{\gamma_6^{21/4}[2(\sigma+\varepsilon_0)]^{7/2}}{a^{21/8}(\lambda M_0)^{1/8}}\right]\mu^2 > 0,$$

which is a contradiction. Thus, we also obtain that $u_{n_k} \to u$ in $H_V^1(\mathbb{R}^3)$ when $\lambda > \tilde{\lambda}_0$.

We now further assume that $\alpha = 1$ to give a more precise estimate of $\tilde{\lambda}_0$ defined as in (3.15). We define a function $\eta(x) = \max\{1-|x|, 0\}$, then $\eta \in H_0^1(\mathbb{R}^3) \subseteq H_V^1(\mathbb{R}^3)$. One can estimate that

$$\begin{aligned} J(t\eta) &= \frac{1}{2}at^2\int_{\mathbb{R}^3}|\nabla\eta|^2 + \frac{1}{4}bt^4\left(\int_{\mathbb{R}^3}|\nabla\eta|^2\right)^2 + \frac{1}{5}lt^5\int_{\mathbb{R}^3}(I*|\eta|^{2/5})|\eta|^{2/5} - \int_{\mathbb{R}^3}F(t\eta) \\ &= \frac{2}{3}\pi a t^2 + \frac{4}{9}\pi^2 b t^4 + \frac{1}{5}lt^5\int_{|x|<1}(I*|\eta|^{2/5})|\eta|^{2/5} - \frac{2}{11}t^{11/2}\int_{|x|<1}\eta^{11/2}. \end{aligned} \tag{3.16}$$

From Lemma 2.1, we have

$$\begin{aligned} \int_{|x|<1}(I*|\eta|^{5/2})|\eta|^{5/2} &\leqslant C(3,2)\left(|\eta^{5/2}|_{L^{3/2}(\mathbb{R}^3)}\right)^2 = C(3,2)\left(\int_\Omega(1-|x|)^{15/4}\right)^{4/3} \\ &= 4\pi C(3,2)\left(\int_0^1 s^2(1-s)^{15/4}\mathrm{d}s\right)^{4/3} = 4\pi C(3,2)\left(\frac{128}{11799}\right)^{4/3}, \end{aligned} \tag{3.17}$$

$$C(3,2) = \pi\frac{\Gamma(1/2)}{\Gamma(2)}\left[\frac{\Gamma(3/2)}{\Gamma(3)}\right]^{-1/3} = 4^{1/3}\pi^{4/3} \approx 7.304 \tag{3.18}$$

and

$$\int_{|x|<1}\eta^{11/2} = \int_{|x|<1}(1-|x|)^{11/2} = 4\pi\int_0^1 s^{11/2}(1-s)^2\mathrm{d}s = \frac{64}{3315}\pi. \tag{3.19}$$

Combining (3.16), (3.17), (3.18) and (3.19), we obtain that

$$J(t\eta) \leqslant \frac{2}{3}\pi a t^2 + \frac{4}{9}\pi^2 b t^4 + \frac{4^{4/3}}{5}\pi^{7/3} l\left(\frac{128}{11799}\right)^{4/3} t^5 - \frac{128\pi}{36465}t^{11/2}. \tag{3.20}$$

We set

$$a = \frac{3}{4\pi}, \quad b = \frac{171}{4\pi^2}, \quad l = \frac{105(11799)^{4/3}}{8(128)^{4/3}\pi C(3,2)} \approx 238.178, \tag{3.21}$$

then (3.20) becomes

$$J(t\eta) \leqslant \frac{1}{2}t^2 + 19t^4 + \frac{21}{2}t^5 - \frac{128\pi}{36465}t^{11/2}, \qquad t \in [0, \infty).$$

Through numerical calculation, we obtain that

$$c_0 \leq \max_{t\in\mathbb{R}_+} J(t\eta) \approx 0.12.$$

Finally, we estimate that $\gamma_6 \leqslant K$, where K is the Talenti's best constant [1, Remark 4.32, p. 103–104]:

$$\begin{aligned} K &= \sup_{u\in\mathcal{D}^{1,2}\setminus\{0\}} \frac{|u|_6}{\|u\|_{\mathcal{D}^{1,2}}} = \pi^{-1/2}N^{-1/2}\left[\frac{\Gamma(N)}{\Gamma(N/2)}\right]^{1/3} \\ &= \pi^{-1/2}3^{-1/2}\left[\frac{\Gamma(3)}{\Gamma(3/2)}\right]^{1/3} = 3^{-1/2}4^{1/3}\pi^{-2/3} \approx 0.427. \end{aligned}$$

Now since $M_0 = 3$, $p' = 5/3$ and $\sigma = (2p'c_0)^{1/2}$, we have

$$\frac{\gamma_6^{42}(2\sigma)^{28}}{a^{21}M_0} \leq \frac{K^{42}(2\sigma)^{28}}{a^{21}M_0} \leq \frac{2^{70}5^{14}c_0^{14}}{3^{36}\pi^{28}a^{21}} \approx 0.858. \tag{3.22}$$

Evidently, $0.858 > a$. Thus, $\tilde{\lambda}_0 \leq 0.858$. This implies that when (3.14), (3.21) are satisfied and $\alpha = 1$, then (KH) has a nontrivial nonnegative ground state solution if $\lambda > 0.858$.

4. Concentration of nontrivial nonnegative solutions

In this section, we study the concentration behavior of nontrivial nonnegative ground state solutions obtained in Sect. 3 and prove part *(ii)* of Theorem 1.1, which is similar to the one in [6].

Proof of Theorem 1.1 (ii). From the existence result of nontrivial nonnegative ground state solutions to (KH) proved in part (i), for any sequence $\{\lambda_n\} \subset [\lambda_0, \infty)$ with $\lambda_n \to \infty$, there is a sequence of critical points $\{u_n\}$ of J_{λ_n} which are nontrivial nonnegative ground states of (KH). Similarly to the calculation in the proof of Lemmas 2.3 and 3.1, we obtain that

$$\frac{1}{2p'}\|u_n\|_{\lambda_n}^2 \leqslant J_{\lambda_n}(u_n) - \frac{1}{2p}\langle J'_{\lambda_n}(u_n), u_n\rangle = c_{\lambda_n} \leqslant c_0,$$

which implies that $\{\|u_n\|_{\lambda_n}\}$ is bounded, and indeed

$$\sup_{n\in\mathbb{N}} \|u_n\|_{\lambda_n}^2 \leqslant 2p'c_0. \tag{4.1}$$

Then without loss of generality, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u_0 \ \text{ in } H_V^1(\mathbb{R}^3), \\ &u_n \to u_0 \ \text{ in } L_{\mathrm{loc}}^s(\mathbb{R}^3), \quad s \in [1, 6), \\ &u_n \to u_0, \ \text{ a.e. on } \mathbb{R}^3. \end{aligned}$$

It follows from $V \geqslant 0$, the Fatou lemma and (4.1) that

$$\int_{\mathbb{R}^3} V u_0^2 \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^3} V u_n^2 \leqslant \liminf_{n\to\infty} \frac{\|u_n\|_{\lambda_n}^2}{\lambda_n} = 0.$$

By the condition (V_1), we get that $u_0(x) = 0$, a.e. $x \in \mathbb{R}^3 \setminus V^{-1}(0)$. It follows from $J'_{\lambda_n}(u_n) = 0$ that for any $v \in H_0^1(\Omega)$,

$$\int_\Omega a\nabla u_n \cdot \nabla v + b\int_{\mathbb{R}^3} |\nabla u_n|^2 \int_\Omega \nabla u_n \cdot \nabla v + l\int_\Omega (I * |u_n|^p)|u_n|^{p-2}u_n v - \int_\Omega f(u_n)v = 0. \tag{4.2}$$

In order to prove that u_0, nontrivial nonnegative function, satisfies (1.5), it is sufficient to show that for any $v \in H^1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} a\nabla u_0 \cdot \nabla v + b\int_{\mathbb{R}^3} |\nabla u_0|^2 \int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla v + l\int_{\mathbb{R}^3} (I * |u_0|^p)|u_0|^{p-2}u_0 v - \int_{\mathbb{R}^3} f(u_0)v = 0. \tag{4.3}$$

In fact, passing to a subsequence if necessary, we may assume that $A^2 := \lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2$ exists. It follows that $\int_{\mathbb{R}^3} |\nabla u_0|^2 \leqslant A^2$. By (4.2), we have

$$\int_{\mathbb{R}^3} a\nabla u_0 \cdot \nabla v + bA^2 \int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla v + l\int_{\mathbb{R}^3} (I * |u_0|^p)|u_0|^{p-2}u_0 v - \int_{\mathbb{R}^3} f(u_0)v = 0. \tag{4.4}$$

From (4.3) and (4.4), it is sufficient to prove that $A^2 = \int_{\mathbb{R}^3} |\nabla u_0|^2 = \int_{\Omega} |\nabla u_0|^2$. Since $J'_{\lambda_n}(u_n) = 0$, then

$$\int_{\mathbb{R}^3} \left[a|\nabla u_n|^2 + \lambda_n V u_n^2\right] + b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 + l\int_{\mathbb{R}^3} (I * |u_n|^p)|u_n|^p = \int_{\mathbb{R}^3} f(u_n)u_n. \tag{4.5}$$

Applying the Lions vanishing lemma ([33, Lemma 1.1 (ii)] or [46, Thoerem 1.21, p. 16]), we obtain that $u_n \to u_0$ in $L^s(\mathbb{R}^3)$ for $s \in (2, 6)$. Suppose that to the contrary, there exist $r > 0, \delta > 0$ and $\{x_n\} \subset \mathbb{R}^3$ such that

$$\int_{B_r(x_n)} (u_n - u_0)^2 \geqslant \delta. \tag{4.6}$$

Since $H^1_V(\mathbb{R}^3) \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^3)$ is compact, it follows that $\{x_n\}$ is unbounded. We then may assume that $|x_n| \to \infty$. It is easy to check that $m(B_r(x_n) \cap \Lambda) \to 0$ as $n \to \infty$ and consequently

$$\int_{B_r(x_n)\cap\Lambda} |u_n - u_0|^2 \leqslant |u_n - u_0|_4^2 [m(B_r(x_n) \cap \Lambda)]^{1/2} \to 0, \quad n \to \infty. \tag{4.7}$$

By (4.6) and (4.7), we get

$$\begin{aligned}
\|u_n\|_{\lambda_n}^2 &\geqslant \lambda_n M_0 \int_{B_r(x_n)\cap\Lambda^c} u_n^2 = \lambda_n M_0 \int_{B_r(x_n)\cap\Lambda^c} |u_n - u_0|^2 \\
&= \lambda_n M_0 \left(\int_{B_r(x_n)} |u_n - u_0|^2 - \int_{B_r(x_n)\cap\Lambda} |u_n - u_0|^2\right) \\
&= \lambda_n M_0 \left(\int_{B_\tau(x_n)} |u_n - u_0|^2 + o(1)\right) \to \infty.
\end{aligned}$$

This contradicts with (4.1). Hence we have $A^2 = \int_{\mathbb{R}^3} |\nabla u_0|^2 = \int_{\Omega} |\nabla u_0|^2$, and (4.3) holds.

Since $u_n \to u_0$ in $L^q(\mathbb{R}^3)$ as $n \to \infty$, it follows from (2.7) that $f(u_n) \to f(u_0)$ in $L^{q/(q-1)}(\Omega)$ as $n \to \infty$. Thus, for any given $\varepsilon > 0$,

$$\begin{aligned}
&\left| \int\limits_{\mathbb{R}^3} f(u_n)u_n - \int\limits_{\mathbb{R}^3} f(u_0)u_0 \right| \\
&\quad \leqslant \int\limits_{\mathbb{R}^3} |f(u_n)||u_n - u_0| + \int\limits_{\mathbb{R}^3} |f(u_n) - f(u_0)||u_0| \\
&\quad \leqslant \varepsilon \int\limits_{\mathbb{R}^3} |u_n||u_n - u_0| + C_\varepsilon \int\limits_{\mathbb{R}^3} |u_n|^{q-1}|u_n - u_0| + \int\limits_{\Omega} |f(u_n) - f(u_0)||u_0| \\
&\quad \leqslant \varepsilon |u_n|_2 |u_n - u_0|_2 + C_\varepsilon |u_n|_q^{q-1} |u_n - u_0|_q + |f(u_n) - f(u_0)|_{q/(q-1)} |u_0|_q \\
&\quad = \varepsilon |u_n|_2 |u_n - u_0|_2 + o(1).
\end{aligned}$$

This implies that

$$\int\limits_{\mathbb{R}^3} f(u_n)u_n \to \int\limits_{\mathbb{R}^3} f(u_0)u_0, \quad n \to \infty. \tag{4.8}$$

From the fact that $u_n \to u_0$ in $L^s(\mathbb{R}^3)$ as $n \to \infty$ for $s \in (2, 6)$, Lemma 3.2 (i) and Lemma 2.1, we can make the claim that

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^3} (I * |u_n|^p)|u_n|^p = \int\limits_{\mathbb{R}^3} (I * |u_0|^p)|u_0|^p. \tag{4.9}$$

Hence, by (4.8) and (4.9), letting $n \to \infty$ for (4.5), we have

$$(a + bA^2)A^2 + \lim_{n\to\infty} \int\limits_{\mathbb{R}^3} \lambda_n V u_n^2 + l \int\limits_{\mathbb{R}^3} (I * |u_0|^p)|u_0|^p = \int\limits_{\mathbb{R}^3} f(u_0)u_0. \tag{4.10}$$

Taking $v = u_0$ in (4.4), we have

$$\int\limits_{\mathbb{R}^3} a|\nabla u_0|^2 + bA^2 \int\limits_{\mathbb{R}^3} |\nabla u_0|^2 + l \int\limits_{\mathbb{R}^3} (I * |u_0|^p)|u_0|^p = \int\limits_{\mathbb{R}^3} f(u_0)u_0. \tag{4.11}$$

It follows from (4.10) and (4.11) that $\lim_{n\to\infty} \int\limits_{\mathbb{R}^3} \lambda_n V u_n^2 = 0$ and $\int\limits_{\mathbb{R}^3} |\nabla u_0|^2 = A^2$.

Finally, we show that $u_n \to u_0$ in $H_V^1(\mathbb{R}^3)$. Indeed, it follows from $J'_{\lambda_n}(u_n) = 0$ that

$$a\|u_n\|^2 \leqslant \|u_n\|_{\lambda_n}^2 = \int\limits_{\mathbb{R}^3} f(u_n)u_n - b\left(\int\limits_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 - l \int\limits_{\mathbb{R}^3} (I * |u_n|^p)|u_n|^p, \tag{4.12}$$

$$\langle u_n, u_0 \rangle_{\lambda_n} = \int\limits_{\mathbb{R}^3} f(u_n)u_0 - b \int\limits_{\mathbb{R}^3} |\nabla u_n|^2 \int\limits_{\mathbb{R}^3} \nabla u_n \cdot \nabla u_0 - l \int\limits_{\mathbb{R}^3} (I * |u_n|^p)|u_n|^{p-2}u_n u_0.$$

Since $\lim_{n\to\infty} \int\limits_{\mathbb{R}^3} |\nabla u_n|^2 = \int\limits_{\mathbb{R}^3} |\nabla u_0|^2$, it follows from (4.11) that

$$a\|u_0\|^2 = \int\limits_{\mathbb{R}^3} f(u_0)u_0 - b\left(\int\limits_{\mathbb{R}^3} |\nabla u_0|^2 \right)^2 - l \int\limits_{\mathbb{R}^3} (I * |u_0|^p)|u_0|^p. \tag{4.13}$$

Hence, we obtain from (4.12) and (4.13) that

$$\begin{aligned}\limsup_{n\to\infty} a\|u_n\|^2 &\leqslant \limsup_{n\to\infty}\left[\int_{\mathbb{R}^3} f(u_n)u_n - b\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^2 - l\int_{\mathbb{R}^3}(I*|u_n|^p)|u_n|^p\right]\\ &= \int_{\mathbb{R}^3} f(u_0)u_0 - b\left(\int_{\mathbb{R}^3}|\nabla u_0|^2\right)^2 - l\int_{\mathbb{R}^3}(I*|u_0|^p)|u_0|^p\\ &= a\|u_0\|^2.\end{aligned}$$

Thus, $u_n \to u_0$ in $H^1_V(\mathbb{R}^3)$ and also in $H^1(\mathbb{R}^3)$ as $n\to\infty$. Hence, $\int_{\mathbb{R}^3} F(u_n) \to \int_{\Omega} F(u_0)$. The fact that $c_\lambda \in [\beta, c_0]$ implies that u_0 is a nontrivial nonnegative function. This completes the proof. □

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