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Calculus of Variations

Standing waves for a coupled nonlinear Hartree equations with nonlocal interaction

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Abstract Standing wave solutions of coupled nonlinear Hartree equations with nonlocal interaction are considered. Such systems arises from mathematical models in Bose–Einstein condensates theory and nonlinear optics. The existence and non-existence of positive ground state solutions are proved under optimal conditions on parameters, and various qualitative properties of ground state solutions are shown. The uniqueness of the positive solution or the positive ground state solution are also obtained in some cases.

Mathematics Subject Classification 35J61 · 35J20 · 35Q55 · 49J40

1 Introduction and main results

In the present paper we study the coupled nonlinear Hartree equations with nonlocal interaction in the following form:

$$\begin{cases} -\Delta u + \lambda_1 u = \mu \left(\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \right) u + \beta \left(\int_{\mathbb{R}^3} \frac{v^2(y)}{|x-y|} dy \right) u, & x \in \mathbb{R}^3, \\ -\Delta v + \lambda_2 v = \nu \left(\int_{\mathbb{R}^3} \frac{v^2(y)}{|x-y|} dy \right) v + \beta \left(\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \right) v, & x \in \mathbb{R}^3, \\ u, v \in H^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

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where $\lambda_1, \lambda_2, \mu, \nu > 0$, and $\beta \in \mathbb{R}$ is a coupling constant describing attractive or repulsive interactions.

The consideration of (1.1) is motivated by recent studies on the nonlinear Schrödinger equation (NLSE)

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\Delta\psi + V\psi - \chi|\psi|^2\psi. \tag{1.2}$$

The NLSE is a canonical way of studying the nonlinear wave propagation in various physical situations such as nonlinear optics and quantum physics. But in many situations the nonlinear interaction can be of nonlocal nature.

For example, for identical and non-relativistic basic particles (such as bosons or electrons) under the influence of an external potential and also two-body attractive interaction between two particles, the condensate in the mean field regime is governed by the nonlinear Hartree equation (see [16,17,19,20])

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\Delta\psi + V\psi - \chi\left(C(x) * |\psi|^2\right)\psi, \quad x \in \mathbb{R}^3. \tag{1.3}$$

Here ψ is a radially symmetric two-body potential function defined in $\mathbb{R}^3$ and $*$ denotes the convolution in $\mathbb{R}^3$. The most typical external potential is the Coulomb function $C(x) = |x|^{-1}$. (1.3) is also used in the description of the Bose–Einstein condensates, in which V is the trapping potential and the nonlocal term describes the interaction between the bosons in the condensate [13,45,49]. When $V = 0$, (1.3) is also known as nonlinear Choquard equation [27,32,37], and the Eq. (1.3) with $V = 0$ also arises from the model of wave propagation in a media with a large response length [1,23].

With the recent experimental advances in multi-component Bose–Einstein condensates [4], systems of coupled nonlinear Schrödinger equations or Hartree equations have been the focus of many recent theoretical studies. The two-component nonlinear Schrödinger system with nonlocal Hartree type interaction can be written in the following form(see [62]):

$$\begin{cases} -i\dfrac{\partial \Phi_1}{\partial t} + V_1(x)\Phi_1 = \dfrac{\hbar^2}{2m}\Delta\Phi_1 + \mu(C(x) * |\Phi_1|^2)\Phi_1 + \beta(C(x) * |\Phi_2|^2)\Phi_1, & x \in \mathbb{R}^3, \\ -i\dfrac{\partial \Phi_2}{\partial t} + V_2(x)\Phi_2 = \dfrac{\hbar^2}{2m}\Delta\Phi_2 + \nu(C(x) * |\Phi_2|^2)\Phi_2 + \beta(C(x) * |\Phi_1|^2)\Phi_2, & x \in \mathbb{R}^3, \\ \Phi_j = \Phi_j(x,t) \in \mathbb{C}, \ \Phi_j(x,t) \to 0, \quad \text{as } |x| \to \infty, \ t > 0, \ j = 1, 2, \end{cases} \tag{1.4}$$

where i is the imaginary unit, m is the mass of the particles, $\hbar$ is the Plank constant, $\mu, \nu > 0$, and $\beta \neq 0$ is a coupling constant which describes the scattering length of the attractive or repulsive interaction, $V_1(x)$ and $V_2(x)$ are the external potentials, and $C(x)$ is the response function which possesses information on the mutual interaction between the particles.

The system (1.4) can also arise from the studies of nonlinear optics. Physically, the solution Φ_i denotes the i-th component of the beam in Kerr-like photorefractive media. Experiments have showed the existence of self-trapping of incoherent beam in a nonlinear medium [40,41]. Such findings are significant since optical pulses propagating in a linear medium have a natural tendency to broaden in time (dispersion) and space (diffraction). In the context of optical propagation, Φ_i in (1.4) denotes the i-th component of the beam in Kerr-like photorefractive media; the positive constants μ, ν indicate the self-focusing strength in the component of the beam; and the coupling constant β measures the interaction between the two components of the beam. The sign of β determines whether the interactions of states are repulsive or attractive.

A standing wave solution of (1.4) is a solution of the form

$$(\Phi_1(x,t), \Phi_2(x,t)) = \left(e^{-iEt}u(x), e^{-iEt}v(x)\right), \quad u, v \in H^1(\mathbb{R}^3). \tag{1.5}$$

Substituting (1.5) into (1.4), and renaming the parameters by

$$\varepsilon = \sqrt{\frac{\hbar^2}{2m}}, \quad \lambda_1(x) = V_1(x) - E \quad \text{and} \quad \lambda_1(x) = V_2(x) - E, \tag{1.6}$$

we obtain the following semilinear elliptic system with nonlocal nonlinearities:

$$\begin{cases} -\varepsilon^2\Delta u + \lambda_1(x)u = \mu(C * u^2)u + \beta(C * v^2)u, & x \in \mathbb{R}^3, \\ -\varepsilon^2\Delta v + \lambda_2(x)v = \nu(C * v^2)v + \beta(C * u^2)v, & x \in \mathbb{R}^3, \\ u, v \in H^1(\mathbb{R}^3). \end{cases} \tag{1.7}$$

Similar systems of equations are also considered in the basic quantum chemistry model of small number of electrons interacting with static nucleii which can be approximated by Hartree or Hartree–Fock minimization problems (see [24,29,34]). The Euler–Lagrange equations corresponding to such Hartree problem are

$$-\Delta u_i + V(x)u_i + \sum_{j\neq i}\left(\int_{\mathbb{R}^3}\frac{u_j^2(y)}{|x-y|}dy\right)u_i + \varepsilon_i u_i = 0, \quad x \in \mathbb{R}^3, \ 1 \le i \le k, \tag{1.8}$$

where $k \in \mathbb{N}$, $V(x)$ describes the attractive interaction between the electrons and the nucleii, the integral term shows the repulsive Coulomb interaction between the electrons, and $-\varepsilon_i$ are the Lagrange multipliers. As pointed out in [34], very often restricted Hartree equations are considered where some of the u_i are taken to be equal. For example, when $k = 2$ and $u_1 = u_2$, then (1.8) is reduced to a scalar equation

$$-\Delta u + V(x)u + \left(\int_{\mathbb{R}^3}\frac{u^2(y)}{|x-y|}dy\right)u + \varepsilon u = 0, \quad x \in \mathbb{R}^3. \tag{1.9}$$

The solutions of (1.9) were considered in, for example, [18,33,34]. We notice that in (1.8), the interaction between electrons is repulsive while the one in (1.3) is attractive. When $k = 4$, $u_1 = u_2$ and $u_3 = u_4$, then we also obtain (1.7) with Coulomb potential.

If the response function is a Dirac-delta function, i.e. $C(x) = \delta(x)$, then the nonlinear response is local and it has been more extensively considered in recent years. In this case, the system (1.4) arises in the theory of Bose–Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ (see [15,53]), where Φ_1 and Φ_2 are the corresponding condensate amplitudes. The standing waves corresponding to (1.7) in this case becomes a semilinear elliptic system with local nonlinearities:

$$\begin{cases} -\varepsilon^2\Delta u + \lambda_1(x)u = \mu u^3 + \beta v^2 u, & x \in \mathbb{R}^3, \\ -\varepsilon^2\Delta v + \lambda_2(x)v = \nu v^3 + \beta u^2 v, & x \in \mathbb{R}^3, \\ u, v \in H^1(\mathbb{R}^3). \end{cases} \tag{1.10}$$

The existence, multiplicity and concentration of positive solutions of (1.10) have been the subject of extensive mathematical studies in recent years, for example, [3,5,6,11,12,14,22, 30,31,35,36,38,42,48,50–52,56,58] and references therein.

In this paper we consider the system (1.7) with a response function of Coulomb type $C(x) = |x|^{-1}$. That is

$$\begin{cases} -\varepsilon^2 \Delta u + \lambda_1(x) u = \mu \left(\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \right) u + \beta \left(\int_{\mathbb{R}^3} \frac{v^2(y)}{|x-y|} dy \right) u, & x \in \mathbb{R}^3, \\ -\varepsilon^2 \Delta v + \lambda_2(x) v = \nu \left(\int_{\mathbb{R}^3} \frac{v^2(y)}{|x-y|} dy \right) v + \beta \left(\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \right) v, & x \in \mathbb{R}^3. \end{cases} \tag{1.11}$$

The system (1.11) was recently considered in [62]. Under some conditions for the potential function $\lambda_i(x)$, $i = 1, 2$, the existence of a ground state solution of (1.11) for $\varepsilon > 0$ small and $\beta > 0$ sufficiently large was proved. Here we are concerned with the case $\lambda_i(x) = \lambda_i =$ constant. Without loss of generality we assume that $\varepsilon = 1$, then (1.11) reduces to (1.1). Our goal here is to prove the existence of positive ground state solutions of (1.1) for all possible range of coupling constant β, and our work is mainly motivated by [51] for the corresponding results in the local case with $C(x) = \delta(x)$.

For any $\beta \in \mathbb{R}$, the system (1.1) possesses a trivial solution (0, 0) and a pair of semi-trivial solutions with one component being zero. These solutions have the form $(U, 0)$ or $(0, V)$, where each of U and V is the positive radial solution of

$$-\Delta w + \sigma w = \tau \left(\int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy \right) w, \quad w \in H^1(\mathbb{R}^3), \tag{1.12}$$

with $(\sigma, \tau) = (\lambda_1, \mu)$ for U, and $(\sigma, \tau) = (\lambda_2, \nu)$ for V respectively. It is well known that (1.12) is related to the stationary solution of Choquard equation (see [27,29,32,37]). Also (1.12) was introduced by Penrose in his discussion on the self gravitational collapse of a quantum mechanical wave-function (see [46,47]). The Eq. (1.12) is also called the Schrödinger–Newton equation [54,59]. According to [27,37], we know that (1.12) has a unique positive solution $w_{\sigma,\tau} \in H^1(\mathbb{R}^3)$ that is radially symmetric for any $\sigma, \tau > 0$.

We look for solutions of (1.1) which are different from the preceding ones. A solution (u, v) of (1.1) is nontrivial if $u \neq 0$ and $v \neq 0$. A solution (u, v) with $u > 0$ and $v > 0$ is called a positive solution. A solution is called a ground state solution (or positive ground state solution) if its energy is minimal among all the nontrivial solutions (or all the positive solutions) of (1.1). Here the energy functional corresponding to (1.1) is defined by

$$\begin{aligned} \mathscr{L}_{\lambda_1\lambda_2}(u, v) = &\frac{1}{2} \int_{\mathbb{R}^3} \left[|\nabla u(x)|^2 + \lambda_1 u^2(x) + |\nabla v(x)|^2 + \lambda_2 v^2(x) \right] dx \\ &- \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\mu u^2(x) u^2(y) + 2\beta u^2(x) v^2(y) + \nu v^2(x) v^2(y)}{|x-y|} dx dy, \end{aligned} \tag{1.13}$$

for $(u, v) \in E \equiv H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. Note that if we consider our problem in the subspace of radially symmetric functions $E_r \equiv H_r^1(\mathbb{R}^3) \times H_r^1(\mathbb{R}^3)$, where $H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) :$ u is radially symmetric$\}$, then we say that $(u, v) \in E_r$ is a radial ground state solution (or positive radial ground state solution) if its energy is minimal among all the nontrivial radial solutions (or all the positive radial solutions) of (1.1).

First we state that when the coupling constant β is positive, then a positive solution of (1.1) is necessarily radially symmetric.

Theorem 1.1 *Assume that $\mu, \nu, \lambda_1, \lambda_2 > 0$, and $\beta \geq 0$, then every positive solution of* (1.1) *is radially symmetric and decreasing in radial direction.*

The proof of Theorem 1.1 is based on the celebrated moving plane method for cooperative integral-differential equations (see [10,37]). For the local interaction case (1.10), this property is well-known (also for the cooperative case $\beta \geq 0$), see [7]. On the other hand, it is also known that when $\beta < 0$, a positive solution of (1.10) may not be radially symmetric, see [5,36]. For the existence and nonexistence of positive (ground state) solutions of (1.1), we have the following main results.

Theorem 1.2 *Assume that $\mu, \nu > 0$ and $\lambda_2 \geq \lambda_1 > 0$ are fixed.*

(i) Let χ_0 be the smaller root of the equation

$$\lambda^{-\frac{5}{4}}(2-\lambda^{-\frac{5}{4}})y^2 - (\mu_1+\nu_1)y + \mu_1\nu_1 = 0, \tag{1.14}$$

where

$$\mu_1 = \lambda^{\frac{3}{4}}\mu, \quad \nu_1 = \lambda^{-\frac{3}{4}}\nu, \quad \text{and } \lambda = \frac{\lambda_2}{\lambda_1}. \tag{1.15}$$

If $-\infty < \beta < \chi_0$, then (1.1) possesses a positive radial ground state solution $z \in E_r$. Moreover, if $0 < \beta < \chi_0$, then z is also a positive ground state solution.

(ii) If $\beta > \max\{\lambda^2\mu, \lambda^{\frac{1}{2}}\nu\} = \lambda^{\frac{5}{4}}\max\{\mu_1, \nu_1\}$, then (1.1) possesses a positive radial ground state solution which is also a positive ground state solution.

(iii) If $\beta \in [\nu, \mu]$ and $\nu < \mu$, then (1.1) has no positive solution.

(iv) For $\beta \in (-\infty, \chi_0) \cup (\max\{\lambda^2\mu, \lambda^{\frac{1}{2}}\nu\}, \infty)$, there exists $M = M(\mu, \nu, \lambda_1, \lambda_2, \beta)$ such that any positive ground state solution (u, v) of (1.1) satisfies $\|u\|_\infty + \|v\|_\infty \leq M$.

In part (i) of Theorem 1.2, the existence of positive radial ground state solution is shown for $\beta < \chi_0$. Indeed we can prove the existence of positive radial ground state solution for $\beta \in (-\infty, \chi_1)$ where $\chi_1 > \chi_0$ defined above (see Lemma 2.7), but the expression of χ_1 is more complicated so it is deferred to Sect. 2. In Sect. 2, we will also show that $\chi_0 < \chi_1 < \min\{\mu_1, \nu_1\}$. Note that $\beta \in (-\infty, \chi_1)$ covers all negative β value, which corresponds to the repulsive interaction case. For the repulsive case, whether the ground state solution is radially symmetric is still not known as the method in Theorem 1.1 requires $\beta > 0$. On the other hand, for the attractive case $\beta > 0$, any positive solution is necessarily radially symmetric from Theorem 1.1, hence any ground state solution must be radial.

In the special case of $\lambda_1 = \lambda_2$, the ground state solution of (1.1) can be constructed from the solution of the scalar equation (1.12), and a more explicit expression of the positive ground state solutions can be obtained as follows.

Theorem 1.3 *Let $w \in H^1(\mathbb{R}^3)$ be the unique positive solution of (1.12) with $\sigma = \tau = 1$. Assume that $\lambda_1 = \lambda_2$.*

(i) If $\beta \in (0, \min\{\mu, \nu\}) \cup (\max\{\mu, \nu\}, \infty)$, then $(\sqrt{\kappa}w, \sqrt{\ell}w)$ is a positive ground state solution of (1.1), where $\kappa > 0$ and $\ell > 0$ satisfy

$$\mu\kappa + \beta\ell = 1, \quad \beta\kappa + \nu\ell = 1. \tag{1.16}$$

(ii) If $\beta \in [\min\{\mu, \nu\}, \max\{\mu, \nu\}]$ and $\mu \neq \nu$, then (1.1) does not have a positive solution.

Some remarks on these results are in order:

1. It is easy to see that the conditions in Theorem 1.2 reduce to the ones in Theorem 1.3 when $\lambda = \lambda_2/\lambda_1 = 1$. While there is a gap between the existence and nonexistence ranges of β for $\lambda \neq 1$ in Theorem 1.2, the β-range for the existence and nonexistence of solution when $\lambda = 1$ is optimal.

2. In the proof of Theorems 1.2–1.3, the main difficulty is to exclude the semitrivial solutions of (1.1). In the local case, many work overcome this difficulty by using different variational methods, for instance, see [3,5,6,11,12,14,22,30,31,35,36,38,42,48,50–52,58] and references therein. In the nonlocal case, some ideas of the papers [30,31,51] can still be adapted to our case. However, many difficulties arise due to the presence of the non-local terms, some new techniques and a more careful analysis of the interaction depending on parameter β are required for the proof given here.
3. To the best of our knowledge, Theorems 1.2–1.3 are the first rigorous results for the existence of nontrivial solution of (1.1). These existence results of (1.1) could play an important role in studying singular perturbation problem of (1.1) as in [62]. Results in this nature for the local case (1.10) have been proved in [30,51]. For example, in [51, Theorem 2], the existence of a positive radial ground state solution was also shown under a similar assumption as in Theorem 1.2 (i).

For the uniqueness of positive solution or positive ground state solution of (1.1), we have the following results.

Theorem 1.4 *Suppose that* $\lambda_1, \lambda_2, \mu, \nu > 0$ *are fixed.*

(i) There exists $\beta_0 > 0$ *such that if* $0 < \beta < \beta_0$*, up to a translation,* (1.1) *has a unique positive solution* (u_β, v_β)*, which is radially symmetric and decreasing in the radial direction, and is non-degenerate in* E_r*. Moreover as* $\beta \to 0^+$*,* $(u_\beta, v_\beta) \to (u_0, v_0)$ *strongly in* E_r*, where* (u_0, v_0) *is the positive ground state solution of* (1.1) *with* $\beta = 0$ *and same center.*

(ii) When $\lambda_1 = \lambda_2$*, if* $0 < \beta < \min\{\mu, \nu\}$ *or* $\beta > \max\{\mu, \nu\}$*, then* $(\sqrt{\kappa} w, \sqrt{\ell} w)$ *is the unique positive ground state solution of* (1.1) *up to a translation.*

The first uniqueness result in Theorem 1.4 is of perturbation nature. At $\beta = 0$, the non-degeneracy of the positive ground state solution is reduced to that of the scalar equation (1.12), and we follow the approach in [21,43,55] to prove the non-degeneracy. Together with the a priori estimate in Theorem 1.2 part (iv), the uniqueness of the positive solution can be shown. The second uniqueness result in Theorem 1.4 takes advantage of $\lambda_1 = \lambda_2$. The uniqueness of positive solution or positive ground state solution for other cases is still open. For the local interaction case, the uniqueness of positive solution of (1.10) when $\lambda_1 = \lambda_2$ and $\beta > \max\{\mu, \nu\}$ was proved in [60]. More partial uniqueness results for the case that $\beta \in (0, \min\{\mu, \nu\})$ and $\lambda_1 = \lambda_2$ in the local interaction situation were also proved in [11,12,60]. On the other hand, the uniqueness of positive solution of (1.12) was proved in [27,37]. We conjecture that under the conditions of (ii) of Theorem 1.4, $(\sqrt{\kappa} w, \sqrt{\ell} w)$ is the unique positive solution to (1.1). The uniqueness of positive solution of (1.1) when $\beta < 0$ is not expected, as for the local interaction case (1.10), multiple positive solutions have been found via bifurcation methods [5]. Showing the existence of multiple positive solutions of (1.1) is another interesting open question.

Our last result concerns with the limiting behavior of the positive ground state solutions of (1.1) as $\beta \to -\infty$.

Theorem 1.5 *Assume that* $\mu, \nu, \lambda_1, \lambda_2 > 0$ *are fixed. Let* $\{\beta_n\}$ *be a sequence satisfying* $\beta_n < 0$ *and* $\beta_n \to -\infty$ *as* $n \to \infty$*, and let* $(u_{\beta_n}, v_{\beta_n})$ *be any nonnegative nontrivial radial ground state solution of* (1.1) *with* $\beta = \beta_n$*. Then as* $n \to \infty$*, at least one of* $\|u_{\beta_n}\|^2_{\lambda_1}$ *and* $\|v_{\beta_n}\|^2_{\lambda_2}$ *goes to infinity, and* $C_r^{\beta_n} \to \infty$ *as* $n \to \infty$ *where* $C_r^{\beta_n}$ *is the least energy level of* (1.1) *with* $\beta = \beta_n$*.*

Our result here implies that when β goes to negative infinity, at least one of component of the ground state solution blow up, hence the separation of phases does not occur in the case of nonlocal interaction. For the Eq. (1.10) with local interaction, the phase separation behavior when $\beta \to -\infty$ has been proved in, for example, [39,44,57,58]. In that case, the profile of components of solution of the limiting equation tend to separate in different regions of the underlying domain.

The paper is structured this way: In Sect. 2, we provide preliminary results, including the proof of Theorem 1.3; we prove the existence of positive ground state when $\beta < \chi_0$ (part (i) of Theorem 1.2) in Sect. 3, and we prove the existence of positive ground state for large β (part (ii) of Theorem 1.2) and nonexistence for intermediate β (part (iii) of Theorem 1.2) in Sect. 4; the a priori estimate of the positive ground state solutions (part (iv) of Theorem 1.2) and the asymptotic behavior of the ground state solutions as $\beta \to -\infty$ (Theorem 1.5) are proved in Sect. 5, and the uniqueness of positive solution (Theorem 1.4) is shown in Sect. 6. The proof of the radial symmetry property in Theorem 1.1 is not directly related to other parts, so we prove it in Sect. 7.

2 Preliminary results

Throughout the paper, we use the following notation:

- $\|\cdot\|$ is the norm of $H^1(\mathbb{R}^3)$ defined by $\|u\|^2 = \int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)$;
- $\|\cdot\|_M$ is an equivalent norm of $H^1(\mathbb{R}^3)$ defined by $\|u\|_M^2 = \int_{\mathbb{R}^3}(|\nabla u|^2 + M|u|^2)$, for a positive function or constant M;
- $E = H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, and $E_r = H_r^1(\mathbb{R}^3) \times H_r^1(\mathbb{R}^3)$ where $H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|)\}$;
- For $z = (u, v) \in E = H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, $\|z\|_E^2 = \|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2$, where $\lambda_1, \lambda_2 > 0$ are the parameters in (1.1);
- $|\cdot|_p$ is the norm of $L^p(\mathbb{R}^3)$ defined by $|u|_p = \left(\int_{\mathbb{R}^3}|u|^p\right)^{1/p}$ for $0 < p \leq \infty$.
- $S = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \dfrac{|\nabla u|_2^2}{|u|_{2^*}^2}$ where $2^* = 6$ here as $N = 3$.

We first recall the following classical Hardy–Littlewood–Sobolev inequality (see [28, Theorem 4.3]).

Lemma 2.1 *Assume that $f \in L^p(\mathbb{R}^3)$ and $g \in L^q(\mathbb{R}^3)$. Then one has*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f(x)g(y)}{|x-y|^t}dxdy \leq c(p,q,t)|f|_p|g|_q,$$

where $1 < p, q < \infty$, $0 < t < 3$ *and* $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{t}{3} = 2$.

The following basic inequality is of fundamental importance for considering (1.1). It is well-known but we include a proof here for reader's convenience.

Lemma 2.2 *For $u, v \in L^{\frac{12}{5}}(\mathbb{R}^3)$, we have that*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)v^2(y)}{|x-y|}dxdy \leq \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}dxdy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{v^2(x)v^2(y)}{|x-y|}dxdy\right)^{\frac{1}{2}}. \tag{2.1}$$

In particular, if $u, v \in H^1(\mathbb{R}^3)$, then (2.1) holds.

Proof From Lemma 2.1, when $u, v \in L^{\frac{12}{5}}(\mathbb{R}^3)$, the integrals in (2.1) are all convergent. First we claim that for any $x, y \in \mathbb{R}^3$, there exists a constant $K > 0$ independent of x, y such that

$$\frac{1}{|x-y|} = K\int_{\mathbb{R}^3}\frac{1}{|x-z|^2}\cdot\frac{1}{|y-z|^2}dz. \tag{2.2}$$

Indeed the right hand side of (2.2) can be considered as a function $h(x, y)$. Then h is translation and rotation invariant in (x, y), it follows that h depends only on $|x-y|$. Furthermore $h(s|x-y|) = s^{-1}|x-y|$ and the left hand side of (2.2) satisfies the same scaling. Therefore they must agree up to a constant factor.

From (2.2), Fubini's theorem and the Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)v^2(y)}{|x-y|}dxdy\\
&= K\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{u^2(x)}{|x-z|^2}dx\right)\left(\int_{\mathbb{R}^3}\frac{v^2(y)}{|y-z|^2}dy\right)dz\\
&\leq \left[K\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{u^2(x)}{|x-z|^2}dx\right)^2dz\right]^{\frac{1}{2}}\left[K\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{v^2(y)}{|y-z|^2}dy\right)^2dz\right]^{\frac{1}{2}}\\
&= \left[K\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{u^2(x)}{|x-z|^2}dx\right)\left(\int_{\mathbb{R}^3}\frac{u^2(y)}{|y-z|^2}dy\right)dz\right]^{\frac{1}{2}}\\
&\quad\times\left[K\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{v^2(x)}{|x-z|^2}dx\right)\left(\int_{\mathbb{R}^3}\frac{v^2(y)}{|y-z|^2}dy\right)dz\right]^{\frac{1}{2}}\\
&= \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}dxdy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{v^2(x)v^2(y)}{|x-y|}dxdy\right)^{\frac{1}{2}}.
\end{aligned}$$

□

In the following, to simplify our notation, we define

$$\phi_f(x) = \int_{\mathbb{R}^3}\frac{f^2(y)}{|x-y|}dy, \ \text{for} \quad f \in H^1(\mathbb{R}^3). \tag{2.3}$$

Apparently we have the following symmetry property of ϕ_f for $u, v \in H^1(\mathbb{R}^3)$:

$$\int_{\mathbb{R}^3}\phi_u(x)v^2(x)dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)v^2(y)}{|x-y|}dxdy = \int_{\mathbb{R}^3}\phi_v(x)u^2(x)dx. \tag{2.4}$$

To study the solutions of (1.1) in a variational framework, we consider the following two cases: (1) β is negative or β is positive and small; (2) β is positive and large.

For the case (1) we consider the energy functional $\mathscr{L}_{\lambda_1\lambda_2}$ on the set

$$\mathscr{N}_{\lambda_1\lambda_2} = \{z = (u, v) \in E : u \not\equiv 0, v \not\equiv 0, F_1(z) = 0 \ \text{ and } \ F_2(z) = 0\}, \tag{2.5}$$

where

$$F_1(z) = \int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda_1 u^2)dx - \mu \int_{\mathbb{R}^3} \phi_u u^2 dx - \beta \int_{\mathbb{R}^3} \phi_u v^2 dx,$$
$$F_2(z) = \int_{\mathbb{R}^3} (|\nabla v|^2 + \lambda_2 v^2)dx - \nu \int_{\mathbb{R}^3} \phi_v v^2 dx - \beta \int_{\mathbb{R}^3} \phi_v u^2 dx. \tag{2.6}$$

We define

$$C = \inf_{z\in\mathscr{N}_{\lambda_1\lambda_2}} \mathscr{L}_{\lambda_1\lambda_2}(z) \quad \text{and} \quad C_r = \inf_{z\in\mathscr{N}^r_{\lambda_1\lambda_2}} \mathscr{L}_{\lambda_1\lambda_2}(z), \tag{2.7}$$

where $\mathscr{N}^r_{\lambda_1\lambda_2} = \mathscr{N}_{\lambda_1\lambda_2} \cap E_r$.

For the case (2), we also consider the problem on the entire Nehari manifold

$$\mathscr{M}_{\lambda_1\lambda_2} = \{z = (u, v) \in E \setminus \{(0, 0)\} : \mathscr{L}'_{\lambda_1\lambda_2}(z)[z] = F_1(z) + F_2(z) = 0\}, \tag{2.8}$$

and corresponding critical values are

$$C^0 = \inf_{z\in\mathscr{M}_{\lambda_1\lambda_2}} \mathscr{L}_{\lambda_1\lambda_2}(z), \quad \text{and} \quad C^0_r = \inf_{z\in\mathscr{M}^r_{\lambda_1\lambda_2}} \mathscr{L}_{\lambda_1\lambda_2}(z), \tag{2.9}$$

where $\mathscr{M}^r_{\lambda_1\lambda_2} = \mathscr{M}_{\lambda_1\lambda_2} \cap E_r$. Some standard arguments show that any nontrivial solution of (1.1) is on both the sets $\mathscr{N}_{\lambda_1\lambda_2}$ and $\mathscr{M}_{\lambda_1\lambda_2}$. Following the idea of [30,51], we shall prove the infimums C and C^0 are attained by a nontrivial solution of (1.1). It is also clear that $\mathscr{N}_{\lambda_1\lambda_2} \subset \mathscr{M}_{\lambda_1\lambda_2}$ and $\mathscr{N}^r_{\lambda_1\lambda_2} \subset \mathscr{M}^r_{\lambda_1\lambda_2}$, which imply that $C^0 \le C$ and $C^0_r \le C_r$. From the above definitions, we know that if $z = (u, v) \in \mathscr{N}_{\lambda_1\lambda_2}$(or $\in \mathscr{M}_{\lambda_1\lambda_2}$, and $u \neq 0$ and $v \neq 0$) satisfies $\mathscr{L}_{\lambda_1\lambda_2}(z) = C$ (or $= C^0$), and z is a solution of (1.1), then z is a ground state solution of (1.1). Similarly if $z = (u, v) \in \mathscr{N}^r_{\lambda_1\lambda_2}$ (or $\in \mathscr{M}^r_{\lambda_1\lambda_2}$, and $u \neq 0$, $v \neq 0$) satisfies $\mathscr{L}_{\lambda_1\lambda_2}(z) = C_r$ (or $= C^0_r$), and z is a solution of (1.1), then z is a radial ground state solution of (1.1).

If C^0 or C^0_r is attained by $z \in \mathscr{M}_{\lambda_1\lambda_2}$, then z is a solution of (1.1) (see, for example, [51, Proposition 3.5] or [61, Chapter 4]). The following lemma shows that when $\beta < \sqrt{\mu\nu}$, if C or C_r is attained by $z \in \mathscr{N}_{\lambda_1\lambda_2}$, then z is also a solution of (1.1).

Lemma 2.3 *Suppose that $-\infty < \beta < \sqrt{\mu\nu}$. If C (or C_r) is attained by $z = (u, v) \in \mathscr{N}_{\lambda_1\lambda_2}$ (or $\in \mathscr{N}^r_{\lambda_1\lambda_2}$), then z is a solution of (1.1).*

Proof We only show that if C is attained by $z = (u, v) \in \mathscr{N}_{\lambda_1\lambda_2}$, then z is a solution of (1.1). One needs to prove that any minimizer of $\mathscr{L}_{\lambda_1\lambda_2}$ restricted to $\mathscr{N}_{\lambda_1\lambda_2}$ satisfies $\mathscr{L}'_{\lambda_1\lambda_2}(z)[\phi] = 0$ for any $\phi \in E$.

Let F_i $(i = 1, 2)$ be defined as in (2.6). We claim that if $z \in \mathscr{N}_{\lambda_1\lambda_2}$ satisfying $\mathscr{L}_{\lambda_1\lambda_2}(z) = C$, then $F'_1(z)$ and $F'_2(z)$ are linear independent. Assume that for $K_1, K_2 \in \mathbb{R}$ such that $K_1 F'_1(z) + K_2 F'_2(z) = 0$. Since $F_1(z) = 0$, it follows from $(K_1 F'_1(z) + K_2 F'_2(z))[(u, 0)] = 0$ that

$$K_1\mu \int_{\mathbb{R}^3} \phi_u u^2 dx + K_2\beta \int_{\mathbb{R}^3} \phi_u v^2 dx = 0. \tag{2.10}$$

Similarly it follows from $F_2(z) = 0$ and $(K_1 F'_1(z) + K_2 F'_2(z))[(0, v)] = 0$ that

$$K_1\beta \int_{\mathbb{R}^3} \phi_v u^2 dx + K_2\nu \int_{\mathbb{R}^3} \phi_v v^2 dx = 0. \tag{2.11}$$

Set

$$A = \begin{pmatrix} \mu \int_{\mathbb{R}^3} \phi_u u^2 dx & \beta \int_{\mathbb{R}^3} \phi_u v^2 dx \\ \beta \int_{\mathbb{R}^3} \phi_v u^2 dx & \nu \int_{\mathbb{R}^3} \phi_v v^2 dx \end{pmatrix}. \tag{2.12}$$

It follows from Lemma 2.2 and $0 < \beta < \sqrt{\mu\nu}$ that

$$det(A) = \mu\nu \int_{\mathbb{R}^3} \phi_u u^2 dx \int_{\mathbb{R}^3} \phi_v v^2 dx - \beta^2 \left(\int_{\mathbb{R}^3} \phi_u v^2 dx \right)^2 > 0. \tag{2.13}$$

That is, A is positively definite, which implies that $K_1 = K_2 = 0$. Thus $F_1'(z)$ and $F_2'(z)$ are linear independent. Since z is a minimizer of $\mathscr{L}_{\lambda_1\lambda_2}$ restricted on $\mathscr{N}_{\lambda_1\lambda_2}$, then according to [8, Corollary 4.1.2], there exist two Lagrange multipliers $H_1, H_2 \in \mathbb{R}$ such that

$$\mathscr{L}'_{\lambda_1\lambda_2}(z) + H_1 F_1'(z) + H_2 F_2'(z) = 0. \tag{2.14}$$

So, by using the same arguments as in (2.10)–(2.13), one can prove that $H_1 = H_2 = 0$. For the case of $\beta < 0$ we can use the idea of the proof of [31, Lemma 2.1] to prove the conclusion. Here we omit the details. □

It follows from Lemma 2.3 that in order to prove the conclusion (i) of Theorem 1.2, we need to show that C_r is attained by a positive $z \in \mathscr{N}^r_{\lambda_1\lambda_2}$ for $-\infty < \beta < \chi_0$, and $C = C_r$ is attained by a positive $z \in \mathscr{N}^r_{\lambda_1\lambda_2}$ for $0 < \beta < \chi_0$, where χ_0 is given in Theorem 1.2. For this purpose we shall make good use of the unique positive solution of (1.12). Let w be the unique positive solution of (1.12) with $\sigma = \tau = 1$. Define

$$w_{\sigma,\tau}(x) = \frac{\sigma}{\sqrt{\tau}} w(\sqrt{\sigma} x), \quad w_\sigma(x) = w_{\sigma,1}(x). \tag{2.15}$$

Then $w_{\sigma,\tau}$ is the unique positive solution of (1.12).

Since $w_{\sigma,\tau}$ is the unique positive solution of (1.12), one can verify the following facts (see [37, Theorem 2], and [51, Section 3.4] or [30, Lemma 2]).

Lemma 2.4 *Consider the the minimization problems*

$$\mathscr{S}_{\sigma,\tau} = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|_\sigma^2}{(\int_{\mathbb{R}^3} \tau\phi_u u^2)^{\frac{1}{2}}} \text{ and } \mathscr{T}_{\sigma,\tau} = \inf_{u \in \mathcal{M}_0} \left\{ \frac{1}{2}\|u\|_\sigma^2 - \frac{1}{4}\int_{\mathbb{R}^3} \tau\phi_u u^2 \right\}, \tag{2.16}$$

where $\mathcal{M}_0 = \left\{ u \in H^1(\mathbb{R}^3) : u \neq 0, \|u\|_\sigma^2 = \int_{\mathbb{R}^3} \tau\phi_u u^2 \right\}$. *Then the function* $w_{\sigma,\tau}(x)$ *is a minimizer of* $\mathscr{T}_{\sigma,\tau}$ *and the unique positive solution of* (1.12). *Moreover, we have*

$$\mathscr{T}_{\sigma,\tau} = \frac{1}{4}\mathscr{S}^2_{\sigma,\tau} \quad \text{and} \quad \mathscr{S}_{\sigma,\tau} = \frac{\sigma^{\frac{3}{4}}}{\sqrt{\tau}} \mathscr{S}_{1,1} = \frac{\sigma^{\frac{3}{4}}}{\sqrt{\tau}} \mathscr{S}_1, \tag{2.17}$$

where $\mathscr{S}_{1,1} = \mathscr{S}_1 = \left(\int_{\mathbb{R}^3} \phi_w w^2 \right)^{\frac{1}{2}}$.

We introduce a function $\theta : [1, \infty) \to \mathbb{R}^+$ defined by

$$\theta(\lambda) = \frac{\displaystyle\int_{\mathbb{R}^3} \phi_w(x) w_\lambda^2(x) dx}{\displaystyle\int_{\mathbb{R}^3} \phi_w(x) w^2(x) dx}, \tag{2.18}$$

The following lemma gives some estimates of $\theta(\lambda)$.

Lemma 2.5 *Let* $\theta(\lambda)$ *be defined as in* (2.18). *Then for any* $\lambda \geq 1$ *we have*

$$\lambda^{-\frac{1}{2}} \leq \theta(\lambda) \leq \lambda^{\frac{3}{4}}. \tag{2.19}$$

Proof Since $w \in H_r^1(\mathbb{R}^3)$ is radial and is strictly decreasing in $r = |x|$, it follows that

$$w(x) \geq w(\sqrt{\lambda}x) \quad \text{for } \lambda \geq 1,\ x \in \mathbb{R}^3.$$

So we infer from Lemma 2.2 and change of variables that

$$\begin{aligned}
\int_{\mathbb{R}^3} \phi_w(x) w_\lambda^2(x) dx &= \lambda^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(x) w^2(\sqrt{\lambda} y)}{|x-y|} dxdy \\
&\leq \lambda^2 \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(x) w^2(y)}{|x-y|} dxdy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(\sqrt{\lambda}x) w^2(\sqrt{\lambda} y)}{|x-y|} dxdy \right)^{\frac{1}{2}} \\
&= \lambda^{\frac{3}{4}} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(x) w^2(y)}{|x-y|} dxdy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(x) w^2(y)}{|x-y|} dxdy \right)^{\frac{1}{2}} \\
&= \lambda^{\frac{3}{4}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(x) w^2(y)}{|x-y|} dxdy.
\end{aligned}$$

That is, $\theta(\lambda) \leq \lambda^{\frac{3}{4}}$. Furthermore we see that

$$\begin{aligned}
\int_{\mathbb{R}^3} \phi_w(x) w_\lambda^2(x) dx &= \lambda^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(y) w^2(\sqrt{\lambda} x)}{|x-y|} dxdy \\
&= \lambda^{-\frac{1}{2}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(\frac{h}{\sqrt{\lambda}}) w^2(z)}{|h-z|} dhdz \\
&\geq \lambda^{-\frac{1}{2}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(h) w^2(z)}{|h-z|} dhdz = \lambda^{-\frac{1}{2}} \int_{\mathbb{R}^3} \phi_w(x) w^2(x) dx,
\end{aligned} \tag{2.20}$$

which implies the lower bound of $\theta(\lambda)$. □

Next we use the function w to provide some estimates for C and C_r.

Lemma 2.6 *Let $\theta(\lambda)$ be defined as in (2.18). If $\kappa, \ell > 0$ satisfy*

$$\begin{cases} \mu\kappa + \beta\theta(\lambda)\ell = 1, \\ \beta\theta(\lambda)\kappa + \lambda^{\frac{3}{2}}\nu\ell = \lambda^{\frac{3}{2}}, \end{cases} \tag{2.21}$$

then we have $(\sqrt{\kappa} w_{\lambda_1}, \sqrt{\ell} w_{\lambda_2}) \in \mathcal{N}^r_{\lambda_1\lambda_2}$. That is, $\mathcal{N}^r_{\lambda_1\lambda_2} \neq \emptyset$ and $\mathcal{N}_{\lambda_1\lambda_2} \neq \emptyset$. Moreover, there exists $\rho_0 > 0$ such that

$$\begin{aligned}
0 < \rho_0 \leq C \leq C_r \leq \mathscr{L}_{\lambda_1\lambda_2}(\sqrt{\kappa} w_{\lambda_1}, \sqrt{\ell} w_{\lambda_2}) &= \frac{1}{4} \left(\kappa\lambda_1^{\frac{3}{2}} + \ell\lambda_2^{\frac{3}{2}} \right) \|w\|^2 \\
&= \frac{1}{4} \left(\kappa\lambda_1^{\frac{3}{2}} + \ell\lambda_2^{\frac{3}{2}} \right) \int_{\mathbb{R}^3} \phi_w w^2 dx.
\end{aligned} \tag{2.22}$$

Proof To prove $(\sqrt{\kappa} w_{\lambda_1}, \sqrt{\ell} w_{\lambda_2}) \in \mathcal{N}^r_{\lambda_1\lambda_2}$, it suffices to show that $(u, v) = (\sqrt{\kappa} w_{\lambda_1}, \sqrt{\ell} w_{\lambda_2})$ satisfy

$$\|u\|_{\lambda_1}^2 = \mu \int_{\mathbb{R}^3} \phi_u u^2 + \beta \int_{\mathbb{R}^3} \phi_v u^2, \quad \|v\|_{\lambda_2}^2 = \nu \int_{\mathbb{R}^3} \phi_v v^2 + \beta \int_{\mathbb{R}^3} \phi_u v^2. \tag{2.23}$$

A direct computation shows that

$$\begin{aligned}\|\sqrt{\kappa}w_{\lambda_1}\|_{\lambda_1}^2 &= \kappa\left(\lambda_1^3\int_{\mathbb{R}^3}|\nabla w(\sqrt{\lambda_1}x)|^2dx + \lambda_1^3\int_{\mathbb{R}^3}w^2(\sqrt{\lambda_1}x)dx\right)\\ &= \kappa\lambda_1^{\frac{3}{2}}\left(\int_{\mathbb{R}^3}|\nabla w(y)|^2dy + \int_{\mathbb{R}^3}w^2(y)dy\right) = \kappa\lambda_1^{\frac{3}{2}}\int_{\mathbb{R}^3}\phi_w w^2dx.\end{aligned} \tag{2.24}$$

On the other hand, by the changes of variables $x \mapsto \frac{x}{\sqrt{\lambda_1}}$ and $y \mapsto \frac{y}{\sqrt{\lambda_1}}$, one sees that

$$\begin{aligned}&\mu\int_{\mathbb{R}^3}\phi_{\sqrt{\kappa}w_{\lambda_1}}(\sqrt{\kappa}w_{\lambda_1})^2 + \beta\int_{\mathbb{R}^3}\phi_{\sqrt{\ell}w_{\lambda_2}}(\sqrt{\kappa}w_{\lambda_1})^2\\ &= \mu\kappa^2\lambda_1^4\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{w^2(\sqrt{\lambda_1}x)w^2(\sqrt{\lambda_1}y)}{|x-y|}dydx + \beta\kappa\ell\lambda_1^2\lambda_2^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{w^2(\sqrt{\lambda_1}x)w^2(\sqrt{\lambda_2}y)}{|x-y|}dydx\\ &= \mu\kappa^2\lambda_1^{\frac{3}{2}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{w^2(x)w^2(y)}{|x-y|}dydx + \beta\kappa\ell\frac{\lambda_2^2}{\sqrt{\lambda_1}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{w^2(x)w^2(\sqrt{\frac{\lambda_2}{\lambda_1}}y)}{|x-y|}dydx\\ &= \kappa\lambda_1^{\frac{3}{2}}[\mu\kappa + \beta\ell\theta(\lambda)]\int_{\mathbb{R}^3}\phi_w w^2dx.\end{aligned} \tag{2.25}$$

So if $\mu\kappa + \beta\theta(\lambda)\ell = 1$, then one has that the quantity in (2.24) equals to the one in (2.25). That is, the first equality in (2.23) is satisfied. Similarly, by using $\beta\theta(\lambda)\lambda^{-\frac{3}{2}}\kappa + \nu\ell = 1$, the second equality in (2.23) is also satisfied.

Next we prove the second part of the lemma. Since for each $z = (u, v) \in \mathcal{N}_{\lambda_1\lambda_2}$, we have

$$\int_{\mathbb{R}^3}(|\nabla u|^2 + \lambda_1u^2 + |\nabla v|^2 + \lambda_2v^2) = \int_{\mathbb{R}^3}(\mu\phi_u u^2 + \beta\phi_u v^2 + \beta\phi_v u^2 + \nu\phi_v v^2). \tag{2.26}$$

By Lemma 2.1, for some $c_1 > 0$ independent of u, v, one has that

$$\int_{\mathbb{R}^3}\phi_u(x)u^2dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(y)u^2(x)}{|x-y|}dydx \le c_1\left(\int_{\mathbb{R}^3}u^{\frac{12}{5}}\right)^{\frac{5}{3}} = c_1|u|_{\frac{12}{5}}^4. \tag{2.27}$$

Similarly, one can also prove that

$$\int_{\mathbb{R}^3}\phi_v(x)v^2dx \le c_1|u|_{\frac{12}{5}}^4 \quad\text{and}\quad \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)v^2(y)}{|x-y|}dydx \le c_1|u|_{\frac{12}{5}}^2|v|_{\frac{12}{5}}^2. \tag{2.28}$$

Substituting (2.27)–(2.28) into (2.26) and using Sobolev embedding, we obtain

$$\begin{aligned}\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2 &\le c_1\mu|u|_{\frac{12}{5}}^4 + c_1\nu|v|_{\frac{12}{5}}^4 + 2c_1\beta|u|_{\frac{12}{5}}^2|v|_{\frac{12}{5}}^2\\ &\le c_2(\|u\|_{\lambda_1}^4 + \|v\|_{\lambda_2}^4 + 2\|u\|_{\lambda_1}^2\|v\|_{\lambda_2}^2),\end{aligned} \tag{2.29}$$

for some $c_2 > 0$. Furthermore, for each $z = (u, v) \in \mathcal{N}_{\lambda_1\lambda_2}$, one has

$$\mathcal{L}_{\lambda_1\lambda_2}(u, v) = \frac{1}{4}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + \lambda_1u^2 + |\nabla v|^2 + \lambda_2v^2\right) = \frac{1}{4}(\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2) \ge \frac{1}{4c_2}, \tag{2.30}$$

which implies that $C \ge \rho_0 > 0$ for some $\rho_0 > 0$.

Finally, since $\kappa, \ell > 0$ satisfy (2.21), we have that $(\sqrt{\kappa} w_{\lambda_1}, \sqrt{\ell} w_{\lambda_2}) \in \mathcal{N}_{\lambda_1\lambda_2}$, and

$$\begin{aligned} C_r &\leq \mathcal{L}_{\lambda_1\lambda_2}(\sqrt{\kappa} w_{\lambda_1}, \sqrt{\ell} w_{\lambda_2}) \\ &= \frac{\kappa}{4} \int_{\mathbb{R}^3} (|\nabla w_{\lambda_1}|^2 + \lambda_1 w_{\lambda_1}^2) + \frac{\ell}{4} \int_{\mathbb{R}^3} (|\nabla w_{\lambda_2}|^2 + \lambda_2 w_{\lambda_2}^2) \\ &= \frac{1}{4} (\kappa \lambda_1^{\frac{3}{2}} + \ell \lambda_2^{\frac{3}{2}}) \|w\|^2 = \frac{1}{4} (\kappa \lambda_1^{\frac{3}{2}} + \ell \lambda_2^{\frac{3}{2}}) \int_{\mathbb{R}^3} \phi_w w^2. \end{aligned} \tag{2.31}$$

This finishes the proof of lemma. □

In the following we shall discuss the solvability of (2.21). From elementary calculation, we know that $\kappa > 0$ and $\ell > 0$ if either

$$det(A_\lambda) = \lambda^{\frac{3}{2}} \mu\nu - \beta^2\theta^2(\lambda) > 0 \quad \text{and} \quad \beta\theta(\lambda) < \min\{\nu, \lambda^{\frac{3}{2}}\mu\}, \tag{2.32}$$

or

$$det(A_\lambda) = \lambda^{\frac{3}{2}} \mu\nu - \beta^2\theta^2(\lambda) < 0 \quad \text{and} \quad \beta\theta(\lambda) > \max\{\nu, \lambda^{\frac{3}{2}}\mu\}, \tag{2.33}$$

where

$$A_\lambda = \begin{pmatrix} \mu & \beta\theta(\lambda) \\ \beta\theta(\lambda) & \lambda^{\frac{3}{2}}\nu \end{pmatrix}.$$

By Lemma 2.5 and further direct computation, we have that (2.32) is satisfied if

$$-\sqrt{\mu\nu} < \beta < \lambda^{-\frac{3}{4}} \min\{\nu, \lambda^{\frac{3}{2}}\mu\} = \min\{\nu_1, \mu_1\}, \tag{2.34}$$

where μ_1 and ν_1 are defined in (1.15). Similarly (2.33) is satisfied if

$$\beta > \max\{\lambda\mu, \lambda^{-\frac{1}{2}}\nu\} = \lambda^{\frac{1}{4}} \max\{\nu_1, \mu_1\}. \tag{2.35}$$

When (2.34) or (2.35) is satisfied, we can solve that

$$\kappa = \frac{\lambda^{\frac{3}{2}}(\nu - \beta\theta(\lambda))}{\lambda^{\frac{3}{2}}\mu\nu - \beta^2\theta^2(\lambda)} \quad \text{and} \quad \ell = \frac{\lambda^{\frac{3}{2}}\mu - \beta\theta(\lambda)}{\lambda^{\frac{3}{2}}\mu\nu - \beta^2\theta^2(\lambda)} \tag{2.36}$$

Define

$$a(\lambda) = h(\lambda)(2 - h(\lambda)) \quad \text{where} \quad h(\lambda) = \lambda^{-\frac{3}{4}}\theta(\lambda). \tag{2.37}$$

From Lemma 2.5 we obtain that

$$\lambda^{-\frac{5}{4}} \leq h(\lambda) \leq 1, \quad \text{and} \quad \lambda^{-\frac{5}{4}}(2 - \lambda^{-\frac{5}{4}}) \leq a(\lambda) \leq 1 \quad \text{for} \quad \lambda \geq 1. \tag{2.38}$$

Now we are ready to study the behaviour of minimizing sequences of $\mathcal{L}_{\lambda_1\lambda_2}$ on $\mathcal{N}_{\lambda_1\lambda_2}$ by using some ideas from [51].

Lemma 2.7 *Suppose that $\lambda = \lambda_2/\lambda_1 \geq 1$. Let $\theta(\lambda)$, $h(\lambda)$ and $a(\lambda)$ be defined as in (2.18) and (2.37) respectively, and let χ_1 be the smaller root of the quadratic equation*

$$a(\lambda)y^2 - (\mu_1 + \nu_1)y + \mu_1\nu_1 = 0,$$

where μ_1 and ν_1 are defined in (1.15). Suppose that

$$-\infty < \beta < \chi_1. \tag{2.39}$$

Let $\{z_n = (u_n, v_n)\} \subset \mathcal{N}_{\lambda_1\lambda_2}$ be a sequence such that

$$\mathcal{L}_{\lambda_1\lambda_2}(z_n) \to C, \quad as \quad n \to \infty. \tag{2.40}$$

Then there exists a constant $\delta > 0$ *such that for all* $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \geq \delta, \quad \textit{and} \quad \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \geq \delta.$$

Proof Let $\{z_n\} \subset \mathscr{N}_{\lambda_1\lambda_2}$ be a sequence satisfying (2.40). First, it follows from $z_n = (u_n, v_n) \in \mathscr{N}_{\lambda_1\lambda_2}$ that $\mathscr{L}'_{\lambda_1\lambda_2}(z_n)[(u_n, 0)] = 0$ and $\mathscr{L}'_{\lambda_1\lambda_2}(z_n)[(0, v_n)] = 0$. That is,

$$\lambda_1^{\frac{3}{4}} \mathscr{S}_1 \left(\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \right)^{\frac{1}{2}} \leq \|u_n\|_{\lambda_1}^2 = \mu \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 + \beta \int_{\mathbb{R}^3} \phi_{u_n} v_n^2, \tag{2.41}$$

and

$$\lambda_2^{\frac{3}{4}} \mathscr{S}_1 \left(\int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \right)^{\frac{1}{2}} \leq \|v_n\|_{\lambda_2}^2 = \nu \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 + \beta \int_{\mathbb{R}^3} \phi_{v_n} u_n^2. \tag{2.42}$$

So one infers from Lemma 2.2 to obtain that

$$\int_{\mathbb{R}^3} \phi_{u_n} v_n^2 = \int_{\mathbb{R}^3} \phi_{v_n} u_n^2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x) v_n^2(y)}{|x-y|} dxdy \leq y_{n,1} y_{n,2}, \tag{2.43}$$

where

$$y_{n,1} = \left(\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \right)^{\frac{1}{2}} \quad \text{and} \quad y_{n,2} = \left(\int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \right)^{\frac{1}{2}}. \tag{2.44}$$

Substituting (2.43) into (2.40)–(2.41), we have

$$\begin{aligned} \lambda_1^{\frac{3}{4}} \mathscr{S}_1 y_{n,1} &\leq \|u_n\|_{\lambda_1}^2 \leq \mu y_{n,1}^2 + \beta^+ y_{n,1} y_{n,2}, \\ \lambda_2^{\frac{3}{4}} \mathscr{S}_1 y_{n,2} &\leq \|v_n\|_{\lambda_2}^2 \leq \nu y_{n,2}^2 + \beta^+ y_{n,1} y_{n,2}. \end{aligned} \tag{2.45}$$

where $\beta^+ = \{\beta, 0\}$. Thus the conclusion of this lemma for $\beta \leq 0$ follows from (2.45).

Next we shall consider the case of $\beta > 0$. It follows from (2.22), (2.41) and (2.42) that

$$\begin{aligned} \mathscr{S}_1(\lambda_1^{\frac{3}{4}} y_{n,1} + \lambda_2^{\frac{3}{4}} y_{n,2}) &\leq \mu \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 + \nu \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 + 2\beta \int_{\mathbb{R}^3} \phi_{u_n} v_n^2 \\ &= 4C + o(1) \leq (\kappa \lambda_1^{\frac{3}{2}} + \ell \lambda_2^{\frac{3}{2}}) \mathscr{S}_1^2 + o(1), \end{aligned} \tag{2.46}$$

where (κ, ℓ) is the root of (2.21). Let

$$h_{n,1} = \frac{y_{n,1}}{\lambda_1^{\frac{3}{4}} \mathscr{S}_1}, \quad h_{n,2} = \frac{y_{n,2}}{\lambda_1^{\frac{3}{4}} \mathscr{S}_1}.$$

Then it follows from (2.45) and (2.46) that $(h_{n,1}, h_{n,2})$ satisfies the system of inequalities:

$$\begin{cases} h_{n,1} + \lambda^{\frac{3}{4}} h_{n,2} \leq \kappa + \ell \lambda^{\frac{3}{2}} + o(1), \\ 1 \leq \mu h_{n,1} + \beta h_{n,2}, \\ \lambda^{\frac{3}{4}} \leq \beta h_{n,1} + \nu h_{n,2}. \end{cases} \tag{2.47}$$

Define a triangular region by

$$\Gamma = \{(h_1, h_2) \in \mathbb{R}^2 : h_1 + \lambda^{\frac{3}{4}} h_2 \leq \kappa + \ell \lambda^{\frac{3}{2}}, \ 1 \leq \mu h_1 + \beta h_2, \ \lambda^{\frac{3}{4}} \leq \beta h_1 + \nu h_2\}.$$

To prove the two sequences $\{h_{n,1}\}$ and $\{h_{n,2}\}$ staying uniformly away from zero, we only need to show that the triangular region Γ is entirely in the interior of the first quadrant of $\mathbb{R}^2$. This can be achieved if the following set of conditions are met:

$$\begin{cases} \beta < \lambda^{\frac{3}{4}}\mu = \mu_1 \quad \text{and} \quad \beta < \lambda^{-\frac{3}{4}}\nu = \nu_1, & (2.48)\\ \mu(\kappa + \lambda^{\frac{3}{2}}\ell) > 1, & (2.49)\\ \nu(\kappa + \lambda^{\frac{3}{2}}\ell) > \lambda^{\frac{3}{2}}, & (2.50)\\ \beta(\kappa + \lambda^{\frac{3}{2}}\ell) < \lambda^{\frac{3}{4}}. & (2.51) \end{cases}$$

First, (2.48) holds if (2.34) is satisfied. Secondly substituting the expression of (κ, ℓ) in (2.36), one has

$$\mu(\kappa + \lambda^{\frac{3}{2}}\ell) - 1 = \frac{(\lambda^{\frac{3}{2}}\mu - \beta\theta(\lambda))^2}{\lambda^{\frac{3}{2}}\mu\nu - \beta^2\theta^2(\lambda)} > 0 \tag{2.52}$$

from (2.32), and this implies that (2.49) holds. Similarly (2.50) also holds. Finally (2.51) is equivalent to

$$\left[\frac{2\theta(\lambda)\lambda^{\frac{3}{4}} - \theta^2(\lambda)}{\lambda^{\frac{3}{2}}}\right]\beta^2 - \left(\lambda^{\frac{3}{4}}\mu + \lambda^{-\frac{3}{4}}\nu\right)\beta + \mu\nu > 0. \tag{2.53}$$

That is,

$$a(\lambda)\beta^2 - (\mu_1 + \nu_1)\beta + \mu_1\nu_1 > 0. \tag{2.54}$$

where μ_1 and ν_1 are defined in (1.15), and $a(\lambda)$ is defined in (2.37). Therefore by the definition of χ_1, one sees that (2.48)-(2.51) are satisfied if $0 < \beta < \min\{\chi_1, \mu_1, \nu_1\}$. From the estimate in (2.38), we know that $a(\lambda) \leq 1$ so $\chi_1 < \min\{\mu_1, \nu_1\}$. Hence the conclusion of the lemma holds if $0 < \beta < \chi_1$. This finishes the proof of this lemma. □

Remark 2.8 Recall the constant χ_0 defined in Theorem 1.2 and (1.14). We can see that $\chi_1 \geq \chi_0$ from (2.38). Indeed in the next section we shall show results in Theorem 1.2 part (i) hold when $\beta < \chi_1$. We can also see that for all $\lambda \geq 1$,

$$\chi_1 \in \left(\frac{\mu_1\nu_1}{\mu_1 + \nu_1}, \min\{\mu_1, \nu_1\}\right]. \tag{2.55}$$

When $\lambda = 1$ ($\lambda_1 = \lambda_2$), we have $\chi_1 = \min\{\mu, \nu\}$. Indeed in this case, a sharper result can be obtained as follows.

Lemma 2.9 *Suppose that $\lambda_1 = \lambda_2$. Let $\{z_n = (u_n, v_n)\} \subset \mathcal{N}_{\lambda_1\lambda_1}$ be a sequence satisfying* (2.40). *If β satisfies*

$$-\infty < \beta < \min\{\mu, \nu\} \quad or \quad \beta > \max\{\mu, \nu\}, \tag{2.56}$$

then

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\right)^{\frac{1}{2}} = \lambda_1^{\frac{3}{4}}\mathcal{S}_1\kappa, \quad and \quad \lim_{n\to\infty}\left(\int_{\mathbb{R}^3}\phi_{v_n}v_n^2\right)^{\frac{1}{2}} = \lambda_1^{\frac{3}{4}}\mathcal{S}_1\ell, \tag{2.57}$$

where (κ, ℓ) satisfies (1.16).

Proof We infer from $\lambda = \lambda_2/\lambda_1 = 1$ that (2.21) and (2.47) are now (1.16) and

$$\begin{cases} h_{n,1} + h_{n,2} \le \kappa + \ell + o(1), \\ 1 \le \mu h_{n,1} + \beta h_{n,2}, \\ 1 \le \beta h_{n,1} + \nu h_{n,2}. \end{cases} \tag{2.58}$$

We set $w_{n,1} = h_{n,1} - \kappa$ and $w_{n,2} = h_{n,2} - \ell$. Then from (1.16) and (2.58), we obtain that

$$\begin{cases} w_{n,1} + w_{n,2} \le o(1), \\ \mu w_{n,1} + \beta w_{n,2} \ge 0, \\ \beta w_{n,1} + \nu w_{n,2} \ge 0. \end{cases} \tag{2.59}$$

Note that the region

$$\Gamma_n' = \{(w_{n,1}, w_{n,2}) : w_{n,1} + w_{n,2} \le o(1),\ \mu w_{n,1} + \beta w_{n,2} \ge 0,\ \beta w_{n,1} + \nu w_{n,2} \ge 0\}$$

represented by (2.59) is a triangle with vertex $(0, 0)$ and it is of diameter $o(1)$ as $n \to \infty$. Thus when (2.56) is satisfied, we have $\kappa > 0$ and $\ell > 0$, so we have $h_{n,1} \to \kappa$ and $h_{n,2} \to \ell$ as $n \to \infty$ which implies (2.57). □

The convergence result in Lemma 2.9 enables us to give the proof of Theorem 1.3.

Proof of Theorem 1.3 Passing the limit in (2.46) and using Lemma 2.9, we obtain that for $\lambda_1 = \lambda_2$,

$$C \ge \frac{1}{4}\mathscr{S}_1^2 \lambda_1^{\frac{3}{2}}(\kappa + \ell). \tag{2.60}$$

On the other hand, it follows from Lemma 2.4 and Lemma 2.6 that for $\lambda_1 = \lambda_2$,

$$C \le \frac{1}{4}\mathscr{S}_1^2 \lambda_1^{\frac{3}{2}}(\kappa + \ell). \tag{2.61}$$

This implies that

$$C = \frac{1}{4}\mathscr{S}_1^2 \lambda_1^{\frac{3}{2}}(\kappa + \ell) = \mathscr{L}_{\lambda_1\lambda_1}(\sqrt{\kappa} w_{\lambda_1}, \sqrt{\ell} w_{\lambda_1}), \tag{2.62}$$

and this proves part (i). For part (ii), multiplying the u-equation in (1.1) by v, the v-equation by u, subtracting and integrating over $\mathbb{R}^3$, we get

$$(\mu - \beta)\int_{\mathbb{R}^3} \phi_u uv + (\beta - \nu)\int_{\mathbb{R}^3} \phi_v uv = 0. \tag{2.63}$$

Thus, any non-negative solution (u, v) for $\beta \in [\min\{\mu, \nu\}, \max\{\mu, \nu\}]$ satisfies $u(x)v(x) \equiv 0$ for $x \in \mathbb{R}^3$. From the strong maximum principle, we must have $u \equiv 0$ or $v \equiv 0$. This proves part (ii). □

3 Existence of positive ground state for negative and small positive β

In this section we prove the existence of positive ground state solution for negative or small positive β, that is, part (i) of Theorem 1.2. For that propose, for $\vartheta, \omega > 0$, we set

$$\Phi(u) = \frac{1}{4}\|u\|_\vartheta^2 = \frac{1}{4}\int_{\mathbb{R}^3} (|\nabla u|^2 + \vartheta u^2),$$

and

$$\mathcal{M}_{\vartheta,\omega} = \left\{ u \in H^1(\mathbb{R}^3) : \|u\|_\vartheta^2 = \omega \int_{\mathbb{R}^3} \phi_u u^2 \right\},$$
$$\widetilde{\mathcal{M}}_{\vartheta,\omega} = \left\{ u \in H^1(\mathbb{R}^3) : \|u\|_\vartheta^2 \le \omega \int_{\mathbb{R}^3} \phi_u u^2 \right\}.$$

Motivated by [30, Lemmas 2 and 3], we prove the following result on the infimum of $\Phi(u)$.

Lemma 3.1 *Let $\vartheta, \omega > 0$ and recall that $w_{\vartheta,\omega}$ is defined in Lemma 2.4. Then*

(i) $B_1 = \inf_{u \in \mathcal{M}_{\vartheta,\omega}} \Phi(u)$ *is attained only by $w_{\vartheta,\omega}$;*
(ii) $B_2 = \inf_{u \in \widetilde{\mathcal{M}}_{\vartheta,\omega}} \Phi(u) = B_1$ *is also attained only by $w_{\vartheta,\omega}$.*

Proof The conclusion (i) follows from Lemma 2.4 and that w is the unique positive solution of (1.12) with $\sigma = \tau = 1$. For part (ii), let u_n be a minimizing sequence for B_2. Observing that $\Phi(u_n) = \Phi(|u_n|)$ and $\{|u_n|\} \subset \mathcal{M}_{\vartheta,\omega}$, so without loss of generality we can assume that $\{u_n\}$ is a nonnegative minimizing sequence. Let u_n^* be the Schwartz symmetrization of u_n. Then by the property of Schwartz symmetrization (see [28, Theorem 3.7]), we have

$$\int_{\mathbb{R}^3} (|\nabla u_n^*|^2 + \vartheta (u_n^*)^2) \le \int_{\mathbb{R}^3} (|\nabla u_n|^2 + \vartheta u_n^2) \le \omega \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \le \omega \int_{\mathbb{R}^3} \phi_{u_n^*} (u_n^*)^2 dx$$

and

$$\Phi(u_n^*) \le \Phi(u_n).$$

So we may also assume that u_n is radially symmetric and decreasing in the radial direction. Since $B_2 \le B_1$, it follows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. We assume that (up to a subsequence) $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$, and $u_n \to u$ in $L_r^p(\mathbb{R}^3)$ for $p \in (2, 6)$, and the limit u is also radially symmetric and decreasing in the radial direction. Moreover one can check that as $n \to \infty$, $\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \to \int_{\mathbb{R}^3} \phi_u u^2 dx$. So we infer from Fatou's Lemma that $u \in \widetilde{\mathcal{M}}_{\vartheta,\omega}$ and B_2 can be attained by u. We then claim that $u \in \mathcal{M}_{\vartheta,\omega}$. Suppose that to the contrary, u is in the interior of $\tilde{\mathcal{M}}_{\vartheta,\omega}$, we have

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + \vartheta u^2) < \omega \int_{\mathbb{R}^3} \phi_u u^2. \tag{3.1}$$

Since u is a minimizer of Φ in the interior, then u is a critical point of Φ, i.e. $\nabla\Phi(u) = 0$, and it implies that

$$0 = 2\langle \nabla\Phi(u), u\rangle = \int_{\mathbb{R}^3} (|\nabla u|^2 + \vartheta u^2).$$

Thus $u \equiv 0$, and it contradicts with (3.1). Hence $u \in \mathcal{M}_{\vartheta,\omega}$. By the conclusion (i), we must have $u = w_{\vartheta,\omega}$. □

Now we are ready to give the proof of part (i) of Theorem 1.2. From Lemma 2.3, it can be accomplished by the following lemma. Indeed the following result is stronger as $\chi_0 \le \chi_1$.

Lemma 3.2 *Suppose that $\lambda_1 \le \lambda_2$.*

(i) If $-\infty < \beta < \chi_1$ where χ_1 is defined in Lemma 2.7, then C_r is attained by some positive $z = (u, v) \in \mathcal{N}_{\lambda_1\lambda_2}^r$.
(ii) If $0 < \beta < \chi_1$, then C is attained by some positive $z = (u, v) \in \mathcal{N}_{\lambda_1\lambda_2}$.

Proof We first prove part (i). Let $\{\tilde{z}_n = (\tilde{u}_n, \tilde{v}_n)\} \subset \mathscr{N}^r_{\lambda_1\lambda_2}$ be a minimizing sequence such that $\mathscr{L}_{\lambda_1\lambda_2}(\tilde{z}_n) \to C_r$ as $n \to \infty$. By applying the Ekeland's variational principle (see [61]) on $\mathscr{N}^r_{\lambda_1\lambda_2}$, one obtains a sequence $\{z_n = (u_n, v_n)\} \subset \mathscr{N}^r_{\lambda_1\lambda_2}$ satisfying

1. $\mathscr{L}_{\lambda_1\lambda_2}(u_n, v_n) \leq C_r + \frac{1}{n},$ (3.2)
2. $\mathscr{L}_{\lambda_1\lambda_2}(u, v) \geq \mathscr{L}_{\lambda_1\lambda_2}(u_n, v_n) - \frac{1}{n}\|(u_n, v_n) - (u, v)\|_E, \quad \forall (u, v) \in \mathscr{N}^r_{\lambda_1\lambda_2},$ (3.3)
3. $\|(\tilde{u}_n, \tilde{v}_n) - (u_n, v_n)\|_E \to 0,$ as $n \to \infty$. (3.4)

We claim that

$$\mathscr{L}'_{\lambda_1\lambda_2}(u_n, v_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.5}$$

First one deduces from $(u_n, v_n) \in \mathscr{N}^r_{\lambda_1\lambda_2}$ and (3.2) that

$$C_r + \frac{1}{n} \geq \mathscr{L}_{\lambda_1\lambda_2}(u_n, v_n) = \frac{1}{4}(\|u_n\|^2_{\lambda_1} + \|v_n\|^2_{\lambda_2}). \tag{3.6}$$

So the sequence $\{(u_n, v_n)\}$ is bounded. Moreover, as in Lemma 2.6 one has that $\|(u_n, v_n)\|_E \geq \delta > 0$ for some $\delta > 0$. For a fixed $(\varphi, \phi) \in E$ and $\|\varphi\|, \|\phi\| \leq 1$, we define

$$\begin{aligned} &G_n(t, \omega, s) = (G^1_n(t, \omega, s), G^2_n(t, \omega, s)) \\ &= \left(F_1\left(u_n + t\varphi + \frac{\omega}{2}u_n, v_n + t\phi + \frac{s}{2}v_n\right), F_2\left(u_n + t\varphi + \frac{\omega}{2}u_n, v_n + t\phi + \frac{s}{2}v_n\right)\right), \end{aligned} \tag{3.7}$$

where F_1 and F_2 are defined in (2.6). Clearly, $G_n(\Theta) = (0, 0)$ and $G_n \in C^1(\mathbb{R}^3, \mathbb{R}^2)$, where $\Theta = (0, 0, 0)$. A direct computation shows that

$$B_n = \begin{pmatrix} \frac{\partial G^1_n}{\partial \omega}(\Theta) & \frac{\partial G^1_n}{\partial s}(\Theta) \\ \frac{\partial G^2_n}{\partial \omega}(\Theta) & \frac{\partial G^2_n}{\partial s}(\Theta) \end{pmatrix} \equiv \begin{pmatrix} -\mu \int_{\mathbb{R}^3} \phi_{u_n} u^2_n & -\beta \int_{\mathbb{R}^3} \phi_{u_n} v^2_n \\ -\beta \int_{\mathbb{R}^3} \phi_{v_n} u^2_n & -\nu \int_{\mathbb{R}^3} \phi_{v_n} v^2_n \end{pmatrix}. \tag{3.8}$$

When $0 \leq \beta < \chi_1$, from Lemmas 2.2 and 2.7 one deduces that

$$det(B_n) \geq (\mu\nu - \beta^2) \int_{\mathbb{R}^3} \phi_{u_n} u^2_n \int_{\mathbb{R}^3} \phi_{v_n} v^2_n \geq c > 0, \tag{3.9}$$

where c is independent of n. For $\beta \leq 0$, since $(u_n, v_n) \in \mathscr{N}_{\lambda_1\lambda_2}$, we infer from (2.25), Lemmas 2.4 and 2.7 that

$$\begin{aligned} det(B_n) &= \left(|\beta| I_n + \|u_n\|^2_{\lambda_1}\right)\left(|\beta| I_n + \|v_n\|^2_{\lambda_2}\right) - \beta^2 I^2_n \\ &\geq \|u_n\|^2_{\lambda_1}\|v_n\|^2_{\lambda_2} \geq \mathscr{S}_{\lambda_1,1}\mathscr{S}_{\lambda_2,1}\left(\int_{\mathbb{R}^3} \phi_{u_n} u^2_n\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3} \phi_{v_n} v^2_n\right)^{\frac{1}{2}} \geq c > 0, \end{aligned} \tag{3.10}$$

where $I_n = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2_n(x)v^2_n(y)}{|x-y|}dxdy$ and c is independent of n.

So $det(B_n) \geq c > 0$ for $-\infty < \beta < \chi_1$, and by the implicit function theorem, there exist C^1 functions $\omega_n(t)$ and $s_n(t)$ defined on some interval $(-\tau_n, \tau_n)$ where $\tau_n > 0$, such that $s_n(0) = \omega_n(0) = 0$ and

$$G_n(t, \omega_n(t), s_n(t)) = 0, \quad t \in (-\tau_n, \tau_n). \tag{3.11}$$

Differentiating (3.11) in t at $t = 0$, we obtain

$$\frac{\partial G_n^i}{\partial t}(\Theta) + \frac{\partial G_n^i}{\partial \omega}(\Theta)\omega_n'(0) + \frac{\partial G_n^i}{\partial s}(\Theta)s_n'(0) = 0, \quad i = 1, 2, \tag{3.12}$$

which together with (3.8) implies that

$$\begin{aligned} \omega_n'(0) &= \frac{\frac{\partial G_n^1}{\partial s}(\Theta)\frac{\partial G_n^2}{\partial t}(\Theta) - \frac{\partial G_n^1}{\partial t}(\Theta)\frac{\partial G_n^2}{\partial s}(\Theta)}{det(B_n)}, \\ s_n'(0) &= \frac{\frac{\partial G_n^1}{\partial t}(\Theta)\frac{\partial G_n^2}{\partial \omega}(\Theta) - \frac{\partial G_n^1}{\partial \omega}(\Theta)\frac{\partial G_n^2}{\partial t}(\Theta)}{det(B_n)}. \end{aligned} \tag{3.13}$$

It follows from the boundedness of $\{z_n = (u_n, v_n)\}$ and (φ, ϕ) that

$$\left|\frac{\partial G_n^1}{\partial t}(\Theta)\right| = 2\left|\int_{\mathbb{R}^3}(\nabla u_n \nabla \varphi + \lambda_1 u_n \varphi) - \int_{\mathbb{R}^3}(2\mu\phi_{u_n}u_n\varphi - \beta\phi_{v_n}u_n\varphi - \beta\phi_{u_n}v_n\phi)\right| \le c,$$

where c is independent of n. Similarly, we also have that

$$\left|\frac{\partial G_n^1}{\partial s}(\Theta)\right|, \left|\frac{\partial G_n^1}{\partial \omega}(\Theta)\right|, \left|\frac{\partial G_n^2}{\partial t}(\Theta)\right|, \left|\frac{\partial G_n^2}{\partial s}(\Theta)\right|, \left|\frac{\partial G_n^2}{\partial \omega}(\Theta)\right| \le c. \tag{3.14}$$

Hence together with $det(B_n) \ge c > 0$ and (3.13), we obtain that

$$|s_n'(0)|, \ |\omega_n'(0)| \le c. \tag{3.15}$$

Let

$$\bar{\varphi}_{n,t} = t\varphi + \frac{\omega_n(t)}{2}u_n, \ \bar{\phi}_{n,t} = t\phi + \frac{s_n(t)}{2}v_n, \ \varphi_{n,t} = u_n + \bar{\varphi}_{n,t}, \ \phi_{n,t} = v_n + \bar{\phi}_{n,t}.$$

Then it follows from (3.11) that $(\varphi_{n,t}, \phi_{n,t}) \in \mathcal{N}^r_{\lambda_1\lambda_2}$ for $t \in (-\tau_n, \tau_n)$. Furthermore, we deduce from (3.3) that

$$\mathcal{L}_{\lambda_1\lambda_2}(\varphi_{n,t}, \phi_{n,t}) - \mathcal{L}_{\lambda_1\lambda_2}(u_n, v_n) \ge -\frac{1}{n}\left\|\left(\bar{\varphi}_{n,t}, \bar{\phi}_{n,t}\right)\right\|_E. \tag{3.16}$$

Note that $\mathcal{L}'_{\lambda_1\lambda_2}(u_n, v_n)[(u_n, 0)] = \mathcal{L}'_{\lambda_1\lambda_2}(u_n, v_n)[(0, v_n)] = 0$. From Taylor expansion we have that

$$\begin{aligned} &\mathcal{L}_{\lambda_1\lambda_2}(\varphi_{n,t}, \phi_{n,t}) - \mathcal{L}_{\lambda_1\lambda_2}(u_n, v_n) \\ =&\mathcal{L}'_{\lambda_1\lambda_2}(u_n, v_n)[(\bar{\varphi}_{n,t}, \bar{\phi}_{n,t})] + K(n,t) = t\mathcal{L}'_{\lambda_1\lambda_2}(u_n, v_n)[(\varphi, \phi)] + K(n,t), \end{aligned} \tag{3.17}$$

where $K(n,t) = o(\|(\bar{\varphi}_{n,t}, \bar{\phi}_{n,t})_E\|) = o(t)$ as $t \to 0$. It follows from (3.15) that

$$\limsup_{n\to\infty} \|(\bar{\varphi}_{n,t}, \bar{\phi}_{n,t})\|_E \le c, \tag{3.18}$$

where c is independent of n. Thus, $K(n,t) = o(t)$ as $t \to 0$. One can deduce from (3.16)–(3.18) that

$$|\mathcal{L}'_{\lambda_1\lambda_2}(u_n, v_n)[(\varphi, \phi)]| \le \frac{c}{n}, \quad n \to \infty. \tag{3.19}$$

That is, the claim (3.5) holds.

Since $\{(u_n, v_n)\}$ is bounded in E, we may assume that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in E. By the compact embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ for $p \in (2, 6)$, we may also assume that the sequence $\{(u_n, v_n)\}$ converges (up to a subsequence) weakly in E_r and strongly in $L^p(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ to a function (u, v). From (3.5) one can conclude that that

$$\mathscr{L}'_{\lambda_1\lambda_2}(u, v) = 0. \tag{3.20}$$

On the other hand, from Lemma 2.7 and Hölder inequality we infer that

$$0 < \delta \leq \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \leq c \left(\int_{\mathbb{R}^3} u_n^{\frac{12}{5}} \right)^{\frac{5}{6}}, \tag{3.21}$$

and it follows from Brezis–Lieb lemma (see [61]) that $\int_{\mathbb{R}^3} u_n^{\frac{12}{5}} \to \int_{\mathbb{R}^3} u^{\frac{12}{5}}$ as $n \to \infty$, and $u \not\equiv 0$ on $\mathbb{R}^3$. Similarly one can prove that $v \not\equiv 0$. These together with (3.21) imply that $(u, v) \in \mathscr{N}^r_{\lambda_1\lambda_2}$ and $C_r \leq \mathscr{L}_{\lambda_1\lambda_2}(u, v)$. Furthermore, we have

$$\begin{aligned} C_r &= \lim_{n\to\infty} \mathscr{L}_{\lambda_1\lambda_2}(u_n, v_n) = \lim_{n\to\infty} \left[\mathscr{L}_{\lambda_1\lambda_2}(u_n, v_n) - \frac{1}{4} \mathscr{L}'_{\lambda_1\lambda_2}(u_n, v_n)[(u_n, v_n)] \right] \\ &= \lim_{n\to\infty} (\|u_n\|^2_{\lambda_1} + \|v_n\|^2_{\lambda_2}) \geq \lim_{n\to\infty} (\|u\|^2_{\lambda_1} + \|v\|^2_{\lambda_2}) = \mathscr{L}_{\lambda_1\lambda_2}(u, v). \end{aligned} \tag{3.22}$$

Finally we can choose (u, v) to be nonnegative since $\mathscr{L}_{\lambda_1\lambda_2}(|u|, |v|) = \mathscr{L}_{\lambda_1\lambda_2}(u, v) = C_r$ and $(|u|, |v|) \in \mathscr{N}^r_{\lambda_1\lambda_2}$. Moreover from the strong maximum principle we infer that $|u|, |v| > 0$. Thus $(u, v) = (|u|, |v|)$ is a positive solution of (1.1) which attains C_r.

Next we prove part (ii) and assume that $0 < \beta < \chi_1$. Let $\{z_n = (u_n, v_n)\} \subset \mathscr{N}_{\lambda_1\lambda_2}$ be a minimizing sequence such that $\mathscr{L}_{\lambda_1\lambda_2}(z_n) \to C$ as $n \to \infty$. Again we may assume that $\{z_n\}$ is a nonnegative minimizing sequence as $\mathscr{L}_{\lambda_1\lambda_2}(|u_n|, |v_n|) = \mathscr{L}_{\lambda_1\lambda_2}(u_n, v_n)$. Let $z_n^* = (u_n^*, v_n^*)$ be the Schwartz symmetrization of (u_n, v_n). Then by the property of symmetrization (see [28]), we have that

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \leq \int_{\mathbb{R}^3} \phi_{u_n^*} (u_n^*)^2, \ \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \leq \int_{\mathbb{R}^3} \phi_{v_n^*} (v_n^*)^2, \ \int_{\mathbb{R}^3} \phi_{u_n} v_n^2 \leq \int_{\mathbb{R}^3} \phi_{u_n^*} (v_n^*)^2,$$
$$\|u_n^*\|_{\lambda_1} \leq \|u_n\|_{\lambda_1}, \quad \|v_n^*\|_{\lambda_1} \leq \|v_n\|_{\lambda_1},$$

So we have that

$$\begin{aligned} &\int_{\mathbb{R}^3} (|\nabla u_n^*|^2 + \lambda_1 (u_n^*)^2) \leq \mu \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 + \beta \int_{\mathbb{R}^3} \phi_{u_n^*} (v_n^*)^2, \\ &\int_{\mathbb{R}^3} (|\nabla v_n^*|^2 + \lambda_2 (v_n^*)^2) \leq \nu \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 + \beta \int_{\mathbb{R}^3} \phi_{v_n^*} (u_n^*)^2. \end{aligned}$$

Therefore, one sees that

$$C = \inf_{(u,v)\in\mathscr{N}_{\lambda_1\lambda_2}} \mathscr{L}_{\lambda_1\lambda_2}(u, v) = \inf_{(u,v)\in\mathscr{N}_{\lambda_1\lambda_2}} \Phi_{\lambda_1\lambda_2}(u, v) \geq \tilde{C} = \inf_{(u,v)\in\tilde{\mathscr{N}}_{\lambda_1\lambda_2}} \Phi_{\lambda_1\lambda_2}(u, v),$$

where

$$\begin{aligned} \Phi_{\lambda_1\lambda_2}(u, v) &= \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda_1 u^2) + \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla v|^2 + \lambda_2 v^2), \\ \tilde{\mathscr{N}}_{\lambda_1\lambda_2} &= \{z = (u, v) \in E : u \not\equiv 0, v \not\equiv 0, F_1(z) \leq 0, F_2(z) \leq 0\}, \end{aligned}$$

where F_1, F_2 are defined in (2.6).

Let $\{\tilde{z}_n = (\tilde{u}_n, \tilde{v}_n)\} \subset \tilde{\mathscr{N}}_{\lambda_1\lambda_2}$ be a minimizing sequence such that $\mathscr{L}_{\lambda_1\lambda_2}(\tilde{z}_n) \to \tilde{C}$ as $n \to \infty$. From the argument above, we may assume that $\tilde{u}_n, \tilde{v}_n$ are radially symmetric and decreasing in radial direction. By using the proof of Lemma 3.1, we conclude that up to a subsequence, $(\tilde{u}_n, \tilde{v}_n)$ converges weakly in E_r and strongly in $L^p(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$

for $p \in (2,6)$, to a minimizer (u,v) for $\tilde{C}$. Clearly (u,v) is also radially symmetric and decreasing in radial direction. Since $0 < \beta < \chi_1$, it follows from Lemma 2.7 that there exists $\delta > 0$ such that

$$\int_{\mathbb{R}^3} \phi_u u^2 \geq \delta, \quad \int_{\mathbb{R}^3} \phi_v v^2 \geq \delta.$$

In the following we shall prove that $(u,v) \in \mathcal{N}_{\lambda_1\lambda_2}$. First, we prove that if $0 < \beta < \chi_1 \leq \min\{\mu_1, \nu_1\} \leq \sqrt{\mu_1\nu_1} = \sqrt{\mu\nu}$, there exists a unique pair point (t_1,t_2) such that $(\sqrt{t_1}u, \sqrt{t_2}v) \in \mathcal{N}_{\lambda_1\lambda_2}$. Set

$$\begin{aligned} L(s,t) &= \mathcal{L}_{\lambda_1\lambda_2}(\sqrt{s}u, \sqrt{t}v) \\ &= \frac{s}{2}\|u\|^2_{\lambda_1} + \frac{t}{2}\|v\|^2_{\lambda_1} - \frac{\mu s^2}{4}\int_{\mathbb{R}^3}\phi_u u^2 - \frac{\nu t^2}{4}\int_{\mathbb{R}^3}\phi_v v^2 - \frac{\beta st}{2}\int_{\mathbb{R}^3}\phi_u v^2, \end{aligned}$$

for $s,t \geq 0$. If $0 < \beta < \sqrt{\mu\nu}$, then we know that the matrix A is positively definite, where A is given in (2.12). Hence, the quadratic form

$$K(s,t) = \frac{\mu s^2}{4}\int_{\mathbb{R}^3}\phi_u u^2 + \frac{\nu t^2}{4}\int_{\mathbb{R}^3}\phi_v v^2 + \frac{\beta st}{2}\int_{\mathbb{R}^3}\phi_u v^2.$$

is positively definite, which implies that $L(s,t)$ is concave in $\mathbb{R}^2_+ = \{(s,t) : s \geq 0, t \geq 0\}$ and $L(s,t) \to -\infty$ as $|s|+|t| \to \infty$. Hence it has a unique (local) maximum point $(t_1,t_2) \in \mathbb{R}^2_+$ such that $(\sqrt{t_1}u, \sqrt{t_2}v) \in \mathcal{N}_{\lambda_1\lambda_2}$. Thus, it follows that

$$\begin{aligned} &\int_{\mathbb{R}^3}(|\nabla u|^2 + \lambda_1 u^2) = t_1\mu\int_{\mathbb{R}^3}\phi_u u^2 + t_1\beta\int_{\mathbb{R}^3}\phi_u v^2 dx, \\ &\int_{\mathbb{R}^3}(|\nabla v|^2 + \lambda_2 v^2) = t_2\nu\int_{\mathbb{R}^3}\phi_v v^2 + t_2\beta\int_{\mathbb{R}^3}\phi_v u^2 dx. \end{aligned} \tag{3.23}$$

Furthermore, since $(\sqrt{t_1}u, \sqrt{t_2}v) \in \mathcal{N}_{\lambda_1\lambda_2} \subset \tilde{\mathcal{N}}_{\lambda_1\lambda_2}$, one has that

$$\tilde{C} \leq \Phi_{\lambda_1\lambda_2}(\sqrt{t_1}u, \sqrt{t_2}v),$$

and hence

$$\int_{\mathbb{R}^3}(|\nabla u|^2+\lambda_1 u^2+|\nabla v|^2+\lambda_2 v^2) \leq t_1\int_{\mathbb{R}^3}(|\nabla u|^2+\lambda_1 u^2)+t_2\int_{\mathbb{R}^3}(|\nabla v|^2+\lambda_2 v^2). \tag{3.24}$$

Substituting (3.23) into (3.24) we obtain that

$$t_1 F_1(u,v) + t_2 F_2(u,v) \geq 0. \tag{3.25}$$

Since $(u,v) \in \tilde{\mathcal{N}}_{\lambda_1\lambda_2}$ and $t_1, t_2 > 0$, then we must have $F_1(u,v) = 0$ and $F_2(u,v) = 0$ thus $(u,v) \in \mathcal{N}_{\lambda_1\lambda_2}$, Therefore $\tilde{C} = C = C_r = \mathcal{L}_{\lambda_1\lambda_2}(u,v)$. According to Lemma 2.3 we know that (u,v) is a critical point of $\mathcal{L}_{\lambda_1\lambda_2}$. By using the same argument as in the case of $-\infty < \beta < \chi_1$, we can prove (u,v) is a positive radial ground state solution of (1.1). □

4 Existence of positive ground state for large β

In order to prove part (ii) of Theorem 1.2, we consider our problem on the entire Nehari manifold given by (2.8). First we show that when $\beta \geq 0$, the critical value C_0 is attained by a radially symmetric solution of (1.1) .

Lemma 4.1 *Suppose that $\beta \geq 0$, and let C^0 and C_r^0 be defined as in (2.8) and (2.9). Then $C^0 = C_r^0 > 0$ is attained by a nonnegative (possibly semi-trivial) radially symmetric solution of (1.1).*

Proof Let $\{z_n = (u_n, v_n)\} \subset \mathscr{M}_{\lambda_1\lambda_2}$ be a minimizing sequence such that $\mathscr{L}_{\lambda_1\lambda_2}(z_n) \to C^0$ as $n \to \infty$. It is easy to check that $\{z_n\}$ is bounded in E, and as in the proof of Lemma 3.2 we may assume that z_n is nonnegative. Let $z_n^* = (u_n^*, v_n^*)$ be the Schwartz symmetrization of z_n, and clearly $\{z_n^*\}$ is also bounded in E. Hence we may assume that (up to a subsequence) $z_n^* \rightharpoonup z^* = (u^*, v^*)$ converges weakly in E and strongly in $L^p(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ for $p \in (2, 6)$. By using the fact that $z_n \in \mathscr{M}_{\lambda_1\lambda_2}$ and the property of Schwartz symmetrization functions we have that

$$\begin{aligned}\|z^*\|_E^2 &\leq \liminf_{n\to\infty} \|z_n^*\|_E^2 \leq \liminf_{n\to\infty} \|z_n\|_E^2 \\ &\leq \lim_{n\to\infty} \int_{\mathbb{R}^3} (\mu\phi_{u_n} u_n^2 + 2\beta\phi_{u_n} v_n^2 + \nu\phi_{v_n} v_n)dx \\ &\leq \lim_{n\to\infty} \int_{\mathbb{R}^3} [\mu\phi_{u_n^*} (u_n^*)^2 + 2\beta\phi_{u_n^*} (v_n^*)^2 + \nu\phi_{v_n^*} (v_n^*)^2]dx \\ &\leq \int_{\mathbb{R}^3} [\mu\phi_{u^*} (u^*)^2 + 2\beta\phi_{u^*} (v^*)^2 + \nu\phi_{v^*} (v^*)^2]dx,\end{aligned} \tag{4.1}$$

and

$$\mathscr{L}_{\lambda_1\lambda_2}(z^*) \leq \liminf_{n\to\infty} \mathscr{L}_{\lambda_1\lambda_2}(z_n^*) \leq \lim_{n\to\infty} \mathscr{L}_{\lambda_1\lambda_2}(z_n) = C^0.$$

Thus, by the Sobolev's inequality, Lemma 2.2 and $z_n \in \mathscr{M}_{\lambda_1\lambda_2}$, we have

$$\begin{aligned}\|z_n^*\|_E^2 &\leq \int_{\mathbb{R}^3} [\mu\phi_{u_n^*} (u_n^*)^2 + 2\beta\phi_{u_n^*} (v_n^*)^2 + \nu\phi_{v_n^*} (v_n^*)^2]dx \\ &\leq C_1 \int_{\mathbb{R}^3} [\phi_{u_n^*} (u_n^*)^2 + \phi_{v_n^*} (v_n^*)^2]dx,\end{aligned} \tag{4.2}$$

for some constant $C_1 > 0$. Moreover we infer from Lemma 2.4 that there exists a constant $C_2 > 0$ such that

$$\|z_n^*\|_E^2 \geq C_2 \left(\int_{\mathbb{R}^3} [\phi_{u_n^*} (u_n^*)^2 + \phi_{v_n^*} (v_n^*)^2]dx \right)^{\frac{1}{2}}. \tag{4.3}$$

So one deduces from (4.2) and (4.3) that $\|z^*\|_E > 0$ hence $z^* \neq (0, 0)$. From (4.1) we can take $s \in (0, 1]$ such that $\tilde{z} = sz^* \in \mathscr{M}_{\lambda_1\lambda_2}$. If $s < 1$, then we infer from $z_n \in \mathscr{M}_{\lambda_1\lambda_2}$ that

$$\mathscr{L}_{\lambda_1\lambda_2}(\tilde{z}) = \frac{1}{4}\|\tilde{z}\|_E^2 < \frac{1}{4}\|z^*\|_E^2 \leq \frac{1}{4} \liminf_{n\to\infty} \|z_n^*\|_E^2 \leq \frac{1}{4} \liminf_{n\to\infty} \|z_n\|_E^2 = C^0.$$

This contradicts with the definition of C^0. Thus $s = 1$ and z^* is a minimizer achieving C^0, and there exists a Lagrange multiplier $L \in \mathbb{R}$ such that

$$\mathscr{L}'_{\lambda_1\lambda_2}(z^*) + LG'_{\lambda_1\lambda_2}(z^*) = 0, \tag{4.4}$$

where $G_{\lambda_1\lambda_2}(z^*) = \mathscr{L}'_{\lambda_1\lambda_2}(z^*)z^*$. Multiplying the Eq. (4.4) by z^* and integrating over $\mathbb{R}^3$, we obtain that $L = 0$ hence $z^* \neq (0, 0)$ is a solution of (1.1). By using the same argument as in Lemma 3.2, one can prove that z^* is nonnegative. Clearly one has $C^0 = C_r^0$. This ends the proof of the lemma. □

Now we are ready to prove part (ii) of Theorem 1.2.

Proof of part (ii) of Theorem 1.2 From Lemma 4.1, we know that $C^0 = \mathscr{L}_{\lambda_1\lambda_2}(u, v)$ for some $z = (u, v) \in \mathscr{M}_{\lambda_1\lambda_2}$. So, in the following we only need to check that $u \neq 0$ and $v \neq 0$. In order to prove this result, it is sufficient to prove that

$$C^0 < \min\{\mathscr{L}_{\lambda_1\lambda_2}(w_{\lambda_1,\mu}, 0), \mathscr{L}_{\lambda_1\lambda_2}(0, w_{\lambda_2,\nu})\}, \tag{4.5}$$

where $w_{\sigma,\tau}$ is defined in (2.15).

In the following we use an approach similar to the one in [51] to prove that (4.5) holds. Define

$$\mathscr{J}_{\lambda_1\lambda_2}(z) = \mathscr{J}_{\lambda_1\lambda_2}(u, v) = \frac{\left(\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2\right)^2}{4\displaystyle\int_{\mathbb{R}^3}(\mu\phi_u u^2 + 2\beta\phi_u v^2 + \nu\phi_v v^2)dx} \tag{4.6}$$

As in Lemma 3.3 of [51], one can check that

$$C^0 = \inf_{z\in E\setminus\{(0,0)\}} \mathscr{J}_{\lambda_1\lambda_2}(z) = \inf_{z\in E_r\setminus\{(0,0)\}} \mathscr{J}_{\lambda_1\lambda_2}(z). \tag{4.7}$$

We define a function

$$g_1(s, t) = \mathscr{J}_{\lambda_1\lambda_2}(\sqrt{s}w_{\lambda_1}, \sqrt{t}w_{\lambda_2}) = \frac{(\mathscr{S}_1^2 s\lambda_1^{\frac{3}{2}} + \mathscr{S}_1^2 t\lambda_2^{\frac{3}{2}})^2}{4\mathscr{S}_1^2(s^2\mu\lambda_1^{\frac{3}{2}} + t^2\nu\lambda_2^{\frac{3}{2}} + 2st\beta\lambda_1^{\frac{3}{2}}\theta(\lambda))}, \tag{4.8}$$

where $(s, t) \in \mathcal{D} = \{(s, t) : s \geq 0, t \geq 0, (s, t) \neq (0, 0)\}$, and w_σ and $\theta(\lambda)$ are defined in (2.15) and (2.18). It is easy to verify that

$$g_1(s, 0) = \frac{\mathscr{S}_1^2\lambda_1^{\frac{3}{2}}}{4\mu} = \mathscr{L}_{\lambda_1\lambda_2}(w_{\lambda_1,\mu}, 0) \text{ and } g_1(0, t) = \frac{\mathscr{S}_1^2\lambda_2^{\frac{3}{2}}}{4\nu} = \mathscr{L}_{\lambda_1\lambda_2}(0, w_{\lambda_2,\nu}). \tag{4.9}$$

To prove (4.2), it is sufficient to show that g does not attain its minimum over $\mathcal{D}$ on the lines $s = 0$ or $t = 0$. For this purpose we define the quadratic form

$$k(s, t) = \frac{(as + bt)^2}{cs^2 + 2dst + et^2}. \tag{4.10}$$

The quadratic form $k(s, t)$ does not attain its minimum in $\mathcal{D}$ on the axes if and only if

$$ad - bc > 0 \quad\text{and}\quad bd - ae > 0. \tag{4.11}$$

Applying this to $g_1(s, t)$, we have

$$\beta\theta(\lambda) > \lambda^{\frac{3}{2}}\mu \quad\text{and}\quad \beta\theta(\lambda) > \nu, \tag{4.12}$$

which is true provided that

$$\beta > \lambda^{\frac{5}{4}} \max\{\mu_1, \nu_1\}. \tag{4.13}$$

from Lemma 2.5. This completes the proof of part (ii) of Theorem 1.2. □

As in [51], one can also find other conditions to guarantee (4.3) holds. We omit the details and leave it for interested readers.

Proof of part (iii) of Theorem 1.2 Assume that $\nu < \mu$ and $\beta \in [\nu, \mu]$, and (1.1) has a positive solution (u, v). Multiplying the u-equation by v, the v-equation by u, and integrating over $\mathbb{R}^3$, we find

$$\int_{\mathbb{R}^3}[\lambda_2 - \lambda_1 + (\mu - \beta)\phi_u + (\beta - \nu)\phi_v]uv = 0. \tag{4.14}$$

If $\beta \in [\nu, \mu]$, then we have that $\lambda_2 - \lambda_1 \geq 0$, while $\mu - \beta$ and $\beta - \nu$ are all positive. This implies that $u = v \equiv 0$. Moreover, for $\beta < \sqrt{\mu\nu}$, if C or C_r is attained, then according to Lemma 2.3, there is a positive solution of (1.1). This gives a contradiction with the nonexistence result above when $\beta \in [\nu, \sqrt{\mu\nu}]$ is satisfied. □

5 A priori estimate and asymptotic behavior of ground states

In this section we first prove the a priori estimates for the positive ground state solutions of (1.1).

Proof of part (iv) of Theorem 1.2 Let (u, v) be a positive ground state solution of (1.1) for a fixed set of parameters $(\lambda_1, \lambda_2, \mu, \nu, \beta)$. Then

$$C = \mathscr{L}_{\lambda_1\lambda_2}(u, v) = \frac{1}{4}(\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2),$$

which implies that (u, v) is bounded in E by a constant only depending on the parameter set. We prove that ϕ_u and ϕ_v are bounded. Indeed, it follows from Hölder and Hardy–Littlewood–Sobolev inequalities that for any $x \in \mathbb{R}^3$,

$$\begin{aligned}\phi_u(x) &= \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}dy = \int_{\mathbb{R}^3} \frac{u^2(t+x)}{|t|}dt \leq \left(\int_{\mathbb{R}^3} u^2(t+x)dt\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \frac{u^2(t+x)}{|t|^2}dt\right)^{\frac{1}{2}} \\ &\leq C_1|u|_2 \left(\int_{\mathbb{R}^3} |\nabla u(t+x)|^2 dt\right)^{\frac{1}{2}} = C_1|u|_2|\nabla u|_2 \leq C_2,\end{aligned} \tag{5.1}$$

where $C_2 > 0$ is independent of x. Similarly one can prove the boundedness of ϕ_v. Since $\phi_u, \phi_v \in L^\infty(\mathbb{R}^3)$, and

$$-\Delta u + (\lambda_1 - \mu\phi_u - \beta\phi_v)u = 0, \text{ in } \mathbb{R}^3,$$

then by using standard elliptic regularity results, we know that there exists a $C_3 > 0$ only depending on the parameter set such that $\|u\|_\infty \leq C_3$. Similarly we have $\|v\|_\infty \leq C_3$. □

Next we study the asymptotic behavior of positive ground state solutions of (1.1) as $\beta \to -\infty$. To emphasize the dependence on β, in the following we use $C^\beta, C_r^\beta, \mathscr{L}_{\lambda_1\lambda_2}^\beta$, $\mathscr{N}_{\lambda_1\lambda_2}^\beta$ and $\mathscr{N}_{\lambda_1\lambda_2}^{r,\beta}$ to denote the same meaning of $C, C_r, \mathscr{L}_{\lambda_1\lambda_2}, \mathscr{N}_{\lambda_1\lambda_2}$ and $\mathscr{N}_{\lambda_1\lambda_2}^r$ (see (1.13), (2.5)–(2.7)).

Proof of Theorem 1.5 Let $\{\beta_n\}$ be a sequence satisfying $\beta_n < 0$ and $\beta_n \to -\infty$ as $n \to \infty$. Let $(u_{\beta_n}, v_{\beta_n}) \in \mathscr{N}_{\lambda_1\lambda_2}^{r,\beta_n}$ be a nonnegative nontrivial ground state solution of (1.1) with $\beta = \beta_n$. We use the contradiction arguments. Assume that $\|(u_{\beta_n}, v_{\beta_n})\|_{E_r}$ is bounded. Without loss of generality we may assume that $(u_{\beta_n}, v_{\beta_n}) \rightharpoonup (u_0, v_0)$ in E_r and $(u_{\beta_n}, v_{\beta_n}) \to (u_0, v_0)$ in $L_r^p(\mathbb{R}^3) \times L_r^p(\mathbb{R}^3)$ for $p \in (2, 6)$. Here $u_0, v_0 \geq 0$ in $\mathbb{R}^3$. It follows from the boundedness of $\{(u_{\beta_n}, v_{\beta_n})\}$ that

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_0^2(x)v_0^2(y)}{|x-y|}dxdy = \lim_{n\to\infty}\int_{\mathbb{R}^3} \phi_{v_{\beta_n}} u_{\beta_n}^2 \\ &= \lim_{n\to\infty} \beta_n^{-1}\left(\|u_{\beta_n}\|_{\lambda_1}^2 - \mu\int_{\mathbb{R}^3} \phi_{u_{\beta_n}} u_{\beta_n}^2\right) = 0.\end{aligned} \tag{5.2}$$

Hence we have $u_0(x) \equiv 0$ or $v_0(y) \equiv 0$. On the other hand, we deduce from $(u_{\beta_n}, v_{\beta_n}) \in \mathcal{N}^{r,\beta_n}_{\lambda_1\lambda_2}$ and $\beta_n < 0$ that

$$\begin{aligned} \|u_{\beta_n}\|^2_{\lambda_1} &= \mu\int_{\mathbb{R}^3}\phi_{u_{\beta_n}}u^2_{\beta_n} + \beta_n\int_{\mathbb{R}^3}\phi_{v_{\beta_n}}u^2_{\beta_n} \le \mu\int_{\mathbb{R}^3}\phi_{u_{\beta_n}}u^2_{\beta_n} \\ \|v_{\beta_n}\|^2_{\lambda_2} &= \nu\int_{\mathbb{R}^3}\phi_{v_{\beta_n}}v^2_{\beta_n} + \beta_n\int_{\mathbb{R}^3}\phi_{v_{\beta_n}}u^2_{\beta_n} \le \nu\int_{\mathbb{R}^3}\phi_{v_{\beta_n}}v^2_{\beta_n}. \end{aligned} \tag{5.3}$$

Similar to (2.45), one infers that

$$\int_{\mathbb{R}^3}\phi_{u_{\beta_n}}u^2_{\beta_n}, \int_{\mathbb{R}^3}\phi_{v_{\beta_n}}v^2_{\beta_n} \ge c > 0. \tag{5.4}$$

Then we deduce that

$$\int_{\mathbb{R}^3}\phi_{u_0}u_0^2, \int_{\mathbb{R}^3}\phi_{v_0}v_0^2 \ge c > 0. \tag{5.5}$$

This contradicts $u_0(x) \equiv 0$ or $v_0(y) \equiv 0$. So, we get $\|(u_{\beta_n}, v_{\beta_n})\|_{E_r} \to \infty$ as $n \to \infty$. That is, at least one of $\|u_{\beta_n}\|^2_{\lambda_1}$ and $\|v_{\beta_n})\|^2_{\lambda_2}$ goes to infinity, as $n \to \infty$. Moreover, $C_r^{\beta_n} = \frac{1}{4}\left(\|u_{\beta_n}\|^2_{\lambda_1} + \|v_{\beta_n}\|^2_{\lambda_2}\right) \to \infty$ as $n \to \infty$. This ends the proof of Theorem 1.5. □

6 Uniqueness of positive solution

In this section we give the proof of Theorem 1.4. We first prove the following Liouville type result.

Lemma 6.1 *The elliptic inequality*

$$-\Delta u \ge \phi_u u \tag{6.1}$$

does not posses any positive solution $u \in H^1(\mathbb{R}^3)$.

Proof Suppose that $u \in H^1(\mathbb{R}^3)$ and $u \ge 0$ is a solution of (6.1). Let $\eta(x)$ be a cutoff function satisfying $\eta \in C^\infty(\mathbb{R}^3)$, $0 \le \eta(x) \le 1$ for $x \in \mathbb{R}^3$, and $\eta(x) = 1$ for $|x| \le 1/2$. Define $\psi_R(x) = [\eta(x/R)]^3$ for $R > 0$. From

$$-\Delta\psi_R(x) = 3R^{-2}\left[\eta^2\Delta\eta + 2\eta|\nabla\eta|^2\right]\left(\frac{x}{R}\right),$$

we have

$$|\Delta\psi_R| \le cR^{-2}\eta\chi_{\{|x|>\frac{R}{2}\}} = cR^{-2}\psi_R^{\frac{1}{3}}\chi_{\{|x|>\frac{R}{2}\}},$$

for some constant $c > 0$. Multiplying (6.1) by ψ_R one gets

$$\int_{\mathbb{R}^3}\phi_u u\psi_R \le -\int_{\mathbb{R}^3}u\Delta\psi_R \le cR^{-2}\int_{\frac{R}{2}<|x|<R}u\psi_R^{\frac{1}{3}} \le c\left(\int_{\frac{R}{2}<|x|<R}u^3\psi_R\right)^{\frac{1}{3}}. \tag{6.2}$$

Since $u \in H^1(\mathbb{R}^3)$, it follows that $\int_{\mathbb{R}^3}\phi_u u = 0$ as $R \to \infty$ in (6.2). That is, $u \equiv 0$. □

Next we show that for $\beta \in (0, \beta_M]$ with any fixed $\beta_M > 0$, all positive solutions of (1.1) are a priori bounded. Note that the a priori estimate in part (iv) of Theorem 1.2 is only for ground state solutions.

Lemma 6.2 *For any $\beta_M > 0$ there exists a constant $C_{\beta_M} > 0$ such that, if (u, v) is a positive solution of* (1.1) *with $\beta \in (0, \beta_M]$, then*

$$|u|_\infty + |v|_\infty \le C_{\beta_M}. \tag{6.3}$$

Proof We argue by contradiction. Assume that there exist a sequence of positive solutions of (1.1) $\{z_n = (u_n, v_n)\}$ with $\beta_n \in (0, \beta_M]$ such that $\beta_n \to \tilde{\beta}$ and $|v_n|_\infty \le |u_n|_\infty \to \infty$ as $n \to \infty$. We set

$$\kappa_n = \frac{1}{|u_n|_\infty}, \quad (w_n(x), h_n(x)) = (\kappa_n u_n(\sqrt{\kappa_n}x), \kappa_n v_n(\sqrt{\kappa_n}x)). \tag{6.4}$$

From Theorem 1.1 and $\beta_n > 0$, we know that u_n and v_n are radially symmetric and decreasing in the radial direction. Hence $|h_n|_\infty \le |w_n|_\infty = w_n(0) = 1$. It is easy to verify that (w_n, h_n) satisfies

$$\begin{cases} -\Delta w_n + \lambda_1 \kappa_n w_n = \mu\phi_{w_n} w_n + \beta_n \phi_{h_n} w_n, \\ -\Delta h_n + \lambda_2 \kappa_n h_n = \nu\phi_{h_n} h_n + \beta_n \phi_{w_n} h_n. \end{cases} \tag{6.5}$$

By the standard elliptic argument, we may assume that, subject to a subsequence, $(w_n, h_n) \to (w_0, h_0)$ in $C^2_{loc}(\mathbb{R}^3)$ as $n \to \infty$, where (w_0, h_0) is a nonnegative solution of

$$\begin{cases} -\Delta w_0 = \mu\phi_{w_0} w_0 + \tilde{\beta}\phi_{h_0} w_0, \\ -\Delta h_0 = \nu\phi_{h_0} h_0 + \tilde{\beta}\phi_{w_0} h_0. \end{cases} \tag{6.6}$$

Since $w_0(0) = 1$, then $w_0 \not\equiv 0$, and by the strong maximum principle, we have $w_0(x) > 0$ for $x \in \mathbb{R}^3$. So, it follows from the first equation of (6.6) that w_0 satisfies

$$-\Delta w_0 \ge \mu\phi_{w_0} w_0, \quad x \in \mathbb{R}^3. \tag{6.7}$$

This contradicts to Lemma 6.1, so the conclusion holds. □

Let (u, v) be a positive solution of (1.1) with $\beta > 0$. Then from Theorem 1.1, (u, v) is radially symmetric thus it satisfies:

$$\begin{cases} -u''(r) - \dfrac{2}{r}u'(r) + \lambda_1 u(r) = \mu\phi_u(r)u(r) + \beta\phi_v(r)u(r), \quad r \in (0, \infty), \\ -v''(r) - \dfrac{2}{r}v'(r) + \lambda_1 v(r) = \nu\phi_v(r)v(r) + \beta\phi_u(r)v(r), \quad r \in (0, \infty), \\ u'(0) = v'(0) = 0,\ u(0) > 0,\ v(0) > 0. \end{cases} \tag{6.8}$$

Clearly $u(r), v(r) \to 0$ as $r \to \infty$. Furthermore, as shown in [27,37], by using Newton's theorem we have that $\phi_u(r), \phi_v(r) \to 0$ as $r \to \infty$. To be more precise, one has the following result on the exponential decay of u and v.

Lemma 6.3 *For any $\beta_M > 0$ there exist constants $C_1, C_2 > 0$ only depending on β_M such that, if (u, v) is a positive solution of* (1.1) *with $\beta \in (0, \beta_M]$, then*

$$|u(r)| + |v(r)|,\ |u'(r)| + |v'(r)| \le C_1 e^{-C_2 r}. \tag{6.9}$$

Moreover if $\{z_n = (u_n, v_n)\}$ is a sequence of positive solutions of (1.1) *with $\beta = \beta_n \in (0, \beta_M]$, then it possesses a subsequence $\{z_{n_k}\}$ converges strongly in E_r.*

Proof Define $S_{\beta_M} := \{(u, v) : (u, v)$ is a radial positive solution of (1.1) with $\beta \in (0, \beta_M]\}$. Assume that $(u, v) \in S_{\beta_M}$, then (u, v) satisfies (6.8). By using the Newton's theorem, we know that

$$\phi_u(r) = 4\pi r^{-1} \int_0^r u^2(s)s^2 ds - 4\pi \int_r^\infty u^2(s)s ds. \tag{6.10}$$

From (6.3) we also have

$$|u|_2 + |v|_2 \le C_{\beta_M}. \tag{6.11}$$

Then (6.10) and (6.11) together imply that $\phi_u(r) \to 0$ as $r \to \infty$, and the convergence is uniform for any $(u, v) \in S_{\beta_M}$. Similarly, one has $\phi_v(r) \to 0$ as $r \to \infty$ uniformly for any $(u, v) \in S_{\beta_M}$. Hence, there exists $R_0 = R_0(\beta_M) > 0$ such that for $r \ge R_0$,

$$u(r), v(r) \le 1, \quad \text{and} \quad 0 < \frac{\lambda_1}{2} \le \lambda_1 - \mu_1\phi_u(r) - \beta\phi_v(r). \tag{6.12}$$

So we infer from (6.8) that for $r \ge R_0$,

$$0 = -u''(r) - \frac{2}{r}u'(r) + (\lambda_1 - \mu\phi_u(r) - \beta\phi_v(r))u(r) \ge -u''(r) - \frac{2}{r}u'(r) + \frac{\lambda_1}{2}u(r). \tag{6.13}$$

Fix a $\sigma > 0$ satisfying $\max\{\sigma^2, 2\sigma\} < \lambda_1/4$ and set

$$h_{R,\sigma}(r) = e^{-\sigma(r-R_0)} + e^{-\sigma(R-r)}, \quad , r > R_0, \tag{6.14}$$

where $R > R_0$. Then for $R_0 \le r \le R$, we have

$$-h''_{R,\sigma}(r) - \frac{2}{r}h'_{R,\sigma}(r) + \frac{\lambda_1}{2}h_{R,\sigma}(r) \ge 0. \tag{6.15}$$

By the Sturm comparison lemma, we infer from $h_{R,\sigma}(R_0) \ge 1$, $h_{R,\sigma}(R) \ge 1$ and (6.13)–(6.15) that

$$u(r) \le h_{R,\sigma}(r) \quad \text{for all } r \in [R_0, R]. \tag{6.16}$$

Since $R > R_0$ is arbitrary, taking $R \to \infty$, then we have

$$u(r) \le e^{-\sigma(r-R_0)} \quad \text{for all } r \ge R_0. \tag{6.17}$$

Hence $u(r)$ decays exponentially with $C_1 = e^{\sigma R_0}$ and $C_2 = \sigma$ which only depend on β_M and λ_1. Furthermore, by (6.8) we conclude that

$$r^2u'(r) = \int_0^r (s^2u'(s))'ds = \int_0^r s^2u(r)(\lambda_1 - \phi_u(s) - \phi_v(s))ds \tag{6.18}$$

Since the integrand of right-hand side of (6.18) decays exponentially, then we conclude that $u'(r)$ also decays exponentially. Similarly, we can show that $v(r)$ and $v'(r)$ decay exponentially.

Finally, the existmate in (6.9) implies that $\{z_n\}$ is bounded in E_r, then by using some standard arguments (see for example [21, Corollary 2.4]), one can prove that $\{z_n = (u_n, v_n)\}$ possesses a subsequence $\{z_{n_k}\}$ converges strongly in E_r. □

Next we state a nondegeneracy result for the positive solution of (1.1) when $\beta = 0$. Recall the scalar equation (1.12). We say that a radially symmetric solution $w_{\sigma,\tau}$ of (1.12) is *non-degenerate* in $H_r^1(\mathbb{R}^3)$ if the following linearized equation has only the trivial solution $\psi = 0$:

$$-\Delta\psi + \sigma\psi = \tau\phi_{w_{\sigma,\tau}}\psi + 2\tau\left(\int_{\mathbb{R}^3} \frac{w_{\sigma,\tau}(y)\psi(y)}{|x-y|}dy\right)w_{\sigma,\tau}, \quad \psi \in H_r^1(\mathbb{R}^3). \tag{6.19}$$

We have the following nondegeneracy result for the unique positive solution $w_{\sigma,\tau}$ of (1.12) and corresponding positive solution for (1.1) when $\beta = 0$.

Lemma 6.4 *For $\sigma, \tau > 0$, let $w_{\sigma,\tau}(x)$ be the unique positive radially symmetric solution of (1.12). Then $w_{\sigma,\tau}$ is nondegenerate in $H_r^1(\mathbb{R}^3)$. Moreover $z = (w_{\lambda_1,\mu}, w_{\lambda_2,\nu}) \in E_r$ is the unique positive solution of (1.1) with $\beta = 0$ centered at 0, and z is nondegenerate in E_r.*

Proof The non-degeneracy of $w_{\sigma,\tau}$ is proved in Wei and Winter [59, Theorem III.1] for the case that $\sigma = \tau = 1$, and their proof is still valid for $\sigma, \tau > 0$. Another proof is given in [25]. For the system (1.1), $z(x) = (w_{\lambda_1,\mu}(x), w_{\lambda_2,\nu}(x)) \in E_r$ is the unique positive solution centered at 0 with $\beta = 0$. Linearizing (1.1) at z and $\beta = 0$, we obtain that

$$\begin{cases} -\Delta\varphi + \lambda_1\varphi = \mu\phi_{w_{\lambda_1,\mu}}\varphi + 2\mu\left(\displaystyle\int_{\mathbb{R}^3}\frac{w_{\lambda_1,\mu}(y)\varphi(y)}{|x-y|}dy\right)w_{\lambda_1,\mu}, & x \in \mathbb{R}^3, \\ -\Delta\psi + \lambda_2\psi = \nu\phi_{w_{\lambda_2,\nu}}\psi + 2\nu\left(\displaystyle\int_{\mathbb{R}^3}\frac{w_{\lambda_2,\nu}(y)\psi(y)}{|x-y|}dy\right)w_{\lambda_2,\nu}, & x \in \mathbb{R}^3, \\ \varphi, \psi \in H_r^1(\mathbb{R}^3). \end{cases} \tag{6.20}$$

Then z is non-degenerate if (6.20) has only the trivial solution. Since (6.20) can be reduced to two separate equations in form of (6.19), then the non-degeneracy of z follows from the non-degeneracy of $w_{\sigma,\tau}$. □

Now we are ready to prove the uniqueness of positive solution of (1.1) for small $\beta > 0$.

Proof of part (i) of Theorem 1.4 We first prove that if $\beta > 0$ is small enough, then any positive solution (u, v) of (1.1) is close to $(w_{\lambda_1,\mu}, w_{\lambda_2,\nu})$. Let $\{z_{\beta_n} = (u_{\beta_n}, v_{\beta_n})\}$ be a sequence of positive solutions of (1.1) with $\beta = \beta_n > 0$ and $\beta \to 0$ as $n \to \infty$. We assume that $u_{\beta_n}(0) = \max_{x\in\mathbb{R}^3} u_{\beta_n}(x)$ and $v_{\beta_n}(0) = \max_{x\in\mathbb{R}^3} v_{\beta_n}(x)$. By Lemma 6.3, we may assume that $(u_{\beta_n}, v_{\beta_n}) \to z_0 = (u_0, v_0)$ in E_r and $u_0, v_0 \geq 0$. Since $x = 0$ is the maximum point of z_{β_n}, it follows from the maximum principle that

$$0 < \lambda_1 \leq \mu\phi_{u_{\beta_n}}(0) + \beta_n\phi_{v_{\beta_n}}(0), \quad 0 < \lambda_2 \leq \nu\phi_{v_{\beta_n}}(0) + \beta_n\phi_{u_{\beta_n}}(0). \tag{6.21}$$

From $\beta_n \to 0$ and $(u_{\beta_n}, v_{\beta_n}) \to (u_0, v_0)$ in E_r, we obtain that

$$\phi_{u_0}(0) = \int_{\mathbb{R}^3}\frac{u_0^2(y)}{|y|}dy \geq \frac{\lambda_1}{\mu} > 0, \quad \phi_{v_0}(0) = \int_{\mathbb{R}^3}\frac{v_0^2(y)}{|y|}dy \geq \frac{\lambda_2}{\nu} > 0.$$

So $u_0, v_0 \not\equiv 0$, and by the strong maximum principle, we have $u_0, v_0 > 0$ in $\mathbb{R}^3$ and $u_0(0) = \max_{x\in\mathbb{R}^3} u_0(x)$, $v_0(0) = \max_{x\in\mathbb{R}^3} v_0(x)$. Thus from Lemma 6.4, we must have $(u_0, v_0) = (w_{\lambda_1,\mu}, w_{\lambda_2,\nu})$ and $(u_{\beta_n}, v_{\beta_n}) \to (w_{\lambda_1,\mu}, w_{\lambda_2,\nu})$ in E_r as $n \to \infty$. Since the above argument holds for any sequence $\{(u_{\beta_n}, v_{\beta_n})\}$, then we conclude that for any $\varepsilon > 0$, there exists $\bar{\beta} = \bar{\beta}(\varepsilon) > 0$ such that for any $(0, \bar{\beta}]$ and any nontrivial positive solution $z_\beta = (u_\beta, v_\beta)$ of (1.1) with $\beta \in (0, \bar{\beta})$ satisfies

$$\|(u_\beta, v_\beta) - (w_{\lambda_1,\mu}, w_{\lambda_2,\nu})\|_E < \varepsilon. \tag{6.22}$$

Now we define a mapping

$$F(\beta, z) = (\mathscr{L}^{\beta}_{\lambda_1\lambda_2})'(z) : \mathbb{R} \times E_r \to E_r^*, \tag{6.23}$$

where E_r^* is the dual space of E_r. Let $z_0 = (w_{\lambda_1,\mu}, w_{\lambda_2,\nu})$. Clearly $F(0, z_0) = 0$. Moreover from Lemma 6.4, we have that $F_z(0, z_0) = (\mathscr{L}^{0}_{\lambda_1\lambda_2})''(z_0)$ is invertible. Then by the implicit function theorem, there exist $\tilde{\beta} > 0$, $k_0 > 0$ and $\theta : (-\tilde{\beta}, \tilde{\beta}) \to B_{k_0}(z_0) \equiv \{z \in E_r : \|z - z_0\|_E \le k_0\}$ such that the solution set of $F(\beta, z) = 0$ in $(-\tilde{\beta}, \tilde{\beta}) \times B_{k_0}(z_0)$ is exactly a smooth curve $\{(\beta, \theta(\beta)) : |\beta| < \tilde{\beta}\}$. This implies that for any $\beta \in (-\tilde{\beta}, \tilde{\beta})$, (1.1) has a unique positive solution near z_0. Together with the property (6.22) shown above for $\beta \in (0, \bar{\beta})$, we conclude that (1.1) has a unique positive solution for $\beta \in (0, \beta_0)$ where $\beta_0 = \min\{\bar{\beta}, \tilde{\beta}\}$. The nondegeneracy of the unique positive solution for $\beta \in (0, \beta_0)$ follows from the nondegeneracy of z_0 in Lemma 6.4. □

Finally we give the proof of the uniqueness of positive ground state solution when $\lambda_1 = \lambda_2$.

Proof of part (ii) of Theorem 1.4 Assume that $\lambda_1 = \lambda_2 > 0$. In the proof of Theorem 1.3, we have obtained that (see (2.62)):

$$C = \frac{1}{4}(\kappa + \ell)\lambda_1^{\frac{3}{2}} \int_{\mathbb{R}^3} \phi_w w^2,\ \kappa = \frac{\lambda_1^{\frac{3}{2}}(\nu - \beta)}{\lambda_1^{\frac{3}{2}}\mu\nu - \beta^2},\ \ell = \frac{\lambda_1^{\frac{3}{2}}(\mu - \beta)}{\lambda_1^{\frac{3}{2}}\mu\nu - \beta^2}, \tag{6.24}$$

where w is the unique positive solution of (1.12) with $\sigma = \tau = 1$. Let (u_0, v_0) be a positive ground state solution of (1.1). By using the same arguments as in [11], one can prove that

$$\begin{aligned}&\int_{\mathbb{R}^3} \phi_{u_0} u_0^2 = \kappa^2 \int_{\mathbb{R}^3} \phi_w w^2,\ \int_{\mathbb{R}^3} \phi_{v_0} v_0^2 = \ell^2 \int_{\mathbb{R}^3} \phi_w w^2,\ \int_{\mathbb{R}^3} \phi_{u_0} v_0^2 = \ell\kappa \int_{\mathbb{R}^3} \phi_w w^2,\\ &\int_{\mathbb{R}^3} \phi_{u_0} v_0^2 = \frac{\ell}{\kappa} \int_{\mathbb{R}^3} \phi_{u_0} u_0^2 = \frac{\kappa}{\ell} \int_{\mathbb{R}^3} \phi_{v_0} v_0^2.\end{aligned} \tag{6.25}$$

Set $(\hat{u}, \hat{v}) = (\kappa^{-1/2} u_0, \ell^{-1/2} v_0)$. We deduce from (6.24) and (u_0, v_0) being a positive ground state solution of (1.1) that

$$\int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \lambda_1 \hat{u}^2) = \int_{\mathbb{R}^3} \phi_{\hat{u}} \hat{u}^2,\ \int_{\mathbb{R}^3} (|\nabla \hat{v}|^2 + \lambda_1 \hat{v}^2) = \int_{\mathbb{R}^3} \phi_{\hat{v}} \hat{v}^2. \tag{6.26}$$

Since w is the unique positive ground state solution of (1.12), it follows that

$$\int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \lambda_1 \hat{u}^2) \ge \lambda_1^{\frac{3}{2}} \mathscr{S}_1^2,\ \int_{\mathbb{R}^3} (|\nabla \hat{v}|^2 + \lambda_1 \hat{v}^2) \ge \lambda_1^{\frac{3}{2}} \mathscr{S}_1^2 \tag{6.27}$$

and

$$\begin{aligned}C &= \frac{1}{4}(\kappa + \ell)\lambda_1^{\frac{3}{2}} \mathscr{S}_1^2 = \frac{1}{4}\int_{\mathbb{R}^3} (|\nabla u_0|^2 + \lambda_1 u_0^2 + |\nabla v_0|^2 + \lambda_1 v_0^2)\\ &= \frac{\kappa}{4}\int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \lambda_1 \hat{u}^2) + \frac{\ell}{4}\int_{\mathbb{R}^3} (|\nabla \hat{v}|^2 + \lambda_1 \hat{v}^2) \ge \frac{1}{4}(\kappa + \ell)\lambda_1^{\frac{3}{2}} \mathscr{S}_1^2.\end{aligned} \tag{6.28}$$

Thus, we have

$$\begin{aligned}&\int_{\mathbb{R}^3} (|\nabla \hat{u}|^2 + \lambda_1 \hat{u}^2) = \lambda_1^{\frac{3}{2}} \mathscr{S}_1^2 = \lambda_1^{\frac{3}{2}} \int_{\mathbb{R}^3} \phi_w w^2,\\ &\int_{\mathbb{R}^3} (|\nabla \hat{v}|^2 + \lambda_1 \hat{v}^2) = \lambda_1^{\frac{3}{2}} \mathscr{S}_1^2 = \lambda_1^{\frac{3}{2}} \int_{\mathbb{R}^3} \phi_w w^2.\end{aligned} \tag{6.29}$$

So, it follows from (6.26) and (6.29) that $\hat{u}$ and $\hat{v}$ are positive ground state solutions of (1.1). As in [37], we know that w is the unique positive solution of (1.12). So, $\hat{u} = \hat{v} = w$ up to a translation. That is, $(u_0, v_0) = (\sqrt{\kappa}\hat{u}, \sqrt{\ell}\hat{v}) = (\sqrt{\kappa} w, \sqrt{\ell} w)$ up to a translation. □

7 Radial symmetry

In this section we prove Theorem 1.1. We shall use the moving plane method introduced by Chen et. al. [10], see also [9,26,27,37] for related results. We point out that the system (1.1) is quite different from the system studied by [9,10], but related to the Schrödinger systems considered in the papers [26,37].

For the convenience of the readers, we first prepare some basic properties of Yukawa potential [28]. The Yukawa potential is given by

$$\mathscr{G}_y^{\gamma}(x) = \mathscr{G}^{\gamma}(x-y) = \int_0^{\infty} (4\pi t)^{-\frac{3}{2}} \exp\left\{-\frac{|x-y|^2}{4t} - \gamma t\right\} dt, \tag{7.1}$$

where $\gamma > 0$. Moreover, $\mathscr{G}_y^{\lambda}(x)$ satisfies the equation

$$(-\Delta + \gamma)\mathscr{G}_y^{\gamma} = \delta_y, \quad x \in \mathbb{R}^3,$$

where δ_y is Dirac's delta measure at y [often written as $\delta(x-y)$]. Let $\mathscr{I}_\gamma = (-\Delta+\gamma)^{-1}$ be the inverse operator of the positive operator $-\Delta+\gamma$ in the Sobolev space $H^1(\mathbb{R}^3)$. Obviously, for $f \in H^1(\mathbb{R}^3)$, one sees that

$$\mathscr{I}_\gamma(f) = \mathscr{G}^{\gamma} * f,$$

where $*$ denotes the convolution in $\mathbb{R}^3$. In addition, by the Sobolev embedding theorem (see [2]), we obtain the estimate

$$|\mathscr{I}_\gamma(f)|_r \le C_{r,s,3}|f|_s, \quad f \in L^s(\mathbb{R}^3), \tag{7.2}$$

where $0 \le \frac{1}{s} - \frac{2}{3} \le \frac{1}{r} \le \frac{1}{s}$ (see [28]). The estimate (7.2) will play a key role in our arguments below.

From the above arguments, we can transform the system (1.1) of differential equations into a system of integral equations involving the Yukawa potential:

$$\begin{cases} u = \mathscr{G}^{\lambda_1} * (\mu u u_1 + \beta u v_1), \\ v = \mathscr{G}^{\lambda_2} * (\nu v u_1 + \beta v v_1), \\ u_1 = \frac{1}{|x|} * u^2, \quad v_1 = \frac{1}{|x|} * v^2. \end{cases} \tag{7.3}$$

In the following, we only deal with the system (7.3) of integral equations. For a given real number t, we define

$$\begin{aligned} &\Sigma_t = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \ge t\}, \\ &\Sigma_t^u = \{x \in \Sigma_t : u_t(x) > u(x)\}, \quad \Sigma_t^v = \{x \in \Sigma_t : v_t(x) > v(x)\}, \end{aligned} \tag{7.4}$$

where

$$x^t = (2t - x_1, x_2, x_3), \quad u_t(x) = u(x^t) \quad \text{and} \quad v_t(x) = v(x^t). \tag{7.5}$$

Then we have the following lemma concerns with the decomposition of $u_t - u$.

Lemma 7.1 *Let $z = (u, v)$ denote a positive solution of (1.1). Then for any $x \in \mathbb{R}^3$, we have*

$$\begin{aligned} u_t(x) - u(x) = \int_{\Sigma_t} &(\mathscr{G}^{\lambda_1}(x-y) - \mathscr{G}^{\lambda_1}(x^t - y))[\mu(u_t(y)u_{1,t}(y) - u(y)u_1(y)) \\ &+ \beta(u_t(y)v_{1,t}(y) - u(y)v_1(y))]dy, \end{aligned} \tag{7.6}$$

and

$$v_t(x)-v(x)=\int_{\Sigma_t}(\mathscr{G}^{\lambda_2}(x-y)-\mathscr{G}^{\lambda_2}(x^t-y))[\mu(v_t(y)v_{1,t}(y)-v(y)v_1(y)) \\ +\beta(v_t(y)u_{1,t}(y)-v(y)u_1(y))]dy, \tag{7.7}$$

where $u_{1,t}(y)=u_1(y^t)$ *and* $v_{1,t}(y)=v_1(y^t)$.

Proof Since $|x^t-y|=|x-y^t|$, it follows from (7.3) and (7.4) that

$$\begin{aligned} u(x) &= \int_{\Sigma_t}\mathscr{G}^{\lambda_1}(x-y)[\mu u(y)u_1(y)+\beta u(y)v_1(y)]dy \\ &\quad+\int_{\mathbb{R}^3\setminus\Sigma_t}\mathscr{G}^{\lambda_1}(x-y)[\mu u(y)u_1(y)+\beta u(y)v_1(y)]dy \\ &= \int_{\Sigma_t}\mathscr{G}^{\lambda_1}(x-y)[\mu u(y)u_1(y)+\beta u(y)v_1(y)]dy \\ &\quad+\int_{\Sigma_t}\mathscr{G}^{\lambda_1}(x^t-y)[\mu u_t(y)u_{1,t}(y)+\beta u_t(y)v_{1,t}(y)]dy. \end{aligned} \tag{7.8}$$

Substituting x by x^t, one has that

$$\begin{aligned} u_t(x) &= \int_{\Sigma_t}\mathscr{G}^{\lambda_1}(x^t-y)[\mu u(y)u_1(y)+\beta u(y)v_1(y)]dy \\ &\quad+\int_{\Sigma_t}\mathscr{G}^{\lambda_1}(x-y)[\mu u_t(y)u_{1,t}(y)+\beta u_t(y)v_{1,t}(y)]dy. \end{aligned} \tag{7.9}$$

From (7.8) and (7.9), we obtain (7.6). Similarly, one can prove the equality (7.7). □

Similarly, one can obtain the decomposition of $u_{1,t}-u_1$ and $v_{1,t}-v_1$ below.

Lemma 7.2 *Let* $z=(u,v)$ *be a positive solution of* (1.1)*, and let* u_1*,* v_1 *be defined as in* (7.3)*. Then for* $x\in\mathbb{R}^3$*, we have*

$$u_{1,t}(x)-u_1(x)=\int_{\Sigma_t}\left(\frac{1}{|x-y|}-\frac{1}{|x^t-y|}\right)(u_t^2(y)-u^2(y))dy, \tag{7.10}$$

$$v_{1,t}(x)-v_1(x)=\int_{\Sigma_t}\left(\frac{1}{|x-y|}-\frac{1}{|x^t-y|}\right)(v_t^2(y)-v^2(y))dy. \tag{7.11}$$

Next we prove the following result for the sets Σ_t^u and Σ_t^v which initiate the process of moving plane.

Lemma 7.3 *There exists* $T>0$ *sufficiently large such that for all* $t\le -T$*,* $\Sigma_t^u=\Sigma_t^v=\emptyset$.

Proof In order to obtain the conclusion, we prove some more precise estimates for the quantities in (7.6)–(7.7) and (7.10)–(7.11). Since $|x-y|\le|x^t-y|$ for all $x,y\in\Sigma_t$, it follows from the expression of $\mathscr{G}^{\lambda_1}$ that

$$\mathscr{G}^{\lambda_1}(|x-y|)-\mathscr{G}^{\lambda_1}(|x^t-y|)\ge 0. \tag{7.12}$$

So we derive from Lemma 7.1 that

$$\begin{aligned}
&u_t(x)-u(x)\\
&=\int_{\Sigma_t}(\mathscr{G}^{\lambda_1}(x-y)-\mathscr{G}^{\lambda_1}(x^t-y))\\
&\quad\cdot[\mu(u_t(y)u_{1,t}(y)-u(y)u_1(y))+\beta(u_t(y)v_{1,t}(y)-u(y)v_1(y))]dy\\
&\le\int_{\Sigma_t\cap\{u_tu_{1,t}>u(y)u_1\}}\mathscr{G}^{\lambda_1}(x-y)[\mu(u_t(y)u_{1,t}(y)-u(y)u_1(y))]dy\\
&\quad+\int_{\Sigma_t\cap\{u_tv_{1,t}>u(y)v_1\}}\mathscr{G}^{\lambda_1}(x-y)[\beta(u_t(y)v_{1,t}(y)-u(y)v_1(y))]dy\\
&\le\mu\int_{\Sigma_t^u}\mathscr{G}^{\lambda_1}(x-y)u_{1,t}(y)(u_t(y)-u(y))dy+\mu\int_{\Sigma_t^{u_1}}\mathscr{G}^{\lambda_1}(x-y)u(y)(u_{1,t}(y)-u_1(y))dy\\
&\quad+\beta\int_{\Sigma_t^u}\mathscr{G}^{\lambda_1}(x-y)v_{1,t}(y)(u_t(y)-u(y))dy+\beta\int_{\Sigma_t^{v_1}}\mathscr{G}^{\lambda_1}(x-y)u(y)(v_{1,t}(y)-v_1(y))dy.
\end{aligned}\tag{7.13}$$

We infer from (7.2) and Hölder inequality that

$$\begin{aligned}
&|u_t(x)-u(x)|_{L^2(\Sigma_t^u)}\\
&\quad\le c\mu|u_{1,t}(y)(u_t(y)-u(y))|_{L^{\frac{3}{2}}(\Sigma_t^u)}+c\mu|u(y)(u_{1,t}(y)-u_1(y))|_{L^{\frac{3}{2}}(\Sigma_t^{u_1})}\\
&\quad\quad+c\beta|v_{1,t}(y)(u_t(y)-u(y))|_{L^{\frac{3}{2}}(\Sigma_t^u)}+c\beta|u(y)(v_{1,t}(y)-v_1(y))|_{L^{\frac{3}{2}}(\Sigma_t^{v_1})}\\
&\quad\le c|u_{1,t}|_{L^6(\Sigma_t^u)}|u_t(x)-u(x)|_{L^2(\Sigma_t^u)}+c|u|_{L^2(\Sigma_t^{u_1})}|u_{1,t}(x)-u_1(x)|_{L^6(\Sigma_t^{u_1})}\\
&\quad\quad+c|v_{1,t}|_{L^6(\Sigma_t^u)}|u_t(x)-u(x)|_{L^2(\Sigma_t^u)}+c|u|_{L^2(\Sigma_t^{v_1})}|v_{1,t}(x)-v_1(x)|_{L^6(\Sigma_t^{v_1})}.
\end{aligned}\tag{7.14}$$

Similarly, we deduce from (7.2), (7.7) and Hölder inequality that

$$\begin{aligned}
&v_t(x)-v(x)\\
&\le\nu\int_{\Sigma_t^v}\mathscr{G}^{\lambda_2}(x-y)v_{1,t}(y)(v_t(y)-v(y))dy+\nu\int_{\Sigma_t^{v_1}}\mathscr{G}^{\lambda_2}(x-y)v(y)(v_{1,t}(y)-v_1(y))dy\\
&+\beta\int_{\Sigma_t^v}\mathscr{G}^{\lambda_2}(x-y)u_{1,t}(y)(v_t(y)-(y))dy+\beta\int_{\Sigma_t^{u_1}}\mathscr{G}^{\lambda_2}(x-y)v(y)(u_{1,t}(y)-u_1(y))dy.
\end{aligned}\tag{7.15}$$

and

$$\begin{aligned}
&|v_t(x)-v(x)|_{L^2(\Sigma_t^v)}\\
&\quad\le c|v_{1,t}|_{L^6(\Sigma_t^v)}|v_t(x)-v(x)|_{L^2(\Sigma_t^v)}+c|v|_{L^2(\Sigma_t^{v_1})}|v_{1,t}(x)-v_1(x)|_{L^6(\Sigma_t^{v_1})}\\
&\quad+c|u_{1,t}|_{L^6(\Sigma_t^v)}|v_t(x)-v(x)|_{L^2(\Sigma_t^v)}+c|v|_{L^2(\Sigma_t^{u_1})}|u_{1,t}(x)-u_1(x)|_{L^6(\Sigma_t^{u_1})}.
\end{aligned}\tag{7.16}$$

On the other hand, a direct computation shows that (7.10) equals to

$$u_{1,t}(x)-u_1(x)\le 2\int_{\Sigma_t^u}(\frac{1}{|x-y|})u_t(y)(u_t(y)-u(y))dy,\tag{7.17}$$

So we have that

$$\begin{aligned}
|u_{1,t}-u_1|_{L^6(\Sigma_t^{u_1})}&\le c|u_t(u_t(x)-u(x))|_{L^{\frac{6}{5}}(\Sigma_t^u)}\\
&\le c|u_t|_{L^3(\Sigma_t^u)}|u_t(x)-u(x)|_{L^2(\Sigma_t^u)}.
\end{aligned}\tag{7.18}$$

Similarly we have that

$$|v_{1,t} - v_1|_{L^6(\Sigma_t^{v_1})} \leq c|v_t|_{L^3(\Sigma_t^v)}|v_t - v|_{L^2(\Sigma_t^v)}. \tag{7.19}$$

Substituting (7.18) and (7.19) into (7.14) and (7.16), we obtain that

$$\begin{aligned}&|u_t - u|_{L^2(\Sigma_t^u)}\\ \leq & c|u_{1,t}|_{L^6(\Sigma_t^u)}|u_t - u|_{L^2(\Sigma_t^u)} + c|u|_{L^2(\Sigma_t^{u_1})}|u_t|_{L^3(\Sigma_t^u)}|u_t - u|_{L^2(\Sigma_t^u)}\\ &+ c|v_{1,t}|_{L^6(\Sigma_t^u)}|u_t - u|_{L^2(\Sigma_t^u)} + c|u|_{L^2(\Sigma_t^{v_1})}|v_t|_{L^3(\Sigma_t^v)}|v_t - v|_{L^2(\Sigma_t^v)}.\end{aligned} \tag{7.20}$$

and

$$\begin{aligned}&|v_t - v|_{L^2(\Sigma_t^v)}\\ \leq & c|v_{1,t}|_{L^6(\Sigma_t^v)}|v_t - v|_{L^2(\Sigma_t^v)} + c|v|_{L^2(\Sigma_t^{v_1})}|v_t|_{L^3(\Sigma_t^v)}|v_t - v|_{L^2(\Sigma_t^v)}\\ &+ c|u_{1,t}|_{L^6(\Sigma_t^v)}|v_t - v|_{L^2(\Sigma_t^v)} + c|v|_{L^2(\Sigma_t^{u_1})}|u_t|_{L^3(\Sigma_t^u)}|u_t - u|_{L^2(\Sigma_t^u)}.\end{aligned} \tag{7.21}$$

Accordingly, we can choose $T > 0$ sufficiently large such that for $t \leq -T$, we have

$$\begin{aligned}&c|u_{1,t}|_{L^6(\Sigma_t^u)},\ c|u|_{L^2(\Sigma_t^{u_1})}|u_t|_{L^3(\Sigma_t^u)},\ c|v_{1,t}|_{L^6(\Sigma_t^u)},\ c|u|_{L^2(\Sigma_t^{v_1})}|v_t|_{L^3(\Sigma_t^v)} \leq \frac{1}{8},\\ &c|v_{1,t}|_{L^6(\Sigma_t^v)},\ c|v|_{L^2(\Sigma_t^{v_1})}|v_t|_{L^3(\Sigma_t^v)},\ c|u_{1,t}|_{L^6(\Sigma_t^v)},\ c|v|_{L^2(\Sigma_t^{u_1})}|u_t|_{L^3(\Sigma_t^u)} \leq \frac{1}{8}.\end{aligned} \tag{7.22}$$

Then (7.20) and (7.21) reduce to

$$\begin{aligned}|u_t - u|_{L^2(\Sigma_t^u)} &\leq \frac{3}{8}|u_t - u|_{L^2(\Sigma_t^u)} + \frac{1}{8}|v_t - v|_{L^2(\Sigma_t^v)},\\ |v_t - v|_{L^2(\Sigma_t^v)} &\leq \frac{3}{8}|v_t - v|_{L^2(\Sigma_t^v)} + \frac{1}{8}|u_t - u|_{L^2(\Sigma_t^u)}.\end{aligned} \tag{7.23}$$

These imply that $|u_t - u|_{L^2(\Sigma_t^u)} = 0$ and $|v_t - v|_{L^2(\Sigma_t^v)} = 0$. Therefore, Σ_t^u and Σ_t^v must be measure zero and hence empty. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 First, from Lemma 7.2, we infer that for $t \leq -T$,

$$u_t(x) \leq u(x) \quad \text{and} \quad v_t(x) \leq v(x), \quad \forall x \in \Sigma_t. \tag{7.24}$$

Starting from such a $t < -T$, one can move the plane $x_1 = t$ to the right as long as (7.24) holds. Suppose that there exists a $t_0 < 0$ such that, for $x \in \Sigma_{t_0}$, we have

$$u_{t_0}(x) \leq u(x) \quad \text{and} \quad v_{t_0}(x) \leq v(x),\ \text{but } u_{t_0}(x) \not\equiv u(x) \text{ or } v_{t_0}(x) \not\equiv v(x), \tag{7.25}$$

then we can continue this process further to the right. More precisely, we prove that if (7.25) holds, then there exists an $\epsilon > 0$ such that

$$u_t(x) \leq u(x) \quad \text{and} \quad v_t(x) \leq v(x),\ x \in \Sigma_t \text{ for all } t \in [t_0, t_0 + \epsilon). \tag{7.26}$$

Without loss of generality we assume that

$$u_{t_0}(x) \not\equiv u(x), \quad x \in \Sigma_{t_0}. \tag{7.27}$$

From (7.10) we infer that $u_1 > u_{1,t_0}$. This together with (7.7) imply that $v > v_{t_0}$ in the interior of Σ_{t_0}. Set

$$\Sigma_{t_0}^{\hat{u}} = \{x \in \Sigma_{t_0} : u(x) \leq u_{t_0}(x)\} \quad \text{and} \quad \Sigma_{t_0}^{\hat{v}} = \{x \in \Sigma_{t_0} : v(x) \leq v_{t_0}(x)\}. \tag{7.28}$$

Thus, $\Sigma_{t_0}^{\hat{u}}$ has measure zero, and $\lim_{t\to t_0}\Sigma_{t_0}^{u}\subset\Sigma_{t_0}^{\hat{u}}$. Similarly, we deduce from (7.11) and (7.6) that $v_1>v_{1,t_0}$, and $u>u_{t_0}$ in the interior of Σ_{t_0}. So, the above conclusion is still true for that of v. Let Ω^* be the reflection of the set Ω about the plane $x_1=t$. From (7.20), we deduce that

$$\begin{aligned}&|u_t-u|_{L^2(\Sigma_t^u)}\\ \le& c|u_1|_{L^6((\Sigma_t^u)^*)}|u_t-u|_{L^2(\Sigma_t^u)}+c|u|_{L^2(\Sigma_t^{u_1})}|u|_{L^3((\Sigma_t^u)^*)}|u_t-u|_{L^2(\Sigma_t^u)}\\ &+c|v_1|_{L^6((\Sigma_t^u)^*)}|u_t-u|_{L^2(\Sigma_t^u)}+c|u|_{L^2(\Sigma_t^{v_1})}|v|_{L^3((\Sigma_t^u)^*)}|v_t-v|_{L^2(\Sigma_t^v)}.\end{aligned}\tag{7.29}$$

Since $u,v\in L^3(\mathbb{R}^3)$ and $u_1,v_1\in L^6(\mathbb{R}^3)$, it follows that

$$c|u_1|_{L^6((\Sigma_t^u)^*)},\ c|u|_{L^2(\Sigma_t^{u_1})}|u|_{L^3((\Sigma_t^u)^*)},\ c|v_1|_{L^6((\Sigma_t^u)^*)},\ c|u|_{L^2(\Sigma_t^{v_1})}|v|_{L^3((\Sigma_t^u)^*)}\le\frac{1}{8}.\tag{7.30}$$

Substituting (7.30) into (7.29), we obtain that

$$|u_t-u|_{L^2(\Sigma_t^u)}\le\frac{3}{8}|u_t-u|_{L^2(\Sigma_t^u)}+\frac{1}{8}|v_t-v|_{L^2(\Sigma_t^v)}.\tag{7.31}$$

By using the same arguments as in (7.29) and (7.30) one has

$$|v_t-v|_{L^2(\Sigma_t^v)}\le\frac{3}{8}|v_t-v|_{L^2(\Sigma_t^v)}+\frac{1}{8}|u_t-u|_{L^2(\Sigma_t^u)}.\tag{7.32}$$

We deduce from (7.31) and (7.32) that $|u_t-u|_{L^2(\Sigma_t^u)}=0$ and $|v_t-v|_{L^2(\Sigma_t^v)}=0$. Therefore, Σ_t^u and Σ_t^v must be of measure zero and hence empty. This verifies (7.26). Thus, we have proved that when the moving plane process stops, we must have $u\equiv u_{t_0}$, and $u_t\le u$ in Σ_t when $t<t_0$.

By a translation, we may assume that $u(0)=\max\limits_{x\in\mathbb{R}^3}u(x)$ and $v(0)=\max\limits_{x\in\mathbb{R}^3}u(x)$. Then it follows that the moving plane process from any direction must stop at the origin. Hence u and v must be radially symmetric and monotone decreasing in the radial direction. This ends the proof of Theorem 1.1. □

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