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UNIQUENESS OF POSITIVE SOLUTIONS TO SOME COUPLED COOPERATIVE VARIATIONAL ELLIPTIC SYSTEMS

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Abstract. The uniqueness of positive solutions to some semilinear elliptic systems with variational structure arising from mathematical physics is proved. The key ingredient of the proof is the oscillatory behavior of solutions to linearized equations for cooperative semilinear elliptic systems of two equations on one-dimensional domains, and it is shown that the stability of the positive solutions for such a semilinear system is closely related to the oscillatory behavior.

1. Introduction

Systems of nonlinear elliptic type partial differential equations arise from many models in mathematical physics, such as the nonlinear static Chern-Simons-Higgs equations of classical field theory [9, 12, 16, 19–21, 59, 60, 68], and standing wave solutions of coupled nonlinear Schödinger equations from Bose–Einstein condensation [1, 11, 47, 58, 63, 65]. In the case of two interacting particles or waves, the static equation is in the form

$$\Delta u_1 + f(u_1,u_2) = 0, \quad \Delta u_2 + g(u_1,u_2) = 0, \quad x \in \Omega, \tag{1.1}$$

where Ω is $\mathbb{R}^n$ or a bounded domain in $\mathbb{R}^n$. While the existence of positive solutions to (1.1) have been obtained through various variational or other methods, the uniqueness or exact multiplicity of solutions have been mostly open.

In this article, we provide a rather general approach for proving the uniqueness of positive solutions to the system in one-dimensional space. To achieve this, we prove some general properties of the associated linearized system which resembles the classic Sturm comparison principle, and with these properties, for some important systems with a variational structure, we prove the uniqueness of the solution of

$$\begin{cases} u_1'' + f(u_1,u_2) = 0, & x \in \mathbb{R}, \\ u_2'' + g(u_1,u_2) = 0, & x \in \mathbb{R}, \\ u_1(x) > 0, \ \ u_2(x) > 0, & x \in \mathbb{R}, \\ u_1(x) \to 0, \ \ u_2(x) \to 0, & |x| \to \infty, \end{cases} \tag{1.2}$$

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or the solution of the related Dirichlet boundary value problem:

$$\begin{cases} u_1'' + f(u_1,u_2) = 0, & -R < x < R, \\ u_2'' + g(u_1,u_2) = 0, & -R < x < R, \\ u_1(x) > 0, \ \ u_2(x) > 0, & -R < x < R, \\ u_1(\pm R) = 0, \ \ u_2(\pm R) = 0. \end{cases} \tag{1.3}$$

Define $\mathbb{R}^2_+ = \{(u_1,u_2) : u_1 \geq 0, u_2 \geq 0\}$. Throughout the paper, we assume that the nonlinear functions f, g in (1.2) and (1.3) satisfy

(f1) $f, g \in C^1(\mathbb{R}^2_+)$.

(f2) (Cooperativeness) Define the Jacobian of the vector field (f,g) to be

$$J(u_1,u_2) = \begin{pmatrix} \dfrac{\partial f}{\partial u_1}(u_1,u_2) & \dfrac{\partial f}{\partial u_2}(u_1,u_2) \\ \dfrac{\partial g}{\partial u_1}(u_1,u_2) & \dfrac{\partial g}{\partial u_2}(u_1,u_2) \end{pmatrix} \equiv \begin{pmatrix} f_1(u_1,u_2) & f_2(u_1,u_2) \\ g_1(u_1,u_2) & g_2(u_1,u_2) \end{pmatrix}. \tag{1.4}$$

Then (f,g) is said to be *cooperative* if $f_2(u_1,u_2) \geq 0$ and $g_1(u_1,u_2) \geq 0$ for $(u_1,u_2) \in \mathbb{R}^2_+$, and $f_2(u_1,u_2) > 0$ and $g_1(u_1,u_2) > 0$ for $(u_1,u_2) \in int(\mathbb{R}^2_+)$.

Under conditions (f1) and (f2), it is well known that a positive solution $(u_1(x), u_2(x))$ of (1.3) must be an even function in the sense that $u_i(-x) = u_i(x)$ and $u_i'(x) < 0$ for $x \in (0,R)$ (see [64]), and the symmetry properties for positive solutions to (1.2) have also been established in [8,33,50] under some additional assumptions on f and g at $(u_1,u_2) = (0,0)$. These works are natural extensions of the classical results in [37,38] for the scalar equation since the maximum principle also holds for elliptic systems with cooperative nonlinearities [28,32,61].

Our first result is a Sturm comparison type result for positive solutions to system (1.2) or (1.3). We note that the Sturm comparison lemma can be regarded as another aspect of a maximum principle in one-dimensional space. A simplified version of the classical Sturm comparison lemma is: suppose that $w_1(x)$ and $w_2(x)$ are two linear independent solutions of $w'' + q(x)w = 0$, where q is continuous on $[a,b]$ and $w_1(a) = w_1(b) = 0$; then w_2 has a zero in (a,b). A straightforward application of this lemma is for a solution of

$$u'' + g(u) = 0, \ \ x \in (0,R), \ \ u'(0) = 0, \ u(0) = \alpha \tag{1.5}$$

such that $u(x) > 0$, $u'(x) < 0$ in $(0,R)$. Then any solution ϕ of the linearized equation

$$\phi'' + g'(u(x))\phi = 0, \ \ x \in (0,R) \tag{1.6}$$

changes sign at most once in $(0,R)$ since $u'(x)$ is also a solution of (1.6), $u'(0) = 0$, and $u'(x) < 0$ in $(0,R)$. Our result for solutions of a linearized equation around a positive solution to (1.2) or (1.3) resembles the one above for the scalar equation. More precisely, suppose that f, g satisfy (f1) and (f2), (u_1,u_2) is a positive solution of (1.2) or (1.3), and f_1, f_2, g_1, g_2 are as defined in (1.4). We prove that if (ϕ_1,ψ_1) (resp., (ϕ_2,ψ_2)) is a solution of

$$\begin{cases} \phi'' + f_1\phi + f_2\psi = 0, & 0 < x < R, \\ \psi'' + g_1\phi + g_2\psi = 0, & 0 < x < R, \\ \phi'(0) = \psi'(0) = 0, & \\ (\phi(0),\psi(0)) = (1,0), \ \ (\text{resp.,} \ \ (0,1)), & \end{cases} \tag{1.7}$$

then each of ϕ_i and ψ_i ($i = 1, 2$) changes sign at most once in $(0, R)$ (see Lemma 2.2 and Corollary 2.3 for more precise statements). It is well known that oscillatory properties of solutions to the linearized equation is critical for the stability and uniqueness of positive solutions of semilinear elliptic equations [3, 44, 53, 54]. Hence the nonoscillatory property for solutions of (1.7) is very useful for the stability and uniqueness of positive solutions to (1.2) or (1.3). We remark that such a property usually does not hold for the higher-dimensional radial Laplacian $Lu = r^{1-n}(r^{n-1}u')'$ even for the scalar case; hence the spatial dimension $n = 1$ is a critical assumption here.

Our second result is related to the stability of a positive solution to (1.2) or (1.3). It is known that for (1.5), the number of sign-changes of the solution of the linearized equation is related to the stability of a positive solution. If the solution of the linearized equation does not change sign, then the positive solution is linearly stable; while if the solution of the linearized equation changes sign once, then the positive solution is linearly unstable. Here we also establish such a connection between the number of sign-changes of the solution to the linearized equation (1.7) and the stability of a positive solution of the system (1.2). More precisely, we show that if f, g satisfy (f1) and (f2), a positive solution (u_1, u_2) of (1.3) is stable if there exists a real number $c > 0$ such that $\phi_1 + c\phi_2 > 0$ and $\psi_1 + c\psi_2 > 0$ in $(0, R]$, where (ϕ_i, ψ_i) is the solution of (1.7), and (u_1, u_2) is unstable if and only if for all $c \geq 0$, at least one of $\phi_1 + c\phi_2$ or $\psi_1 + c\psi_2$ is not positive in $(0, R)$ (see Proposition 2.8).

The nonoscillatory results above are proved under rather general conditions (f1) and (f2) on (f, g), and these results pave the way for the stability, nondegeneracy, and uniqueness of the positive solution to (1.2) or (1.3) from a wide range of applications. Two additional structures on (f, g) would be needed for these further results: (i) the growth rate of functions f and g; and (ii) a variational structure for the vector field (f, g).

The growth rate of the functions f and g plays an important role in the qualitative behavior of the solutions to (1.2) and (1.3). Here we define several conditions on the growth rate of f and g:

(f3) (Superlinear) The vector field (f, g) is said to be *superlinear* if for all $(u_1, u_2) \in \mathbb{R}^2_+$,

$$f_1u_1 + f_2u_2 - f \geq 0, \quad g_1u_1 + g_2u_2 - g \geq 0. \tag{1.8}$$

(f3') (Strongly superlinear) The vector field (f, g) is said to be *strongly superlinear* if for all $(u_1, u_2) \in \mathbb{R}^2_+$,

$$f_1u_1 - f \geq 0, \quad g_2u_2 - g \geq 0. \tag{1.9}$$

(f4) (Sublinear) The vector field (f, g) is said to be *sublinear* if for all $(u_1, u_2) \in \mathbb{R}^2_+$,

$$f_1u_1 + f_2u_2 - f \leq 0, \quad g_1u_1 + g_2u_2 - g \leq 0. \tag{1.10}$$

(f4') (Weakly sublinear) The vector field (f, g) is said to be *weakly sublinear* if for all $(u_1, u_2) \in \mathbb{R}^2_+$,

$$f_1u_1 - f \leq 0, \quad g_2u_2 - g \leq 0. \tag{1.11}$$

We remark that all definitions above are actually for a single function $f : \mathbb{R}^2_+ \to \mathbb{R}$, but we assume that f and g have the same type of growth rate in this article. Note that under the cooperativeness assumption (f2), condition (f3') implies (f3), while condition (f4) implies (f4'), which is the reason for the "strongly" and "weakly" in the definition. On the other hand, we notice that a function f can be both weakly sublinear and superlinear. A prototypical example of a cooperative type function f is

$$f(u_1,u_2) = \lambda u_1 + \mu u_2 + \sum_{i=1}^{k} a_i u_1^{p_i} u_2^{q_i}, \tag{1.12}$$

where $\lambda \in \mathbb{R}$, $\mu \in [0,\infty)$, $a_i > 0$, and $p_i, q_i \geq 0$ for all $1 \leq i \leq k$. Under these general conditions, it is easy to verify that

$p_i + q_i \geq 1 \Rightarrow f$ is superlinear; $p_i \geq 1$, $\mu = 0 \Rightarrow f$ is strongly superlinear;

$p_i + q_i \leq 1 \Rightarrow f$ is sublinear; $p_i \leq 1 \Rightarrow f$ is weakly sublinear.

Hence f is both weakly sublinear and superlinear if $0 \leq p_i \leq 1$, $0 \leq q_i$, and $p_i + q_i \geq 1$, which always holds when $p_i = 1$. The notion of superlinear and sublinear growth rate for a uni-variable function $g : \mathbb{R}_+ \to \mathbb{R}$ was considered in [54], and the definition here can be considered as a generalization of the definition in [54] to multivariable functions. Similar definitions of multi-variable sublinear/superlinear functions can also be found in [18, 25, 57]. It is known that sublinear/superlinear properties are related to the stability of positive solutions of (1.3). We prove that when (f,g) is sublinear, then any positive solution of (1.3) is stable, while when (f,g) is superlinear, then any positive solution of (1.3) is unstable (see Lemma 2.5 and also [25]).

The final assumption for the nondegeneracy and uniqueness of positive solutions is the variational structure on the vector field (f,g). Here two possible variational structures can be defined as in [29, 30]. The system (1.2) or (1.3) is a *Hamiltonian system* if there exists a differentiable function $H(u_1,u_2)$ such that

$$f(u_1,u_2) = \frac{\partial H(u_1,u_2)}{\partial u_2} \quad \text{and} \quad g(u_1,u_2) = \frac{\partial H(u_1,u_2)}{\partial u_1}. \tag{1.13}$$

Clearly a Hamiltonian system satisfies $f_1 = g_2$. For a Hamiltonian system, if $(u_1(x), u_2(x))$ is a solution of (2.3), we define

$$H_0(x) = u_1'(x)u_2'(x) + H(u_1(x), u_2(x)). \tag{1.14}$$

Then $H_0'(x) = 0$ and hence $H_0(x) \equiv H_0(0)$ for $x > 0$. On the other hand, the system (1.2) or (1.3) is a *gradient system* if there exists a differentiable function $F(u_1,u_2)$ such that

$$f(u_1,u_2) = \frac{\partial F(u_1,u_2)}{\partial u_1} \quad \text{and} \quad g(u_1,u_2) = \frac{\partial F(u_1,u_2)}{\partial u_2}. \tag{1.15}$$

Clearly a gradient system satisfies $f_2 = g_1$; hence the Jacobian matrix is symmetric and the corresponding linearized equation is self-adjoint. For a gradient system, if $(u_1(x), u_2(x))$ is a solution of (2.3), we define

$$F_0(x) = \frac{1}{2}[u_1'(x)]^2 + \frac{1}{2}[u_2'(x)]^2 + F(u_1(x), u_2(x)). \tag{1.16}$$

Then $F_0'(x) = 0$ and hence $F_0(x) \equiv F_0(0)$ for $x > 0$. Both the energy functions H_0 and F_0 are generalizations of the energy function $G_0(x) = \frac{1}{2}[u'(x)]^2 + G(u(x))$ for (1.5) where $G(u) = \int_0^u g(s)ds$. For the scalar equation

$$u'' + g(u) = 0,\ x \in \mathbb{R},\ u'(0) = 0,\ \lim_{|x|\to\infty} u(x) = 0, \tag{1.17}$$

the energy function G_0 alone can guarantee the uniqueness of positive solutions to (1.17), which in general is not the case for the system (1.3). But with the Hamiltonian or gradient structure, the uniqueness of positive solutions to (1.3) can be proved by combining with oscillatory property.

Our third key result is that if we assume (u_1, u_2) is a positive solution of (1.3), (f, g) satisfies (f1) and (f2), (f, g) is superlinear (thus (u_1, u_2) is unstable), and in addition, if (f, g) is weakly sublinear and is a Hamiltonian or gradient system, then (u_1, u_2) must be nondegenerate, which often suggests uniqueness. Combining the cooperative and variational structure, and the weakly sublinear and superlinear properties, we can prove that the positive solution to (1.3) for certain (f, g) is unique for any given $R > 0$, and it also implies the uniqueness of positive solutions to (1.2) when it exists. More precisely, it can be shown that all solutions of (1.3) can be represented by a curve $N = \{(\alpha, \beta(\alpha), R(\alpha)) : \alpha \in A\}$, where A is a subset of $\mathbb{R}_+$. For a given α in the admissible set A, there is a unique $\beta(\alpha) > 0$ such that (1.3) has a unique positive solution (u_1, u_2) satisfying $u_1(0) = \alpha$, $u_2(0) = \beta(\alpha)$, and $R = R(\alpha)$. Moreover, it is shown that $\beta(\alpha)$ is a strictly increasing function of α and that $R(\alpha)$ is a strictly increasing (decreasing) function of α when (f, g) is sublinear (superlinear). The monotonicity of $R(\alpha)$ implies the uniqueness of positive solutions to (1.3) for a given $R > 0$ (see Section 3 for more precise results).

In Section 3, this program of proving uniqueness of positive solutions to (1.3) is achieved for

(A) (Schrödinger type [31, 33, 39]) $f(u_1, u_2) = -u_1 + u_2^q$, $g(u_1, u_2) = -u_2 + u_1^p$, where $p, q > 1$.
(B) (optics model [48, 49, 69]) $f(u_1, u_2) = -bu_1 + u_1u_2$, $g(u_1, u_2) = -cu_2 + u_1^2/2$, where $b, c > 0$.

Here system (A) is a Hamiltonian system and (B) is a gradient system. The more general results established in Section 2 can be applied to prove the uniqueness of positive solutions of (1.3) as long as (f, g) is (i) cooperative, (ii) weakly sublinear and superlinear, and (iii) Hamiltonian or gradient.

The uniqueness of positive solutions of (1.3) for the sublinear (f, g) case holds for a general bounded domain in $\mathbb{R}^n$ with $n \geq 1$; see [4, 18, 25]. Some methods for the uniqueness of the positive solutions of semilinear elliptic systems used in this paper were also used in some of our earlier work [13, 15–17, 19, 20], but the oscillatory results for the linearized equation here are the most general ones as they do not rely on specific algebraic form of the nonlinearities f and g. We expect these results will be useful for the uniqueness of the positive solutions to other cooperative systems. Previously the uniqueness of the positive solutions was generally only known for the Lane–Emden system in which the nonlinearities are power functions; see [26, 27, 41, 43, 56]. Recently the uniqueness of positive solutions to the coupled Schrödinger equations in some special cases was proved in [11, 40, 51, 67].

The remaining part of the paper is organized as follows. In Section 2, we consider the properties of solutions to linearized equations, and the relation between the stability of solutions and nodal properties of solutions to linearized equations is considered. The uniqueness of positive solutions to several semilinear cooperative elliptic systems with weakly sublinear and superlinear nonlinearities are studied in Section 3.

2. Linearized equations and stability

2.1. **Preliminaries.** In the following we always assume that (f, g) satisfies the conditions (f1) and (f2). With a possible translation, a solution of (1.2) or (1.3) is necessarily symmetric with respect to $x = 0$ and decreasing when $x > 0$ [8, 64]; hence it satisfies either

$$\begin{cases} u_1'' + f(u_1, u_2) = 0, & 0 < x < \infty, \\ u_2'' + g(u_1, u_2) = 0, & 0 < x < \infty, \\ u_1(x) > 0, \;\; u_2(x) > 0, \;\; u_1'(x) < 0, \;\; u_2'(x) < 0, & 0 < x < \infty, \\ u_1'(0) = u_2'(0) = 0, & \\ u_1(x) \to 0, \;\; u_2(x) \to 0, & x \to \infty \end{cases} \tag{2.1}$$

or

$$\begin{cases} u_1'' + f(u_1, u_2) = 0, & 0 < x < R, \\ u_2'' + g(u_1, u_2) = 0, & 0 < x < R, \\ u_1(x) > 0, \;\; u_2(x) > 0, \;\; u_1'(x) < 0, \;\; u_2'(x) < 0, & 0 < x < R, \\ u_1'(0) = u_2'(0) = 0, & \\ u_1(R) = 0, \;\; u_2(R) = 0. & \end{cases} \tag{2.2}$$

A solution $(u_1(x), u_2(x))$ of (2.1) or (2.2) is a solution of the initial value problem

$$\begin{cases} u_1'' + f(u_1, u_2) = 0, & x > 0, \\ u_2'' + g(u_1, u_2) = 0, & x > 0, \\ u_1'(0) = u_2'(0) = 0, & \\ u_1(0) = \alpha, \;\; u_2(0) = \beta, & \end{cases} \tag{2.3}$$

where $\alpha > 0$ and $\beta > 0$. The local existence and uniqueness of the solution of (2.3) can be proved via a standard argument. We denote the solution of (2.3) by $(u_1(x; \alpha, \beta), u_2(x; \alpha, \beta))$ or simply $(u_1(x), u_2(x))$ when there is no confusion. We will only consider a solution of (2.3) satisfying $u_i(x) > 0$ and $u_i'(x) < 0$ for $x \in (0, \delta)$, $i = 1, 2$. This requires the initial value (α, β) to satisfy $f(\alpha, \beta) \geq 0$ and $g(\alpha, \beta) \geq 0$, and at least one of them must be strictly greater than zero. A solution of (2.2) is a *crossing solution* of (2.3), and a solution of (2.1) is a *ground state solution*. The local solution $(u_1(x), u_2(x))$ of (2.3) can be extended to a maximal interval $(0, R(\alpha, \beta))$ so that $u_i(x) > 0$ and $u_i'(x) < 0$ in $(0, R(\alpha, \beta))$, $i = 1, 2$. In the following we will use $R = R(\alpha, \beta)$ when there is no confusion.

If $R < \infty$, then, for technical reasons, we need to extend the solution beyond $x = R$. We extend the definition of (f, g) to $\mathbb{R}^2$ so that $f, g \in C^1(\mathbb{R}^2)$. Hence the solution $(u_1(x), u_2(x))$ can be extended to $x \in [0, R + \varepsilon]$ for a positive $\varepsilon = \varepsilon(\alpha, \beta)$ with u_i or $-u_i'$ possibly negative in $(R, R + \varepsilon]$. In the following we shall always assume that the domain of a solution $(u_1(x), u_2(x))$ is $x \in [0, R + \varepsilon]$ when $R < \infty$.

To consider the dependence of solutions on the initial values, we consider the linearized equation of $(u_1(x), u_2(x))$ with respect to the initial value (α, β). Let $W(x)$ be a 2×2 matrix function defined as

$$W(x) \equiv \begin{pmatrix} \dfrac{\partial u_1(x; \alpha, \beta)}{\partial \alpha} & \dfrac{\partial u_1(x; \alpha, \beta)}{\partial \beta} \\ \dfrac{\partial u_2(x; \alpha, \beta)}{\partial \alpha} & \dfrac{\partial u_2(x; \alpha, \beta)}{\partial \beta} \end{pmatrix}. \tag{2.4}$$

Then $W(x)$ satisfies

$$\begin{cases} W'' + J(u_1(x), u_2(x))W = 0, & x > 0, \\ W(0) = I, \quad W'(0) = 0. \end{cases} \tag{2.5}$$

Here I is the identity matrix, J is defined in (1.4), and W' and W'' are the matrices of derivatives of each entry W_{ij} of W with respect to x. Equivalently, we can write the componentwise equation: let $(\phi_1, \psi_1) = \left(\frac{\partial u_1(x;\alpha,\beta)}{\partial\alpha}, \frac{\partial u_2(x;\alpha,\beta)}{\partial\alpha}\right)$; then (ϕ_1, ψ_1) satisfies

$$\begin{cases} \phi_1'' + f_1\phi_1 + f_2\psi_1 = 0, & x > 0, \\ \psi_1'' + g_1\phi_1 + g_2\psi_1 = 0, & x > 0, \\ \phi_1(0) = 1, \quad \phi_1'(0) = 0, \\ \psi_1(0) = 0, \quad \psi_1'(0) = 0; \end{cases} \tag{2.6}$$

and let $(\phi_2, \psi_2) = \left(\frac{\partial u_1(x;\alpha,\beta)}{\partial\beta}, \frac{\partial u_2(x;\alpha,\beta)}{\partial\beta}\right)$; then (ϕ_2, ψ_2) satisfies

$$\begin{cases} \phi_2'' + f_1\phi_2 + f_2\psi_2 = 0, & x > 0, \\ \psi_2'' + g_1\phi_2 + g_2\psi_2 = 0, & x > 0, \\ \phi_2(0) = 0, \quad \phi_2'(0) = 0, \\ \psi_2(0) = 1, \quad \psi_2'(0) = 0. \end{cases} \tag{2.7}$$

To study the oscillatory behavior of (ϕ_i, ψ_i) $(i = 1, 2)$, we first consider two auxiliary equations. Let (ϕ_3, ψ_3) be the unique solution of

$$\begin{cases} \phi_3'' + f_1\phi_3 = 0, & x > 0, \\ \psi_3'' + g_2\psi_3 = 0, & x > 0, \\ \phi_3(0) = 1, \quad \phi_3'(0) = 0, \\ \psi_3(0) = 1, \quad \psi_3'(0) = 0. \end{cases} \tag{2.8}$$

The oscillatory behavior of (ϕ_3, ψ_3) can be obtained as follows.

Lemma 2.1. *Let (u_1, u_2) be a solution of* (2.3) *such that $u_1(x) > 0$, $u_2(x) > 0$, $u_1'(x) < 0$, and $u_2'(x) < 0$ for $x \in (0, R)$ (R may be ∞), and let (ϕ_3, ψ_3) be defined as in* (2.8). *Assume that (f, g) is cooperative as defined in* (f2). *Then*

(1) *Each of ϕ_3 and ψ_3 has at most one zero in $(0, R]$.*
(2) *If, in addition, (f, g) is weakly sublinear and (u_1, u_2) is a solution of* (2.2), *then each of ϕ_3 and ψ_3 is positive in $(0, R]$.*
(3) *If, in addition, (f, g) is strongly superlinear and (u_1, u_2) is a solution of* (2.2), *then each of ϕ_3 and ψ_3 changes sign exactly once in $(0, R)$, and $\phi_3(R) < 0$, $\psi_3(R) < 0$.*

Proof.
(1) We only prove it for ϕ_3, and the proof for ψ_3 is similar. Suppose that ϕ_3 changes sign more than once. Let $0 < x_1 < x_2 \le R$ be the first two zeros of ϕ_3. From the equation which ϕ_3 satisfies, each zero of ϕ_3 is a simple one. Then $\phi_3(x) < 0$ in (x_1, x_2), $\phi_3'(x_1) < 0$, and $\phi_3'(x_2) > 0$. Notice that (u_1', u_2') satisfies

$$\begin{cases} (u_1')'' + f_1u_1' + f_2u_2' = 0, & x > 0, \\ (u_2')'' + g_1u_1' + g_2u_2' = 0, & x > 0, \\ u_1'(0) = 0, \quad u_2'(0) = 0. \end{cases} \tag{2.9}$$

Then multiplying the equation of ϕ_3 in (2.8) by u_1', multiplying the equation of u_1' in (2.9) by ϕ_3 and subtracting and integrating on (x_1, x_2), we obtain

$$\phi_3'(x_2)u_1'(x_2) - \phi_3'(x_1)u_1'(x_1) = \int_{x_1}^{x_2} f_2\phi_3 u_2' dx. \tag{2.10}$$

The left-hand side of (2.10) is nonpositive since $u_1'(x) < 0$ in $(0, R)$, while the right-hand side of (2.10) is positive since $f_2 > 0$, $\phi_3 < 0$, and $u_2' < 0$ in (x_1, x_2). That is a contradiction. Hence ϕ_3 cannot have more than one zero in $(0, R]$.

(2) Next we assume that (f, g) is weakly sublinear and (u_1, u_2) is a solution of (2.2). Suppose that ϕ_3 has a zero $x_1 \in (0, R]$; then $\phi_3(x) > 0$ for $x \in (0, x_1)$ and $\phi_3(x_1) = 0$. We rewrite the equations in (2.2) to

$$\begin{cases} u_1'' + f_1 u_1 = f_1 u_1 - f, & x > 0, \\ u_2'' + g_2 u_2 = g_2 u_2 - g, & x > 0. \end{cases} \tag{2.11}$$

Multiplying the equation of ϕ_3 in (2.8) by u_1, multiplying the equation of u_1 in (2.11) by ϕ_3, and subtracting and integrating on $(0, x_1)$, we have

$$\phi_3'(x_1)u_1(x_1) = -\int_0^{x_1} (f_1 u_1 - f)\phi_3 dx. \tag{2.12}$$

Then the left-hand side of (2.12) is negative, while the right-hand side of (2.12) is nonnegative. This is a contradiction, so $\phi_3(x) > 0$ for $x \in (0, R]$.

(3) Now we also assume (f, g) is strongly superlinear and (u_1, u_2) is a solution of (2.2). Suppose that ϕ_3 does not change sign; then $\phi_3(x) > 0$ for $x \in (0, R)$. Similar to (2), we have

$$-\phi_3(R)u_1'(R) = -\int_0^{R} (f_1 u_1 - f)\phi_3 dx. \tag{2.13}$$

Then the left-hand side of (2.13) is nonnegative, while the right-hand side of (2.13) is negative. This contradiction implies that ϕ_3 has to change sign in $(0, R)$, and $\phi_3(R) < 0$ since ϕ_3 cannot have more than one zero from the previous proof. □

The following result provides a key oscillatory result for the solutions of the linearized equations (2.6) and (2.7).

Lemma 2.2. *Let (u_1, u_2) be a solution of (2.3) such that $u_1(x) > 0$, and $u_2(x) > 0$ for $x \in (0, R)$, and $u_1'(x) < 0$ and $u_2'(x) < 0$ for $x \in (0, R]$ (when R is ∞, then $u_i'(x) < 0$ for $x \in (0, R)$), and let ϕ_i, ψ_i $(i = 1, 2, 3)$ be defined as in (2.6), (2.7) and (2.8). Assume that (f, g) is cooperative as defined in (f2). Then:*

(1) *$\phi_1(x)$ changes sign at most once, and $\psi_1(x) < 0$ for $x \in (0, R)$; if in addition $\phi_3(x) > 0$, then $\phi_1(x) > 0$ for $x \in (0, R)$;*
(2) *$\psi_2(x)$ changes sign at most once, and $\phi_2(x) < 0$ for $x \in (0, R)$; if in addition $\psi_3(x) > 0$, then $\psi_2(x) > 0$ for $x \in (0, R)$.*

Proof. If $R < \infty$, then from our earlier comment, we extend the definition of the solution to $[0, R + \varepsilon]$, and since $u_i'(R) < 0$, then we can assume that $u_i'(x) < 0$ for $x \in (0, R + \varepsilon]$. In the following we only prove the result for ϕ_1 and ψ_1, and the proof for the other case is similar. Since $\phi_1(0) > 0$, $\psi_1(0) = 0$, $\psi_1'(0) = 0$, and $\psi_1''(0) = -g_2\psi_1(0) - g_1\phi_1(0) < 0$. Then for some $x_0 > 0$, $\phi_1(x) > 0$ and $\psi_1(x) < 0$ in $(0, x_0)$. Define

$$x_1 = \sup\{0 < x < R : \phi_1(r) > 0 \text{ and } \psi_1(r) < 0 \text{ in } (0, x)\}. \tag{2.14}$$

If $x_1 = R$, then the result holds. So we assume $x_1 < R$. Then either $\phi_1(x_1) = 0$ or $\psi_1(x_1) = 0$. If $\psi_1(x_1) = 0$, then $\psi_1(x) < 0$ in $(0, x_1)$ and $\psi_1'(x_1) \geq 0$. Then multiplying the equation of ψ_1 in (2.6) by u_2', multiplying the equation of u_2' in (2.9) by ψ_1, and subtracting and integrating on $(0, x_1)$, we obtain

$$[\psi_1' u_2' - \psi_1 u_2'']|_0^{x_1} = -\int_0^{x_1} g_1(\phi_1 u_2' - \psi_1 u_1')dx. \tag{2.15}$$

The left-hand side is $\psi_1'(x_1)u_2'(x_1) \leq 0$, and the right-hand side is positive since $g_1 > 0$, $\phi_1 > 0$, $\psi_1 < 0$, $u_1' < 0$, and $u_2' < 0$ in $(0, x_1)$. That is a contradiction.

Hence $\psi_1(x_1) = 0$ does not hold and one must have $\phi_1(x_1) = 0$, which implies that $\psi_1(x) < 0$ in $(0, x_1]$, $\phi_1(x) > 0$ in $(0, x_1)$, and $\phi_1'(x_1) \leq 0$. We claim that in this case ϕ_3 defined in (2.8) changes sign in $(0, x_1)$. If not, $\phi_3(x) > 0$ in $(0, x_1)$. Then from the equation of ϕ_1 and ϕ_3, we obtain that

$$\phi_3(x_1)\phi_1'(x_1) = -\int_0^{x_1} f_2\psi_1\phi_3 dx, \tag{2.16}$$

which is a contradiction since the left-hand side is nonpositive and the right-hand side is positive. Hence ϕ_3 has to change sign before ϕ_1. From Lemma 2.1, ϕ_3 changes sign at most once in $(0, R)$. If ϕ_3 does not change sign in $(0, R)$, then ϕ_1 and ψ_1 do not change sign either; thus the conclusion in the lemma holds.

So now we assume that ϕ_3 changes sign exactly once at $x_0 \in (0, R)$; then $0 < x_0 < x_1$, and $\phi_3(x) < 0$ in $[x_0, R]$. We claim that ϕ_1 cannot have more than one zeros in $(0, R)$. We define another pair of auxiliary functions $(\phi_{4,k}, \psi_{4,k})$ which is defined by $\phi_{4,k} = \phi_1 + ku_1'$ and $\psi_{4,k} = \psi_1 + ku_2'$ for $k \geq 0$. (ϕ_1, ψ_1) and (u_1', u_2') satisfy the same linear equations, and so does $(\phi_{4,k}, \psi_{4,k})$; namely,

$$\begin{cases} \phi_{4,k}'' + f_1\phi_{4,k} + f_2\psi_{4,k} = 0, & x > 0, \\ \psi_{4,k}'' + g_1\phi_{4,k} + g_2\psi_{4,k} = 0, & x > 0, \\ \phi_{4,k}(0) = 1, \quad \psi_{4,k}(0) = 0. \end{cases} \tag{2.17}$$

When $k > 0$ is large enough, $\psi_{4,k}(x) < 0$ for $x \in (0, R)$, and $\phi_{4,k}$ has exactly one zero, $x_{1,k} \in (0, R)$, which approaches to 0 as $k \to \infty$ since $\phi_{4,k}'(0) = ku_1''(0) < 0$. Hence we can define

$$\begin{aligned} k_* = \inf\{k_0 > 0 :\ & \psi_{4,k}(x) < 0 \text{ for } x \in (0, R), \\ & \text{and } \phi_{4,k} \text{ has exactly one zero, } x_{1,k} \in (0, R), \text{ for all } k \geq k_0\}. \end{aligned} \tag{2.18}$$

Such a k_* clearly exists as for any $k < 0$, $\psi_{4,k}(x) > 0$ near $x = 0$. Hence $k_* \geq 0$. Suppose that $k_* > 0$. Then $\phi_{4,k_*}(x) > 0$ and $\psi_{4,k_*}(x) < 0$ for small x, and one of the following alternatives occurs:

(i) there exists $x_2 > 0$ such that $\psi_{4,k_*}(x) < 0$ in $(0, x_2)$ and $\psi_{4,k_*}(x_2) = 0$;
(ii) ϕ_{4,k_*} has more than one zero in $(0, R)$;
(iii) $\phi_{4,k}$ has exactly one zero, $x_{1,k} \in (0, R)$, for $k > k_*$, but $\phi_{4,k}$ has no zero in $(0, R)$ when $k = k_*$.

If case (iii) occurs, then $\phi_1(x) > 0$ for $x \in (0, R)$ since $\phi_{4,k}$ decreases in k. Then the conclusion in the lemma holds. It is obvious that if (iii) occurs, then (ii) cannot occur. Also notice that if (iii) occurs, then (i) cannot occur. On the contrary, suppose cases (i) and (iii) both occur. Similar to (2.15), if a, b are two consecutive zeros of $\psi_{4,k}$, then

$$[\psi'_{4,k}u'_2 - \psi_{4,k}u''_2]\,|_a^b = -\int_a^b g_1(\phi_{4,k}u'_2 - \psi_{4,k}u'_1)dx. \tag{2.19}$$

For $k = k_*$, let $a = 0$ and $b = x_2$. Then the right side of (2.19) is positive and the left side is nonnegative. This is a contradiction.

Suppose that case (i) occurs but (ii) does not. Similar to the discussion of the last paragraph, we must have $x_2 > x_{1,k_*}$, the unique zero of ϕ_{4,k_*}. In particular, this implies that $\phi_{4,k_*}(x_2) < 0$. Also, x_2 is necessarily a local maximum point of ψ_{4,k_*}. However, from (2.17),

$$\psi''_{4,k_*}(x_2) = -g_1\phi_{4,k_*}(x_2) > 0,$$

and we reach another contradiction.

Next we consider that the case (ii) occurs but not (i); hence $\psi_{4,k_*}(x) < 0$ in $(0, R)$. Let $x_3 < x_4$ be the first two zeros of ϕ_{4,k_*}. By using the equations of $\phi_{4,k}$ and u'_1, if a, b are two consecutive zeros of $\phi_{4,k}$, then

$$[\phi'_{4,k}u'_1 - \phi_{4,k}u''_1]\,|_a^b = -\int_a^b f_2(\psi_{4,k}u'_1 - \phi_{4,k}u'_2)dx. \tag{2.20}$$

Since $\phi_{4,k_*}(0) = 1 > 0$, then $\phi_{4,k_*}(x) < 0$ in (x_3, x_4), and x_4 is necessarily a local maximum point. However, from (2.17),

$$\phi''_{4,k_*}(x_4) = -f_2\psi_{4,k_*}(x_2) > 0,$$

a contradiction.

Finally we consider that the cases (i) and (ii) occur simultaneously; that is, ϕ_{4,k_*} has the first two zeros $x_5 < x_6$, and ψ_{4,k_*} has a first zero at x_2. Again we have $\phi_{4,k_*}(x) < 0$ in (x_5, x_6). Apparently $x_2 > x_5$ from the proof above. If $x_5 < x_2 < x_6$, then the argument for case (i) still works, and we can use (2.19) to get a contradiction; if $x_2 > x_6$, then the argument for case (ii) still works, and we can use (2.20) to get a contradiction. Finally if $x_2 = x_6$, we will have $\phi_{4,k_*}(x_2) = \phi'_{4,k_*}(x_2) = 0$ and $\psi_{4,k_*}(x_2) = \psi'_{4,k_*}(x_2) = 0$, but that would imply $\phi_{4,k_*} \equiv \psi_{4,k_*} \equiv 0$ for $x \in (0, R)$ from the uniqueness of second order linear ODE.

Since we reach contradictions in both cases (i) and (ii), then we cannot have $k_* > 0$. Therefore $k_* = 0$ and the proof is completed. $\square$

For the weakly sublinear case, Lemmas 2.1 and 2.2 imply the following result which will be used later.

Corollary 2.3. *Let (u_1, u_2) be a solution of* (2.2) *such that $u'_1(R) < 0$ and $u'_2(R) < 0$, and let (ϕ_i, ψ_i) ($i = 1, 2, 3$) be the solution of* (2.6), (2.7), *and* (2.8), *respectively. Assume that (f, g) is cooperative as defined in (f2) and is weakly sublinear; then both ϕ_1 and ψ_2 are positive on* $[0, R]$.

2.2. **Stability.** Let (u_1, u_2) be a solution of (1.3). The stability of (u_1, u_2) is determined by the eigenvalue problem

$$\begin{cases} \xi'' + f_1\xi + f_2\eta = -\mu\xi, & x \in I, \\ \eta'' + g_1\xi + g_2\eta = -\mu\eta, & x \in I, \\ \xi(x) = 0, \quad \eta(x) = 0, & x \in \partial I. \end{cases} \tag{2.21}$$

Here $I = (-R, R)$. Notice that (2.21) can be written as

$$L\mathbf{u} = H\mathbf{u} + \mu\mathbf{u}, \tag{2.22}$$

where

$$\mathbf{u} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad L\mathbf{u} = \begin{pmatrix} -\xi'' - f_1\xi \\ -\eta'' - g_2\eta \end{pmatrix}, \quad \text{and} \ \ H = \begin{pmatrix} 0 & f_2 \\ g_1 & 0 \end{pmatrix}. \tag{2.23}$$

Again we assume that (f, g) is cooperative. Then the system of (2.22) and (2.23) is a linear elliptic system of cooperative type, and the maximum principles hold for such systems. Here we recall some known results.

Lemma 2.4. *Suppose that L, H are given in* (2.23), $\mathbf{u} \in X \equiv [W^{2,2}(I) \bigcap W_0^{1,2}(I)]^2$, *and (f, g) is cooperative as defined in* (f2). *Then:*

(1) $\mu_1 = \inf\{\mu \in Re(spt(L-H))\}$ *is a real-value eigenvalue of $L - H$, where $spt(L-H)$ is the spectrum of $L - H$.*
(2) *For $\mu = \mu_1$, there exists a unique (up to a constant multiple) eigenfunction $\mathbf{u}_1 \in Y \equiv [L^2(I)]^2$, and $\mathbf{u}_1 > 0$ in I.*
(3) *For $\mu < \mu_1$, the equation $L\mathbf{u} = H\mathbf{u} + \mu\mathbf{u} + \mathbf{f}$ is uniquely solvable for any $\mathbf{f} \in Y$, and $\mathbf{u} > 0$ as long as $\mathbf{f} \geq 0$.*
(4) *(Maximum principle) For $\mu \leq \mu_1$, suppose that $\mathbf{u} \in [W^{2,2}(I)]^2 f$ satisfies $L\mathbf{u} \geq H\mathbf{u} + \mu\mathbf{u}$ in I and $\mathbf{u} \geq 0$ on ∂I; then $\mathbf{u} \geq 0$ in I.*
(5) *If there exists $\mathbf{u} \in [W^{2,2}(I)]^2$ satisfying $L\mathbf{u} \geq H\mathbf{u}$ and $\mathbf{u} \geq 0$ in I, and either $\mathbf{u} \not\equiv 0$ on ∂I or $L\mathbf{u} \not\equiv H\mathbf{u}$ in I, then $\mu_1 > 0$.*
(6) *The principal eigenvalue can be characterized by*

$$\mu_1 = \sup\{\mu : \exists \mathbf{w} \in [W_{loc}^{2,1}(I)]^2 \ \ \textit{such that} \ \ (L-H)\mathbf{w} \geq \mu\mathbf{w}, \ \ \textit{and} \ \ \mathbf{w} > 0\}. \tag{2.24}$$

For the result and proofs of parts (1)–(5) in Lemma 2.4, see Sweers [61], Proposition 3.1 and Theorems 1.1. For part (6), see Birindelli et al. [7], Theorem 1 (which extends the result in Berestycki et al. [6] for scalar equations). Moreover, from a standard compactness argument, the linear operator $L - H$ has countably many eigenvalues μ_i $(i = 1, 2, \ldots)$, and the eigenvalues can be ordered so that $\mathrm{Re}(\mu_i - \mu_1) \to \infty$ as $i \to \infty$. A solution (u_1, u_2) of (1.3) is called *stable* if $\mu_1 > 0$, *neutrally stable* if $\mu_1 = 0$, and it is *unstable* if $\mu_1 < 0$. The *Morse index* $M(u_1, u_2)$ of (u_1, u_2) is the number of eigenvalues with negative real parts counting multiplicity. Also, (u_1, u_2) is *nondegenerate* if $\mu_i \neq 0$ for all $i = 1, 2, \ldots$.

One useful fact is that, from part (6) of Lemma 2.4, if (f, g) is cooperative as defined in $(f2)$, then (2.24) implies that for the same (f_1, f_2, g_1, g_2) defined for $I = (-R_1, R_1)$,

$$\mu_1(R_1) \leq \mu_1(R_2) \ \ \text{if} \ \ R_1 > R_2 > 0, \tag{2.25}$$

where $\mu_1(R_j)$ is the corresponding principal eigenvalue of (2.21) with $I = (-R, R)$ and $R = R_j$ $(j = 1, 2)$.

For a given solution (u_1, u_2) of (1.3), we can also define another eigenvalue problem:

$$\begin{cases} \xi'' + f_1\xi + f_2\eta = -\mu\xi, & x \in (0, R), \\ \eta'' + g_1\xi + g_2\eta = -\mu\eta, & x \in (0, R), \\ \xi'(0) = \xi(R) = 0, \ \ \eta'(0) = \eta(R) = 0. \end{cases} \tag{2.26}$$

If (μ, ξ, η) is an eigen-triple of (2.26), then it also satisfies (2.21) via an even extension from $[0, R]$ to $[-R, R]$. This also implies that if (2.26) has a positive eigenvector (ξ_1, η_1) for an eigenvalue μ_1, then extended (μ_1, ξ_1, η_1) is also the principal eigen-triple for (2.21). On the other hand, if (μ_1, ξ_1, η_1) is the principal eigen-triple for (2.21) such that $\xi_1, \eta_1 > 0$, then $(\xi_1(-x), \eta_1(-x))$ is also a positive eigenvector. From the simplicity of the eigenvector in Lemma 2.4, it is necessary that $(\xi_1(-x), \eta_1(-x)) = (\xi_1(x), \eta_1(x))$ for $x \in I$. This implies that (ξ_1, η_1) is symmetric with respect to 0, thus (μ_1, ξ_1, η_1) satisfies (2.26). Therefore the principal eigenvalues of (2.21) and (2.26) are equivalent. In particular, a solution (u_1, u_2) of (2.2) is stable if the principal eigenvalue μ_1 of (2.26) is positive.

The stability of a positive crossing solution is known if (f, g) is sublinear or superlinear. The following lemma is from Theorem 2.3 and Theorem 2.4 of [25].

Lemma 2.5. *Let (u_1, u_2) be a solution of* (2.2) *such that $u_1'(R) < 0$ and $u_2'(R) < 0$. Assume that (f, g) is cooperative as defined in* (f2).

(1) *If (f, g) is sublinear, then the principal eigenvalue μ_1 of* (2.26) *is positive, i.e., (u_1, u_2) is stable.*
(2) *If (f, g) is superlinear, then the principal eigenvalue μ_1 of* (2.26) *is negative, i.e., (u_1, u_2) is unstable.*

Next we observe a connection between the principal eigenvalue of (2.26) (or equivalently (2.21)) and two scalar eigenvalue problems. Consider the eigenvalue problems

$$\begin{cases} \xi_3'' + f_1\xi_3 = -\kappa\xi_3, & 0 < x < R, \\ \xi_3'(0) = 0, \quad \xi_3(R) = 0 \end{cases} \tag{2.27}$$

and

$$\begin{cases} \eta_3'' + g_2\eta_3 = -\nu\eta_3, & 0 < x < R, \\ \eta_3'(0) = 0, \quad \eta_3(R) = 0. \end{cases} \tag{2.28}$$

Then we have the following relation.

Proposition 2.6. *Let (u_1, u_2) be a solution of* (2.2) *such that $u_1'(R) < 0$ and $u_2'(R) < 0$, and assume that (f, g) is cooperative as defined in (f2). Let μ_1 be the principal eigenvalue of* (2.26)*, let κ_1 and ν_1 be the principal eigenvalues of* (2.27) *and* (2.28)*, respectively, and let (ϕ_3, ψ_3) be the solution of* (2.8)*. Then*

$$\mu_1 < \min\{\kappa_1, \nu_1\}. \tag{2.29}$$

In particular, if one of ϕ_3 and ψ_3 changes sign in $(0, R)$, then (u_1, u_2) is unstable.

Proof. Let $L_1(\phi) = -\phi'' - f_1\phi - \mu_1\phi$ for $\phi \in H_0^1(I)$. Then $L_1(\xi_1) = f_2\eta_1 \geq (\not\equiv)0$ for $x \in I$, and $\xi_1 = 0$ for $x = \pm R$. Hence from the maximum principle (part (5) of Lemma 2.4 for L_1), the principal eigenvalue $\mu_1(L_1) = \kappa_1 - \mu_1 > 0$. The same proof can be applied to $L_2(\phi) = -\phi'' - g_2\phi - \mu_1\phi$ for $\phi \in H_0^1(I)$ to prove $\mu_1 < \nu_1$. It is well known that κ_1 (or ν_1) is positive if and only if ϕ_3 (or ψ_3) is positive in $[0, R]$ from the Sturm comparison lemma. Thus if one of ϕ_3 and ψ_3 changes sign in $(0, R)$, then $\mu_1 < \min\{\kappa_1, \nu_1\} < 0$. ∎

Remark 2.7.

(1) Some stability results similar to Lemma 2.5 for a positive solution $u(x)$ of the scalar elliptic equation

$$\Delta u + h(u) = 0, \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega \tag{2.30}$$

were proved in [54] for the general bounded spatial domain $\Omega \subseteq \mathbb{R}^n$. If $h(u)$ is sublinear, that is, $h'(u)u - h(u) \le 0$ for $u \ge 0$, then u is stable; and if $h(u)$ is superlinear, that is, $h'(u)u - h(u) \ge 0$ for $u \ge 0$, then u is unstable.

(2) If (f, g) is weakly sublinear and (u_1, u_2) is a positive solution of (1.3), then both ϕ_3 and ψ_3 are positive on $[0, R]$. But (u_1, u_2) may not be stable with respect to (1.3). For example, consider the boundary value problem

$$\begin{cases} u_1'' + u_1 - u_1^3 + u_1 u_2^2 = 0, & x \in (-\pi/2, \pi/2), \\ u_2'' + u_2 - u_2^3 + u_1^2 u_2 = 0, & x \in (-\pi/2, \pi/2), \\ u_1(x) > 0, \quad u_2(x) > 0, & x \in (-\pi/2, \pi/2), \\ u_1(\pm\pi/2) = u_2(\pm\pi/2) = 0. \end{cases} \tag{2.31}$$

It is easy to verify that $(u_1(x), u_2(x)) = (\cos(x), \cos(x))$ is a positive solution to (2.31). This solution is neutrally stable as $\mu_1 = 0$ and $(\xi_1, \eta_1) = (\cos(x), \cos(x))$. Since $f_1 u_1 - f = -2u_1^3 < 0$ and $g_2 u_2 - g = -2u_2^3 < 0$, then (f, g) is weakly sublinear. But (f, g) is not sublinear since $f_1 u_1 + f_2 u_2 - f = 2u_1(u_2^2 - u_1^2)$ and $g_1 u_1 + g_2 u_2 - g = 2u_2(u_1^2 - u_2^2)$.

2.3. **Nondegeneracy.** With the oscillatory properties of (ϕ_i, ψ_i) $(i = 1, 2)$ proved in Lemma 2.2 and Corollary 2.3, we consider the solution (A_c, B_c) of the following initial value problem:

$$\begin{cases} A_c'' + f_1 A_c + f_2 B_c = 0, & 0 < x < R, \\ B_c'' + g_1 A_c + g_2 B_c = 0, & 0 < x < R, \\ A_c(0) = 1, \quad A_c'(0) = 0, \\ B_c(0) = c > 0, \quad B_c'(0) = 0, \end{cases} \tag{2.32}$$

for $c \ge 0$. Clearly $(A_c, B_c) = (\phi_1, \psi_1) + c(\phi_2, \psi_2)$. If $(u_1(x; \alpha, \beta), u_2(x; \alpha, \beta))$ is a crossing solution of (2.2), then the behavior of the solution (A_c, B_c) roughly shows the nodal behavior of $(u_1(x; \alpha + \varepsilon, \beta + c\varepsilon) - u_1(x; \alpha, \beta), u_2(x; \alpha + \varepsilon, \beta + c\varepsilon) - u_2(x; \alpha, \beta))$. Thus (A_c, B_c) can be viewed as the directional derivative of (u_1, u_2) along the direction $(1, c)$.

We first show a general result on the connection between the stability of a solution and the variational equation (2.32).

Proposition 2.8. *Let (u_1, u_2) be a solution of (2.2) such that $u_1'(R) < 0$ and $u_2'(R) < 0$, and let (A_c, B_c) be defined as in (2.32). Assume that (f, g) is cooperative as defined in (f2).*

(1) *(u_1, u_2) is stable if and only if for some $c > 0$, $A_c(x) > 0$ and $B_c(x) > 0$ in $(0, R]$.*

(2) *(u_1, u_2) is neutrally stable if and only if for some $c > 0$, $A_c(x) > 0$ and $B_c(x) > 0$ in $(0, R)$ and $A_c(R) = B_c(R) = 0$.*

(3) *(u_1, u_2) is unstable if and only if for all $c \ge 0$, at least one of $A_c(x)$ or $B_c(x)$ is not positive in $(0, R)$.*

Proof. For part (1), if for some $c > 0$, $A_c(x) > 0$ and $B_c(x) > 0$ in $(0, R]$, then we extend (A_c, B_c) to $[-R, R]$. Then $\mu_1 > 0$ follows from part (5) of Lemma 2.4. On the other hand, if (u_1, u_2) is stable, then (2.21) has a positive principal eigenvalue $\mu_1(R) > 0$ with a positive eigenvector (ξ_1, η_1). From the uniqueness of principal eigenvector (Lemma 2.4 (2)) and the fact that (u_1, u_2) is symmetric with respect to $x = 0$, then (ξ_1, η_1) is also symmetric with respect to $x = 0$; hence (μ_1, ξ_1, η_1) necessarily satisfies (2.26). Recall that we have extended the definition of (f, g) to $\mathbb{R}^2$ so it is C^1; thus we can also extend $f_i(u_1(x), u_2(x))$, $g_i(u_1(x), u_2(x))$, $i = 1, 2$, to $x \in \mathbb{R}^2$. Consider the eigenvalue problem

$$\begin{cases} \xi'' + f_1\xi + f_2\eta = -\mu\xi, & x \in I_r, \\ \eta'' + g_1\xi + g_2\eta = -\mu\eta, & x \in I_r, \\ \xi(x) = 0, \;\; \eta(x) = 0, & x \in \partial I_r, \end{cases} \tag{2.33}$$

where $I_r = (-r, r)$ for $r > 0$. It is well known that the principal eigenvalue $\mu_1(r)$ is continuous with respect to r. Moreover, if (f, g) is cooperative, then $\mu_1(r)$ is nonincreasing with respect to r by (2.25). We claim that there exists some $c > 0$ such that $A_c(x) > 0$ and $B_c(x) > 0$ in $(0, R]$. On the contrary, suppose for any $c > 0$, $A_c(x)$ or $B_c(x)$ changes sign at least once in $(0, R]$. Then the graphs of $c_1(x)$ and $c_2(x)$ must intersect in $(0, R]$. Let x_* be the smallest $x > 0$ such that $c_1(x) = c_2(x) = c > 0$. We obtain that the principal eigenvalue of (2.33) is zero, i.e., $\mu_1(x_*) = 0$. It implies that $\mu_1(R) \le 0$ since $x_* \le R$. This contradicts $\mu_1(R) > 0$.

Part (2) follows easily from part (2) of Lemma 2.4 and the definition of neutral stability. Part (3) also follows easily from parts (1) and (2) as (u_1, u_2) is unstable if it is neither stable nor neutrally stable. □

Proposition 2.8 shows that the oscillatory property of functions A_c and B_c is closely related to the stability of a positive solution (u_1, u_2) of (2.2). The sign-changing of the functions A_c and B_c can be tracked by the functions

$$c_1(x) = -\frac{\phi_1(x)}{\phi_2(x)}, \;\; 0 < x \le R, \qquad c_2(x) = -\frac{\psi_1(x)}{\psi_2(x)}, \;\; 0 < x \le R, \; \psi_2(x) \ne 0. \tag{2.34}$$

From Lemma 2.1, $c_1(x)$ is well defined for all $x \in (0, R]$; $c_2(x)$ is well defined for all $x \in (0, R]$ if $\psi_2(x) > 0$ for $x \in (0, R]$, and $c_2(x)$ has a vertical asymptote at some $x_* \in (0, R)$ if $\psi_2(x_*) = 0$ for some $x_* \in (0, R)$. For a given $c > 0$, the roots of $c_1(x) = c$ and $c_2(x) = c$ give the zeros of the functions $A_c(x)$ and $B_c(x)$.

First we show that the stability/instability result for the sublinear/superlinear cases shown in Lemma 2.5 can be rephrased using $c_1(x)$ and $c_2(x)$ as follows.

Lemma 2.9. *Let (u_1, u_2) be a solution of* (2.2) *such that $u_1'(R) < 0$ and $u_2'(R) < 0$, and let $c_i(x)$ $(i = 1, 2)$ be defined as in* (2.34). *Assume that (f, g) is cooperative as defined in* (f2).

(1) *If (f, g) is sublinear, then $c_1(x)$ is strictly decreasing and $c_2(x)$ is strictly increasing for $x \in (0, R)$, and $c_1(R) > c_2(R)$, i.e., the graphs of $c_1(x)$ and $c_2(x)$ do not intersect in $(0, R]$.*
(2) *If (f, g) is superlinear, then there exists $x^* \in (0, R)$ such that the graphs of $c_1(x)$ and $c_2(x)$ intersect at $x = x^*$, and $c_1(x)$ is strictly decreasing and $c_2(x)$ is strictly increasing for $x \in (0, x^*)$.*

Proof.

(1) If (f,g) is sublinear, then $\phi_1(x) > 0$ and $\psi_2(x) > 0$ for all $x \in [0,R]$ by Corollary 2.3. Thus $c_1(x), c_2(x) > 0$ for $x \in (0,R]$. We prove that $c_1(x) > c_2(x)$ for all $x \in [0,R]$. When $x > 0$ is small, we have $c_1(x) > c_2(x)$ since $\lim_{x\to 0^+} c_1(x) = \infty$ and $c_2(0) = 0$. Let x_* be the smallest $x > 0$ such that $c_1(x) = c_2(x) = c^* > 0$. From the definition of $c_i(x)$, we have

$$
(2.35)\qquad
\begin{aligned}
c_1'(x) &= \frac{\phi_1\phi_2' - \phi_2\phi_1'}{\phi_2^2} = \frac{\int_0^x f_2(\psi_1\phi_2 - \psi_2\phi_1)ds}{\phi_2^2} = \frac{\int_0^x f_2\phi_2\psi_2(c_1 - c_2)ds}{\phi_2^2},\\
c_2'(x) &= \frac{\psi_1\psi_2' - \psi_2\psi_1'}{\psi_2^2} = \frac{\int_0^x g_1(\psi_2\phi_1 - \psi_1\phi_2)ds}{\psi_2^2} = \frac{\int_0^x g_1\phi_2\psi_2(c_2 - c_1)ds}{\psi_2^2}.
\end{aligned}
$$

Since $c_1(x) - c_2(x) > 0$ for $x \in (0,x_*)$, then $c_1'(x) < 0$, $c_2'(x) > 0$, and $c_1(x) > c^* > c_2(x)$ for $x \in (0,x_*)$. This also implies that for $c = c^*$, $A_{c^*}(x) > 0$ and $B_{c^*}(x) > 0$ for $x \in [0,x_*)$ and $A_{c^*}(x_*) = B_{c^*}(x_*) = 0$.

Consider an eigenvalue problem

$$
(2.36)\qquad
\begin{cases}
\xi'' + f_1\xi + f_2\eta = -\mu\xi, & x \in (0,x_*),\\
\eta'' + g_1\xi + g_2\eta = -\mu\eta, & x \in (0,x_*),\\
\xi'(0) = \xi(x_*) = 0, \quad \eta'(0) = \eta(x_*) = 0.
\end{cases}
$$

From the above discussion and Lemma 2.4, we know that the principal eigenvalue of (2.36) is zero, i.e., $\mu_1(x_*) = 0$ and $(A_{c^*}(x),\ B_{c^*}(x))$ with $x \in (0,x_*)$ is the corresponding positive eigenfunction. On the other hand, by (2.25), we have $\mu_1(R) \le \mu_1(x_*) = 0$ since $x_* \le R$. But since (f,g) is sublinear, then $\mu_1(R) > 0$ from part (1) of Lemma 2.5. That is a contradiction. Hence if (f,g) is sublinear, then $c_1(x) > c_2(x)$ for all $x \in [0,R]$. It implies that for any $c \in (c_2(R), c_1(R))$, $A_c(x) > 0$ and $B_c(x) > 0$ in $(0,R]$.

(2) Next we assume that (f,g) is superlinear. We prove that $c_1(x)$ and $c_2(x)$ must intersect at some $x^* \in (0,R)$. Suppose not; then $c_1(x) > c_2(x)$ for $x \in (0,R)$ and $c_1(R) \ge c_2(R)$. Let $c = c_1(R)$; then $A_c(x) > 0$ and $B_c(x) > 0$ in $(0,R)$. If $c_1(R) > c_2(R)$, then $B_c(R) > 0$, so $\mu_1(R) > 0$ from part (5) of Lemma 2.4. If $c_1(R) = c_2(R)$, then $\mu_1(R) = 0$. Therefore we get $\mu_1(R) \ge 0$, which contradicts part (2) of Lemma 2.5. □

To prove the nondegeneracy of positive solutions for the superlinear problem, we would need the information of $c_1(x)$ and $c_2(x)$ beyond the intersection point shown in Lemma 2.9. While in general it is difficult to classify such behavior, in the following key lemma, we prove that the graphs of $c_1(x)$ and $c_2(x)$ cannot intersect again if additional structure is imposed on the vector field (f,g): (i) it is weakly sublinear; and (ii) it is variational (gradient or Hamiltonian).

Lemma 2.10. *Let (u_1,u_2) be a solution of* (2.2) *such that $u_1'(R) < 0$ and $u_2'(R) < 0$, and let $c_i(x)$ $(i = 1,2)$ be defined as in* (2.34). *Assume that (f,g) is cooperative as defined in* (f2). *If (f,g) is superlinear and weakly sublinear, and* (2.2) *is a Hamiltonian system or a gradient system, then the graphs of $c_1(x)$ and $c_2(x)$ intersect only at $x = x^*$ in $(0,R]$, each of $c_1(x)$ and $c_2(x)$ has at most one critical point in $(0,R]$, and $c_2(x) > c_1(x)$ for $x \in (x^*,R]$.*

Proof. We assume that (f,g) is superlinear and weakly sublinear. From part (2) of Lemma 2.9, the graphs of $c_1(x)$ and $c_2(x)$ intersect at some $x^* \in (0,R)$. We prove that the graphs of $c_1(x)$ and $c_2(x)$ do not intersect in $(x^*, R]$. Since (f, g) is weakly sublinear, then $\phi_1(x) > 0$ and $\psi_2(x) > 0$ for all $x \in [0, R]$ by Corollary 2.3. From (2.35), $c_1'(x) < 0$ and $c_2'(x) > 0$ for $x \in (0, x^*]$, and thus there exists a small $\epsilon > 0$ such that $c_1(x) < c_2(x)$ for $x \in (x^*, x^* + \epsilon]$. Suppose that the graphs of $c_1(x)$ and $c_2(x)$ have another intersection point. Let x^{**} be the smallest $x \in (x^*, R]$ such that $c_1(x) = c_2(x) = c^{**} > 0$. Then $c_1(x) < c_2(x)$ for $x \in (x^*, x^{**})$.

Note that

$$
\begin{aligned}
c_1''(x) &= \frac{f_2\phi_2\psi_2(c_1 - c_2)\phi_2^2 - 2\phi_2\phi_2' \int_0^x f_2\phi_2\psi_2(c_1 - c_2)ds}{\phi_2^4}, \\
c_2''(x) &= \frac{g_1\phi_2\psi_2(c_2 - c_1)\psi_2^2 - 2\psi_2\psi_2' \int_0^x g_1\phi_2\psi_2(c_2 - c_1)ds}{\psi_2^4}.
\end{aligned} \tag{2.37}
$$

If there exists $x_1 \in (x^*, x^{**})$ such that $c_1'(x_1) = 0$, then

$$c_1''(x_1) = \frac{f_2\phi_2\psi_2(c_1 - c_2)}{\phi_2^2}\Big|_{x=x_1} > 0,$$

since $c_1(x_1) < c_2(x_1)$ and $f_2(x) > 0$, $\phi_2(x) < 0$, and $\psi_2(x) > 0$ for $x \in (0, R]$. Similarly, if there exists $x_2 \in (x^*, x^{**})$ such that $c_2'(x_2) = 0$, then

$$c_2''(x_2) = \frac{g_1\phi_2\psi_2(c_2 - c_1)}{\phi_2^2}\Big|_{x=x_2} < 0.$$

This combining with $c_1'(x) < 0$ and $c_2'(x) > 0$ in $(0, x^*]$ implies that $c_1(x)$ has at most one critical point which is a local minimum and that $c_2(x)$ has at most one critical point which is a local maximum. In particular, the horizontal line $c = c^{**}$ intersects each of $c = c_1(x)$ and $c = c_2(x)$ at most once for $x \in [0, x^{**})$ or, equivalently, each of $A_{c^{**}}(x)$ and $B_{c^{**}}(x)$ changes sign in $(0, x^{**})$ at most once. Then there are the following three possible cases:

(i) Both of $A_{c^{**}}(x)$ and $B_{c^{**}}(x)$ change sign in $(0, x^{**})$ exactly once.

(ii) $A_{c^{**}}(x)$ changes sign in $(0, x^{**})$ exactly once, and $B_{c^{**}}(x)$ does not change sign in $(0, x^{**})$.

(iii) $B_{c^{**}}(x)$ changes sign in $(0, x^{**})$ exactly once, and $A_{c^{**}}(x)$ does not change sign in $(0, x^{**})$.

In the following, we discuss these three cases when one of two types of additional variational structure (Hamiltonian or gradient) is imposed on system (2.2).

Type I: First we suppose (2.2) is a Hamiltonian system. Then

$$f_1(u_1, u_2) \equiv g_2(u_1, u_2), \quad \text{for } (u_1, u_2) \in \mathbb{R}_+^2. \tag{2.38}$$

If case (i) occurs, then

$$A_{c^{**}}(x^{**}) = B_{c^{**}}(x^{**}) = 0, \quad A_{c^{**}}'(x^{**}) \geq 0, \quad B_{c^{**}}'(x^{**}) \geq 0. \tag{2.39}$$

Using the equation of A_c with $c = c^{**}$ and the equation of u_2', we obtain

$$A_{c^{**}}''u_2' - (u_2')''A_{c^{**}} + f_2 B_{c^{**}} u_2' - g_1 u_1' A_{c^{**}} = 0, \tag{2.40}$$

since (2.38) holds. Similarly we have

$$B_{c^{**}}''u_1' - (u_1')''B_{c^{**}} + g_1 u_1' A_{c^{**}} - f_2 u_2' B_{c^{**}} = 0. \tag{2.41}$$

Adding (2.40) and (2.41), we get

$$A_{c^{**}}''u_2' - (u_2')''A_{c^{**}} + B_{c^{**}}''u_1' - (u_1')''B_{c^{**}} = 0. \tag{2.42}$$

Define a function

$$P(x) = A'_{c^{**}}(x)u'_2(x) - u''_2(x)A_{c^{**}}(x) + B'_{c^{**}}(x)u'_1(x) - u''_1(x)B_{c^{**}}(x), \quad x \in [0, R].$$

Then (2.42) implies that $P'(x) \equiv 0$ for $x \in (0, R)$. Hence, for $x \in [0, R]$,

$$P(x) \equiv P(0) = -u''_2(0)A_{c^{**}}(0) - u''_1(0)B_{c^{**}}(0) = g(\alpha, \beta) + f(\alpha, \beta)c^{**} > 0. \tag{2.43}$$

However, from (2.39) we have

$$P(x^{**}) = A'_{c^{**}}(x^{**})u'_2(x^{**}) + B'_{c^{**}}(x^{**})u'_1(x^{**}) \le 0, \tag{2.44}$$

which is a contradiction with (2.43).

If case (ii) occurs, suppose the unique zero of $A_{c^{**}}$ in $(0, x^{**})$ is x_1. Then

$$A_{c^{**}}(x_1) = A_{c^{**}}(x^{**}) = B_{c^{**}}(x^{**}) = 0, \quad A_{c^{**}}(x) < 0,\ B_{c^{**}}(x) > 0 \text{ in } (x_1, x^{**}) \tag{2.45}$$

and

$$A'_{c^{**}}(x_1) \le 0, \quad A'_{c^{**}}(x^{**}) \ge 0. \tag{2.46}$$

Then multiplying the equation of $A_{c^{**}}$ by u'_1, multiplying the equation of u'_1 in (2.9) by $A_{c^{**}}$, and subtracting and integrating on (x_1, x^{**}), we obtain

$$A'_{c^{**}}(x^{**})u'_1(x^{**}) - A'_{c^{**}}(x_1)u'_1(x_1) = \int_{x_1}^{x^{**}} f_2 u'_2(A_{c^{**}} - B_{c^{**}})dx. \tag{2.47}$$

The left-hand side of (2.47) is nonpositive since $u'_1(x) < 0$ in $(0, R)$ and (2.46) holds, while the right-hand side of (2.47) is positive since $f_2 > 0$, $u'_2 < 0$ in $(0, R)$ and (2.45) holds. That is a contradiction. If case (iii) occurs, we can derive a similar contradiction as in case (ii).

Summarizing the discussion above, we have proved that if (f, g) is a Hamiltonian system, then $c_1(x) < c_2(x)$ for all $x \in (x^*, R]$, $c_1(x)$ has at most one critical point which is a local minimum, and $c_2(x)$ has at most one critical point which is a local maximum for $x \in (0, R)$.

Type II: Secondly, we suppose (2.2) is a gradient system. Then

$$f_2(u_1, u_2) \equiv g_1(u_1, u_2), \quad \text{for } (u_1, u_2) \in \mathbb{R}^2_+. \tag{2.48}$$

If case (i) occurs, then again (2.39) holds. Using the equation of A_c with $c = c^{**}$ and the equation of u'_1, we obtain

$$A''_{c^{**}}u'_1 - (u'_1)''A_{c^{**}} + f_2 B_{c^{**}}u'_1 - f_2 u'_2 A_{c^{**}} = 0, \tag{2.49}$$

since (2.48) holds. Similarly we have

$$B''_{c^{**}}u'_2 - (u'_2)''B_{c^{**}} + g_1 u'_2 A_{c^{**}} - g_1 u'_2 B_{c^{**}} = 0. \tag{2.50}$$

Adding (2.49) and (2.50), we get that

$$A''_{c^{**}}u'_1 - (u'_1)''A_{c^{**}} + B''_{c^{**}}u'_2 - (u'_2)''B_{c^{**}} = 0. \tag{2.51}$$

Define a function

$$Q(x) = A'_{c^{**}}(x)u'_1(x) - u''_1(x)A_{c^{**}}(x) + B'_{c^{**}}(x)u'_2(x) - u''_2(x)B_{c^{**}}(x), \quad x \in [0, R].$$

Then we have $Q'(x) \equiv 0$ for $x \in (0, R)$ from (2.51), which implies that for any $x \in [0, R]$,

$$Q(x) \equiv Q(0) = -u''_1(0)A_{c^{**}}(0) - u''_2(0)B_{c^{**}}(0) = f(\alpha, \beta) + g(\alpha, \beta)c^{**} > 0. \tag{2.52}$$

On the other hand, from (2.39) we have

$$Q(x_*) = A'_{c^{**}}(x_*)u'_1(x_*) + B'_{c^{**}}(x_*)u'_2(x_*) \le 0, \tag{2.53}$$

a contradiction. The proofs for cases (ii) and (iii) are the same as the ones in the proof of the Hamiltonian case. This completes the proof for the gradient system case. □

In the above proof, we show that the superlinearity of the vector field (f, g) ensures that $c_1(x)$ and $c_2(x)$ intersect at least once for $x \in (0, R]$, while the weak sublinearity and variational structure (gradient or Hamiltonian) of (f, g) ensure that $c_1(x)$ and $c_2(x)$ intersect only once for $x \in (0, R]$. Both of these two aspects guarantee the nondegeneracy of a positive solution to problem (2.2). In Section 3 we will give two examples of vector fields with variational structure (gradient or Hamiltonian) and which simultaneously satisfy the superlinear and weakly sublinear conditions.

Define

$$c_1 = c_1(R) = \begin{cases} -\dfrac{\phi_1(R)}{\phi_2(R)} & \text{if } \phi_1(R) > 0, \\ 0 & \text{if } \phi_1(R) \le 0; \end{cases} \quad c_2 = c_2(R) = \begin{cases} -\dfrac{\psi_1(R)}{\psi_2(R)} & \text{if } \psi_2(R) > 0, \\ \infty & \text{if } \psi_2(R) \le 0. \end{cases} \tag{2.54}$$

The properties of $c_1(x)$ and $c_2(x)$ in Lemma 2.9 imply the following corollary, which is the key for obtaining the uniqueness of positive solutions in the next section.

Corollary 2.11. *Let (u_1, u_2) be a solution of (2.2) such that $u_1'(R) < 0$ and $u_2'(R) < 0$. Assume that (2.3) is a Hamiltonian system or is a gradient system and that (f, g) is cooperative as defined in (f2). Then*

(1) *If (f, g) is sublinear, then $c_1 > c_2$, and for any $c_2 < c < c_1$, each of $A_c(x)$ and $B_c(x)$ is positive in $(0, R]$.*
(2) *If (f, g) is superlinear and weakly sublinear, then $c_1 < c_2$, and for any $c_1 < c < c_2$, each of $A_c(x)$ and $B_c(x)$ changes sign exactly once in $(0, R)$ and $A_c(R) < 0$, $B_c(R) < 0$. Moreover, for any $c \ge c_2$ or $c \le c_1$, $A_c(R)B_c(R) \le 0$.*

Proof.
(1) If (f, g) is sublinear, then $c_1 = c_1(R) > c_2(R) = c_2$ from part (1) of Lemma 2.9. Clearly for $c \in (c_2, c_1)$, each of A_c and B_c is positive from definition.

(2) If (f, g) is superlinear and weakly sublinear, then $c_1 = c_1(R) < c_2(R) = c_2$ from part (2) of Lemma 2.9. Let $c_m = \min_{x \in (0,R]} c_1(x)$ and $c_M = \max_{x \in (0,R]} c_2(x)$. Then from part (2) of Lemma 2.9, we have $c_m \le c_1 < c_2 \le c_M$. If $c \in (c_2, c_1)$, then each of $c_1(x)$ and $c_2(x)$ equals c exactly once for $x \in (0, R]$, and hence each of $A_c(x)$ and $B_c(x)$ changes sign exactly once in $(0, R)$. If $c \ge c_2$, then we always have that A_c changes sign exactly once and $A_c(R) < 0$. For B_c, there are several cases: (i) if $c > c_M$, then B_c is positive; (ii) if $c = c_M$, then B_c is positive except at one point; (iii) if $c_2 < c < c_M$, then B_c changes sign exactly twice. In all three cases, we have $B_c(R) > 0$. Thus $A_c(R)B_c(R) < 0$ if $c > c_2$. When $c = c_2$, we have $B_c(R) = 0$, and hence $A_c(R)B_c(R) = 0$. Similarly we can also show that for any $c \le c_1$, $A_c(R)B_c(R) \le 0$. □

For the sublinear case, the functions $c_1(x)$ and $c_2(x)$ defined in (2.34) have been used in [13, 14] for special Hamiltonian nonlinearity but for the radially symmetric solutions in higher dimension, and again one can show that the graphs of $c_1(x)$ and $c_2(x)$ do not intersect. Here we prove such a result holds for any sublinear system in Lemma 2.9. The results in Lemma 2.10 and Corollary 2.11 appear to be the first ones to consider $c_1(x)$ and $c_2(x)$ intersecting, and we obtain the nondegeneracy results under three conditions on the vector field: (i) superlinear; (ii) weakly sublinear; and (iii) Hamiltonian or gradient. Applications of these results will be given in Section 3.

3. Uniqueness for weakly sublinear and superlinear systems

3.1. **A Hamiltonian Schrödinger system.** In this subsection we consider (1.2) and (1.3) with the following Hamiltonian functional:

$$H(u_1,u_2) = -u_1u_2 + H_1(u_1) + H_2(u_2), \tag{3.1}$$

where for $i=1,2$, $H_i : \mathbb{R}_+ \to \mathbb{R}_+$ is C^2 function satisfying $H_i(0)=0$. Let $h_i(u_i) = H_i'(u_i)$. Then the corresponding elliptic system on a bounded interval is

$$\begin{cases} u_1'' - u_1 + h_2(u_2) = 0, & x\in(0,R),\\ u_2'' - u_2 + h_1(u_1) = 0, & x\in(0,R),\\ u_1(x)>0,\ \ u_2(x)>0, & x\in(0,R),\\ u_1'(0)=u_2'(0)=0,\ \ u_1(R)=u_2(R)=0, \end{cases} \tag{3.2}$$

and the ground state solutions satisfy

$$\begin{cases} u_1'' - u_1 + h_2(u_2) = 0, & x\in(0,\infty),\\ u_2'' - u_2 + h_1(u_1) = 0, & x\in(0,\infty),\\ u_1(x)>0,\ u_2(x)>0,\ u_1'(x)<0,\ u_2'(x)<0, & x\in(0,\infty),\\ u_1'(0)=u_2'(0)=0. \end{cases} \tag{3.3}$$

In this subsection we assume that for $i=1,2$,

$$h_i(0)=0,\quad h_i'(u_i)>0,\quad \text{and}\quad h_i'(u_i)u_i - h_i(u_i) > 0 \text{ for } \quad u_i>0 \tag{3.4}$$

and

$$h_i'(0)=0,\quad \lim_{u_i\to\infty} h_i'(u_i) = \infty. \tag{3.5}$$

Notice that (3.4) implies (f,g) is superlinear but not strongly superlinear. Also (f,g) is weakly sublinear.

We consider the initial value problem

$$\begin{cases} u_1'' - u_1 + h_2(u_2) = 0, & x>0,\\ u_2'' - u_2 + h_1(u_1) = 0, & x>0,\\ u_1'(0)=u_2'(0)=0,\\ u_1(0)=\alpha>0,\ \ u_2(0)=\beta>0. \end{cases} \tag{3.6}$$

Define

$$f(u_1,u_2) = -u_1 + h_2(u_2) \quad \text{and} \quad g(u_1,u_2) = -u_2 + h_1(u_1). \tag{3.7}$$

According to the signs of f and g, we define the following regions in $\mathbb{R}^2_+$:

$$\begin{aligned} I &= \{(u_1,u_2)\in\mathbb{R}^2_+ : f(u_1,u_2)>0, g(u_1,u_2)>0\},\\ II &= \{(u_1,u_2)\in\mathbb{R}^2_+ : f(u_1,u_2)<0, g(u_1,u_2)<0\},\\ III &= \{(u_1,u_2)\in\mathbb{R}^2_+ : f(u_1,u_2)<0, g(u_1,u_2)>0\},\\ IV &= \{(u_1,u_2)\in\mathbb{R}^2_+ : f(u_1,u_2)>0, g(u_1,u_2)<0\}. \end{aligned} \tag{3.8}$$

Since we assume that h_i satisfies (3.4) and (3.5), then the curves $f(u_1, u_2) = 0$ and $g(u_1, u_2) = 0$ are monotone ones, and they have a unique intersection point (u_1^*, u_2^*) (see Figure 1).

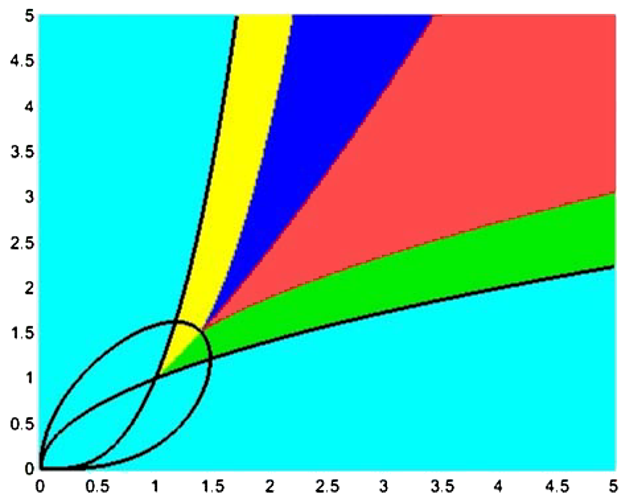

FIGURE 1. Numerical bifurcation diagram for (3.20) for $q = 2$ and $p = 3$. Here the horizontal axis is $u_1(0) = \alpha \in [0, 5]$ and the vertical axis is $u_2(0) = \beta \in [0, 5]$. Regions II, III, and IV are colored in cyan, while region I is according to the behavior of solutions at $R(\alpha, \beta)$: $\mathcal{B}$ (blue), $\mathcal{G}$ (green), $\mathcal{R}$ (red), and $\mathcal{Y}$ (yellow). The curves $f(u_1, u_2) = 0$ (bordering $\mathcal{G}$), $g(u_1, u_2) = 0$ (bordering $\mathcal{Y}$), and $H(u_1, u_2) = 0$ are plotted in black.

For $(\alpha, \beta) \in II \cup III \cup IV$, $u_1' > 0$ or $u_2' > 0$ in $(0, \delta)$; hence it cannot be a solution of (3.2). For $(\alpha, \beta) \in I$, $u_1' < 0$ and $u_2' < 0$ in $(0, \delta)$. We recall $R = R(\alpha, \beta)$ to be the right endpoint of the maximal interval $(0, R(\alpha, \beta))$ so that $u_i(x) > 0$ and $u_i'(x) < 0$ in $(0, R(\alpha, \beta))$, $i = 1, 2$. We partition I into the following classes:

$$
\begin{aligned}
&\mathcal{B} = \{(\alpha, \beta) \in I : R < \infty, u_1(R) = 0, u_1'(R) < 0, u_2(R) > 0, u_2'(R) < 0\},\\
&\mathcal{G} = \{(\alpha, \beta) \in I : R < \infty, u_1(R) > 0, u_1'(R) = 0, u_2(R) > 0, u_2'(R) < 0\},\\
&\mathcal{R} = \{(\alpha, \beta) \in I : R < \infty, u_1(R) > 0, u_1'(R) < 0, u_2(R) = 0, u_2'(R) < 0\},\\
&\mathcal{Y} = \{(\alpha, \beta) \in I : R < \infty, u_1(R) > 0, u_1'(R) < 0, u_2(R) > 0, u_2'(R) = 0\},\\
&\mathcal{S} = \{(\alpha, \beta) \in I : R < \infty, u_1(R) = 0, u_1'(R) < 0, u_2(R) = 0, u_2'(R) < 0\},\\
&\mathcal{Q} = \{(\alpha, \beta) \in I : R = \infty, \lim_{x\to\infty} u_1(x) = \lim_{x\to\infty} u_2(x) = 0\},\\
&\mathcal{P} = I \backslash (\mathcal{B} \cup \mathcal{G} \cup \mathcal{R} \cup \mathcal{Y} \cup \mathcal{S} \cup \mathcal{Q}).
\end{aligned}
\tag{3.9}
$$

It is clear that if $(\alpha, \beta) \in \mathcal{S}$, then the corresponding solution (u_1, u_2) is a solution of (3.2), while each element in $\mathcal{Q}$ defines a ground state solution in the whole space. The elements in $\mathcal{S}$ and $\mathcal{Q}$ can be characterized as follows.

Lemma 3.1. *Consider equation* (3.6), *and let* $H(u_1, u_2)$ *be defined as in* (3.1).

(1) *If* $(\alpha, \beta) \in \mathcal{S}$, *then* $H(\alpha, \beta) > 0$, $u_1'(R) < 0$, *and* $u_2'(R) < 0$.
(2) *If* $(\alpha, \beta) \in \mathcal{Q}$, *then* $H(\alpha, \beta) = 0$, *and* $u_1(x)$, $u_2(x)$, $u_1'(x)$, *and* $u_2'(x) \to 0$ *as* $x \to \infty$.

Proof. For part (1), if $(\alpha, \beta) \in \mathcal{S}$, then $(u_1(x), u_2(x))$ satisfies

$$\begin{aligned} &u_1'' - u_1 = -h_2(u_2) < 0, \quad u_2'' - u_2 = -h_1(u_1) < 0, \quad x \in (-R, R), \\ &u_1 = u_2 = 0, \quad |x| = R. \end{aligned} \tag{3.10}$$

The maximum principle and the Hopf boundary lemma hold, hence $u_1'(R) < 0$ and $u_2'(R) < 0$. Recall $H_0(x)$ defined in (1.14). Since $H_0(x) \equiv H_0(0) = H(\alpha, \beta)$, then $H(\alpha, \beta) = H_0(R) = u_1'(R)u_2'(R) > 0$.

For part (2), assume that $(\alpha, \beta) \in \mathcal{Q}$. It is clear that $u_i'(x)$, $u_i''(x) \to 0$, and $(u_1(x), u_2(x)) \to (a, b) \in \mathbb{R}_+^2$ as $x \to \infty$. Since $u_i''(x) \to 0$ as $x \to \infty$, then $f(a, b) = g(a, b) = 0$. Thus (a, b) is either $(0, 0)$ or (u_1^*, u_2^*). We claim that the latter case is not possible. In fact, from $H_0(x) \equiv H_0(0) = H(\alpha, \beta)$, and $u_i'(x) \to 0$ as $x \to \infty$, we conclude that $H(a, b) = H(\alpha, \beta)$. But (u_1^*, u_2^*) is the global minimum of the energy functional H, and for any $(\alpha, \beta) \in \mathbb{R}_+^2$ and $(\alpha, \beta) \neq (u_1^*, u_2^*)$, $H(\alpha, \beta) > H(u_1^*, u_2^*)$. Thus $(\alpha, \beta) = (u_1^*, u_2^*)$ if $(a, b) = (u_1^*, u_2^*)$. That is a contradiction. Hence $(a, b) = (0, 0)$. Also, $0 = \lim_{x \to \infty} H_0(x) \equiv H_0(0)$ implies that $H(\alpha, \beta) = H_0(0) = 0$. □

For the purpose of identifying the sets $\mathcal{S}$ and $\mathcal{Q}$, we will use a more coarse partition of $\mathbb{R}_+^2$ than the one given in (3.9). For $(\alpha, \beta) \in \mathbb{R}_+^2$, define

$$\hat{R} = \hat{R}(\alpha, \beta) = \sup\{r > 0 : u_1(x) > 0, u_2(x) > 0, x \in (0, r)\}. \tag{3.11}$$

Then $R(\alpha, \beta) \leq \hat{R}(\alpha, \beta)$ for any $(\alpha, \beta) \in \mathbb{R}_+^2$. Indeed it is easy to see that if $(\alpha, \beta) \in \mathcal{B} \cup \mathcal{R}$, then $R(\alpha, \beta) = \hat{R}(\alpha, \beta)$, while if $(\alpha, \beta) \in \mathcal{G} \cup \mathcal{Y}$, then $R(\alpha, \beta) < \hat{R}(\alpha, \beta)$ as the solution can be extended beyond $R(\alpha, \beta)$ with $u_i(x) > 0$. For $\hat{R}$, we define

(3.12)
$$\mathcal{U} = \{(\alpha, \beta) \in \mathbb{R}_+^2 : \hat{R} < \infty, u_1 > 0, u_2 > 0, x \in (0, \hat{R}),\ u_1(\hat{R}) > 0, u_2(\hat{R}) = 0\},$$
$$\mathcal{V} = \{(\alpha, \beta) \in \mathbb{R}_+^2 : \hat{R} < \infty, u_1 > 0, u_2 > 0, x \in (0, \hat{R}),\ u_1(\hat{R}) = 0, u_2(\hat{R}) > 0\}.$$

Note that $\mathcal{U}$ and $\mathcal{V}$ are not restricted to I. Next we use this definition to show the behavior of solutions with initial values in III and IV.

Proposition 3.2. *Let* $\mathcal{U}$ *and* $\mathcal{V}$ *be defined as in* (3.12). *Then* $\mathcal{U}$ *and* $\mathcal{V}$ *are open subsets of* $\mathbb{R}_+^2$ *such that* $\mathcal{U} \supset \overline{III} \backslash \{(u_1^*, u_2^*)\}$ *and a portion of* I *and* II *adjacent to* III, *and* $\mathcal{V} \supset \overline{IV} \backslash \{(u_1^*, u_2^*)\}$ *and a portion of* I *and* II *adjacent to* IV.

Proof. We only prove the result for $\mathcal{U}$, and the one for $\mathcal{V}$ also follows by symmetry. If $(\alpha, \beta) \in III$, then $f(\alpha, \beta) < 0$ and $g(\alpha, \beta) > 0$, and $u_1' > 0$ and $u_2' < 0$ for $x \in (0, \delta)$ from the equations. Let

$$R_1 = \sup\{r > 0 : u_1 > 0, u_2 > 0, u_1' > 0, u_2' < 0, x \in (0, r)\}. \tag{3.13}$$

If $R_1 < \infty$, we claim that $u_2(R_1) = 0$ and $u_1(R_1) > \alpha$. Indeed, for $x \in (0, R_1)$, $u_1' > 0$ and $u_2' < 0$; hence $(u_1(x), u_2(x))$ remains in III so $u_1(x) > \alpha$ for $x \in (0, R_1]$. Also for $x \in (0, R_1]$,

$$u_1'(x) = -\int_0^x f(u_1, u_2)dt > 0 \quad \text{and} \quad u_2'(x) = -\int_0^x g(u_1, u_2)dt < 0. \tag{3.14}$$

Thus $u_2(R_1) = 0$. If $R_1 = \infty$, then $\lim_{x\to\infty} u_2(x) \geq 0$ exists and $u_2'(x) \to 0$ as $x \to \infty$, but $g(u_1(x), u_2(x)) > g(\alpha, \beta) \geq \delta_1 > 0$. From the second equation in (3.14), $u_2'(x) < -\delta_1 x$, a contradiction. Hence $R_1 = \infty$ is not possible, and $\hat{R} = R_1 < \infty$ satisfying $u_1(\hat{R}) > 0$ and $u_2(\hat{R}) = 0$. Thus $\mathcal{U} \supset III$.

If (α, β) is a boundary point of III but not (u_1^*, u_2^*), without loss of generality we assume that $f(\alpha, \beta) = 0$ and $g(\alpha, \beta) > 0$. Then $u_2' < 0$ in $(0, \delta)$ and the orbit $(u_1(x), u_2(x))$ is in the interior of III for $x \in (0, \delta)$. Thus we can proceed with the proof as above. The openness of $\mathcal{U}$ follows from the continuous dependence of solutions of (3.6) on the initial conditions. Since $\mathcal{U} \supset \overline{III} \backslash \{(u_1^*, u_2^*)\}$, it also contains a portion in I and II adjacent to $\partial III \backslash \{(u_1^*, u_2^*)\}$ from the openness of $\mathcal{U}$. □

The result in Proposition 3.2 shows that $\mathcal{U}$ and $\mathcal{V}$ are not empty, and the following result shows that there is an order for the elements in $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{S}$, where the cooperativeness of the system is a key.

Proposition 3.3.

(1) *Suppose that* $(\alpha_0, \beta_0) \in \mathcal{S}$; *then* $(\alpha, \beta_0) \in \mathcal{V}$ *for any* $0 < \alpha < \alpha_0$ *and* $(\alpha_0, \beta) \in \mathcal{V}$ *for any* $\beta > \beta_0$.

(2) *Suppose that* $(\alpha_0, \beta_0) \in \mathcal{S}$; *then* $(\alpha, \beta_0) \in \mathcal{U}$ *for any* $\alpha > \alpha_0$ *and* $(\alpha_0, \beta) \in \mathcal{U}$ *for any* $0 < \beta < \beta_0$.

Proof. Again we only prove the first case. Assume that $(\alpha_0, \beta_0) \in \mathcal{S}$. Then for $u_1(x) = u_1(x; \alpha_0, \beta_0)$ and $u_2(x) = u_2(x; \alpha_0, \beta_0)$, we have $u_1, u_2 > 0$ and $u_1', u_2' < 0$ for $x \in (0, \hat{R}(\alpha_0, \beta_0))$. We claim that $\hat{R}(\alpha, \beta_0) < \hat{R}(\alpha_0, \beta_0)$. Define $\phi(x) = u_1(x; \alpha_0, \beta_0) - u_1(x; \alpha, \beta_0)$ and $\psi(x) = u_2(x; \alpha_0, \beta_0) - u_2(x; \alpha, \beta_0)$. Then (ϕ, ψ) satisfies

$$\begin{cases} \phi'' - \phi + h_2'(U_2)\psi = 0, & x \in (0, R_*), \\ \psi'' + h_1'(U_1)\phi - \psi = 0, & x \in (0, R_*), \\ \phi(0) = \alpha_0 - \alpha > 0, \quad \phi'(0) = 0, & \\ \psi(0) = 0, \quad \psi'(0) = 0, & \end{cases}$$

where $R_* = \min\{\hat{R}(\alpha, \beta_0), \hat{R}(\alpha_0, \beta_0)\}$, $U_1 = t_1(x)u_1(x; \alpha_0, \beta_0) + (1-t_1(x))u_1(x; \alpha, \beta_0) > 0$, and $U_2 = t_2(x)u_2(x; \alpha_0, \beta_0) + (1 - t_2(x))u_2(x; \alpha, \beta_0) > 0$. Then by comparing with $(u_1'(x; \alpha_0, \beta_0), u_2'(x; \alpha_0, \beta_0))$ and using the same proof as in Lemma 2.2, we can prove that $\psi(x) < 0$ for $x \in (0, R_*]$ and that ϕ has at most one zero in $(0, R_*]$. In fact, $\phi(x) > 0$ for $x \in (0, R_*]$, since the solution ϕ_3 of $\phi_3'' - \phi_3 = 0$ satisfying $\phi_3(0) = 1$ and $\phi_3'(0) = 0$ is clearly positive in $(0, R_*)$. This implies that at R_*, $u_1(R_*; \alpha_0, \beta_0) > u_1(R_*; \alpha, \beta_0)$ and $u_2(R_*; \alpha_0, \beta_0) < u_2(R_*; \alpha, \beta_0)$. Thus $R_* = \hat{R}(\alpha, \beta_0)$, $u_1(R_*; \alpha, \beta_0) = 0$, and $(\alpha, \beta_0) \in \mathcal{V}$. Similarly, by comparing the solutions initiating from (α, β) and (α, β_0), where $\alpha < \alpha_0$ and $\beta > \beta_0$, we can conclude that $(\alpha, \beta) \in \mathcal{V}$ as well. □

The result in Proposition 3.3 suggests that the set $\mathcal{S}$ of initial values generating the crossing solutions is on the boundary between the sets $\mathcal{U}$ and $\mathcal{V}$. We now prove that this is the case.

Proposition 3.4. *For* $\alpha > 0$, *define*

$$\Phi_1(\alpha) = \inf\{\beta > 0 : (\alpha, \beta) \in \mathcal{V}\}, \quad \Phi_2(\alpha) = \sup\{\beta > 0 : (\alpha, \beta) \in \mathcal{U}\}. \tag{3.15}$$

Then Φ_i $(i = 1, 2)$ are well defined. Moreover, define

$$\alpha_* = \sup\{\alpha > 0 : H(\alpha, \Phi_1(\alpha)) \le 0\}; \tag{3.16}$$

then for $\alpha > \alpha_*$, $\Phi_1(\alpha) = \Phi_2(\alpha) \equiv \Phi(\alpha)$, where $\Phi : (\alpha_*, \infty) \to \mathbb{R}_+$ is a continuously differentiable, strictly increasing function. Moreover, $\mathcal{S} = \{(\alpha, \Phi(\alpha)) : \alpha > \alpha_*\}$, $\mathcal{V} \supset \{(\alpha, \beta) : \alpha > \alpha_*, \beta > \Phi(\alpha)\}$, and $\mathcal{U} \supset \{(\alpha, \beta) : \alpha > \alpha_*, 0 < \beta < \Phi(\alpha)\}$.

Proof. First $\Phi_1(\alpha)$ is well defined since for fixed $\alpha > 0$, $(\alpha, \beta) \in IV \subset \mathcal{V}$ if $\beta > 0$ is large enough; secondly $\Phi_1(\alpha) > 0$ for any $\alpha > 0$ since for fixed $\alpha > 0$, if $0 < \beta < \beta_0$, then $(\alpha, \beta) \in III \subset \mathcal{U}$, where $f(\alpha, \beta_0) = 0$. Similarly $\Phi_2(\alpha)$ is also well defined.

From Proposition 3.3, $H(\alpha, \Phi_1(\alpha)) < 0$ for $\alpha > 0$ is small. And it is clear that $H(\alpha, \Phi_1(\alpha)) > 0$ when $\alpha > 0$ is large. Thus $\alpha_* > 0$ exists, and it is necessary that $H(\alpha_*, \Phi_1(\alpha_*)) = 0$. For $\alpha > \alpha_*$, $H(\alpha, \Phi_1(\alpha)) > 0$. We claim that if $\alpha > \alpha_*$, then $(\alpha, \Phi_1(\alpha)) \in \mathcal{S}$. To prove the claim, we fix $\alpha > \alpha_*$. Consider $\hat{R}(\alpha, \Phi_1(\alpha))$. There are two cases: (1) $\hat{R}(\alpha, \Phi_1(\alpha)) = \infty$ or (2) $\hat{R}(\alpha, \Phi_1(\alpha)) < \infty$. For case (1), $\hat{R}(\alpha, \Phi_1(\alpha)) = \infty$. Then $u_1(x), u_2(x) > 0$ for all $x \in [0, \infty)$, and there are two subcases: (1a) $R(\alpha, \Phi_1(\alpha)) = \infty$ or (1b) $R(\alpha, \Phi_1(\alpha)) < \infty$. For case (1a), $(\alpha, \Phi_1(\alpha)) \in \mathcal{Q}$, but on the other hand, $H(\alpha, \Phi_1(\alpha)) > 0$, which is a contradiction by Lemma 3.1. For case (1b), let $R_2 = R(\alpha, \Phi_1(\alpha))$; then $u_1(R_2) = u_1^* > 0$ and $u_2(R_2) = u_2^* > 0$, and at least one of $u_1'(R_2)$ or $u_2'(R_2)$ is zero. If $u_1'(R_2) = u_2'(R_2) = 0$; then $f(u_1^*, u_2^*) = -u_1''(R_2) \le 0$ and $g(u_1^*, u_2^*) = -u_2''(R_2) \le 0$, so $(u_1^*, u_2^*) \in \overline{II}$ and $0 \ge H(u_1^*, u_2^*) = H_0(R_2) = H_0(0) = H(\alpha, \Phi_1(\alpha))$, a contraction with $H(\alpha, \Phi_1(\alpha)) > 0$. If only one of $u_1'(R_2)$ or $u_2'(R_2)$ is zero, without loss of generality we assume that $u_1'(R_2) = 0$ and $u_2'(R_2) < 0$. Then we still have $f(u_1^*, u_2^*) = -u_1''(R_2) \le 0$, and we must have $(u_1^*, u_2^*) \in III$. Then we can follow the proof of Proposition 3.2 to show that $(u_1(x), u_2(x)) \in III$ for all $x > R_2$, which leads to $u_2(R_3) = 0$ for some $R_3 > R_2$, but that contradicts $\hat{R}(\alpha, \Phi_1(\alpha)) = \infty$. This shows that case (1) cannot happen.

Hence we must have case (2): $\hat{R}(\alpha, \Phi_1(\alpha)) < \infty$. Since $(\alpha, \Phi_1(\alpha))$ is a boundary point of V, from the continuity, $u_1(\hat{R}(\alpha, \Phi_1(\alpha))) = 0$ and $u_2(\hat{R}(\alpha, \Phi_1(\alpha))) \ge 0$. If $u_2(\hat{R}(\alpha, \Phi_1(\alpha))) > 0$, then $(\alpha, \Phi_1(\alpha)) \in V$, and V is open, so for some small $\epsilon > 0$, $(\alpha, \Phi_1(\alpha) - \epsilon) \in V$. This contradicts the definition of Φ_1. Thus we must have $u_2(\hat{R}(\alpha, \Phi_1(\alpha))) = 0$ and $(\alpha, \Phi_1(\alpha)) \in \mathcal{S}$.

Now since $(\alpha, \Phi_1(\alpha)) \in \mathcal{S}$ for $\alpha > \alpha_*$, then from Proposition 3.3, $(\alpha, \beta) \in \mathcal{V}$ for $\beta > \Phi_1(\alpha)$, and $(\alpha, \beta) \in \mathcal{U}$ for $0 < \beta < \Phi_1(\alpha)$. Similarly we can also prove that $(\alpha, \Phi_2(\alpha)) \in \mathcal{S}$ for $\alpha > \alpha_*$, which implies that we must have $\Phi_1(\alpha) = \Phi_2(\alpha)$ for $\alpha > \alpha_*$ from Proposition 3.3. And we now denote it by $\Phi(\alpha)$. The continuity of $\Phi(\alpha)$ follows from the definition, and the differentiability of $\Phi(\alpha)$ follows from the differentiability of the solutions (u_1, u_2) on the initial values. Again from Proposition 3.3, $\Phi(\alpha)$ is strictly increasing in α since $(\alpha, \Phi_1(\alpha)) \in \mathcal{S}$.

We have proved that $\mathcal{S} \supset \{(\alpha, \Phi(\alpha)) : \alpha > \alpha_*\}$. We show that $\mathcal{S}$ has no other elements other than the ones on $\{(\alpha, \Phi(\alpha)) : \alpha > \alpha_*\}$. For any $\alpha > \alpha_*$, there are no elements of $\mathcal{S}$ since $(\alpha, \beta) \in \mathcal{V}$ if $\beta > \Phi(\alpha)$, and $(\alpha, \beta) \in \mathcal{U}$ if $\beta < \Phi(\alpha)$. The same can be said for $\beta > \Phi(\alpha_*)$ from the symmetry. Hence possible elements of $\mathcal{S}$ can only be in $W = (I \cap \{(\alpha, \beta) : 0 < \alpha \le \alpha_*, 0 < \beta \le \Phi(\alpha_*)\}) \backslash (\mathcal{U} \cup \mathcal{V})$. But for any $(\alpha, \beta) \in W$, $H(\alpha, \beta) \le 0$. Thus $\mathcal{S} \cap W = \emptyset$ from Lemma 3.1. This proves $\mathcal{S} = \{(\alpha, \Phi(\alpha)) : \alpha > \alpha_*\}$. ∎

Now we are in a position to classify all solutions of (3.2) and (3.3) with the aid of the general results established in previous sections.

Proposition 3.5. *Let* $\Phi(\alpha) : (\alpha_*, \infty) \to \mathbb{R}_+$ *be defined as in Proposition* 3.4. *Define* $R(\alpha)$ *to be the common first zero of* $u_1(x;\alpha,\Phi(\alpha))$ *and* $u_2(x;\alpha,\Phi(\alpha))$. *Then* $R(\alpha) : (\alpha_*, \infty) \to \mathbb{R}_+$ *is a continuously differentiable, strictly decreasing function satisfying*

$$\lim_{\alpha\to\alpha_*^+} R(\alpha) = \infty \quad \textit{and} \quad \lim_{\alpha\to\infty} R(\alpha) = 0, \tag{3.17}$$

and $\mathcal{Q} = (\alpha_*, \Phi(\alpha_*))$.

Proof. For $\alpha > \alpha_*$, we define the function $R(\alpha)$ by $R(\alpha) = R(\alpha,\Phi(\alpha)) = \hat{R}(\alpha,\Phi(\alpha))$, which is continuous and differentiable because of the continuous differentiability of $\Phi(\alpha)$ and the solution $u(x;\alpha,\beta)$ on initial values. We prove that $R(\alpha)$ is strictly decreasing. We differentiate $u_i(R(\alpha);\alpha,\Phi(\alpha)) = 0$ with respect to α for $i = 1,2$. Then

$$\begin{aligned} &u_1'(R(\alpha))R'(\alpha) + \phi_1(R(\alpha)) + \Phi'(\alpha)\phi_2(R(\alpha)) = 0, \\ &u_2'(R(\alpha))R'(\alpha) + \psi_1(R(\alpha)) + \Phi'(\alpha)\psi_2(R(\alpha)) = 0, \end{aligned} \tag{3.18}$$

where (ϕ_1,ψ_1) and (ϕ_2,ψ_2) are as defined in (2.6) and (2.7). Let $c = \Phi'(\alpha) > 0$. Then (3.18) is equivalent to

$$u_1'(R(\alpha))R'(\alpha) = -A_c(R(\alpha)), \;\; u_2'(R(\alpha))R'(\alpha) = -B_c(R(\alpha)), \tag{3.19}$$

where $(A_c, B_c) = (\phi_1,\psi_1) + c(\phi_2,\psi_2)$ is defined in (2.32). Since $u_1'(R(\alpha)) < 0$ and $u_2'(R(\alpha)) < 0$, then (3.19) implies that $A_c(R(\alpha))B_c(R(\alpha)) > 0$. Let c_1 and c_2 be defined as in (2.54). If $c = \Phi'(\alpha) > 0$ satisfies either $c \le c_1$ or $c \ge c_2$, then $A_c(R(\alpha))B_c(R(\alpha)) \le 0$ from Corollary 2.11. Hence $c = \Phi'(\alpha)$ satisfies $c_1 < c < c_2$. Also from Corollary 2.11, we have $A_c(R(\alpha)) < 0$ and $B_c(R(\alpha)) < 0$, which implies $R'(\alpha) < 0$ by using (3.19). Since $R(\alpha)$ is decreasing, $\lim_{\alpha\to\alpha_*^+} R(\alpha)$ exists. If the limit is finite, then $(\alpha_*, \Phi(\alpha_*)) \in \mathcal{S}$, but $H(\alpha_*, \Phi(\alpha_*)) = 0$, a contradiction with Lemma 3.1. Thus $\lim_{\alpha\to\alpha_*^+} R(\alpha) = \infty$. From the properties of h_i, $\Phi(\alpha) \to \infty$ as $\alpha \to \infty$. To prove $\lim_{\alpha\to\infty} R(\alpha) = 0$, it suffices to show that for any $R > 0$, there exists a positive solution for (3.2) with $R > 0$. Indeed, by using the existence result in [31] or [39], there exists a positive solution $u(x)$ for (3.2) with h_i satisfying (3.4) and (3.5).

Finally we prove that $\mathcal{Q} = (\alpha_*, \Phi(\alpha_*))$. We have shown that $\lim_{\alpha\to\alpha_*^+} R(\alpha) = \infty$, thus $(\alpha_*, \Phi(\alpha_*)) \in \mathcal{Q}$. By using a similar proof as in that of Proposition 3.3, we can show that if $(\alpha_0, \beta_0) \in \mathcal{Q}$, then for any $(\alpha,\beta) \neq (\alpha_0,\beta_0)$, we have $(\alpha,\beta) \in \mathcal{V}$ if $0 < \alpha \le \alpha_0$ and $\beta \ge \beta_0$, and $(\alpha,\beta) \in \mathcal{U}$ if $\alpha \ge \alpha_0$ and $0 < \beta \le \beta_0$. Suppose that there exists another $(\alpha^*, \beta^*) \in \mathcal{Q}$. Then $H(\alpha^*,\beta^*) = 0$, and $f(\alpha^*,\beta^*) > 0$ and $g(\alpha^*,\beta^*) > 0$. Notice that $\partial H/\partial u_1 = g$ and $\partial H/\partial u_2 = f$; then either $\alpha^* > \alpha_*$ or $\beta^* > \Phi_1(\alpha_*)$. If $\alpha^* > \alpha_*$, then we easily obtain that $\beta^* < \Phi(\alpha_*) < \Phi(\alpha^*)$ and thus $(\alpha^*,\beta^*) \in U$ from the above claim. If $\beta^* > \Phi(\alpha_*)$, by the same argument, we can also get a contradiction. This shows that the set $\mathcal{Q}$ contains only one element, i.e., $\mathcal{Q} = (\alpha_*, \Phi(\alpha_*))$. □

We can now state our main result in this subsection about the existence and uniqueness of positive solutions to (3.2) and (3.3).

Theorem 3.6. *Suppose that h_i $(i = 1, 2)$ satisfy* (3.4) *and* (3.5). *Then for any $R > 0$,* (3.2) *has a unique positive solution $(u_1(x; R), u_2(x; R))$. If $R_1 > R_2$, then $u_1(0; R_1) < u_1(0; R_2)$ and $u_2(0; R_1) < u_2(0; R_2)$. Moreover,* (3.3) *has a unique solution (U_1, U_2).*

Proof. For any $R > 0$, from Proposition 3.5 there exists a unique $\alpha \in (\alpha_*, \infty)$ such that $R(\alpha) = R$. Hence the solution $(u_1(x; \alpha, \Phi(\alpha)), u_2(x; \alpha, \Phi(\alpha)))$ of (3.6) is a positive solution of (3.2). From Proposition 3.4, $\mathcal{S} = \{(\alpha, \Phi(\alpha) : \alpha > \alpha_*\}$, which shows the uniqueness of the positive solutions with any fixed $R > 0$. From Proposition 3.5, the unique solution of (3.3) is the one of (3.6) with initial value $(\alpha_*, \Phi(\alpha_*))$. □

We remark that as a limit of elements in $\mathcal{S}$, the unique ground state solution with $(\alpha, \beta) \in \mathcal{Q}$ is necessarily strictly decreasing, i.e., $u_1'(x) < 0$ and $u_2'(x) < 0$ for $x \in (0, \infty)$. For (α, β) satisfying $H(\alpha, \beta) \le 0$, there are other positive solutions of (3.6) defined for all $x > 0$, but they are oscillatory around (u_1^*, u_2^*) and do not converge to $(0, 0)$. It is possible to classify all these solutions using a similar method, but we will not pursue that here.

Example 3.7. Consider

$$
\begin{cases}
u_1'' - u_1 + u_2^q = 0, & -R < x < R, \\
u_2'' - u_2 + u_1^p = 0, & -R < x < R, \\
u_1(x) > 0, \ \ u_2(x) > 0, & -R < x < R, \\
u_1(\pm R) = u_2(\pm R) = 0,
\end{cases}
\tag{3.20}
$$

where $p, q > 1$. Then the conditions (3.4) and (3.5) are satisfied. Our result shows that for any $R > 0$, (3.20) has a unique solution, which is even and decreasing in $(0, R)$, and there is a unique (up to a translation) ground state solution of (3.20) which is defined on $\mathbb{R}$. Equations like (3.20) were considered in [31, 33, 39], in which the existence and symmetry of the positive solutions for a general bounded domain Ω were proved. Our result appears to be the first uniqueness result for such superlinear Hamiltonian elliptic systems. The existence and uniqueness of positive solutions to (3.20) for the case of $pq < 1$ and Ω being a general bounded domain was proved in [13]. The uniqueness of the positive solution to (3.20) for $pq \ge 1$ and $p > 1 > q$ (or $q > 1 > p$) is still not known.

A numerical bifurcation diagram of (3.20) is shown in Figure 1. One can see that $U \supset III \cup \mathcal{G} \cup \mathcal{R}$ and $V \supset IV \cup \mathcal{Y} \cup \mathcal{B}$. The boundary between $\mathcal{B}$, and $\mathcal{R}$ is the set $\mathcal{S}$ of crossing solutions, and the unique common point of $\mathcal{G}$, $\mathcal{R}$, $\mathcal{Y}$, $\mathcal{B}$, and also $H = 0$ is the element in $\mathcal{Q}$. The numerical bifurcation diagram is plotted using a method described in [36] and `Matlab`.

Remark 3.8.

(1) Our uniqueness result for (3.20) with $p, q > 1$ can also be viewed as a generalization (in the case of $n = 1$) of the uniqueness of positive solutions to a well-known scalar equation:

$$
\begin{cases}
\Delta u - u + u^p = 0, & x \in \mathbb{R}^n, \\
u(x) > 0, & x \in \mathbb{R}^n, \\
u(x) \to 0, & |x| \to \infty,
\end{cases}
\tag{3.21}
$$

where $1 < p < \frac{n+2}{n-2}$ for $n \geq 3$ and $1 < p < \infty$ for $n = 1, 2$. The solution of (3.21) serves as the basic building block of spike layer solutions in a singularly perturbed elliptic problem arising from the pattern formation of hydra (see [46]). The uniqueness of the solution of (3.21) was first proved by Coffman [22] when $n = 3$, and by Kwong [44] for general subcritical power nonlinearity. Simplifications of the proof and generalizations can be found in, for example, [5, 10, 45, 52–54, 62].

(2) Korman [42] proved the uniqueness of positive solution for a Hamiltonian system

$$\begin{cases} u'' + \lambda f(v) = 0, & -1 < x < 1, \\ v'' + \lambda g(u) = 0, & -1 < x < 1, \\ u(\pm 1) = v(\pm 1) = 0 \end{cases} \tag{3.22}$$

under the conditions that $f(t) > 0$, $g(t) > 0$, $f'(t) > 0$, $g'(t) > 0$, and $tf'(t) \geq f(t)$ and $tg'(t) \geq g(t)$ for all $t > 0$. Note that these $f(t)$ and $g(t)$ are superlinear, but they are positive, thus no ground state is possible here. Our approach in this subsection can be adapted to give a proof of the result in [42].

3.2. **A gradient quadratic Schrödinger system.** In this subsection we consider the uniqueness of positive solutions of the $\chi^{(2)}$ Second-Harmonic Generation (SHG) equations from the nonlinear optics models (see [48, 69, 70]).

$$\begin{cases} u_1'' - bu_1 + u_1 u_2 = 0, & x \in (0, R), \\ u_2'' + u_1^2/2 - cu_2 = 0, & x \in (0, R), \\ u_1'(0) = u_2'(0) = u_1(R) = u_2(R) = 0, \end{cases} \tag{3.23}$$

where $b, c > 0$. The system (3.23) also appears in the study of gravity water waves, and it has been studied from the mathematical point of view in the works [23, 24, 34, 35]. When $c = 0$ in (3.23), it is also called the Schrödinger–Newton equation (see [55, 66]). The corresponding equation of (3.23) for $x \in (0, \infty)$ is

$$\begin{cases} u_1'' - bu_1 + u_1 u_2 = 0, & x \in (0, \infty), \\ u_2'' + u_1^2/2 - cu_2 = 0, & x \in (0, \infty), \\ u_1(x) > 0,\ u_2(x) > 0,\ u_1'(x) < 0,\ u_2'(x) < 0, & x \in (0, \infty), \\ u_1'(0) = u_2'(0). \end{cases} \tag{3.24}$$

Here we consider the initial value problem (2.3) with

$$f(u_1, u_2) = -bu_1 + u_1 u_2, \quad g(u_1, u_2) = \frac{1}{2}u_1^2 - cu_2. \tag{3.25}$$

It is easy to verify that (f, g) is superlinear but not strongly superlinear, and it is also weakly sublinear. Clearly it is also cooperative. Define

$$F(u_1, u_2) = -\frac{b}{2}u_1^2 - \frac{c}{2}u_2^2 + \frac{1}{2}u_1^2 u_2. \tag{3.26}$$

Then we can see that (3.23) is a gradient system with energy function defined in (1.16) and (3.26).

We first consider the case $c > 0$ for (3.23). For the f and g defined as in (3.25), we use the definitions of the set I, II, III, and IV as in (3.8), and $\mathcal{U}, \mathcal{V}, \mathcal{S}$, and $\mathcal{Q}$ as in (3.9) and (3.12). Then the results in Lemma 3.1 hold except that $H(\alpha, \beta)$ is replaced by $F(\alpha, \beta)$. Next the results in Propositions 3.2, 3.3, and 3.4 also hold with essentially the same proof, and this leads to the result in Proposition 3.5 in the exact same wording except that $H(\alpha, \beta)$ is replaced by $F(\alpha, \beta)$. Note that here (f, g) is again superlinear and weakly sublinear as in the proof of Proposition 3.5. The existence of a positive solution of (3.23) with $c > 0$ for any $R > 0$ can be proved via a standard variational method using the mountain pass lemma (see [2]) for the functional

$$E(u_1, u_2) = \int_{-R}^{R} \left(\frac{1}{2}[u_1'(x)]^2 + \frac{1}{2}[u_2'(x)]^2 - F(u_1(x), u_2(x)) \right) dx,$$

where F is defined by (3.26) and $(u_1, u_2) \in [H_0^1(-R, R)]^2$. Indeed, one can show that (i) the Palais-Smale condition is satisfied for E; (ii) (u_1, u_2) is a local minimum of the functional E, and (iii) $E(t\phi_1, t\phi_1) < 0$ for $t > 0$ large and $\phi_1(x) = \cos(\pi x/(2R)) > 0$. Hence the mountain pass lemma can be applied. The existence of positive solutions defined for $x \in \mathbb{R}$ was proved in [69] by again using the variational approach but with the concentration-compactness principle due to the lack of compactness. Again the element in $\mathcal{Q}$ is unique as the set $F(\alpha, \beta) = 0$ in the region I is a curve $\{(\alpha, \mathcal{F}(\alpha)\}$ such that $\mathcal{F}'(\alpha) < 0$. In summary we obtain the following uniqueness result for the solutions of (3.23).

Theorem 3.9. *Suppose that $b, c > 0$. Then for any $R > 0$, (3.23) has a unique positive solution $(u_1(x; R), u_2(x; R))$. If $R_1 > R_2$, then $u_1(0; R_1) < u_1(0; R_2)$ and $u_2(0; R_1) < u_2(0; R_2)$. Moreover, (3.24) has a unique solution (U_1, U_2).*

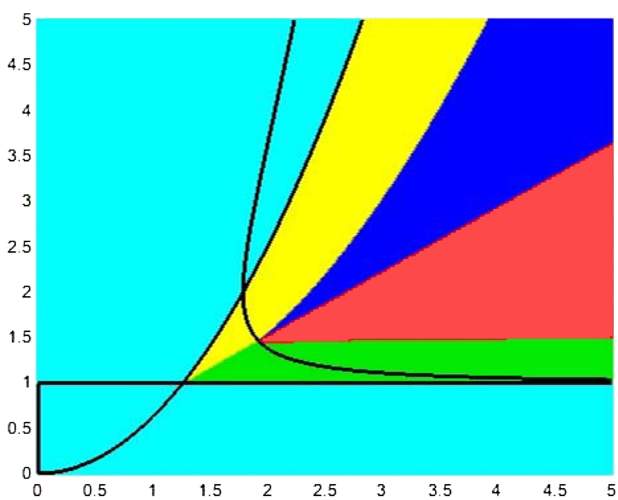

FIGURE 2. Numerical bifurcation diagram for (3.23) for $b = 1$ and $c = 0.8$. Here the horizontal axis is $u_1(0) = \alpha \in [0, 5]$ and the vertical axis is $u_2(0) = \beta \in [0, 5]$. Regions II, III, and IV are colored in cyan, while region I is according to the behavior of solutions at $R(\alpha, \beta)$: $\mathcal{B}$ (blue), $\mathcal{G}$ (green), $\mathcal{R}$ (red), and $\mathcal{Y}$ (yellow). The curves $f(u_1, u_2) = 0$ (bordering $\mathcal{G}$), $g(u_1, u_2) = 0$ (bordering $\mathcal{Y}$), and $F(u_1, u_2) = 0$ are plotted in black.

Remark 3.10. The uniqueness of the positive solution of (3.24) has been proved in Lopes [49] by using his earlier result [48] that the linearized problem of (3.24) has zero as a simple eigenvalue and has exactly one negative eigenvalue when $b, c > 0$. Here we obtain not only the uniqueness of ground state solutions but also the uniqueness of crossing solutions on the bounded intervals. Also our approach is more general and applicable to more problems.

Next we consider the case $c = 0$ for (3.23). Then (3.25) is written as

$$f(u_1, u_2) = -bu_1 + u_1u_2, \quad g(u_1, u_2) = \frac{1}{2}u_1^2. \tag{3.27}$$

We still use the notation defined in (3.8) and (3.9), but we notice that here $g > 0$ for all $(u_1, u_2) \in \mathbb{R}^2_+$, so $II = IV = \emptyset$. Again for a solution of (2.3) with $(\alpha, \beta) \in I$, we define R to be the right endpoint of the maximal interval $(0, R(\alpha, \beta))$ so that $u_i(x) > 0$ and $u_i'(x) < 0$ in $(0, R(\alpha, \beta))$, and we define $\hat{R}$ to be the right endpoint of the maximal interval $(0, \hat{R}(\alpha, \beta))$ so that $u_i(x) > 0$ in $(0, \hat{R}(\alpha, \beta))$, $i = 1, 2$. The following lemma characterizes the sets $\mathcal{U}$, $\mathcal{V}$, $\mathcal{S}$, and $\mathcal{Q}$ for the case of (3.23) with $c = 0$.

Lemma 3.11. *Suppose that f and g are defined as in* (3.27)*, and the sets $\mathcal{U}$, $\mathcal{V}$, $\mathcal{S}$, and $\mathcal{Q}$ are defined as in* (3.12) *and* (3.9)*. Then:*

(1) *$\mathcal{V} \neq \emptyset$, and if $(\alpha, \beta) \in \mathcal{V}$, then $u_1'(x) < 0$ and $u_2'(x) < 0$ for $x \in (0, \hat{R}(\alpha, \beta))$ (so $R(\alpha, \beta) = \hat{R}(\alpha, \beta)$).*

(2) *$\mathcal{U} \neq \emptyset$, and if $(\alpha, \beta) \in \mathcal{U}$, then $u_2'(x) < 0$ for $x \in (0, \hat{R}(\alpha, \beta))$ and either $u_1'(x) < 0$ for $x \in (0, \hat{R}(\alpha, \beta))$ or there exists $R_1 > 0$ such that $u_1'(x)(x - R_1) > 0$ for $x \in (0, \hat{R}(\alpha, \beta)) \backslash \{R_1\}$.*

(3) *$\mathcal{Q} = \emptyset$ and $I = \mathcal{U} \cup \mathcal{V} \cup \mathcal{S}$.*

(4) *If $(\alpha, \beta) \in \mathcal{S}$, then $u_1'(x) < 0$, $u_2'(x) < 0$ for $x \in (0, R(\alpha, \beta)]$.*

Proof.

(1) First we prove that if $(\alpha, \beta) \in \mathcal{V}$, then $u_1'(x) < 0$, $u_2'(x) < 0$ for $x \in (0, \hat{R})$; hence $R = \hat{R}$. Since $u_1(x) > 0$ for $x \in (0, \hat{R})$, then $u_2'(x) < 0$ for $x \in (0, \hat{R})$ from $u_2''(x) = -\frac{1}{2}u_1^2(x) < 0$. Since $\beta > b$, then $u_1'(x) < 0$ for x near 0. Suppose that there exists $R_2 \in (0, \hat{R})$ such that $u_1'(x) < 0$ for $x \in (0, R_2)$ and $u_1'(R_2) = 0$. Then $x = R_2$ is a local minimum of $u_1(x)$, $u_1''(R_2) \geq 0$ and u_1 is increasing for $x \in (R_2, R_2 + \epsilon)$. Since $u_1(\hat{R}) = 0$, then there exists $R_3 \in (R_2, \hat{R})$ such that $u(x)$ achieves a local maximum at $x = R_3$. Also $u_1'(x) > 0$ for $x \in (R_2, R_3)$. This follows that $(u_2(R_2) - b)u_1(R_2) = -u_1''(R_2) \leq 0$ and $(u_2(R_3) - b)u_1(R_3) = -u_1''(R_3) \geq 0$, and one of the inequalities must be strict since $u_1(R_3) > u_1(R_2)$. Then $u_2(R_3) \geq b \geq u_2(R_2)$ and $u_2(R_3) > u_2(R_2)$, which contradicts $u_2'(x) < 0$ for $x \in (0, \hat{R})$. Therefore $u_1'(x) < 0$ for $x \in (0, \hat{R})$.

Next we prove that $\mathcal{V} \neq \emptyset$. Indeed, we prove that any fixed $\beta > b$, there exists a constant $\delta_1 = \delta_1(\beta) > 0$ such that for $0 < \alpha < \delta_1$, $(\alpha, \beta) \in \mathcal{V}$. Let $\beta - b = 2\varepsilon$ and let $v_1(x)$ be the solution to

$$\begin{cases} v''(x) + \varepsilon v(x) = 0, & x > 0, \\ v(0) = \alpha, \ v'(0) = 0. \end{cases} \tag{3.28}$$

Since (3.28) is a linear equation, then there exists $R_4 = \pi/(2\sqrt{\varepsilon}) > 0$ such that $v_1(R_4) = 0$ and $v_1(x) > 0$, $v_1'(x) < 0$ for $x \in (0, R_4)$. Let $(u_1(x), u_2(x))$ be a solution of (2.3) with initial value (α, β) satisfying $0 < \alpha < 2\sqrt{\varepsilon}/R_4 = 4\varepsilon^2/\pi$. We define $R_5 = \sup\{x > 0 : u_1(s) > 0, u_2(s) > b + \varepsilon, \text{ for all } s \in (0, x)\}$ and $R_6 = \min\{R_5, R_4\}$. Then from the second equation of (2.3) and $u_1'(x) < 0$ in $(0, R_5)$, we have for any $x \in (0, R_6)$,

$$-u_2'(x) = \int_0^x \frac{1}{2} u_1^2(s)\, ds \le \frac{1}{2} \int_0^x \alpha^2 ds = \frac{1}{2}\alpha^2 x. \tag{3.29}$$

Integrating (3.29) again on $[0, R_6]$, we obtain

$$-u_2(R_6) + \beta \le \frac{\alpha^2 R_6^2}{4} < \varepsilon,$$

from the assumption on α. This implies that $u_2(R_6) > \beta - \varepsilon = b + \varepsilon$. On the other hand, for $r \in (0, R_6)$,

$$u_1''(x) = -(u_2(x) - b)u_1(x) < -\varepsilon u_1(x).$$

By the comparison principle, $u_1(x) \le v_1(x)$ for all $x > 0$. Since $v_1(R_4) = 0$, then $u_1(x)$ must reach zero for some $x < R_4$. Therefore $R_6 = R_5 < R_4$ and $u_1(R_5) = 0$ and $u_2(x) > b + \varepsilon$ for $x \in (0, R_5)$. This proves that $(\alpha, \beta) \in \mathcal{V}$ when $0 < \alpha < 4\varepsilon^2/\pi$.

(2) First we prove the monotonicity of $u_1(x)$ and $u_2(x)$ when $(\alpha, \beta) \in \mathcal{U}$. Again since $u_1(x) > 0$ for $x \in (0, \hat{R})$, then $u_2'(x) < 0$ for $x \in (0, \hat{R})$ from $u_2''(x) = -\frac{1}{2}u_1^2(x) < 0$. Since $\beta > b$, then $u_1'(x) < 0$ for x near 0. By the arguments above, for $x \in (0, \hat{R})$, one cannot have $0 < R_2 < R_3 < \hat{R}$ such that $x = R_2$ is a local minimum and $x = R_3$ is a local maximum of $u_1(x)$. Hence either $u_1'(x) < 0$ for $x \in (0, \hat{R})$ or there exists $R_1 > 0$ such that $u_1'(x)(x - R_1) > 0$ for $x \in (0, \hat{R}) \backslash \{R_1\}$.

Next we prove that $\mathcal{U} \neq \emptyset$. Indeed we prove that any fixed $\alpha > 0$, there exists a constant $\delta_2 = \delta_2(\alpha) > 0$ such that for $b < \beta < \delta_2$, $(\alpha, \beta) \in \mathcal{U}$. For a fixed $\alpha > 0$, if $\beta = b$, then $u_1'(x) > 0$ and $u_2'(x) < 0$ for small $x > 0$. For a small $\varepsilon > 0$, let $R_7 > 0$ such that $u_1(R_7; \alpha, b) = \alpha + 2\varepsilon$. Then from the continuous dependence of (2.3) on the initial value, there exists $\delta_2 > 0$ such that when $b < \beta < b + \delta_2$, $u_1(R_7; \alpha, \beta) > \alpha + \varepsilon$. In particular, this implies that there exists $R_1 \in (0, R_7)$ such that $u_1'(x; \alpha, \beta)(x - R_1) > 0$ for $x \in (0, \hat{R}) \backslash \{R_1\}$. Then for this $u_1(x)$, we have $u_1(x) \ge u_1(R_1)$ for $x \in (0, \hat{R})$, which implies that

$$-u_2'(x) = \int_0^x \frac{1}{2} u_1^2(s)\, ds \ge \frac{1}{2} \int_0^x u_1^2(R_1)\, ds = \frac{1}{2} u_1^2(R_1) x. \tag{3.30}$$

By integrating (3.30) on $[0, r]$ for $r > 0$, we obtain $u_2(r) \le \beta - u_1^2(R_1) r^2/4$. Then for $r > 2\sqrt{\beta}/u_1(R_1)$, we have $u_2(r) \le 0$, which implies that $(\alpha, \beta) \in \mathcal{U}$.

(3) Suppose that $(\alpha, \beta) \in \mathcal{Q}$; then $u_2(x)$ is a positive subharmonic function in $\mathbb{R}$ by an even extension. Hence $u_2(x)$ must be a constant and $u_1(x) \equiv 0$ which contradicts that $\alpha > 0$. Therefore $\mathcal{Q} = \emptyset$. Since for any $(\alpha, \beta) \in I$, $u_1(x)$ is either strictly decreasing in $(0, \hat{R})$ or there exists $R_1 > 0$ such that $u_1'(x)(x - R_1) > 0$ for $x \in (0, R) \backslash \{R_1\}$. Then $u_2(x)$ is always decreasing in $(0, \hat{R})$. Also we have proved that $\mathcal{Q} = \emptyset$. Then $\hat{R}$ must be finite, and either $u_1(\hat{R}) = 0$ or $u_2(\hat{R}) = 0$. This proves $I = \mathcal{U} \cup \mathcal{V} \cup \mathcal{S}$, and also when $(\alpha, \beta) \in \mathcal{S}$, we must have $u_1'(x) < 0$, $u_2'(x) < 0$ for $x \in (0, \hat{R}]$. $\square$

Secondly we consider the linearized equations (2.6) and (2.7). Here we have

$$f_1 = -b + u_2(x), \quad f_2 = u_1(x), \quad g_1 = u_1(x), \quad g_2 = 0. \tag{3.31}$$

It is clear that the system is cooperative and that (f,g) is superlinear and weakly sublinear. Then the solutions (ϕ_i,ψ_i) $(i=1,2)$ of the linearized equation are of one sign. That is,

$$\phi_1(x)>0,\quad \psi_1(x)<0,\quad \phi_2(x)<0,\quad \psi_2(x)>0. \tag{3.32}$$

Similar to the proof in subsection 3.1, we are able to show that the characterization of the sets $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{S}$ as in Proposition 3.3 also hold here. Since the statement is exactly the same, we omit it here.

Now we can classify all solutions of the initial value problem (2.3) with (f,g) defined in (3.27).

Proposition 3.12. *Consider equation* (2.3) *with* $c=0$*, and* (f,g) *is defined in* (3.27)*.*

(1) *There exists a strictly increasing function* $\Phi:(0,\infty)\to(b,\infty)$ *satisfying*

$$\lim_{\alpha\to 0^+}\Phi(\alpha)=b \quad \text{and} \quad \lim_{\alpha\to\infty}\Phi(\alpha)=\infty \tag{3.33}$$

such that

$$\begin{aligned} &\mathcal{S}=\{(\alpha,\Phi(\alpha)):\alpha>0\},\\ &\mathcal{V}=\{(\alpha,\beta):\alpha>0,\beta>\Phi(\alpha)\},\\ &\mathcal{U}=\{(\alpha,\beta):\alpha>0,b<\beta<\Phi(\alpha))\}. \end{aligned} \tag{3.34}$$

(2) *Define* $R(\alpha)$ *to be the common first zero of* $u_1(x;\alpha,\Phi(\alpha))$ *and* $u_2(x;\alpha,\Phi(\alpha))$*. Then* $R(\alpha):(0,\infty)\to(0,\infty)$ *is a continuously differentiable, strictly decreasing function satisfying* $\lim_{\alpha\to 0^+}R(\alpha)=\infty$ *and* $\lim_{\alpha\to\infty}R(\alpha)=0$*.*

Proof.

(1) Similar to the proof of Proposition 3.4, for $\alpha>0$, we define Φ_1 and Φ_2 as in (3.15). Again from the continuous dependence of solutions of (2.3) on the initial conditions, both $\mathcal{U}$ and $\mathcal{V}$ are open subsets of I. From Lemma 3.11, $\mathcal{Q}=\emptyset$, and for $\alpha>0$, we can show that $(\alpha,\Phi_i(\alpha))\in\mathcal{S}$ for $i=1,2$ using the same argument as in the proof of Proposition 3.4. Therefore $\Phi_1(\alpha)=\Phi_2(\alpha)$ (which we define as $\Phi(\alpha)$). This proves the characterization in (3.34). The properties in Proposition 3.3 imply that $\Phi(\alpha)$ must be strictly increasing. From the proof of Lemma 3.11, the limits in (3.33) hold.

(2) The proof for the properties of $R(\alpha)$ is similar to that of the proof of Proposition 3.5. □

The existence of positive solutions of (3.23) with $c=0$ and any $R>0$ can be proved using the same variational method as in the $c>0$ case. Hence we have the following result.

Theorem 3.13. *Suppose that* $b>0$ *and* $c=0$*. Then for any* $R>0$*,* (3.23) *has a unique positive solution* $(u_1(x;R),u_2(x;R))$*. If* $R_1>R_2$*, then* $u_1(0;R_1)<u_1(0;R_2)$ *and* $u_2(0;R_1)<u_2(0;R_2)$*. Moreover,* (3.24) *has no positive solution.*

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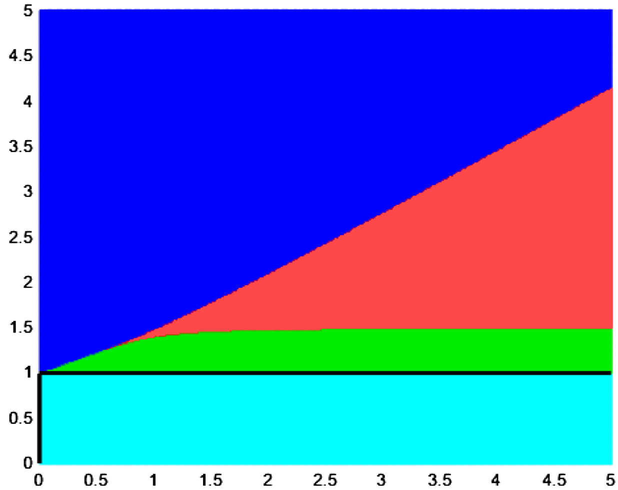

FIGURE 3. Numerical bifurcation diagram for (3.23) for $b = 1$ and $c = 0$. Here the horizontal axis is $u_1(0) = \alpha \in [0, 5]$, the vertical axis is $u_2(0) = \beta \in [0, 5]$. Regions II, III, and IV are colored in cyan, while region I are according to the behavior of solution at $R(\alpha, \beta)$: $\mathcal{B}$ (blue), $\mathcal{G}$ (green), $\mathcal{R}$ (red). The curve $f(u_1, u_2) = 0$ (bordering $\mathcal{G}$) is plotted in black.

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REFERENCES

[1] Antonio Ambrosetti and Eduardo Colorado, *Standing waves of some coupled nonlinear Schrödinger equations*, J. Lond. Math. Soc. (2) **75** (2007), no. 1, 67–82, DOI 10.1112/jlms/jdl020. MR2302730

[2] Antonio Ambrosetti and Paul H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis **14** (1973), 349–381. MR0370183

[3] Werner O. Amrein, Andreas M. Hinz, and David B. Pearson (eds.), *Sturm-Liouville theory: Past and present*, Including papers from the International Colloquium held at the University of Geneva, Geneva, September 15–19, 2003, Birkhäuser Verlag, Basel, 2005. MR2132131

[4] Yulian An, *Uniqueness of positive solutions for a class of elliptic systems*, J. Math. Anal. Appl. **322** (2006), no. 2, 1071–1082, DOI 10.1016/j.jmaa.2005.09.071. MR2250636

[5] Peter W. Bates and Junping Shi, *Existence and instability of spike layer solutions to singular perturbation problems*, J. Funct. Anal. **196** (2002), no. 2, 211–264, DOI 10.1016/S0022-1236(02)00013-7. MR1943093

[6] H. Berestycki, L. Nirenberg, and S. R. S. Varadhan, *The principal eigenvalue and maximum principle for second-order elliptic operators in general domains*, Comm. Pure Appl. Math. **47** (1994), no. 1, 47–92, DOI 10.1002/cpa.3160470105. MR1258192

[7] I. Birindelli, È. Mitidieri, and G. Svirs, *Existence of the principal eigenvalue for cooperative elliptic systems in a general domain* (Russian, with Russian summary), Differ. Uravn. **35** (1999), no. 3, 325–333, 429; English transl., Differential Equations **35** (1999), no. 3, 326–334. MR1726799

[8] Jérôme Busca and Boyan Sirakov, *Symmetry results for semilinear elliptic systems in the whole space*, J. Differential Equations **163** (2000), no. 1, 41–56, DOI 10.1006/jdeq.1999.3701. MR1755067

[9] Hsungrow Chan, Chun-Chieh Fu, and Chang-Shou Lin, *Non-topological multi-vortex solutions to the self-dual Chern-Simons-Higgs equation*, Comm. Math. Phys. **231** (2002), no. 2, 189–221, DOI 10.1007/s00220-002-0691-6. MR1946331

[10] Chiun Chuan Chen and Chang Shou Lin, *Uniqueness of the ground state solutions of $\Delta u + f(u) = 0$ in $\mathbf{R}^n$, $n \geq 3$*, Comm. Partial Differential Equations **16** (1991), no. 8-9, 1549–1572, DOI 10.1080/03605309108820811. MR1132797

[11] Zhijie Chen and Wenming Zou, *Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent*, Arch. Ration. Mech. Anal. **205** (2012), no. 2, 515–551, DOI 10.1007/s00205-012-0513-8. MR2947540

[12] Zhi-You Chen and Jann-Long Chern, *Uniqueness of topological multivortex solutions in the Maxwell-Chern-Simons model*, J. Funct. Anal. **270** (2016), no. 6, 2073–2125, DOI 10.1016/j.jfa.2016.01.024. MR3460235

[13] Zhi-You Chen, Jann-Long Chern, Junping Shi, and Yong-Li Tang, *On the uniqueness and structure of solutions to a coupled elliptic system*, J. Differential Equations **249** (2010), no. 12, 3419–3442, DOI 10.1016/j.jde.2010.09.001. MR2737436

[14] Zhi-You Chen, Jann-Long Chern, and Yong-Li Tang, *On the solutions to a Liouville-type system involving singularity*, Calc. Var. Partial Differential Equations **43** (2012), no. 1-2, 57–81, DOI 10.1007/s00526-011-0403-1. MR2860403

[15] Jann-Long Chern, Zhi-You Chen, Jhih-He Chen, and Yong-Li Tang, *On the classification of standing wave solutions for the Schrödinger equation*, Comm. Partial Differential Equations **35** (2010), no. 2, 275–301, DOI 10.1080/03605300903419767. MR2748625

[16] Jann-Long Chern, Zhi-You Chen, and Chang-Shou Lin, *Uniqueness of topological solutions and the structure of solutions for the Chern-Simons system with two Higgs particles*, Comm. Math. Phys. **296** (2010), no. 2, 323–351, DOI 10.1007/s00220-010-1021-z. MR2608118

[17] Jann-Long Chern, Zhi-You Chen, Yong-Li Tang, and Chang-Shou Lin, *Uniqueness and structure of solutions to the Dirichlet problem for an elliptic system*, J. Differential Equations **246** (2009), no. 9, 3704–3714, DOI 10.1016/j.jde.2009.01.005. MR2515175

[18] Jann-Long Chern, Yong-Li Tang, Chang-Shou Lin, and Junping Shi, *Existence, uniqueness and stability of positive solutions to sublinear elliptic systems*, Proc. Roy. Soc. Edinburgh Sect. A **141** (2011), no. 1, 45–64, DOI 10.1017/S0308210509001115. MR2773438

[19] Jann-Long Chern and Sze-Guang Yang, *Evaluating solutions on an elliptic problem in a gravitational gauge field theory*, J. Funct. Anal. **265** (2013), no. 7, 1240–1263, DOI 10.1016/j.jfa.2013.05.041. MR3073254

[20] Jann-Long Chern and Sze-Guang Yang, *The non-topological fluxes of a two-particle system in the Chern-Simons theory*, J. Differential Equations **256** (2014), no. 10, 3417–3439, DOI 10.1016/j.jde.2014.02.006. MR3177901

[21] Jann-Long Chern and Sze-Guang Yang, *Uniqueness of topological vortex in a self-dual Maxwell-Chern-Simons-Higgs system*, Indiana Univ. Math. J. **64** (2015), no. 6, 1747–1766, DOI 10.1512/iumj.2015.64.5691. MR3436234

[22] Charles V. Coffman, *Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions*, Arch. Rational Mech. Anal. **46** (1972), 81–95, DOI 10.1007/BF00250684. MR0333489

[23] Eduardo Colorado, *Existence of bound and ground states for a system of coupled nonlinear Schrödinger-KdV equations* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **353** (2015), no. 6, 511–516, DOI 10.1016/j.crma.2015.03.011. MR3348984

[24] Eduardo Colorado, *On the existence of bound and ground states for some coupled nonlinear Schrödinger-Korteweg-de Vries equations*, Adv. Nonlinear Anal. DOI: 10.1515/anona-2015-0181 (2016).

[25] Renhao Cui, Ping Li, Junping Shi, and Yunwen Wang, *Existence, uniqueness and stability of positive solutions for a class of semilinear elliptic systems*, Topol. Methods Nonlinear Anal. **42** (2013), no. 1, 91–104. MR3155616

[26] Renhao Cui, Yuwen Wang, and Junping Shi, *Uniqueness of the positive solution for a class of semilinear elliptic systems*, Nonlinear Anal. **67** (2007), no. 6, 1710–1714, DOI 10.1016/j.na.2006.08.010. MR2326023

[27] Robert Dalmasso, *Existence and uniqueness of positive radial solutions for the Lane-Emden system*, Nonlinear Anal. **57** (2004), no. 3, 341–348, DOI 10.1016/j.na.2004.02.018. MR2064095

[28] Djairo G. de Figueiredo, *Monotonicity and symmetry of solutions of elliptic systems in general domains*, NoDEA Nonlinear Differential Equations Appl. **1** (1994), no. 2, 119–123, DOI 10.1007/BF01193947. MR1273345

[29] Djairo G. de Figueiredo, *Semilinear elliptic systems*, Nonlinear functional analysis and applications to differential equations (Trieste, 1997), World Sci. Publ., River Edge, NJ, 1998, pp. 122–152. MR1703530

[30] Djairo G. de Figueiredo, *Semilinear elliptic systems: existence, multiplicity, symmetry of solutions*, Handbook of differential equations: stationary partial differential equations. Vol. V, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008, pp. 1–48, DOI 10.1016/S1874-5733(08)80008-3. MR2497896

[31] Djairo G. de Figueiredo and Patricio L. Felmer, *On superquadratic elliptic systems*, Trans. Amer. Math. Soc. **343** (1994), no. 1, 99–116, DOI 10.2307/2154523. MR1214781

[32] Djairo G. de Figueiredo and Enzo Mitidieri, *Maximum principles for linear elliptic systems* (English, with English and Italian summaries), Rend. Istit. Mat. Univ. Trieste **22** (1990), no. 1-2, 36–66 (1992). MR1210477

[33] Djairo G. De Figueiredo and Jianfu Yang, *Decay, symmetry and existence of solutions of semilinear elliptic systems*, Nonlinear Anal. **33** (1998), no. 3, 211–234, DOI 10.1016/S0362-546X(97)00548-8. MR1617988

[34] João-Paulo Dias, Mário Figueira, and Filipe Oliveira, *Existence of bound states for the coupled Schrödinger-KdV system with cubic nonlinearity* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **348** (2010), no. 19-20, 1079–1082, DOI 10.1016/j.crma.2010.09.018. MR2735011

[35] João-Paulo Dias, Mário Figueira, and Filipe Oliveira, *Well-posedness and existence of bound states for a coupled Schrödinger-gKdV system*, Nonlinear Anal. **73** (2010), no. 8, 2686–2698, DOI 10.1016/j.na.2010.06.049. MR2674102

[36] Michael Essman and Junping Shi, *Bifurcation diagrams of coupled Schrödinger equations*, Appl. Math. Comput. **219** (2012), no. 8, 3646–3654, DOI 10.1016/j.amc.2012.09.061. MR2996805

[37] B. Gidas, Wei Ming Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243. MR544879

[38] B. Gidas, Wei Ming Ni, and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbf{R}^n$, Mathematical analysis and applications, Part A, Adv. in Math. Suppl. Stud., vol. 7, Academic Press, New York-London, 1981, pp. 369–402. MR634248

[39] Josephus Hulshof and Robertus van der Vorst, *Differential systems with strongly indefinite variational structure*, J. Funct. Anal. **114** (1993), no. 1, 32–58, DOI 10.1006/jfan.1993.1062. MR1220982

[40] Norihisa Ikoma, *Uniqueness of positive solutions for a nonlinear elliptic system*, NoDEA Nonlinear Differential Equations Appl. **16** (2009), no. 5, 555–567, DOI 10.1007/s00030-009-0017-x. MR2551904

[41] Philip Korman, *Global solution curves for semilinear systems*, Math. Methods Appl. Sci. **25** (2002), no. 1, 3–20, DOI 10.1002/mma.273. MR1874447

[42] Philip Korman, *Global solution curves for a class of elliptic systems*, Appl. Anal. **89** (2010), no. 6, 849–860, DOI 10.1080/00036811003649140. MR2666544

[43] Philip Korman and Junping Shi, *On Lane-Emden type systems*, Discrete Contin. Dyn. Syst. (suppl.) (2005), 510–517. MR2192709

[44] Man Kam Kwong, *Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$ *in* $\mathbf{R}^n$, Arch. Rational Mech. Anal. **105** (1989), no. 3, 243–266, DOI 10.1007/BF00251502. MR969899

[45] Man Kam Kwong and Li Qun Zhang, *Uniqueness of the positive solution of* $\Delta u + f(u) = 0$ *in an annulus*, Differential Integral Equations **4** (1991), no. 3, 583–599. MR1097920

[46] C.-S. Lin, W.-M. Ni, and I. Takagi, *Large amplitude stationary solutions to a chemotaxis system*, J. Differential Equations **72** (1988), no. 1, 1–27, DOI 10.1016/0022-0396(88)90147-7. MR929196

[47] Tai-Chia Lin and Juncheng Wei, *Ground state of* N *coupled nonlinear Schrödinger equations in* $\mathbf{R}^n$, $n \leq 3$, Comm. Math. Phys. **255** (2005), no. 3, 629–653, DOI 10.1007/s00220-005-1313-x. MR2135447

[48] Orlando Lopes, *Stability of solitary waves of some coupled systems*, Nonlinearity **19** (2006), no. 1, 95–113, DOI 10.1088/0951-7715/19/1/006. MR2191621

[49] Orlando Lopes, *Uniqueness of a symmetric positive solution to an ODE system*, Electron. J. Differential Equations (2009), No. 162, 8. MR2578785 (2011a:34027)

[50] Li Ma and Baiyu Liu, *Symmetry results for decay solutions of elliptic systems in the whole space*, Adv. Math. **225** (2010), no. 6, 3052–3063, DOI 10.1016/j.aim.2010.05.022. MR2729001

[51] Li Ma and Lin Zhao, *Uniqueness of ground states of some coupled nonlinear Schrödinger systems and their application*, J. Differential Equations **245** (2008), no. 9, 2551–2565, DOI 10.1016/j.jde.2008.04.008. MR2455776

[52] Kevin McLeod, *Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbf{R}^n$. II*, Trans. Amer. Math. Soc. **339** (1993), no. 2, 495–505, DOI 10.2307/2154282. MR1201323

[53] Tiancheng Ouyang and Junping Shi, *Exact multiplicity of positive solutions for a class of semilinear problems*, J. Differential Equations **146** (1998), no. 1, 121–156, DOI 10.1006/jdeq.1998.3414. MR1625731

[54] Tiancheng Ouyang and Junping Shi, *Exact multiplicity of positive solutions for a class of semilinear problem. II*, J. Differential Equations **158** (1999), no. 1, 94–151, DOI 10.1016/S0022-0396(99)80020-5. MR1721723

[55] David Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlinear local term*, J. Funct. Anal. **237** (2006), no. 2, 655–674, DOI 10.1016/j.jfa.2006.04.005. MR2230354

[56] James Serrin and Henghui Zou, *Existence of positive solutions of the Lane-Emden system*, Atti Sem. Mat. Fis. Univ. Modena **46** (suppl.) (1998), 369–380. MR1645728

[57] Boyan Sirakov, *Notions of sublinearity and superlinearity for nonvariational elliptic systems*, Discrete Contin. Dyn. Syst. **13** (2005), no. 1, 163–174, DOI 10.3934/dcds.2005.13.163. MR2128797

[58] Boyan Sirakov, *Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$*, Comm. Math. Phys. **271** (2007), no. 1, 199–221, DOI 10.1007/s00220-006-0179-x. MR2283958

[59] Joel Spruck and Yi Song Yang, *The existence of nontopological solitons in the self-dual Chern-Simons theory*, Comm. Math. Phys. **149** (1992), no. 2, 361–376. MR1186034

[60] Joel Spruck and Yi Song Yang, *Topological solutions in the self-dual Chern-Simons theory: existence and approximation* (English, with English and French summaries), Ann. Inst. H. Poincaré Anal. Non Linéaire **12** (1995), no. 1, 75–97, DOI 10.1016/S0294-1449(16)30168-8. MR1320569

[61] Guido Sweers, *Strong positivity in $C(\overline{\Omega})$ for elliptic systems*, Math. Z. **209** (1992), no. 2, 251–271, DOI 10.1007/BF02570833. MR1147817

[62] Moxun Tang, *Uniqueness of positive radial solutions for $\Delta u - u + u^p = 0$ on an annulus*, J. Differential Equations **189** (2003), no. 1, 148–160, DOI 10.1016/S0022-0396(02)00142-0. MR1968317

[63] E. Timmermans, *Phase separation of Bose-Einstein condensates*, Phys. Rev. Lett. **81** (1998), 5718–5721.

[64] William C. Troy, *Symmetry properties in systems of semilinear elliptic equations*, J. Differential Equations **42** (1981), no. 3, 400–413, DOI 10.1016/0022-0396(81)90113-3. MR639230

[65] Jun Wang and Junping Shi, *Standing waves of a weakly coupled Schrödinger system with distinct potential functions*, J. Differential Equations **260** (2016), no. 2, 1830–1864, DOI 10.1016/j.jde.2015.09.052. MR3419747

[66] Juncheng Wei and Matthias Winter, *Strongly interacting bumps for the Schrödinger-Newton equations*, J. Math. Phys. **50** (2009), no. 1, 012905, 22, DOI 10.1063/1.3060169. MR2492602

[67] Juncheng Wei and Wei Yao, *Uniqueness of positive solutions to some coupled nonlinear Schrödinger equations*, Commun. Pure Appl. Anal. **11** (2012), no. 3, 1003–1011. MR2968605

[68] Yisong Yang, *Solitons in field theory and nonlinear analysis*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2001. MR1838682

[69] A. C. Yew, A. R. Champneys, and P. J. McKenna, *Multiple solitary waves due to second-harmonic generation in quadratic media*, J. Nonlinear Sci. **9** (1999), no. 1, 33–52, DOI 10.1007/s003329900063. MR1656377

[70] Leiga Zhao, Fukun Zhao, and Junping Shi, *Higher dimensional solitary waves generated by second-harmonic generation in quadratic media*, Calc. Var. Partial Differential Equations **54** (2015), no. 3, 2657–2691, DOI 10.1007/s00526-015-0879-1. MR3412388

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