Higher dimensional solitary waves generated by second-harmonic generation in quadratic media

Leiga Zhao, Fukun Zhao & Junping Shi

Calculus of Variations and Partial Differential Equations

ISSN 0944-2669
Volume 54
Number 3

Calc. Var. (2015) 54:2657-2691
DOI 10.1007/s00526-015-0879-1

Springer

Calc. Var. (2015) 54:2657–2691
DOI 10.1007/s00526-015-0879-1

Calculus of Variations

Higher dimensional solitary waves generated by second-harmonic generation in quadratic media

Leiga Zhao[1] · **Fukun Zhao**[2] · **Junping Shi**[3]

Received: 23 September 2014 / Accepted: 30 April 2015 / Published online: 28 May 2015

Abstract Schrödinger type soliton waves generated by second-harmonic generation in higher dimensional quadratic optical media are considered. The existence of ground state solutions for spatial dimension from two to five is proved, and the continuous dependence on the parameter and asymptotic behavior of ground state solutions are established. Multi-pulse solutions with certain symmetry are also obtained. In a bounded domain setting, global bifurcation diagram of multi-pulse solutions are shown by using new technique of double saddle-node bifurcation.

Mathematics Subject Classification 35J50 · 35J61 · 58J55 · 81V80

Communicated by P. Rabinowitz.

This research is partially supported by the National Natural Science Foundation of China (11271264,11361078), National Science Foundation of US (DMS-1313243), Beijing Higher Education Young Elite Teacher Project (YETP0515) and China Scholarship Council.

✉ Junping Shi
jxshix@wm.edu

Leiga Zhao
zhaoleiga@163.com

Fukun Zhao
fukunzhao@163.com

1 Department of Mathematics, Beijing University of Chemical Technology, Beijing 100029, People's Republic of China

2 Department of Mathematics, Yunnan Normal University, Kunming 650092, People's Republic of China

3 Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795, USA

1 Introduction and background

In nonlinear optic theory, the cubic nonlinear Schrödinger (NLS) equation

$$i\frac{\partial \psi}{\partial z} + r\nabla^2\psi + \chi|\psi|^2\psi = 0 \tag{1.1}$$

is the basic equation describing the formation and propagation of optical solitons in Kerr-type materials [13,47]. Here ψ is a slowly varying complex envelope of electric field, the real-valued parameters r and χ represent the relative strength and sign of dispersion/diffraction and nonlinearity, respectively, and z is the propagation distance coordinate. The Laplacian operator ∇^2 can either be $\partial^2/\partial\tau^2$ for temporal solitons where τ is the normalized retarded time, or $\nabla^2 = \sum_{i=1}^{N} \partial^2/\partial x_i^2$ for spatial solitons where $x = (x_1, x_2, \ldots, x_N)$ is the spatial coordinate with the spatial dimension $N \geq 1$. Here x is in the direction orthogonal to z. Solitary wave solutions to (1.1) and its generalizations have been proved in, for examples, [7,46].

The invention of lasers in 1960s enabled experimental physical scientists to obtain a powerful source of coherent light so nonlinear optical effects such as Second Harmonic Generation (SHG) were discovered when the optical material has a $\chi^{(2)}$ (i.e. quadratic) nonlinear response instead of conventional Kerr $\chi^{(3)}$ material for which the Eq. (1.1) is based on (see [10,11]). Assuming that we consider a strong parametric interaction of three stationary quasi-plane monochromatic waves with frequencies ω_i ($i = 1, 2, 3$), the wave vectors are in the same direction (assuming to be z-axis), there is no walk-off between harmonic waves, the frequencies of interacting waves are matched exactly ($\omega_1 + \omega_2 = \omega_3$), and corresponding wave vectors are almost matched ($k_1\omega_1 + k_2\omega_2 - k_3\omega_3 = \Delta k \ll k_i$), then with some conventional normalizations and the assumption that $\omega_1 = \omega_2 = \omega_3/2$, one obtains the following system of SHG of type I (see [10, p. 104]):

$$\begin{cases} i\dfrac{\partial v}{\partial z} + r\Delta v - v + wv^* = 0, & x \in \mathbb{R}^N, \\ i\sigma\dfrac{\partial w}{\partial z} + s\Delta w - \alpha w + \dfrac{1}{2}v^2 = 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where v is a renormalized slowly varying complex envelope of wave with frequency ω_1, w is the one with frequency ω_3, $\sigma, \alpha > 0$, and $r, s = \pm 1$. In the spatial soliton case $r = s = 1$, while the temporal case all four combinations for $r, s = \pm 1$ are possible. The physically realistic spatial dimensions are $N = 1$ or $N = 2$. In this paper we only consider the case of $r = s = 1$. Then the chirp-free two-wave (symbiotic) solitons can be found as real-valued solutions of the steady state ($\partial/\partial z = 0$) equation:

$$\begin{cases} \Delta v - v + wv = 0, & x \in \mathbb{R}^N, \\ \Delta w - \alpha w + \dfrac{1}{2}v^2 = 0, & x \in \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} v(x) = \lim\limits_{|x|\to\infty} w(x) = 0. \end{cases} \tag{1.3}$$

Note that the limiting behavior of the soliton solutions at infinity is added to the equation as a typical requirement. On the other hand, the type II SHG can be described by the following renormalized three-wave mixing equation ([10, p. 118]):

$$\begin{cases} i\eta \dfrac{\partial v}{\partial z} + \Delta v - v + wu^* = 0, & x \in \mathbb{R}^N, \\ i(2-\eta)\dfrac{\partial u}{\partial z} + \Delta u - \gamma u + wv^* = 0, & x \in \mathbb{R}^N, \\ 2i\dfrac{\partial w}{\partial z} + \Delta w - \alpha w + uv^* = 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

and its steady state equation is of form

$$\begin{cases} \Delta v - v + wu = 0, & x \in \mathbb{R}^N, \\ \Delta u - \beta u + wv = 0, & x \in \mathbb{R}^N, \\ \Delta w - \gamma w + uv = 0, & x \in \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} v(x) = \lim\limits_{|x|\to\infty} u(x) = \lim\limits_{|x|\to\infty} w(x) = 0, \end{cases} \tag{1.5}$$

where $\beta, \gamma > 0$.

We remark that similar to (1.2) and (1.4), the propagation of solitons in $\chi^{(3)}$ nonlinear fiber couplers can be described by a set of coupled nonlinear Schrödinger equations:

$$i\frac{\partial \psi_j}{\partial z} + r\nabla^2\psi_j + \chi\left(\sum_{i=1}^K |\psi_i|^2\right)\psi_j = 0, \tag{1.6}$$

for $j = 1, 2, \ldots, K$. Here the complex-valued ψ_j denotes the jth component of the light beam, and $\sum |\psi_j|^2$ is the change in refractive index profile created by all the incoherent components in the light beam. The solitary waves of (1.6) satisfies $\phi_j(t, x) = u_j(x)\exp(i\mu_j t)$, and $u_j(x)$ satisfies

$$r\Delta u_j - \mu_j u_j + \chi\left(\sum_{i=1}^K u_i^2\right)u_j = 0, \tag{1.7}$$

for $j = 1, 2, \ldots, K$ (see [28]). The existence of solitary wave solutions to (1.6) has been considered by many authors in recent years, for example, [1,3,4,18,20,28,33,43,45,48] and the references therein, and the same system in a bounded domain was also considered in [15,16,19,37,49–51].

Contrast to the well-studied $\chi^{(3)}$ nonlinear Schrödinger system (1.6) and (1.7), the two-wave $\chi^{(2)}$ SHG systems (1.2) and (1.3) has only been analytically studied by a few authors [54, 55] for the case of $N = 1$, and there is no rigorous result for the three-wave case (1.4) and (1.5) yet. In this paper, we aim to consider the existence, uniqueness, and multiplicity of soliton solutions of (1.2) in higher dimensional case, and we also consider the dependence of positive ground state soliton solutions on the parameter α. We also obtain some preliminary results for the three-wave system (1.5). To fully explore the mathematical structure of solutions of $\chi^{(2)}$ SHG systems (1.3) and (1.5), we will not restrict the spatial dimension N to be only 1 or 2, but a general positive integer. Our results show that $N = 6$ is a critical dimension for the existence of positive solitons as such solutions do not exist when $N \geq 6$.

Our main analytic tool is the variational method. System (1.3) has a variational structure and its energy functional is defined by $I_\alpha : H := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \to \mathbb{R}$ which is defined by

$$I_\alpha(v, w) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2 + |\nabla w|^2 + \alpha w^2\right) dx - \frac{1}{2}\int_{\mathbb{R}^N} v^2 w \, dx. \tag{1.8}$$

It is standard to verify that I_α is well defined when the spatial dimension N satisfies $1 \le N \le 6$, I_α is continuously differentiable and its critical points correspond to the weak solutions of (1.3), see Sect. 2 for more details. The trivial solution of (1.3) is $(v, w) = (0, 0)$, and (1.3) has no semi-trivial solutions in form of $(v, 0)$ or $(0, w)$. Any solution (v, w) other than $(0, 0)$ is a non-trivial solution of (1.3), and a non-trivial solution is called a ground state solution if it has the least energy among all the non-trivial solutions.

For system (1.3), first we have the following existence and nonexistence result for the positive ground state solution in higher dimensional space:

Theorem 1.1 1. *Suppose that $1 \le N \le 5$ and $\alpha > 0$. Then system (1.3) possesses a positive ground state solution $(v_\alpha, w_\alpha) \in H$. Moreover each of v_α and w_α is radially symmetric with respect to a point $x_0 \in \mathbb{R}^N$ and is strictly decreasing in the radial direction.*
2. *Suppose that $N \ge 6$ and $\alpha > 0$. Then system (1.3) has no positive solution in H.*

From a result in [12], all positive solutions of (1.3) are necessarily radially symmetric and strictly decreasing in the radial direction. In the case $N = 1$, the existence of a non-trivial ground state solution of (1.3) was shown in [55] by using a variational approach. It is also known that when $N = 1$, the positive solution of (1.3) is unique up to a spatial translation [34]. Here we state the following partial result on the uniqueness of positive solution of (1.3) in higher dimensional space.

Theorem 1.2 1. *When $\alpha = 1$ and $2 \le N \le 5$, the positive solution (v_1, w_1) of (1.3) is unique up to a translation. Moreover $w_1 = v_1/\sqrt{2}$, and w_1 is the unique positive solution of*

$$\Delta v - v + v^2 = 0, \quad v \in H^1(\mathbb{R}^N). \tag{1.9}$$

2. *When $\alpha = 0$ and $3 \le N \le 5$, (1.3) possesses a unique positive solution $(v_0, w_0) \in H^1(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ up to a translation, and v_0 is the unique positive solution of the nonlinear Choquard equation*

$$\Delta v - v + \frac{1}{2}\left(G_N * v^2\right) v = 0, \quad v \in H^1(\mathbb{R}^N), \tag{1.10}$$

and $w_0 = (1/2)G_N * v_0^2$, where $G_N(x)$ is the Newton potential

$$G_N(x) = \frac{1}{(N-2)\omega_N}|x|^{2-N}, \tag{1.11}$$

and ω_N is the surface area of the unit sphere in $\mathbb{R}^N$. When $\alpha = 0$ and $N = 1, 2$, (1.3) has no positive solution.

The uniqueness of positive solution of (1.9) is a well-known result proved in [24], see also [5,14,25]. According to Theorem 1.2, when $\alpha = 1$, the two components v and w have the same linear term, and the system (1.3) can be reduced to a single equation in a way. A similar uniqueness result for ground state was proved in [23], and other similar uniqueness results for the $\chi^{(3)}$ Schrodinger system (1.7) with $K = 2$ were proved in [16,52]. The uniqueness of positive solution of the nonlinear Choquard equation in Theorem 1.2 follows from recent result in [35] for $N \ge 3$, which improved the earlier one in [26]. For the case of $\alpha \ne 1$ and $2 \le N \le 5$, the uniqueness of positive solution of (1.3) is an interesting open question. By using a similar approach we also have the results on the existence and uniqueness of ground state solution of the three-wave system (1.5), see Theorem 2.7.

Our next result provides information of the dependence of positive ground state solutions of (1.3) on the parameter α, and also the limiting behavior of positive ground state solutions as α approaches to 0 or ∞.

Theorem 1.3 *Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}^+$ and let $(v_{\alpha_n}, w_{\alpha_n})$ be a positive ground state solution of* (1.3) *with $\alpha = \alpha_n$ such that the maximum of v_{α_n} and w_{α_n} is located at origin.*

1. *Suppose that $1 \le N \le 5$ and $\lim_{n\to\infty} \alpha_n = \alpha_* > 0$. Then up to a subsequence, there exists $(v_{\alpha_*}, w_{\alpha_*}) \in H$, a ground state solution of* (1.3) *with $\alpha = \alpha_*$, such that*
$$(v_{\alpha_n}, w_{\alpha_n}) \to (v_{\alpha_*}, w_{\alpha_*}) \text{ in } H, \quad \text{as } n \to \infty.$$
In particular, if $\alpha_* = 1$, then $(v_{\alpha_*}, w_{\alpha_*}) = (v_1, w_1)$.
2. *Suppose that $3 \le N \le 5$ and $\lim_{n\to\infty} \alpha_n = 0$. Then up to a subsequence,*
$$(v_{\alpha_n}, w_{\alpha_n}) \to (v_0, w_0) \text{ in } H^1(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N), \quad \text{as } n \to \infty,$$
where $(v_0, w_0) \in H^1(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ is the unique positive solution of system (1.3) *with $\alpha = 0$.*
3. *Suppose that $\lim_{n\to\infty} \alpha_n = \infty$. If $1 \le N \le 3$, then up to a subsequence,*
$$\frac{1}{\sqrt{2\alpha_n}} v_{\alpha_n} \to v_\infty \text{ in } H^1(\mathbb{R}^N), \quad w_{\alpha_n} \to v_\infty^2 \text{ in } L^p(\mathbb{R}^N), \quad \text{as } n \to \infty,$$
for $2 \le p < 2^*$, 2^* is defined in (2.1)*, and v_∞ is the unique positive solution of*
$$\Delta v - v + v^3 = 0, \quad v \in H^1(\mathbb{R}^N); \tag{1.12}$$
On the other hand, if $N = 4$ or 5, then
$$\frac{1}{\sqrt{2\alpha_n}} |v_{\alpha_n}|_\infty \to \infty, \text{ and } |w_{\alpha_n}|_\infty \to \infty, \quad \text{as } n \to \infty.$$

It is known that the limiting equation (1.12) has only trivial non-negative solution in $H^1(\mathbb{R}^N)$ if $N \ge 4$, so it is necessary to assume $1 \le N \le 3$ for the convergence result when $\alpha \to \infty$ in Theorem 1.3. In the above limiting behavior of the ground state solutions of (1.3), we notice a transition of $(2\alpha)^{-1/2} v$ from the positive solution of (1.9) (with exponent 2) when $\alpha = 1$ to the positive solution of (1.12) (with exponent 3) when α is near infinity. The limiting behavior when α is large is formally known as the cascading limit or effective Kerr limit in physics literature (see [10, p. 105]), and here we give the first rigorous justification of such limiting behavior. All above results are concerned with the ground state solution of (1.3). In the following result, we obtain multiple solutions of (1.3).

Theorem 1.4 *Suppose that $2 \le N \le 5$ and $\alpha > 0$. Then system* (1.3) *possesses infinitely many distinct radially symmetric solutions $\{(v_n, w_n)\}$ satisfying $I_\alpha(v_n, w_n) \to \infty$ as $n \to \infty$.*

The sequence of solutions $\{(v_n, w_n)\}$ obtained in Theorem 1.4 are radially symmetric multi-pulse solutions, and in light of uniqueness of positive solution shown in Theorem 1.2, these $\{(v_n, w_n)\}$ are likely to be sign-changing. Multi-pulse solutions for $N = 1$ were first observed in numerical simulations (see [55]), and the existence of multi-pulse solutions was analytically proved using singular perturbation theory in [54]. Note that the multi-pulse solutions in [54,55] are of two types: (i) v and w are both even functions, i.e. $(v(-x), w(-x)) = (v(x), w(x))$; (ii) v is an odd function and w is an even function, *i.e.* $(v(-x), w(-x)) = (-v(x), w(x))$. Hence our solutions are higher dimensional counterpart of the first type multi-pulse solutions in one-dimensional domain. The existence of second type higher dimensional multi-pulse solutions is another interesting open question. Also note that for any solution (v, w) of (1.3), $w(x)$ must be positive hence it cannot be an odd function. For $N = 1$, multi-pulse solutions were only found when α is less than 1 and near 1 in [54,55]. In comparison, Theorem 1.4 suggests the existence of multi-pulse solutions for all $\alpha > 0$ when $2 \leq N \leq 5$, which shows the effect of spatial dimension on the existence of soliton waves. We also comment that (1.3) possesses another symmetry that if $(v(x), w(x))$ is a solution of (1.3), so is $(-v(x), w(x))$. Hence the results in Theorems 1.1 and 1.3 also hold for ground solution (v, w) such that $v < 0$ and $w > 0$.

The result in Theorem 1.4 is proved by using variational method for a nonlocal elliptic equation which is equivalent to (1.3):

$$\Delta v - v + \frac{1}{2}\left(G_N^{\sqrt{\alpha}} * v^2\right) v = 0, \quad v \in H^1(\mathbb{R}^N), \tag{1.13}$$

where $G_N^{\sqrt{\alpha}}(x)$ is the Bessel kernel or Yukawa potential for the equation $(-\Delta+\alpha)\Phi(x) = 0$ in $\mathbb{R}^N$. Note that equation (1.13) is in a similar form as the nonlinear Choquard Eq. (1.10) except a different convolution kernel, and the existence of infinitely many solutions of (1.10) was obtained in [29], see also [36]. On the other hand, such approach has been used for another similar system: Schrödinger-Poisson system, which has an opposite sign in the nonlocal term (see, for example, [2,6,22,42,56]).

Finally we consider the system (1.3) on a bounded smooth domain Ω:

$$\begin{cases} \Delta v - v + vw = 0, & x \in \Omega, \\ \Delta w - \alpha w + \dfrac{1}{2} v^2 = 0, & x \in \Omega, \\ v = w = 0, & x \in \partial\Omega. \end{cases} \tag{1.14}$$

The energy functional is given by

$$\Phi_\alpha(v, w) = \frac{1}{2}\int_\Omega \left(|\nabla v|^2 + v^2 + |\nabla w|^2 + \alpha w^2\right)\, dx - \frac{1}{2}\int_\Omega v^2 w\, dx,$$

for $v, w \in H_0^1(\Omega)$. The existence and multiplicity results for the solutions of (1.3) (Theorems 1.1 and 1.3) can be proved for (1.14) using similar proof (see Theorem 6.1). But in this setting, more can be said about the structure of all solutions of (1.14). Here we summarize the bifurcation results for (1.14) (precise statements are given in Theorems 6.2 and 6.4):

1. Let (λ_1, φ_1) be the principal eigen-pair of $-\Delta$ on $H_0^1(\Omega)$ such that $\varphi_1 > 0$. Then $(\alpha, v, w) = (-\lambda_1, 0, \mu\varphi_1)$ is a semi-trivial solution of (1.14) for $\mu \in \mathbb{R}$.
2. For $N \geq 1$, there exists an increasing sequence $\mu_n > 0$, $n = 1, 2, 3, \ldots$, such that a continuum Σ_n of non-trivial solutions of (1.14) bifurcates from the the branch of semi-trivial solutions at $\mu = \mu_n$. In particular, Σ_1 contains the positive solutions of (1.14).
3. For $N \geq 6$, the solution continuum Σ_n only exists for $\alpha \in (-\lambda_1, 0)$ when Ω is a star shaped domain; for $N = 2, 3$, the solution continuum Σ_n is either extended to $\alpha = \infty$ or $\Sigma_n = \Sigma_m$ for some $m \neq n$, and Σ_1 exists for $\alpha \in (-\lambda_1, \infty)$; for $N = 1$ and $\Omega = (-L, L)$, each Σ_n exists for $\alpha \in (-\lambda_1, \infty)$ and $\Sigma_n \cap \Sigma_m = \emptyset$, that is, for any $\alpha \in (-\lambda_1, \infty)$, (1.14) has a pair of solutions $(\pm v_n, w_n)$ such that v_n changes sign exactly $n - 1$ times, and $w_n > 0$ (see Fig. 1)

The result in part 3 shows the existence of both type (i) and type (ii) multi-pulse solutions when $N = 1$ and $\Omega = (-L, L)$, since an even v changes sign even number of times in $(-L, L)$ while an odd v changes sign odd number of times. The bifurcation result in part 2 above utilizes a new "double saddle-node" bifurcation theorem proved in [32]. Indeed near

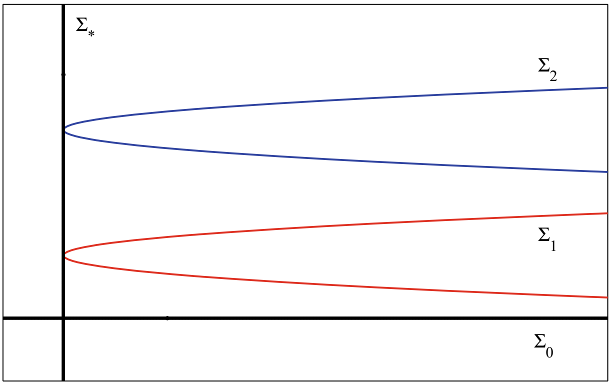

Fig. 1 Illustration of global bifurcation diagram of (1.14). Here the *horizontal* direction is μ, and the *vertical* direction is (v, w)

the bifurcation point, the solution continuum Σ_n is a curve which is tangent to the curve of semi-trivial solutions and the kernel of the linearized operator is two-dimensional, while in the classical Crandall-Rabinowitz bifurcation theorem [17], the solution continuum is transversal to the known branch and the corresponding kernel is one-dimensional. This appears to be the first example of double saddle-node bifurcation occurring in practical mathematical models, and it suggests that such bifurcations may arise in many other situations. Also these results cannot be obtained from variational methods.

The paper is structured this way: in Sect. 2, we prove the existence of ground state solution and prove the existence parts in Theorems 1.1 and 2.7, and the uniqueness of solution is proved in Sect. 3. We study the convergence and asymptotic behavior of ground state solutions in Sect. 4, and prove Theorem Theorem 1.3. The existence of multiple solutions is proved in Sect. 5, and the bounded domain case is considered in Sect. 6. Throughout the paper (except Sect. 6), we denote the norm of $L^p(\mathbb{R}^N)$ by $|v|_p = \left(\int_{\mathbb{R}^N} |v|^p\, dx\right)^{1/p}$ for $1 \le p \le \infty$, and the norm of $H^1(\mathbb{R}^N)$ by $\|v\| = \left(|\nabla v|_2^2 + |v|_2^2\right)^{1/2}$; we also use $\|v\|_\alpha = \left(|\nabla v|_2^2 + \alpha|v|_2^2\right)^{1/2}$ as an equivalent norm of $H^1(\mathbb{R}^N)$ for $\alpha > 0$; and we use $\|u\|_H = (\|v\|^2 + \|w\|^2)^{\frac{1}{2}}$ as the norm for $u = (v, w)$ in H.

2 Existence of ground state solutions

In this section, we prove the existence of a positive ground state solution of system (1.3) with $1 \le N \le 5$ for $\alpha > 0$. To achieve this, we use variational method to the functional I_α defined in (1.12). We note that the existence of a ground state solution was essentially obtained by Brezis and Lieb [8] by using a constrained minimization method. Here, we use a different argument and obtain the mountain pass characterization of the ground state solution, which plays an important role in the characterization of the asymptotical behavior of the ground state solutions depending on the parameter α.

First we observe the following estimate.

Lemma 2.1 *Suppose that $1 \le N \le 6$ and $u, v, w \in H^1(\mathbb{R}^N)$, then there exists $C > 0$ independent of u, v, w such that*

$$\int_{\mathbb{R}^N} uvw\, dx \le C\|u\|\|v\|\|w\|.$$

In particular, for $(v, w) \in H$*, there exists* $C > 0$ *independent of* v, w *such that*

$$\int_{\mathbb{R}^N} v^2 w\, dx \le C\|v\|^2\|w\|.$$

Proof Define

$$2^* = \begin{cases} \frac{2N}{N-2}, & N \ge 3, \\ \infty, & N = 1, 2, \end{cases} \quad \text{and} \quad \widetilde{2}^* = \frac{2N}{N+2}. \tag{2.1}$$

We first consider the case of $3 \le N \le 6$. From Hölder inequality, we have

$$\int_{\mathbb{R}^N} uvw\, dx \le \left(\int_{\mathbb{R}^N} u^{2\widetilde{2}^*} dx\right)^{1/(2\widetilde{2}^*)} \left(\int_{\mathbb{R}^N} v^{2\widetilde{2}^*} dx\right)^{1/(2\widetilde{2}^*)} \left(\int_{\mathbb{R}^N} w^{2^*} dx\right)^{1/2^*}.$$

Since $2\widetilde{2}^* \le 2^*$ for $3 \le N \le 6$, we get the conclusion by the Sobolev inequality. If $N = 1, 2$, it follows from the Sobolev embedding theorems that $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for any $p \in [2, \infty)$. Thus

$$\int_{\mathbb{R}^N} uvw\, dx \le \left(\int_{\mathbb{R}^N} u^3\, dx\right)^{1/3} \left(\int_{\mathbb{R}^N} v^3\, dx\right)^{1/3} \left(\int_{\mathbb{R}^N} w^3\, dx\right)^{1/3} \le C\|u\|\|v\|\|w\|.$$

□

By Lemma 2.1, we see that the functional I_α defined in (1.8) is well defined if $1 \le N \le 6$. Moreover, it is standard to verify that $I_\alpha \in C^1(H, \mathbb{R})$ and its derivative is given by

$$\begin{aligned} \langle I_\alpha(v, w), (\varphi, \psi)\rangle = & \int_{\mathbb{R}^N} (\nabla v \nabla\varphi + v\varphi + \nabla w \nabla\psi + \alpha w\psi)\, dx - \int_{\mathbb{R}^N} vw\varphi\, dx \\ & -\frac{1}{2}\int_{\mathbb{R}^N} v^2\psi\, dx, \end{aligned}$$

for any $(v, w), (\varphi, \psi) \in H$. Therefore, a weak solution of system (1.3) corresponds to a critical point of I_α in H. Since the nonlinearities in (1.3) is sufficiently smooth, then a weak solution is necessarily a classical solution.

For any critical point $(v, w) \in H$ of I_α, it is standard to deduce the following Pohozaev's identity

$$\begin{aligned} & \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{N}{2}\int_{\mathbb{R}^N} v^2 dx + \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla w|^2 dx \\ & + \frac{\alpha N}{2}\int_{\mathbb{R}^N} w^2 dx - \frac{N}{2}\int_{\mathbb{R}^N} v^2 w dx = 0. \end{aligned} \tag{2.2}$$

Combining with $\langle I'_\alpha(v, w), (v, w)\rangle = 0$, we see that

$$\int_{\mathbb{R}^N} v^2\, dx + \alpha\int_{\mathbb{R}^N} w^2\, dx = \frac{6-N}{4}\int_{\mathbb{R}^N} v^2 w\, dx, \tag{2.3}$$

which implies that (v, w) must be zero if $N \ge 6$ and $\alpha > 0$. This proves the nonexistence part in Theorem 1.1.

In what follows, we will focus on the case of $1 \le N \le 5$. We will use the Mountain-Pass Theorem without compactness ([53, Theorem 1.15]) and the concentration-compactness principle [31, Lemma 1.1] to prove the existence of non-trivial critical points of I_α for $\alpha > 0$. We recall that for $c \in \mathbb{R}$, a sequence $\{(v_n, w_n)\} \subset H$ is called a $(\mathrm{PS})_c$ sequence of I_α if

$I_\alpha(v_n, w_n) \to c$ in $\mathbb{R}$ and $I'_\alpha(v_n, w_n) \to 0$ in H^{-1} as $n \to \infty$, where H^{-1} is the dual space of H. We have the following basic variational result:

Theorem 2.2 *Suppose that* $1 \le N \le 5$ *and the functional* I_α *is defined as in* (1.8). *Define* c_α *by*

$$c_\alpha = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\alpha(\gamma(t)) \tag{2.4}$$

where $\Gamma = \{\gamma(t) \in C([0,1], H) : \gamma(0) = 0, I_\alpha(\gamma(1)) < 0\}$. *Then for any* $\alpha > 0$, *the functional* I_α *has a non-trivial critical point* $(\overline{v}_\alpha, \overline{w}_\alpha) \in H$ *such that* $I_\alpha(\overline{v}_\alpha, \overline{w}_\alpha) \le c_\alpha$.

Proof We prove it in the following steps.

Step 1 The functional I_α has a strict local minimum at 0 in H. In fact, from Lemma 2.1, we obtain that

$$\begin{aligned} I_\alpha(v, w) &= \frac{1}{2}\|v\|^2 + \frac{1}{2}\|w\|_\alpha^2 - \frac{1}{2}\int_{\mathbb{R}^N} v^2 w\, dx \\ &\ge \frac{1}{2}\|v\|^2 + \frac{1}{2}\|w\|_\alpha^2 - C\|v\|^2\|w\| \ge \frac{1}{2}\min(1,\alpha)\|u\|_H^2 - C\|u\|_H^3, \end{aligned}$$

where $u = (v, w)$ and $\|u\|_H^2 = \|v\|^2 + \|w\|^2$. Thus we get the desired conclusion by choosing $\|u\|_H = \rho$ small enough. This also implies that c_α defined in (2.4) is strictly positive.

Step 2 For any fixed $(v, w) \in H$ with $v > 0$, $w > 0$, $I_\alpha(t(v, w)) \to -\infty$ as $t \to \infty$. In fact, this follows directly from

$$I_\alpha(t(v, w)) = \frac{t^2}{2}\|v\|^2 + \frac{t^2}{2}\|w\|_\alpha^2 - \frac{t^3}{2}\int_{\mathbb{R}^N} v^2 w\, dx.$$

Therefore, I_α satisfies the assumptions of the mountain pass theorem in H and I_α has a $(PS)_{c_\alpha}$ sequence, i.e. there exists a sequence $\{(v_n, w_n)\} \subset H$ such that

$$I_\alpha(v_n, w_n) \to c_\alpha, \text{ and } I'_\alpha(v_n, w_n) \to 0 \text{ in } H^{-1}. \tag{2.5}$$

Step 3 The sequence $\{(v_n, w_n)\}$ is bounded in H. In fact, it follows from (2.5) that

$$\begin{aligned} c_\alpha + 1 + \left(\|v_n\|^2 + \|w_n\|_\alpha^2\right)^{1/2} &\ge I_\alpha(v_n, w_n) - \frac{1}{3}\langle I'_\alpha(v_n, w_n), (v_n, w_n)\rangle \\ &= \frac{1}{6}\|v_n\|^2 + \frac{1}{6}\|w_n\|_\alpha^2. \end{aligned}$$

Hence, $\{(v_n, w_n)\}$ is bounded in H.

Step 4 The functional I_α has a non-trivial critical point $(\overline{v}_\alpha, \overline{w}_\alpha) \in H$ such that $I_\alpha(\overline{v}_\alpha, \overline{w}_\alpha) \le c_\alpha$. In fact, to get a non-trivial critical point, we use the concentration compactness arguments. Since $\{(v_n, w_n)\}$ is bounded in H, up to a subsequence, we can assume that there exists $\delta \ge 0$ such that

$$\delta = \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} (v_n^2 + w_n^2)\, dx.$$

If $\delta = 0$, i.e. (v_n, w_n) is vanishing, then by Lions Lemma [31, Lemma 1.1], we have $v_n \to 0$ and $w_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*$, where 2^* is defined in (2.1). It follows from Lemma 2.1 that

$$\begin{aligned} c_\alpha &= \lim_{n\to\infty} I_\alpha(v_n, w_n) = \lim_{n\to\infty}\left(I_\alpha(v_n, w_n) - \frac{1}{2}\langle I'_\alpha(v_n, w_n), (v_n, w_n)\rangle\right) \\ &= \frac{1}{4}\lim_{n\to\infty}\int_{\mathbb{R}^N} v_n^2 w_n\, dx = 0, \end{aligned}$$

which is a contradiction. Therefore, the constant $\delta > 0$ and there exists $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_1(y_n)} (v_n^2 + w_n^2)\, dx \geq \frac{\delta}{2}.$$

Set $\widetilde{v}_n(x) = v_n(x - y_n)$ and $\widetilde{w}_n(x) = w_n(x - y_n)$, then

$$\int_{B_1(0)} (\widetilde{v}_n^2 + \widetilde{w}_n^2)\, dx \geq \frac{\delta}{2}. \tag{2.6}$$

Moreover, $(\widetilde{v}_n, \widetilde{w}_n)$ is also a bounded $(\mathrm{PS})_{c_\alpha}$ sequence of I_α due to the invariance of translation of I_α. Therefore, up to a subsequence, we can assume that for some $(\overline{v}_\alpha, \overline{w}_\alpha) \in H$,

$$\begin{aligned} &\widetilde{v}_n \rightharpoonup \overline{v}_\alpha \text{ in } H^1(\mathbb{R}^N), \quad \widetilde{w}_n \rightharpoonup \overline{w}_\alpha \text{ in } H^1(\mathbb{R}^N), \\ &\widetilde{v}_n \to \overline{v}_\alpha \text{ a.e. in } \mathbb{R}^N, \quad \widetilde{w}_n \to \overline{w}_\alpha \text{ a.e. in } \mathbb{R}^N. \end{aligned} \tag{2.7}$$

By (2.6) and the compactness of the embedding of $H^1(\mathbb{R}^N) \hookrightarrow L^2_{loc}(\mathbb{R}^N)$, we see that $(\overline{v}_\alpha, \overline{w}_\alpha) \neq 0$.

Next we check that $(\overline{v}_\alpha, \overline{w}_\alpha)$ is a critical point of I_α. In fact, for any $(\varphi, \psi) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} \langle I_\alpha(\widetilde{v}_n, \widetilde{w}_n), (\varphi, \psi)\rangle = &\int_{\mathbb{R}^N} (\nabla\widetilde{v}_n\nabla\varphi + \widetilde{v}_n\varphi + \nabla\widetilde{w}_n\nabla\psi + \alpha\widetilde{w}_n\psi)\, dx \\ &- \int_{\mathbb{R}^N} \widetilde{v}_n\widetilde{w}_n\varphi\, dx - \frac{1}{2}\int_{\mathbb{R}^N} \widetilde{v}_n^2\psi\, dx. \end{aligned}$$

By (2.2) and (2.7), it suffices to show that

$$\int_{\mathbb{R}^N} \widetilde{v}_n\widetilde{w}_n\varphi\, dx \to \int_{\mathbb{R}^N} \overline{v}_\alpha\overline{w}_\alpha\varphi\, dx, \tag{2.8}$$

and

$$\int_{\mathbb{R}^N} \widetilde{v}_n^2\psi\, dx \to \int_{\mathbb{R}^N} \overline{v}_\alpha^2\psi\, dx. \tag{2.9}$$

Indeed for (2.8) we have

$$\begin{aligned} \int_{\mathbb{R}^N} \widetilde{v}_n\widetilde{w}_n\varphi\, dx - \int_{\mathbb{R}^N} \overline{v}_\alpha\overline{w}_\alpha\varphi\, dx &= \int_{\mathbb{R}^N} \widetilde{v}_n(\widetilde{w}_n - \overline{w}_\alpha)\varphi\, dx + \int_{\mathbb{R}^N} (\widetilde{v}_n - \overline{v}_\alpha)\overline{w}_\alpha\varphi\, dx \\ &\leq |\varphi|_\infty\left(\int_\Omega \widetilde{v}_n^2\, dx\right)^{1/2}\left(\int_\Omega (\widetilde{w}_n - \overline{w}_\alpha)^2\, dx\right)^{1/2} \\ &\quad + |\varphi|_\infty\left(\int_\Omega (\widetilde{v}_n - \overline{v}_\alpha)^2\, dx\right)^{1/2}\left(\int_\Omega \overline{w}_\alpha^2\, dx\right)^{1/2} \to 0 \end{aligned}$$

by (2.7), where $\Omega = suppt(\varphi)$ (the support set of φ) is a bounded domain. Similarly (2.9) holds. Hence $I'_\alpha(\overline{v}_\alpha, \overline{w}_\alpha) = 0$ as $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$.

Now by Fatou's Lemma, we have

$$I_\alpha(\overline{v}_\alpha, \overline{w}_\alpha) = I_\alpha(\overline{v}_\alpha, \overline{w}_\alpha) - \frac{1}{2}\langle I'_\alpha(\overline{v}_\alpha, \overline{w}_\alpha), (\overline{v}_\alpha, \overline{w}_\alpha)\rangle = \frac{1}{4}\int_{\mathbb{R}^N} \overline{v}_\alpha^2 \overline{w}_\alpha \, dx$$
$$\leq \frac{1}{4}\liminf_{n\to\infty} \int_{\mathbb{R}^N} \widetilde{v}_n^2 \widetilde{w}_n \, dx \leq \liminf_{n\to\infty} \left(I_\alpha(\widetilde{v}_n, \widetilde{w}_n) - \frac{1}{2}\langle I'_\alpha(\widetilde{v}_n, \widetilde{w}_n), (\widetilde{v}_n, \widetilde{w}_n)\rangle \right) = c_\alpha.$$

□

Moreover we give a more precise estimate of the mountain pass value c_α defined in (2.4) which is useful for subsequent arguments.

Lemma 2.3 *Let $\alpha > 0$ and let $(v, w) \in H$ satisfying $\int_{\mathbb{R}^N} v^2 w \, dx \neq 0$, then*

$$c_\alpha \leq \frac{2}{27} \frac{\left(\int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2 + |\nabla w|^2 + \alpha w^2 \right) \, dx \right)^3}{\left(\int_{\mathbb{R}^N} v^2 w \, dx \right)^2}.$$

Proof By the definition of c_α and the fact that $I_\alpha(t(v, w)) \to -\infty$ as $t \to \infty$, we have

$$\begin{aligned} c_\alpha &\leq \max_{t\geq 0} I_\alpha(t(v, w)) \\ &= \max_{t\geq 0} \left(\frac{t^2}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2 + |\nabla w|^2 + \alpha w^2 \right) \, dx - \frac{t^3}{2}\int_{\mathbb{R}^N} v^2 w \, dx \right) \\ &= \frac{2}{27} \frac{\left(\int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2 + |\nabla w|^2 + \alpha w^2 \right) \, dx \right)^3}{\left(\int_{\mathbb{R}^N} v^2 w \, dx \right)^2}. \end{aligned}$$

□

Theorem 2.2 shows the existence of a non-trivial solution of (1.3) for any $\alpha > 0$. Note that we can use the same proof to the truncated functional

$$I_\alpha^+(v, w) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2 + |\nabla w|^2 + \alpha w^2 \right) \, dx - \frac{1}{2}\int_{\mathbb{R}^N} (v^+)^2 w \, dx.$$

where $v^+(x) = \max\{v(x), 0\}$, to show the existence of a positive solution using standard argument. It is unclear whether a solution obtained in this way is a ground state solution for (1.3) or not, which we shall further explore in the following.

We note that the Nehari manifold

$$N_\alpha = \{(v, w) \in H\backslash\{0\} : \langle I'_\alpha(v, w), (v, w)\rangle = 0\}, \tag{2.10}$$

plays an important role in the variational analysis as shown in the following lemma.

Lemma 2.4 *Let N_α be defined as in (2.10).*

1. *For $u = (v, w) \in H\backslash\{0\}$ such that $\int_{\mathbb{R}^N} v^2 w dx > 0$, there exists a unique $t_u > 0$ such that $t_u u \in N_\alpha$ and*

$$I_\alpha(t_u(v, w)) = \max_{t\geq 0} I_\alpha(t(v, w)).$$

2. *For* $(v, w) \in N_\alpha$,

$$I_\alpha((v, w)) = \max_{t \geq 0} I_\alpha(t(v, w)).$$

Proof 1. Since 0 is a local minimum of I_α and $f(t) := I_\alpha(tu) \to -\infty$ as $t \to \infty$, it is easy to see that $f(t)$ for $t > 0$ has at least one maximum point $t_u > 0$ with maximum value greater than 0. Clearly, $f'(t_u) = \langle I'_\alpha(t_u u), u\rangle = 0$ and $t_u u \in N_\alpha$. We prove next $f(t)$ has a unique critical point for $t > 0$, which then must be the global maximum point. Consider a critical point of f,

$$0 = f'(t) = t(\|v\|^2 + \|w\|_\alpha^2) - \frac{3}{2}t^2 \int_{\mathbb{R}^N} v^2 w \, dx,$$

then

$$f''(t) = \left(\|v\|^2 + \|w\|_\alpha^2\right) - 3t \int_{\mathbb{R}^N} v^2 w \, dx = -\left(\|v\|^2 + \|w\|_\alpha^2\right) < 0.$$

Therefore if t is a critical point of f, then it must be a strict local maximum point. This implies the uniqueness.

2. Since $\int_{\mathbb{R}^N} v^2 w dx > 0$ for $(v, w) \in N_\alpha$, the conclusion follows from the uniqueness of t_u. □

Note that it is well-known that such geometric property of Nehari manifold in Lemma 2.4 holds for the scalar equation

$$-\Delta u + u = f(u), \quad x \in \mathbb{R}^N,$$

where $f(u)$ is a superlinear function such as $|u|^{p-1}u$ $(p > 1)$, see for example [21]. However the structure of Nehari manifold for the system (1.3) is more complicated hence a direct minimization on the Nehari manifold may not produce desired ground state solution. Our approach in this section is to use the mountain-pass geometry to obtain a solution, then we use Nehari manifold to show the ground state solution can be achieved as well.

For that purpose, we define

$$\begin{aligned} n_\alpha &= \inf\{I_\alpha(v, w) : (v, w) \in N_\alpha\}. \\ m_\alpha &= \inf\{I_\alpha(v, w) : I'_\alpha(v, w) = 0, (v, w) \in H\backslash\{0\}\}. \end{aligned} \tag{2.11}$$

By Theorem 2.2, m_α is well-defined and $n_\alpha \leq m_\alpha \leq c_\alpha$. We will prove m_α can be achieved in H by the minimizing method. In fact, the following result implies the existence of ground state solution in Theorem 1.1.

Theorem 2.5 *Suppose that* $1 \leq N \leq 5$, *and let* m_α *be defined as in* (2.11). *Then* m_α *is achieved at some* $(v_\alpha, w_\alpha) \in H$ *with* $v_\alpha > 0$, $w_\alpha > 0$ *for any* $\alpha > 0$. *That is,* (v_α, w_α) *is a positive ground state solution of* (1.3).

Proof Let m_α and n_α be defined as in (2.11). First we claim that $m_\alpha = n_\alpha = c_\alpha$ for any $\alpha > 0$. By the definition of c_α and Lemma 2.4, we have

$$\begin{aligned} c_\alpha &\leq \inf\{\max_{t \geq 0} I_\alpha(t(v, w)) : (v, w) \in N_\alpha\} \\ &= \inf\{I_\alpha(v, w) : (v, w) \in N_\alpha\} = n_\alpha. \end{aligned}$$

By Theorem 2.2, $n_\alpha \leq m_\alpha \leq c_\alpha$ and we obtain the claim. Let $\{(v_n, w_n)\}$ be a minimizing sequence of m_α, that is

$$I_\alpha(v_n, w_n) \to m_\alpha, \quad I'_\alpha(v_n, w_n) = 0. \tag{2.12}$$

In particular, $\{(v_n, w_n)\}$ is a $(\mathrm{PS})_{m_\alpha}$ sequence, so it is bounded in H. Since $m_\alpha > 0$, by similar arguments as in the proof of Theorem 2.2, $\{(v_n, w_n)\}$ is non-vanishing and there exists $\{y_n\} \subset \mathbb{R}^N$ such that the sequence $\widetilde{v}_n(x) = v_n(x - y_n)$ and $\widetilde{w}_n(x) = w_n(x - y_n)$ has a non-trivial weak limit (v_α, w_α) in H. It is easy to check that (v_α, w_α) is a critical point of I_α. Hence by Fatou's Lemma,

$$m_\alpha \le I_\alpha(v_\alpha, w_\alpha) = \frac{1}{4}\int_{\mathbb{R}^N} v_\alpha^2 w_\alpha \, dx \le \frac{1}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^N} \widetilde{v}_n^2 \widetilde{w}_n \, dx \le \liminf_{n\to\infty} I_\alpha(\widetilde{v}_n, \widetilde{w}_n) = m_\alpha.$$

Therefore, m_α is achieved at (v_α, w_α).

We show next that $w_\alpha > 0$ and v_α can be chosen to be positive. Clearly, $v_\alpha \ne 0$ and $w_\alpha \ne 0$ by the equations of (1.3). Since w_α satisfies

$$-\Delta w_\alpha + \alpha w_\alpha = \frac{1}{2} v_\alpha^2, \quad x \in \mathbb{R}^N,$$

and $w_\alpha \to 0$ as $|x| \to \infty$, we see that $w_\alpha > 0$ by the strong maximum principle. It is easy to check that $(|v_\alpha|, w_\alpha) \in N_\alpha$ and $I_\alpha(|v_\alpha|, w_\alpha) = I_\alpha(v_\alpha, w_\alpha)$, hence $(|v_\alpha|, w_\alpha)$ is also a minimizer of n_α and m_α. Hence, we can assume that v_α is nonnegative. It follows from the strong maximum principle that $v_\alpha > 0$. □

Remark 2.6 If $3 \le N \le 5$, by the above similar arguments and working in $H^1(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, we can obtain a positive ground state solution for system (1.3) with $\alpha = 0$.

We conclude this section by proving the existence of ground state solution for the three-wave Eq. (1.5). The proof is mostly similar to the proof of two-wave Eq. (1.3), so we will only sketch the proof by showing how the proof of two-wave case can be adapted to the three-wave case. It is not hard to see that system (1.5) is reduced to (1.3) if one of the parameters β or γ is 1. The solutions of (1.5) are critical points of the energy functional

$$\begin{aligned} J_{\beta,\gamma}(v,u,w) = &\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + v^2 + |\nabla u|^2 + \beta u^2 + |\nabla w|^2 + \gamma w^2\right) dx \\ &-\frac{1}{2}\int_{\mathbb{R}^N} vuw \, dx, \end{aligned} \tag{2.13}$$

for $(v, u, w) \in [H^1(\mathbb{R}^N)]^3$. We have the following results on the existence and uniqueness of ground state solution of the three-wave system (1.5).

Theorem 2.7 *Suppose that* $1 \le N \le 5$.

1. *For any* $\beta > 0$, $\gamma > 0$, *system* (1.5) *possesses a positive ground state solution* $(v_{\beta,\gamma}, u_{\beta,\gamma}, w_{\beta,\gamma}) \in [H^1(\mathbb{R}^N)]^3$. *Moreover each of* $v_{\beta,\gamma}$, $u_{\beta,\gamma}$ *and* $w_{\beta,\gamma}$ *is radially symmetric with respect to a point* $x_0 \in \mathbb{R}^N$ *and is strictly decreasing in the radial direction.*
2. *If* $\beta = \gamma = 1$, *then the positive solution* $(v_{1,1}, u_{1,1}, w_{1,1})$ *of* (1.5) *is unique up to translation. Moreover* $v_{1,1} = u_{1,1} = w_{1,1}$ *and* $w_{1,1}$ *is the unique positive solution of* (1.9).
3. *If* $\beta = 1$ *and* $\gamma = 0$, $N \ge 3$, *then* (1.5) *possesses a unique positive solution* $(v_{1,0}, u_{1,0}, w_{1,0}) \in [H^1(\mathbb{R}^N)]^2 \times D^{1,2}(\mathbb{R}^N)$ *up to a translation, with* $v_{1,0} = u_{1,0}$ *and* $\sqrt{2}u_{1,0}$ *is the unique positive solution of* (1.10). *A similar result holds for* $\beta = 0$ *and* $\gamma = 1$.

Proof of Theorem 2.7 (*existence*) From Lemma 2.1, the functional $J_{\beta,\gamma}$ defined in (2.13) is well-defined if $1 \leq N \leq 6$ and $J_{\beta,\gamma} \in C^1([H^1(\mathbb{R}^N)]^3, \mathbb{R})$. For $\beta, \gamma > 0$, the proof of Theorem 2.2 can be easily modified to prove the existence of a non-trivial critical point of $J_{\beta,\gamma}$. We define the Nehari manifold

$$N_{\beta,\gamma} = \{(u, v, w) \in [H^1(\mathbb{R}^N)]^3 \backslash \{0\} : \langle J'_{\beta,\gamma}(u, v, w), (u, v, w)\rangle = 0\}, \tag{2.14}$$

and we can define critical energy level $c_{\beta,\gamma}$, $m_{\beta,\gamma}$ and $n_{\beta,\gamma}$ in a similar fashion as in the proof of Theorems 2.2 and 2.5. Then the characterization of $N_{\beta,\gamma}$ in Lemma 2.4 can also be obtained, and the existence of a ground state solution $(v_{\beta,\gamma}, u_{\beta,\gamma}, w_{\beta,\gamma})$ can be proved using the same way as the one for Theorem 2.5. Finally, we use the characterization of the Nehari manifold $N_{\beta,\gamma}$ to get a positive ground state solution. In fact, there exists a $t_{\beta,\gamma} > 0$ such that $t_{\beta,\gamma}(v_{\beta,\gamma}, u_{\beta,\gamma}, w_{\beta,\gamma}) \in N_{\beta,\gamma}$. From the two relations

$$t^2_{\beta,\gamma} \left(\|v_{\beta,\gamma}\|^2 + \|u_{\beta,\gamma}\|^2_\beta + \|w_{\beta,\gamma}\|^2_\gamma\right) = t^3_{\beta,\gamma} \int_{\mathbb{R}^N} |v_{\beta,\gamma}||u_{\beta,\gamma}||w_{\beta,\gamma}|\, dx,$$

$$\|v_{\beta,\gamma}\|^2 + \|u_{\beta,\gamma}\|^2_\beta + \|w_{\beta,\gamma}\|^2_\gamma = \int_{\mathbb{R}^N} v_{\beta,\gamma} u_{\beta,\gamma} w_{\beta,\gamma}\, dx$$

it follows that $t_{\beta,\gamma} \leq 1$. Therefore,

$$\begin{aligned} n_{\beta,\gamma} &\leq J_{\beta,\gamma}\left(t_{\beta,\gamma}(|v_{\beta,\gamma}|, |u_{\beta,\gamma}|, |w_{\beta,\gamma}|)\right) = \frac{1}{6} t^2_{\beta,\gamma} \left(\|v_{\beta,\gamma}\|^2 + \|u_{\beta,\gamma}\|^2_\beta + \|w_{\beta,\gamma}\|^2_\gamma\right) \\ &\leq \frac{1}{6}\left(\|v_{\beta,\gamma}\|^2 + \|u_{\beta,\gamma}\|^2_\beta + \|w_{\beta,\gamma}\|^2_\gamma\right) = J_{\beta,\gamma}(v_{\beta,\gamma}, u_{\beta,\gamma},, w_{\beta,\gamma}) = n_{\beta,\gamma} \end{aligned}$$

then $t_{\beta,\gamma}(|u_{\beta,\gamma}|, |v_{\beta,\gamma}|, |w_{\beta,\gamma}|)$ is a nonnegative ground state solution and each component is positive by the maximum principle. □

3 Uniqueness of the positive solution

In this section we prove the uniqueness of positive solution of (1.3) and (1.5) in some special cases.

Proof of Theorem 1.2 First we assume that $\alpha = 1$. Let $(v, w) \in H$ with $v > 0$ and $w > 0$ be a solution of (1.3). Then

$$-\Delta\left(v - \sqrt{2}w\right) + \left(v - \sqrt{2}w\right) = vw - \frac{1}{\sqrt{2}}v^2.$$

By integration by parts, we get

$$\int_{\mathbb{R}^N} \left(|\nabla\left(v - \sqrt{2}w\right)|^2 + \left(v - \sqrt{2}w\right)^2\right) dx = -\frac{1}{\sqrt{2}} \int_{\mathbb{R}^N} v\left(v - \sqrt{2}w\right)^2 dx \leq 0.$$

Hence, $v = \sqrt{2}w$ and w satisfies a scalar equation (1.9). It is known (see [24]) that equation (1.9) has a unique positive solution up to translation, which implies the uniqueness of positive solution for (1.3).

Secondly we assume that $\alpha = 0$. Then for $N \geq 3$, the equation of w in (1.3) becomes a Poisson's equation, and w can be solved as $w(x) = (1/2)(G_N * v^2)(x)$, where G_N is defined in (1.11). Thus the equation of v in (1.3) becomes (1.10), and the uniqueness of positive solution of (1.10) follows from the result of [35]. If $\alpha = 0$ and $N = 1$ or 2, then the w-equation in (1.3) becomes $\Delta w = -(1/2)v^2 \leq 0$, thus $-w$ is a bounded subharmonic function and consequently $-w$ is constant. Hence $w \equiv 0$ and $v \equiv 0$ since $(v, w) \in H$. □

Proof of Theorem 2.7 (*uniqueness*) When $\beta = \gamma = 1$, we use the same argument as the one in the proof of Theorem 1.2 to obtain

$$\int_{\mathbb{R}^N} \left(|\nabla(v-u)|^2 + (v-u)^2\right) dx = -\int_{\mathbb{R}^N} w(v-u)^2\, dx \leq 0,$$

hence $v \equiv u$. Similarly we obtain $v \equiv w$ which implies $v \equiv u \equiv w$ and it satisfies equation (1.9). When $\beta = 1$ and $\gamma = 0$, we still obtain $v \equiv u$, and the uniqueness follows from Theorem 1.2. □

4 Behavior of ground state solutions

In this section, we consider the continuous dependence of ground state solutions of (1.3) on the parameter α and the asymptotic behavior of ground state solutions as α approaches 0 or ∞.

4.1 Continuous dependence of ground states

To consider the dependence of positive ground state solutions on the parameter α, we first prove the following properties of the ground state energy m_α defined as in (2.11).

Lemma 4.1 *Let m_α be defined as in* (2.11) *and let $\alpha^* > 0$ be fixed. Then*

1. *If $0 \leq \alpha_1 < \alpha_2$, then $m_{\alpha_1} \leq m_{\alpha_2}$.*
2. *$m_\alpha \to m_{\alpha^*}$, as $\alpha \to \alpha^*$.*

Proof 1. Since $m_\alpha = c_\alpha$, it suffices to show that $c_{\alpha_1} \leq c_{\alpha_2}$. Since $I_{\alpha_1}(u) \leq I_{\alpha_2}(u)$ for any fixed $u \in H$, we have $\Gamma_{\alpha_2} \subset \Gamma_{\alpha_1}$, where

$$\Gamma_{\alpha_i} = \{\gamma(t) \in C([0,1], H) : \gamma(0) = 0, I_{\alpha_i}(\gamma(1)) < 0\}, \quad \text{for } \alpha_i > 0,\ i = 1, 2,$$

and

$$\Gamma_0 = \{\gamma(t) \in C([0,1], H^1(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)) : \gamma(0) = 0, I_0(\gamma(1)) < 0\}.$$

Hence,

$$c_{\alpha_1} = \inf_{\gamma\in\Gamma_{\alpha_1}} \max_{t\in[0,1]} I_{\alpha_1}(\gamma(t)) \leq \inf_{\gamma\in\Gamma_{\alpha_1}} \max_{t\in[0,1]} I_{\alpha_2}(\gamma(t)) \leq \inf_{\gamma\in\Gamma_{\alpha_2}} \max_{t\in[0,1]} I_{\alpha_2}(\gamma(t)) = c_{\alpha_2}.$$

2. Let $\{\alpha_n\}$ be a sequence satisfying $\lim_{n\to\infty} \alpha_n = \alpha^*$. We denote $(v_{\alpha_n}, w_{\alpha_n})$ and $(v_{\alpha^*}, w_{\alpha^*})$ a minimizer of m_{α_n} and m_{α^*} respectively. By Lemmas 2.3 and 2.4 part 1, we have

$$\begin{aligned} m_{\alpha_n} = c_{\alpha_n} &\leq \max_{t\geq 0} I_{\alpha_n}(t(v_{\alpha^*}, w_{\alpha^*})) \\ &\leq \frac{2}{27} \frac{\left(\int_{\mathbb{R}^N} \left(|\nabla v_{\alpha^*}|^2 + v_{\alpha^*}^2 + |\nabla w_{\alpha^*}|^2 + \alpha_n w_{\alpha^*}^2\right) dx\right)^3}{\left(\int_{\mathbb{R}^N} v_{\alpha^*}^2 w_{\alpha^*}\, dx\right)^2} \\ &= \frac{2}{27} \frac{\left(\int_{\mathbb{R}^N} \left(|\nabla v_{\alpha^*}|^2 + v_{\alpha^*}^2 + |\nabla w_{\alpha^*}|^2 + \alpha^* w_{\alpha^*}^2\right) dx\right)^3}{\left(\int_{\mathbb{R}^N} v_{\alpha^*}^2 w_{\alpha^*}\, dx\right)^2} + o(1) \\ &= \max_{t\geq 0} I_{\alpha^*}(t(v_{\alpha^*}, w_{\alpha^*})) + o(1) = m_{\alpha^*} + o(1), \end{aligned}$$

which implies that

$$\limsup_{n\to\infty} m_{\alpha_n} \le m_{\alpha^*}. \tag{4.1}$$

On the other hand, by the result in part 1, there exists $\sigma > 0$ such that

$$m_{\alpha^*-\sigma} \le m_{\alpha_n} \le m_{\alpha^*+\sigma}.$$

Hence from

$$\begin{aligned} m_{\alpha_n} &= I_{\alpha_n}(v_{\alpha_n}, w_{\alpha_n}) - \frac{1}{3}\langle I'_{\alpha_n}(v_{\alpha_n}, w_{\alpha_n}), (v_{\alpha_n}, w_{\alpha_n})\rangle \\ &= \frac{1}{6}\left(\|v_{\alpha_n}\|^2 + \|w_{\alpha_n}\|^2_{\alpha_n}\right) \ge C\left(\|v_{\alpha_n}\|^2 + \|w_{\alpha_n}\|^2\right), \end{aligned} \tag{4.2}$$

we see that $\{(v_{\alpha_n}, w_{\alpha_n})\}$ is bounded in H. Similarly, it follows from $m_{\alpha_n} = \frac{1}{4}\int_{\mathbb{R}^N} v^2_{\alpha_n} w_{\alpha_n}\, dx$ that

$$\int_{\mathbb{R}^N} v^2_{\alpha_n} w_{\alpha_n}\, dx \ge 4m_{\alpha^*-\sigma}. \tag{4.3}$$

Therefore, by (4.2), (4.3), Lemmas 2.3 and 2.4, we have

$$\begin{aligned} m_{\alpha^*} = c_{\alpha^*} &\le \max_{t\ge 0} I_{\alpha^*}(t(v_{\alpha_n}, w_{\alpha_n})) \\ &\le \frac{2}{27}\frac{\left(\int_{\mathbb{R}^N}\left(|\nabla v_{\alpha_n}|^2 + v^2_{\alpha_n} + |\nabla w_{\alpha_n}|^2 + \alpha^* w^2_{\alpha_n}\right)\, dx\right)^3}{\left(\int_{\mathbb{R}^N} v^2_{\alpha_n} w_{\alpha_n}\, dx\right)^2} \\ &= \frac{2}{27}\frac{\left(\int_{\mathbb{R}^N}\left(|\nabla v_{\alpha_n}|^2 + v^2_{\alpha_n} + |\nabla w_{\alpha_n}|^2 + \alpha_n w^2_{\alpha_n}\, dx\right) + \int_{\mathbb{R}^N}(\alpha^* - \alpha_n) w_n^2\, dx\right)^3}{\left(\int_{\mathbb{R}^N} v^2_{\alpha_n} w_{\alpha_n}\, dx\right)^2} \\ &= \frac{2}{27}\frac{\left(\int_{\mathbb{R}^N}\left(|\nabla v_{\alpha_n}|^2 + v^2_{\alpha_n} + |\nabla w_{\alpha_n}|^2 + \alpha_n w^2_{\alpha_n}\right)\, dx\right)^3}{\left(\int_{\mathbb{R}^N} v^2_{\alpha_n} w_{\alpha_n}\, dx\right)^2} + o(1) \\ &\le \max_{t\ge 0} I_{\alpha_n}(t(v_{\alpha_n}, w_{\alpha_n})) + o(1) = m_{\alpha_n} + o(1), \end{aligned}$$

which implies

$$m_{\alpha^*} \le \liminf_{n\to\infty} m_{\alpha_n}. \tag{4.4}$$

Clearly the conclusion follows from (4.3) and (4.4). □

Now we are ready to prove the continuous dependence of ground state solution (v_α, w_α) on the parameter α.

Proof of Theorem 1.3 part 1 Let $\{\alpha_n\}$ be a sequence such that $\alpha_n \to \alpha^*$, and let $(v_{\alpha_n}, w_{\alpha_n})$ be a positive ground state solution of (1.3) with $\alpha = \alpha_n$. That is

$$I_{\alpha_n}(v_{\alpha_n}, w_{\alpha_n}) = m_{\alpha_n}, \quad I'_{\alpha_n}(v_{\alpha_n}, w_{\alpha_n}) = 0. \tag{4.5}$$

By the assumption and the result of [12], we see that $v_{\alpha_n}, w_{\alpha_n} \in H^1_r(\mathbb{R}^N)$, the subspace of $H^1(\mathbb{R}^N)$ consisting of radially symmetric functions. From (4.2), $\{(v_{\alpha_n}, w_{\alpha_n})\}$ is bounded in H.

We consider two cases according to the dimension N. If $N \ge 2$, up to a subsequence, there exists $(\widehat{v}_{\alpha^*}, \widehat{w}_{\alpha^*}) \in H^1_r(\mathbb{R}^N)$ such that

$$v_{\alpha_n} \rightharpoonup \widehat{v}_{\alpha^*} \text{ in } H^1_r(\mathbb{R}^N), \;\; w_{\alpha_n} \rightharpoonup \widehat{w}_{\alpha^*} \text{ in } H^1_r(\mathbb{R}^N).$$

By Strauss' compactness embedding theorem [46], we have

$$v_{\alpha_n} \to \widehat{v}_{\alpha^*} \ \text{in } L^p(\mathbb{R}^N),\ w_{\alpha_n} \to \widehat{w}_{\alpha^*} \ \text{in } L^p(\mathbb{R}^N), \tag{4.6}$$

for $2 < p < 2^*$. For any $\varphi, \psi \in C_0^\infty(\mathbb{R}^N)$, by (4.5), we have

$$\int_{\mathbb{R}^N} \left(\nabla v_{\alpha_n} \nabla \varphi + v_{\alpha_n}\varphi + \nabla w_{\alpha_n} \nabla \psi + \alpha_n w_{\alpha_n} \psi\right) dx \\ - \int_{\mathbb{R}^N} v_{\alpha_n} w_{\alpha_n} \varphi \, dx - \frac{1}{2}\int_{\mathbb{R}^N} v_{\alpha_n}^2 \psi \, dx = 0.$$

Taking $n \to \infty$, we see that $(\widehat{v}_{\alpha^*}, \widehat{w}_{\alpha^*})$ is a critical point of I_{α^*}. Next, we show that

$$v_{\alpha_n} \to \widehat{v}_{\alpha^*} \text{ in } H^1(\mathbb{R}^N), \quad w_{\alpha_n} \to \widehat{w}_{\alpha^*} \text{ in } H^1(\mathbb{R}^N). \tag{4.7}$$

In fact, we have

$$\int_{\mathbb{R}^N} \left(|\nabla v_{\alpha_n}|^2 + v_{\alpha_n}^2\right) dx = 2\int_{\mathbb{R}^N} \left(|\nabla w_{\alpha_n}|^2 + \alpha_n w_{\alpha_n}^2\right) dx = \int_{\mathbb{R}^N} v_{\alpha_n}^2 w_{\alpha_n} \, dx, \\ \int_{\mathbb{R}^N} \left(|\nabla \widehat{v}_{\alpha^*}|^2 + \widehat{v}_{\alpha^*}^2\right) dx = 2\int_{\mathbb{R}^N} \left(|\nabla \widehat{w}_{\alpha^*}|^2 + \alpha^* \ \widehat{w}_{\alpha^*}^2\right) dx = \int_{\mathbb{R}^N} \widehat{v}_{\alpha^*}^2 \widehat{w}_{\alpha^*} \, dx,$$

so it suffices to show that

$$\int_{\mathbb{R}^N} v_{\alpha_n}^2 w_{\alpha_n} \, dx \to \int_{\mathbb{R}^N} \widehat{v}_{\alpha^*}^2 \widehat{w}_{\alpha^*} \, dx. \tag{4.8}$$

In fact, by (4.6), the convergence in (4.8) follows from the fact that

$$\int_{\mathbb{R}^N} v_{\alpha_n}^2 w_{\alpha_n} \, dx - \int_{\mathbb{R}^N} \widehat{v}_{\alpha^*}^2 \widehat{w}_{\alpha^*} \, dx \\ = \int_{\mathbb{R}^N} (v_{\alpha_n}^2 - \widehat{v}_{\alpha^*}^2) w_{\alpha_n} \, dx + \int_{\mathbb{R}^N} \widehat{v}_{\alpha^*}^2 (w_{\alpha_n} - \widehat{w}_{\alpha^*}) \, dx \\ \leq |v_{\alpha_n} - \widehat{v}_{\alpha^*}|_3 |v_{\alpha_n} + \widehat{v}_{\alpha^*}|_3 |w_{\alpha_n}|_3 + |\widehat{v}_{\alpha^*}|_3^2 |w_{\alpha_n} - \widehat{w}_{\alpha^*}|_3 \to 0.$$

Hence (4.7) holds. Therefore, by (4.7), (4.8) and Lemma 4.1, we have

$$I_{\alpha^*}(\widehat{v}_{\alpha^*}, \widehat{w}_{\alpha^*}) = \lim_{n\to\infty} I_{\alpha_n}(v_{\alpha_n}, w_{\alpha_n}) = \lim_{n\to\infty} m_{\alpha_n} = m_{\alpha^*}.$$

Hence $(\widehat{v}_{\alpha^*}, \widehat{w}_{\alpha^*})$ is a positive ground state solution of (1.3) with $\alpha = \alpha^*$.

If $N = 1$, the argument as above is invalid due to the lack of compactness on the embedding $H^1(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$. We follow an argument in [23] to prove the convergence (4.7). Let $(\varphi_n, \psi_n) = (v_{\alpha_n}, w_{\alpha_n}) - (\widehat{v}_{\alpha^*}, \widehat{w}_{\alpha^*})$, we claim that

$$I_{\alpha_n}(\varphi_n, \psi_n) = I_{\alpha_n}(v_{\alpha_n}, w_{\alpha_n}) - I_{\alpha^*}(\widehat{v}_{\alpha^*}, \widehat{w}_{\alpha^*}) + o(1) \ \text{in } \mathbb{R}, \tag{4.9}$$

and

$$I'_{\alpha_n}(\varphi_n, \psi_n) = I'_{\alpha_n}(v_{\alpha_n}, w_{\alpha_n}) - I'_{\alpha^*}(\widehat{v}_{\alpha^*}, \widehat{w}_{\alpha^*}) + o(1) \ \text{in } H^{-1}. \tag{4.10}$$

In fact, since $v_{\alpha_n} \rightharpoonup \widehat{v}_{\alpha^*}$ and $w_{\alpha_n} \rightharpoonup \widehat{w}_{\alpha^*}$ in $H^1(\mathbb{R})$, we have

$$\|v_{\alpha_n} - \widehat{v}_{\alpha^*}\|^2 = \|v_{\alpha_n}\|^2 - \|\widehat{v}_{\alpha^*}\|^2 + o(1), \quad \|w_{\alpha_n} - \widehat{w}_{\alpha^*}\|^2 = \|w_{\alpha_n}\|^2 - \|\widehat{w}_{\alpha^*}\|^2 + o(1).$$

Therefore, we have

$$\begin{aligned}&|I_{\alpha_n}(v_{\alpha_n}, w_{\alpha_n}) - I_{\alpha_n}(\varphi_n, \psi_n) - I_{\alpha^*}(\widehat{v}_{\alpha^*}, \widehat{w}_{\alpha^*})|\\ &\le \frac{1}{2}\int_{\mathbb{R}} \left|\varphi_n^2 \widehat{w}_{\alpha^*} + 2\varphi_n \widehat{v}_{\alpha^*}\psi_n + 2\varphi_n \widehat{v}_{\alpha^*}\widehat{w}_{\alpha^*} + \widehat{v}_{\alpha^*}^2\psi_n\right| \, dx + o(1)\\ &\le \frac{1}{2}|\varphi_n \widehat{w}_{\alpha^*}|_2|\varphi_n|_2 + |\varphi_n \widehat{v}_{\alpha^*}|_2|\psi_n|_2 + |\varphi_n \widehat{v}_{\alpha^*}|_2|\widehat{w}_{\alpha^*}|_2 + \frac{1}{2}|\psi_n \widehat{v}_{\alpha^*}|_2|\widehat{v}_{\alpha^*}|_2.\end{aligned}$$

To obtain (4.9), it suffices to show that for any sequence $\{u_n\} \subset H^1(\mathbb{R})$ with $u_n \rightharpoonup 0$ in $H^1(\mathbb{R})$ satisfies $u_n v_0 \to 0$ in $L^2(\mathbb{R})$ for any $v_0 \in H^1(\mathbb{R})$. In fact, for any $R > 0$,

$$\int_{\mathbb{R}} u_n^2 v_0^2 \le \left(\sup_{x\in\mathbb{R}} |v_0(x)|^2\right) \int_{-R}^{R} u_n^2 \, dx + \left(\sup_{|x|\ge R} |v_0(x)|^2\right) \int_{\mathbb{R}} u_n^2 \, dx. \tag{4.11}$$

The first term on the right hand side of (4.11) tends to 0 because of the compactness of the embedding $H^1(\mathbb{R}) \hookrightarrow L^2_{loc}(\mathbb{R})$ and the last term tends to zero as $R \to \infty$ since $v_0 \in C(\mathbb{R})$ and $\lim_{|x|\to\infty} v_0(x) = 0$. Hence (4.9) holds. Similarly we have (4.10). By (4.9), (4.10) and Lemma 4.1 part 2, we see that $\{(\varphi_n, \psi_n)\}$ is a $(\text{PS})_0$ sequence, *i.e.*,

$$I_{\alpha_n}(\varphi_n, \psi_n) = o(1) \text{ in } \mathbb{R}, \quad I'_{\alpha_n}(\varphi_n, \psi_n) = o(1) \text{ in } H^{-1}.$$

Hence, similar to (4.2), we have

$$\|\varphi_n\|^2 + \|\psi_n\|^2 \le C\left(I_{\alpha_n}(\varphi_n, \psi_n) - \frac{1}{3}\langle I'_{\alpha_n}(\varphi_n, \psi_n), (\varphi_n, \psi_n)\rangle\right) \to 0,$$

which implies (4.7). This completes the proof for any $\alpha^* > 0$. If $\alpha^* = 1$, by Theorem 1.2, we see that $(\widehat{v}_1, \widehat{w}_1)$ is necessarily the unique positive ground state solution (v_1, w_1). □

We remark that the convergence $(v_{\alpha_n}, w_{\alpha_n}) \to (\widehat{v}_{\alpha^*}, \widehat{w}_{\alpha^*})$ in $H^1 \times H^1$ may be set in a stronger topology.

4.2 Asymptotic behavior of ground state solutions as $\alpha \to 0$

In this subsection, we study the behavior of the ground state solutions of (1.3) as $\alpha \to 0$ and give the proof of Theorem 1.3 part 2.

First we consider the existence of ground state solutions for (1.3) with $\alpha = 0$ for $3 \le N \le 5$. Compared with the case of $\alpha > 0$, the main difference is that the suitable working space for (1.3) is $H^1(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ when using variational methods, where

$$D^{1,2}(\mathbb{R}^N) = \{w \in L^{2^*}(\mathbb{R}^N) : |\nabla w| \in L^2(\mathbb{R}^N)\} \text{ for } 3 \le N \le 5,$$

with the norm

$$\|w\|_D = \left(\int_{\mathbb{R}^N} |\nabla w|^2 \, dx\right)^{1/2}.$$

By Remark 2.6, a positive ground state solution, denoted by (v_0, w_0), exists in $H^1(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$. Moreover, by the result in Theorem 1.2, (v_0, w_0) is the unique positive solution up to a translation for (1.3) with $\alpha = 0$.

Proof of Theorem 1.3 part 2 We assume that $3 \le N \le 5$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}^+$ satisfying $\alpha_n \to 0$ as $n \to \infty$ and let $(v_{\alpha_n}, w_{\alpha_n})$ be a positive ground state solution of (1.3) with $\alpha = \alpha_n$ and energy m_{α_n} such that the maximum of $(v_{\alpha_n}, w_{\alpha_n})$ is located at origin in $\mathbb{R}^N$.

By Lemma 4.1, we see that

$$m_1 \geq m_{\alpha_n} = \frac{1}{6}\int_{\mathbb{R}^N} \left(|\nabla v_{\alpha_n}|^2 + v_{\alpha_n}^2 + |\nabla w_{\alpha_n}|^2 + \alpha_n w_{\alpha_n}^2\right)\, dx \geq \frac{1}{6}\|v_{\alpha_n}\|^2 + \frac{1}{6}\|w_{\alpha_n}\|_D^2.$$

Hence, the sequence $\{(v_{\alpha_n}, w_{\alpha_n})\}$ is bounded in $H^1(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$. Therefore, up to a subsequence, we may assume that there exists $\widehat{v}_0 \in H^1(\mathbb{R}^N)$, $\widehat{w}_0 \in D^{1,2}(\mathbb{R}^N)$ such that

$$\begin{aligned} &v_{\alpha_n} \rightharpoonup \widehat{v}_0 \text{ in } H^1(\mathbb{R}^N), \quad w_{\alpha_n} \rightharpoonup \widehat{w}_0 \text{ in } D^{1,2}(\mathbb{R}^N),\\ &v_{\alpha_n} \to \widehat{v}_0 \ \text{ a.e. in } \ \mathbb{R}^N, \quad w_{\alpha_n} \to \widehat{w}_0 \ \text{ a.e. in } \mathbb{R}^N. \end{aligned} \tag{4.12}$$

By the same arguments as in the proof of Theorem 2.2, one can see that $(\widehat{v}_0, \widehat{w}_0)$ is a critical point of I_0. Moreover, integrating the equations in (1.3), and by using Hölder inequality and Sobolev inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\nabla v_{\alpha_n}|^2 + v_{\alpha_n}^2 + |\nabla w_{\alpha_n}|^2 + \alpha_n w_{\alpha_n}^2\right)\, dx\\ &= \frac{3}{2}\int_{\mathbb{R}^N} v_{\alpha_n}^2 w_{\alpha_n}\, dx \leq \frac{3}{2}|v_{\alpha_n}|_{2\widetilde{2}^*}^2 |w_{\alpha_n}|_{2^*} \leq \frac{3}{4}\|v_{\alpha_n}\|^4 + \frac{3}{4}\|w_{\alpha_n}\|_D^2. \end{aligned}$$

Hence,

$$\|v_{\alpha_n}\|^2 \geq \frac{4}{3}. \tag{4.13}$$

If $\{v_{\alpha_n}\}$ is vanishing, then $v_{\alpha_n} \to 0$ in $L^{2\widetilde{2}^*}(\mathbb{R}^N)$ and $\|v_{\alpha_n}\| \to 0$ by (4.2), which is a contradiction to (4.13). Therefore, $\{v_{\alpha_n}\}$ is non-vanishing. By the assumption on the maximum of $(v_{\alpha_n}, w_{\alpha_n})$, we see that $\widehat{v}_0 \neq 0$ and $\widehat{w}_0 \neq 0$. Following the same arguments in the proof of Theorem 2.2 and (4.12), we see that $(\widehat{v}_0, \widehat{w}_0)$ is a positive solution of (1.3) with $\alpha = 0$. By the uniqueness result in Theorem 1.2 part 2, $(\widehat{v}_0, \widehat{w}_0)$ is necessarily the unique positive ground state solution (v_0, w_0). Following the arguments in the proof of Theorem 1.3 part 1, we see that $(v_\alpha, w_\alpha) \to (v_0, w_0)$, in $H^1(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ as $\alpha \to 0$. □

We comment that it is useful to consider the convergence property of the least energy m_α as $\alpha \to 0$. By Lemma 2.3, we clearly have $m_\alpha \geq m_0$. To get $m_\alpha \to m_0$ by the method given in Lemma 2.3, we need to show that $w_0 \in H^1(\mathbb{R}^N)$ which is reduced to show that $w_0 \in L^2(\mathbb{R}^N)$. For that purpose, we restrict to the case of $N = 4, 5$ and recall the following Hardy-Littlewood-Sobolev inequality [27, Theorem 4.3].

Lemma 4.2 *Suppose that* $1 < p < q < \infty$, *and* $\frac{1}{r} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)$. *For* $f \in L^p(\mathbb{R}^N)$, *define*

$$I_r f(x) = \int_{\mathbb{R}^N} |x-y|^{-N/r} f(y)\, dy.$$

Then there exists a $C = C(p, q, r, N)$ *such that*

$$|I_r f|_q \leq C|f|_p. \tag{4.14}$$

By using Lemma 4.2, we have the following result on the convergence of m_α to m_0 for $N = 4, 5$, and the convergence for $N = 3$ is not known.

Proposition 4.3 *Suppose that* $N = 4, 5$ *and* $(v_0, w_0) \in H^1(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ *is a positive solution of* (1.3) *with* $\alpha = 0$, *then* $w_0 \in L^2(\mathbb{R}^N)$. *In particular,* $\lim_{\alpha\to 0^+} m_\alpha = m_0$ *where* m_λ *is defined in* (2.11).

Proof It is standard to deduce that $v_0(x)$ has an exponential decay as $|x| \to \infty$. Therefore, $v_0 \in L^p(\mathbb{R}^N)$ for any $p \in [2, \infty]$. Moreover, w_0 can be represented as

$$w_0(x) = \frac{1}{2}(G_N * v_0^2)(x) = \frac{1}{2\omega_N} \int_{\mathbb{R}^N} \frac{v_0^2(y)}{|x-y|^{N-2}}\, dy. \tag{4.15}$$

We apply Hardy-Littlewood-Sobolev inequality (4.14) with $r = N/(N-2)$ and $q = 2$ to deduce $w_0 \in L^2(\mathbb{R}^N)$. If $N = 4$, we can choose $p = 1$ since $v_0^2 \in L^1(\mathbb{R}^N)$. If $N = 5$, we can choose $p = 10/9$ since $v_0^2 \in L^{10/9}(\mathbb{R}^N)$. □

4.3 Asymptotic behavior of ground state solutions as $\alpha \to \infty$

In this subsection, we study the behavior of the ground state solutions of (1.3) as $\alpha \to \infty$ and give the proof of Theorem 1.3 part 3.

Let (v_α, w_α) be a positive ground state solution of (1.3). Define

$$\widetilde{v}_\alpha = \frac{1}{\sqrt{2\alpha}} v_\alpha. \tag{4.16}$$

It is easy to verify that $(\widetilde{v}_\alpha, w_\alpha)$ satisfies the rescaled equation

$$\begin{cases} -\Delta \widetilde{v}_\alpha + \widetilde{v}_\alpha = \widetilde{v}_\alpha w_\alpha, & x \in \mathbb{R}^N, \\ -\Delta w_\alpha + \alpha w_\alpha = \alpha \widetilde{v}_\alpha^2, & x \in \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} \widetilde{v}_\alpha(x) = \lim\limits_{|x|\to\infty} w_\alpha(x) = 0. \end{cases} \tag{4.17}$$

In this subsection we prove the convergence of $(\widetilde{v}_\alpha, w_\alpha)$ to (v_∞, v_∞^2) as $\alpha \to \infty$, where v_∞ is the unique positive solution of (1.12). To achieve that, we need to obtain boundedness of $\widetilde{v}_\alpha$ as $\alpha \to \infty$. First we have the following estimate in $H^1(\mathbb{R}^N)$.

Lemma 4.4 *Let (v_α, w_α) be a positive ground state solution of (1.3), and let $\widetilde{v}_\alpha$ be defined as in (4.16). Then for all $\alpha > 0$, $\widetilde{v}_\alpha$ is uniformly bounded in $H^1(\mathbb{R}^N)$, and w_α is uniformly bounded in $L^2(\mathbb{R}^N)$.*

Proof We first estimate m_α. Applying Lemma 2.3 with $(\sqrt{\alpha} v, w)$, where (v, w) is a fixed element in H such that $\int_{\mathbb{R}^N} v^2 w\, dx \neq 0$, we find that there exists $C > 0$ such that

$$m_\alpha \le \max_{t\ge 0} I_\alpha(t(\sqrt{\alpha} v, w)) \le \frac{2}{27} \frac{\left(\int_{\mathbb{R}^N} \left(\alpha|\nabla v|^2 + \alpha v^2 + |\nabla w|^2 + \alpha w^2\right)\, dx\right)^3}{\left(\int_{\mathbb{R}^N} \alpha v^2 w\, dx\right)^2} \le \alpha C. \tag{4.18}$$

Noticing that

$$\begin{aligned} &\frac{1}{6}\int_{\mathbb{R}^N} \left(|\nabla v_\alpha|^2 + v_\alpha^2 + |\nabla w_\alpha|^2 + \alpha w_\alpha^2\right)\, dx \\ &\quad = I_\alpha(v_\alpha, w_\alpha) - \frac{1}{3}\langle I_\alpha'(v_\alpha, w_\alpha), (v_\alpha, w_\alpha)\rangle = m_\alpha, \end{aligned}$$

together with (4.18), we have

$$\int_{\mathbb{R}^N}\left[\left|\nabla\left(\frac{1}{\sqrt{2\alpha}}v_\alpha\right)\right|^2+\left(\frac{1}{\sqrt{2\alpha}}v_\alpha\right)^2\right]dx+\frac{1}{\alpha}\int_{\mathbb{R}^N}|\nabla w_\alpha|^2\,dx+\int_{\mathbb{R}^N}w_\alpha^2\,dx\le C. \tag{4.19}$$

□

It is not straightforward to obtain the boundedness of $\{w_\alpha\}$ in $H^1(\mathbb{R}^N)$ and the convergence properties of $\alpha^{-1}\int_{\mathbb{R}^N}|\nabla w_\alpha|^2$. In the following we obtain uniform boundedness of $\widetilde{v}_\alpha$ in $L^\infty(\mathbb{R}^N)$ by using Moser iteration and consequently get the boundedness of $\alpha^{-1}\int_{\mathbb{R}^N}|\nabla w_\alpha|^2$ by representing w_α by using Yukawa potential. This approach was first used in [22].

For $f(x)\in L^2(\mathbb{R}^N)$, it is known that the unique solution of linear modified Helmholtz equation

$$-\Delta u+\mu^2u=f(x)$$

has the integral representation (see [27])

$$u(x)=(G_N^\mu * f)(x)=\int_{\mathbb{R}^N}G_N^\mu(x-y)\,f(y)\,dy,$$

where $G_N^\mu(x)$ is the Yukawa potential defined by

$$G_N^\mu(x)=\int_0^\infty\frac{1}{(4\pi t)^{N/2}}e^{-(|x|^2/(4t)+\mu^2t)}\,dt. \tag{4.20}$$

We have the following estimates of the Yukawa potential and $(\widetilde{v}_\alpha, w_\alpha)$.

Lemma 4.5 *Let G_N^μ be the Yukawa potential defined in (4.20), and let $(\widetilde{v}_\alpha, w_\alpha)$ be a solution of (4.17) for $1\le N\le 5$ and $\alpha>0$.*

1. *There exists $C_1=C_1(N,p)>0$ such that*

$$|G_N^\mu|_p\le C_1\mu^{N-2-\frac{N}{p}}, \tag{4.21}$$

   for $1\le p\le\dfrac{N}{N-2}$ if $N\ge 3$ and $1\le p<\infty$ if $N=1,2$.
2. *There exists $C_2>0$ independent of α such that*

$$|\nabla w_\alpha|_2\le C_2|\widetilde{v}_\alpha|_\infty. \tag{4.22}$$

3. *There exists $C_3>0$ independent of α such that*

$$|w_\alpha|_\infty\le C_3|\widetilde{v}_\alpha|_\infty^2, \tag{4.23}$$

   and

$$|w_\alpha|_\infty\le C_3\alpha^{(N-2)/4}|\widetilde{v}_\alpha|_\infty,\quad \text{for } N\ge 3. \tag{4.24}$$

Proof The proof is similar to the ones in [22] and we give the details here for reader's convenience.

1. By the Minkowski inequality for integral, we have

$$|G_N^{\mu}|_p \le \int_0^{\infty} \frac{1}{(4\pi t)^{N/2}} e^{-\mu^2 t} |e^{-\frac{|\cdot|^2}{4t}}|_p \, dt$$
$$= \frac{1}{(4\pi)^{N/2}} \left(\frac{4}{p}\right)^{\frac{N}{2p}} \int_0^{\infty} t^{-\frac{N}{2}+\frac{N}{2p}} e^{-\mu^2 t} \, dt \left(\int_{\mathbb{R}^N} e^{-|y|^2} \, dy\right)^{\frac{1}{p}}$$
$$\le C\mu^{N-2-\frac{N}{p}} \Gamma\left(-\frac{N}{2} + \frac{N}{2p} + 1\right),$$

which implies the conclusion.

2. It is easy to see that $\widetilde{v}_\alpha$ and w_α are exponentially decaying as $|x| \to \infty$. Hence $\widetilde{v}_\alpha, w_\alpha \in L^\infty(\mathbb{R}^N)$. Moreover by part 1 we have $|G_N^{\sqrt{\alpha}}|_1 \le C_1\alpha^{-1}$. Hence from the generalized Young's inequality and part 1, we have

$$|\nabla w_\alpha|_2 = \alpha |G_N^{\sqrt{\alpha}} * \nabla \widetilde{v}_\alpha^2|_2 \le \alpha |G_N^{\sqrt{\alpha}}|_1 |\nabla \widetilde{v}_\alpha^2|_2 \le 2C_1 |\widetilde{v}_\alpha|_\infty |\nabla \widetilde{v}_\alpha|_2.$$

Hence the desired conclusion follows from Lemma 4.4.

3. By part 1, we have

$$|w_\alpha(x)| = \alpha \int_{\mathbb{R}^N} G_N^{\sqrt{\alpha}}(x-y) \widetilde{v}_\alpha^2(y) \, dy \le \alpha |\widetilde{v}_\alpha|_\infty^2 \int_{\mathbb{R}^N} G_N^{\sqrt{\alpha}}(y) \, dy \le C |\widetilde{v}_\alpha|_\infty^2.$$

Moreover, if $N \ge 3$, then by part 2, Hölder inequality and Gagliardo-Nirenberg inequality, we have

$$|w_\alpha(x)| = \alpha \int_{\mathbb{R}^N} G_N^{\sqrt{\alpha}}(x-y) \widetilde{v}_\alpha^2(y) \, dy \le \alpha |\widetilde{v}_\alpha|_\infty \int_{\mathbb{R}^N} G_N^{\sqrt{\alpha}}(x-y) \widetilde{v}_\alpha(y) \, dy$$
$$\le \alpha |\widetilde{v}_\alpha|_\infty |G_N^{\sqrt{\alpha}}|_{2N/(N+2)} |\widetilde{v}_\alpha|_{2^*} \le \alpha^{(N-2)/4} |\widetilde{v}_\alpha|_\infty |\nabla \widetilde{v}_\alpha|_2 \le \alpha^{(N-2)/4} C |\widetilde{v}_\alpha|_\infty.$$

□

In the following lemma, we establish a uniform bound of $\widetilde{v}_\alpha$ in $L^\infty(\mathbb{R}^N)$. The main idea of the proof is the Moser iterations, which is somewhat standard. For the sake of completeness, we give a proof in the appendix. By using this lemma, we prove the part 3 of Theorem 1.3.

Lemma 4.6 *Assume that* $1 \le N \le 3$ *and* $\alpha > 0$, *and* $(\widetilde{v}_\alpha, w_\alpha)$ *is a positive solution of* (4.17). *Then there exists* $C > 0$ *independent of* α *such that*

$$|\widetilde{v}_\alpha|_\infty \le C. \tag{4.25}$$

Proof of Theorem 1.3 part 3 Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}^+$ and let $(v_{\alpha_n}, w_{\alpha_n})$ be a positive ground state solution of (1.3) with $\alpha = \alpha_n$ such that the maximum of v_{α_n} and w_{α_n} is located at origin. We prove the results in several steps.

Step 1 First we consider the case of $1 \le N \le 3$. We prove that, up to a subsequence, for some $(v_\infty, w_\infty) \in H$,

$$\widetilde{v}_{\alpha_n} \rightharpoonup v_\infty \text{ in } H^1(\mathbb{R}^N), \quad w_{\alpha_n} \rightharpoonup w_\infty \text{ in } H^1(\mathbb{R}^N), \quad \text{as } n \to \infty, \tag{4.26}$$

$w_\infty = v_\infty^2$, and v_∞ is a nonnegative solution of (1.12).

By Lemmas 4.4 and 4.5 part 2 and Lemma 4.6, we see that $\{(\widetilde{v}_{\alpha_n}, w_{\alpha_n})\}$ is bounded in H. Hence we can assume the convergence in (4.26) holds up to a subsequence. Since $(\widetilde{v}_{\alpha_n}, w_{\alpha_n})$ satisfies the equation (4.17), then for any $(\varphi, \psi) \in H$, we have

$$\int_{\mathbb{R}^N} \nabla \widetilde{v}_{\alpha_n} \nabla \varphi \, dx + \int_{\mathbb{R}^N} \widetilde{v}_{\alpha_n} \varphi \, dx = \int_{\mathbb{R}^N} \widetilde{v}_{\alpha_n} w_{\alpha_n} \varphi \, dx, \tag{4.27}$$

and

$$\frac{1}{\alpha_n}\int_{\mathbb{R}^N} \nabla w_{\alpha_n} \nabla \psi \; dx + \int_{\mathbb{R}^N} w_{\alpha_n} \psi \, dx = \int_{\mathbb{R}^N} \widetilde{v}_{\alpha_n}^2 \psi \, dx. \tag{4.28}$$

By (4.26), we have

$$\int_{\mathbb{R}^N} w_{\alpha_n} \psi \, dx \to \int_{\mathbb{R}^N} w_\infty \psi \, dx, \quad \int_{\mathbb{R}^N} \widetilde{v}_{\alpha_n}^2 \psi \, dx \to \int_{\mathbb{R}^N} v_\infty^2 \psi \, dx, \quad n \to \infty.$$

From Lemma 4.5 part 2 and Lemma 4.6, we see that $\frac{1}{\alpha_n}\int_{\mathbb{R}^N} |\nabla w_{\alpha_n}|^2 \, dx \to 0$ as $n \to \infty$. Hence (4.28) yields $w_\infty \equiv v_\infty^2$. Let $n \to \infty$ in (4.27), we have

$$\int_{\mathbb{R}^N} (\nabla v_\infty \nabla \varphi + v_\infty \varphi) \, dx = \int_{\mathbb{R}^N} v_\infty w_\infty \varphi \, dx = \int_{\mathbb{R}^N} v_\infty^3 \varphi \, dx.$$

Hence, v_∞ is a solution of the limit Eq. (1.12) and $v_\infty \geq 0$ since $v_{\alpha_n} > 0$.

Step 2 Next we show that

$$\widetilde{v}_{\alpha_n} \to v_\infty \text{ in } H^1(\mathbb{R}^N), \quad w_{\alpha_n} \to w_\infty \text{ in } L^p(\mathbb{R}^N), \quad \text{for } 2 \leq p < 2^*. \tag{4.29}$$

and v_∞ is the unique positive solution of (1.12). In fact, taking $\varphi = \widetilde{v}_{\alpha_n}$ and $\psi = w_{\alpha_n}$ in (4.27) and (4.28), we have

$$\int_{\mathbb{R}^N} \left(|\nabla \widetilde{v}_{\alpha_n}|^2 + \widetilde{v}_{\alpha_n}^2\right) \, dx = \int_{\mathbb{R}^N} \widetilde{v}_{\alpha_n}^2 w_{\alpha_n} \, dx, \tag{4.30}$$

and

$$\frac{1}{\alpha_n}\int_{\mathbb{R}^N} |\nabla w_{\alpha_n}|^2 \, dx + \int_{\mathbb{R}^N} w_{\alpha_n}^2 \, dx = \int_{\mathbb{R}^N} \widetilde{v}_{\alpha_n}^2 w_{\alpha_n} \, dx. \tag{4.31}$$

For $2 \leq N \leq 3$, noticing that $\widetilde{v}_{\alpha_n}, w_{\alpha_n} \in H_r^1(\mathbb{R}^N)$ and $\widetilde{v}_{\alpha_n} \to v_\infty$, $w_{\alpha_n} \to w_\infty$ in $L^3(\mathbb{R}^N)$, one can prove that

$$\int_{\mathbb{R}^N} \widetilde{v}_{\alpha_n}^2 w_{\alpha_n} \, dx \to \int_{\mathbb{R}^N} v_\infty^2 w_\infty \, dx,$$

by the same arguments as (4.8) in the proof of Theorem 1.3 part 1. Hence, it follows from (4.30) and Eq. (1.12) that $\|\widetilde{v}_{\alpha_n}\| \to \|v_\infty\|$ as $n \to \infty$, which implies that $\widetilde{v}_{\alpha_n} \to v_\infty$ in $H^1(\mathbb{R}^N)$. Similarly, it follows from (4.31) that $w_{\alpha_n} \to w_\infty$ in $L^2(\mathbb{R}^N)$. Then $w_{\alpha_n} \to w_\infty$ in $L^p(\mathbb{R}^N)$ for $2 \leq p < 2^*$ by the interpolation inequality in Hölder spaces. Moreover, by (4.30) we have

$$\|\widetilde{v}_{\alpha_n}\|^2 \leq C|\widetilde{v}_{\alpha_n}|_3^2 |w_{\alpha_n}|_3 \leq C\|\widetilde{v}_{\alpha_n}\|^2 |w_{\alpha_n}|_3. \tag{4.32}$$

Hence $|w_\infty|_3 = \lim_{n\to\infty} |w_{\alpha_n}|_3 \geq 1/C$, and $w_\infty \neq 0$, $v_\infty \neq 0$. Thus v_∞ cannot be 0 and $v_\infty > 0$ from the maximum principle. Therefore v_∞ is the unique positive solution of (1.12) with $\max_{x\in\mathbb{R}^N} v_\infty(x) = v(0)$. This completes the proof for $N = 2, 3$.

For $N = 1$, it suffices to show that $\widetilde{v}_{\alpha_n} \to v_\infty$ and $w_{\alpha_n} \to w_\infty$ in $L^3(\mathbb{R}^N)$ to follow the arguments above for the case of $2 \leq N \leq 3$. For this purpose, we apply the concentration-compactness argument. In fact, since $\{(\widetilde{v}_{\alpha_n}, w_{\alpha_n})\}$ is bounded in H and $|w_{\alpha_n}|_3 \geq 1/C$ from (4.32), we may assume that up to a subsequence for some $\rho > 0$,

$$\int_{\mathbb{R}} \left(|\widetilde{v}_{\alpha_n}|^3 + |w_{\alpha_n}|^3\right) \, dx \to \rho, \quad n \to \infty.$$

By the concentration-compactness lemma [30, Lemma 1.1], there is a subsequence, still denoted by $\{(\widetilde{v}_{\alpha_n}, w_{\alpha_n})\}$, such that one of the following three cased occurs: (here $B_R(x)$ is the interval $(x-R, x+R)$, and $B_R^c(x)$ is $\mathbb{R}\backslash B_R(x)$)

(a) (compactness) There exists a sequence $\{x_n\} \subset \mathbb{R}$ satisfying that for any $\varepsilon > 0$ there is an $R > 0$ such that

$$\int_{B_R(x_n)} \left(|\widetilde{v}_{\alpha_n}|^3 + |w_{\alpha_n}|^3\right) dx \geq \rho - \varepsilon, \quad \text{for all } \ n \geq 1.$$

(b) (vanishing) For all $R > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{B_R(y)} \left(|\widetilde{v}_{\alpha_n}|^3 + |w_{\alpha_n}|^3\right) dx = 0.$$

(c) (dichotomy) There exists $\overline{\rho} \in (0, \rho)$ and $\{x_n\} \subset \mathbb{R}$ such that for any $\varepsilon > 0$, there is $R_\varepsilon > 0$, for all $r > R_\varepsilon$, $r' > R_\varepsilon$, have

$$\liminf_{n\to\infty} \int_{B_r(x_n)} \left(|\widetilde{v}_{\alpha_n}|^3 + |w_{\alpha_n}|^3\right) dx \geq \overline{\rho} - \varepsilon,$$
$$\liminf_{n\to\infty} \int_{B^c_{r'}(x_n)} \left(|\widetilde{v}_{\alpha_n}|^3 + |w_{\alpha_n}|^3\right) dx \geq (\rho - \overline{\rho}) - \varepsilon.$$

It is clear that (c) cannot occur since $\widetilde{v}_{\alpha_n}$ and w_{α_n} are radially symmetric and strictly decreasing in the radial direction. If (b) occurs, noting that $\widetilde{v}_{\alpha_n}$ and w_{α_n} are bounded in $H^1(\mathbb{R})$, then we can get $\widetilde{v}_{\alpha_n} \to 0$, $w_{\alpha_n} \to 0$ in $L^3(\mathbb{R})$, which contradicts with the fact $|w_{\alpha_n}|_3 \geq 1/C$. Therefore, the compactness case (a) occurs. Since $\widetilde{v}_{\alpha_n}$ and w_{α_n} are radially symmetric and the maximum of v_{α_n} and w_{α_n} is located at the origin of $\mathbb{R}$, we see that $\{x_n\}$ in (a) is bounded. Hence, we have $\widetilde{v}_{\alpha_n} \to v_\infty$, $w_{\alpha_n} \to w_\infty$ in $L^3(\mathbb{R}^N)$. This completes the proof for the case of $N = 1$.

Step 3 We consider the case of $N = 4, 5$. Suppose on the contrary that there exists $C > 0$ such that $|\widetilde{v}_{\alpha_n}|_\infty \leq C$ or $|w_{\alpha_n}|_\infty \leq C$. We claim that both of $|\widetilde{v}_{\alpha_n}|_\infty$ and $|w_{\alpha_n}|_\infty$ are bounded. In fact, if $|\widetilde{v}_{\alpha_n}|_\infty \leq C$, then Lemma 4.5 part 3 implies $|w_{\alpha_n}|_\infty \leq C$. If $|w_{\alpha_n}|_\infty \leq C$, then by (7.14) and the iteration procedure, we see that $|\widetilde{v}_{\alpha_n}|_\infty \leq C$. Under the assumption that both of $|\widetilde{v}_{\alpha_n}|_\infty$ and $|w_{\alpha_n}|_\infty$ are bounded, one can obtain a positive solution of (1.12) in $H^1(\mathbb{R}^N)$ by using the proof in *Step 1* and *Step 2*, which is a contradiction since (1.12) has no positive solution in $H^1(\mathbb{R}^N)$ when $N = 4, 5$. □

5 The existence of infinitely many solutions

In this section, we prove the existence of multiple non-trivial radially symmetric solutions of the system (1.3). Notice that $I_\alpha(v, w)$ is not an even functional with respect to w, we use a reduced functional. In fact, for any $v \in H^1(\mathbb{R}^N)$, the w-equation in (1.3) has a unique positive solution

$$w_v(x) = \frac{1}{2} \int_{\mathbb{R}^N} G_N^{\sqrt{\alpha}}(x-y) v^2(y)\, dy, \tag{5.1}$$

where $G_N^{\sqrt{\alpha}}(x)$ is the Yukawa potential defined in (4.20). Substituting $w_v(x)$ into the v-equation of system (1.3), we reduce the system (1.3) into a scalar equation

$$-\Delta v + v = v w_v, \quad v \in H^1(\mathbb{R}^N). \tag{5.2}$$

Define a new energy functional $J : H_r^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + v^2)\, dx - \frac{1}{4}\int_{\mathbb{R}^N} v^2 w_v\, dx. \tag{5.3}$$

It is standard to verify that $J \in C^1(H_r^1(\mathbb{R}^N), \mathbb{R})$ and $J'(v) = 0$ is equivalent that (v, w_v) is a solution of (5.2). Since J is an even functional so that $J(-v) = J(v)$, we are able to apply the Mountain-Pass Theorem with symmetry [41, Theorem 9.12] to obtain multiple critical points. To achieve that, we first establish some basic properties of w_v defined in (5.1).

Lemma 5.1 *Let $v \in H^1(\mathbb{R}^N)$, and let w_v be defined as in (5.1). Then*

1. *There exists $C_\alpha > 0$ independent of v such that*

$$\int_{\mathbb{R}^N} v^2 w_v\, dx \le C_\alpha \|v\|^4. \tag{5.4}$$

2.

$$\frac{1}{2}\int_{\mathbb{R}^N} v^3\, dx \le \max\{1, \alpha\}\frac{1}{2}\|v\|^2 + \frac{1}{4}\int_{\mathbb{R}^N} v^2 w_v\, dx \tag{5.5}$$

Proof 1. By integrating the w-equation in (1.3) and Lemma 2.1, we obtain that

$$\|w_v\|_\alpha^2 = \frac{1}{2}\int_{\mathbb{R}^N} v^2 w_v\, dx \le C\|v\|^2\|w_v\| \le C'_\alpha\|v\|^2\|w_v\|_\alpha. \tag{5.6}$$

Hence the estimate (5.4) holds.

2. From the w-equation in (1.3) and the Hölder inequality, we have

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N} v^3\, dx &= \int_{\mathbb{R}^N} (\nabla v \nabla w + \alpha v w)\, dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + \alpha v^2\right)\, dx + \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla w|^2 + \alpha w^2\right)\, dx \\ &\le \max\{1, \alpha\}\frac{1}{2}\|v\|^2 \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^N} v^2 w_v\, dx.\end{aligned}$$

□

Now we prove the existence of multiple solutions of (1.3).

Proof of Theorem 1.4 From Lemma 5.1 part 1, we have

$$J(v) \ge \frac{1}{2}\|v\|^2 - \frac{1}{4}C_\alpha\|v\|^4, \tag{5.7}$$

which implies that J has a strict local minimum at $v = 0$. Secondly we show that for any finite dimensional subspace E of $H_r^1(\mathbb{R}^N)$, there exists $R = R(E) > 0$ such that $J(v) \le 0$ for any $v \in E\backslash B_R(0)$. Indeed by using Lemma 5.1 part 2, we obtain that

$$J(v) = \frac{1}{2}\|v\|^2 - \frac{1}{4}\int_{\mathbb{R}^N} v^2 w_v\, dx \le \frac{1}{2}\left(1 + \max\{1, \alpha\}\right)\|v\|^2 - \frac{1}{2}|v|_3^3, \tag{5.8}$$

which implies that for any finite dimensional subspace E, $J(v) < 0$ for $v \in E\backslash B_R(0)$ as all the norms on a finite dimensional space is equivalent. Finally it is standard to verify that J satisfies the Palais-Smale condition. Hence all conditions of Mountain-Pass Theorem with symmetry [41, Theorem 9.12] are satisfied, and the conclusion follows from that. □

6 Bounded domain case

In this section, we consider the solutions of the corresponding Dirichlet boundary value problem (1.14) on a bounded domain Ω with smooth boundary $\partial\Omega$. Let (λ_1, φ_1) be the principal eigen-pair of

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0, & x \in \Omega, \\ \varphi = 0, & x \in \partial\Omega, \end{cases} \tag{6.1}$$

then it is well-known that $\lambda_1 > 0$ is a simple eigenvalue and φ_1 can be chosen as positive. In the following we assume that $\varphi_1(x) > 0$ for $x \in \Omega$ and $|\varphi_1|_\infty = 1$.

The existence and multiplicity results for the solutions of (1.14) using variational method are as follows. Here we only state the corresponding results for (1.14) without proof as the proof is essentially similar to Theorems 1.1 and 1.3.

Theorem 6.1 *Suppose that $1 \le N \le 5$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$.*

1. *For each $\alpha > -\lambda_1$, system (1.14) possesses a positive ground state solution $(v_\alpha, w_\alpha) \in H_0^1(\Omega) \times H_0^1(\Omega)$.*
2. *For each $\alpha > -\lambda_1$, system (1.14) possesses infinitely many solutions $\{(v_n, w_n)\}$ such that $\Phi_\alpha(v_n, w_n) \to \infty$ as $n \to \infty$.*

We observe that when $\alpha = -\lambda_1$, the system (1.14) has a branch of semi-trivial solutions $\Sigma_* = \{(\alpha, v, w) = (-\lambda_1, 0, \mu\varphi_1) : \mu \in \mathbb{R}\}$ which intersects with the branch of the trivial solutions $\Sigma_0 = \{(\alpha, 0, 0) : \alpha \in \mathbb{R}\}$ at $(-\lambda_1, 0, 0)$. Next we use bifurcation theory to consider the set of solutions of (1.14), especially the solutions near the semi-trivial branch Σ_*. For that purpose, we consider an eigenvalue problem:

$$\begin{cases} \Delta\phi - \phi + \mu\varphi_1\phi = 0, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega, \end{cases} \tag{6.2}$$

where $\varphi_1 > 0$ is the positive principal eigenfunction of (6.1). From a well-known result (see for example [9]), (6.2) has a sequence of eigenvalues $0 < \mu_1 < \mu_2 \le \cdots \le \mu_k \le \cdots \to \infty$, the eigenfunction ϕ_1 corresponding to μ_1 is simple, and ϕ_1 can be chosen as positive. The principal eigenvalue μ_1 can be expressed by the following Rayleigh quotient:

$$\mu_1 = \inf_{\phi \in H_0^1(\Omega), \phi \ne 0} \frac{\displaystyle\int_\Omega (|\nabla\phi|^2 + \phi^2)\, dx}{\displaystyle\int_\Omega \varphi_1\phi^2\, dx}.$$

Let $X = C_0^{2,\theta}(\overline{\Omega})$ for $\theta \in (0, 1)$. Define the set of non-trivial solutions of (1.14) to be

$$\Sigma = \{(\alpha, v, w) \in \mathbb{R} \times X^2 : (\alpha, v, w) \text{ is a solution of(1.14) such that } v \not\equiv 0 \text{ and } w \not\equiv 0\}. \tag{6.3}$$

Here we consider solutions in X instead of in $H_0^1(\Omega)$ since all weak solutions are indeed classical ones because of the smoothness of nonlinearities in the equation (5.2). Our results about existence, nonexistence, multiplicity and bifurcation of solutions of (1.14) in a general bounded domain Ω are as follows:

Theorem 6.2 *Suppose that $N \geq 1$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$.*

1. *For $-\infty < \alpha < -\lambda_1$, the only non-negative solution of system (1.14) is $(v, w) = (0, 0)$.*
2. *For $\alpha = -\lambda_1$, if (v, w) is a solution of (1.14), then $v = 0$ and $w = \mu\varphi_1$ for $\mu \in \mathbb{R}$, that is $(-\lambda_1, v, w) \in \Sigma_*$.*
3. *Assume that all eigenvalues μ_n of (6.2) are simple ones. Then for any $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that for $-\lambda_1 < \alpha < -\lambda_1 + \delta_n$, system (1.14) possesses a pair of non-trivial solutions in form of $(\pm v_n(\alpha, \cdot), w_n(\alpha, \cdot))$ such that*

$$v_n(\alpha, \cdot) = k_n(\alpha + \lambda_1)^2\phi_n + o(|\alpha + \lambda_1|),\ \ w_n(\alpha, \cdot) = \mu_n\varphi_1 + o(|\alpha + \lambda_1|), \quad (6.4)$$

for some constant $k_n > 0$. Moreover for each $n \in \mathbb{N}$, there exists a connected component Σ_n of $\overline{\Sigma}$ such that $(-\lambda_1, 0, \mu_n\varphi_1) \in \Sigma_n$, and either Σ_n is unbounded in $(-\lambda_1, \infty) \times X^2$, or there exists $m \in \mathbb{N}$ and $m \neq n$ such that $(-\lambda_1, 0, \mu_m\varphi_1) \in \Sigma_n$, that is, $\Sigma_n = \Sigma_m$.

Proof 1. Suppose that (v, w) is a non-negative solution of (1.14). Multiplying the w-equation in (1.14) by φ_1, and integrating over Ω, we obtain

$$0 = -(\lambda_1 + \alpha)\int_\Omega w\varphi_1\, dx + \frac{1}{2}\int_\Omega v^2\varphi_1\, dx.$$

Since $\int_\Omega w\phi_1\, dx \geq 0$ and $\int_\Omega v^2\phi_1\, dx \geq 0$, then under the condition that $\alpha < -\lambda_1$, we must have $w \equiv 0$ and $v \equiv 0$ in Ω.

2. For $\alpha = -\lambda_1$, suppose that (v, w) is a solution of (1.14), then we have $-\Delta w - \lambda_1 w = (1/2)v^2 \geq 0$ for $x \in \Omega$, and $w = 0$ on $\partial\Omega$. The equation is solvable only if $\int_\Omega v^2\varphi_1\, dx = 0$, which implies that $v \equiv 0$, and $w = \mu\varphi_1$ for $\mu \in \mathbb{R}$.
3. Assume an eigenvalues μ_n of (6.2) is a simple one. We consider the bifurcation of non-trivial solutions of (1.14) from the semi-trivial branch Σ_* near $(\alpha, v, w) = (-\lambda, 0, \mu_n\varphi_1)$. We use a "double saddle-node" bifurcation theorem in [32]. For that purpose, we define $F : \mathbb{R} \times X^2 \to Y^2$ where $Y = C^\theta(\overline{\Omega})$, and F is defined by

$$F(\alpha, v, w) = \begin{pmatrix} \Delta v - v + wv \\ \Delta w - \alpha w + \dfrac{1}{2}v^2 \end{pmatrix}. \quad (6.5)$$

It is straightforward to calculate the Fréchet derivatives of F:

$$F_{(v,w)}(\mu, v, w)[(\phi, \psi)] = \begin{pmatrix} \Delta\phi - \phi + w\phi + v\psi \\ \Delta\psi - \alpha\psi + v\phi \end{pmatrix}, \quad F_\alpha(\mu, v, w) = \begin{pmatrix} 0 \\ -w \end{pmatrix},$$

$$F_{(v,w)(v,w)}(\mu, v, w)[(\phi_a, \psi_a), (\phi_b, \psi_b)] = \begin{pmatrix} \phi_a\psi_b + \phi_b\psi_a \\ \phi_a\phi_b \end{pmatrix}.$$

In particular, we have

$$L[(\phi, \psi)] := F_{(v,w)}(-\lambda_1, 0, \mu_n\varphi_1)[(\phi, \psi)] = \begin{pmatrix} \Delta\phi - \phi + \mu_n\varphi_1\phi \\ \Delta\psi + \lambda_1\psi \end{pmatrix}.$$

Then the null space $N(L) = span\{(\phi_n, 0), (0, \varphi_1)\}$ where ϕ_n is the eigenfunction of (6.2) associated with $\mu = \mu_n$; and the range space of L is defined by

$$R(L) = \left\{(f, g) \in Y^2 : \int_\Omega f\phi_n\, dx = 0, \text{ and } \int_\Omega g\varphi_1\, dx = 0\right\}.$$

Hence $\dim N(L) = \operatorname{codim} R(L) = 2$. On the other hand,

$$F_\alpha(-\lambda_1, 0, \mu_n\varphi_1) = \begin{pmatrix} 0 \\ -\mu_n\varphi_1 \end{pmatrix} \notin R(L),$$

since $-\int_\Omega \mu_n\varphi_1^2\,dx \neq 0$. Define $v_1, v_2 \in (Y^2)^*$ (the dual space of Y^2) by

$$\langle v_1, (f, g)\rangle = \int_\Omega g\varphi_1\,dx, \quad \langle v_2, (f, g)\rangle = \int_\Omega f\phi_n\,dx.$$

Then $R(L) = \{(f, g) \in Y^2 : \langle v_1, (f, g)\rangle = 0 \text{ and } \langle v_2, (f, g)\rangle = 0\}$, and $\langle v_1, F_\alpha(-\lambda_1, 0, \mu_n\varphi_1)\rangle \neq 0$ and $\langle v_2, F_\alpha(-\lambda_1, 0, \mu_n\varphi_1)\rangle = 0$. Define a 2×2 matrix

$$\begin{aligned} H &= \begin{pmatrix} \langle v_2, F_{(v,w)(v,w)}[(\phi_n, 0), (\phi_n, 0)]\rangle & \langle v_2, F_{(v,w)(v,w)}[(\phi_n, 0), (0, \varphi_1)]\rangle \\ \langle v_2, F_{(v,w)(v,w)}[(\phi_n, 0), (0, \varphi_1)]\rangle & \langle v_2, F_{(v,w)(v,w)}[(0, \varphi_1), (0, \varphi_1)]\rangle \end{pmatrix} \\ &= \begin{pmatrix} 0 & \int_\Omega \phi_n^2\varphi_1\,dx \\ \int_\Omega \phi_n^2\varphi_1\,dx & 0 \end{pmatrix}. \end{aligned}$$

Hence the determinant of H is $-\left(\int_\Omega \phi_n^2\varphi_1\,dx\right)^2 < 0$. Then from [32, Theorem 2.3], the set of solutions of $F = 0$ near $(-\lambda_1, 0, \mu_n\varphi_1)$ is the union of two smooth curves $S_i = \{(\mu^i(t), v^i(t,\cdot), w^i(t,\cdot)) : |t| < \epsilon\}$, $i = 1, 2$, satisfying $\mu^i(0) = -\lambda_1$, $(\mu^i)'(0) = 0$,

$$\begin{aligned} \left(v^1(t,\cdot), w^1(t,\cdot)\right) &= \left(t\phi_n + tx^1(t), \mu_n\varphi_1 + ty^1(t)\right), \\ \left(v^2(t,\cdot), w^2(t,\cdot)\right) &= \left(tx^2(t), \mu_n\varphi_1 + t\varphi_1 + ty^2(t)\right), \end{aligned}$$

where $x^i(0) = y^i(0) = (x^i)'(0) = (y^i)'(0) = 0$. Apparently S_2 is identical to Σ_*, the branch of semi-trivial solutions. Hence $(\mu, v, w) = (-\lambda_1, 0, \mu_n\varphi_1)$ is a bifurcation point so that non-trivial solutions of (1.14) are on S_1. From [32, Propostion 2.4], we get

$$(\mu^1)''(0) = -\frac{\langle v_1, F_{(v,w)(v,w)}[(\phi_n, 0), (\phi_n, 0)]\rangle}{\langle v_1, F_\alpha\rangle} = \frac{\displaystyle\int_\Omega \phi_n^2\varphi_1\,dx}{\mu_n\displaystyle\int_\Omega \varphi_1^2\,dx} > 0.$$

Hence for $\alpha \in (-\lambda_1, -\lambda_1 + \delta_n)$, there exists two non-trivial solutions of (1.14) near $(v, w) = (0, \mu_n\varphi_1)$ as stated.

Let Σ_n be the connected component of $\overline{\Sigma}$ containing $(-\lambda_1, 0, \mu_n\varphi_1)$. Then we can use a similar argument as in [39] or the proof of [57, Theorem 4.1] to prove that either Σ_n is unbounded, or Σ_n contains some $(\tilde\alpha, \tilde v, \tilde w)$ such that $\tilde v \equiv 0$ or $\tilde w \equiv 0$. Note that we can use the result in [39] by converting (5.2) into equivalent integral equation which is compact operator, or we can use the result in [44] for (5.2) directly as Fredholm operator. We assume that Σ_n contains some $(\tilde\alpha, \tilde v, \tilde w)$ such that $\tilde v \equiv 0$ or $\tilde w \equiv 0$. It is easy to see that for a solution (v, w) of (5.2), if $w \equiv 0$, then $v \equiv 0$. So possible forms of $(\tilde v, \tilde w)$ are $(0, 0)$ or $(0, \tilde w)$. If $(\tilde v, \tilde w) = (0, 0)$, then $\tilde\alpha > -\lambda_1$, as near $(-\lambda_1, 0, 0)$ the only solutions of (5.2) are on Σ_* and Σ_0. For $\tilde\alpha > -\lambda_1$, $(\tilde\alpha, 0, 0)$ cannot be a bifurcation point for (5.2) from the form of $F_{(v,w)}(\tilde\alpha, 0, 0)$. Hence $(\tilde v, \tilde w)$ cannot be $(0, 0)$. Thus $(\tilde v, \tilde w) = (0, \tilde w)$ and from the equation, $\tilde w$ must be $k\varphi_p$ for $k \in \mathbb{R}$ and $\alpha = -\lambda_p$, where $m \geq 1$. Since near $\alpha = -\lambda_1$, Σ_n is on the right side of $\alpha = -\lambda_1$, and Σ_n is connected, then Σ_n contains another $(-\lambda_1, 0, \mu_m\varphi_1)$ with $m \neq n$. □

Remark 6.3
1. The bifurcation result in Theorem 6.2 for α near $-\lambda_1$ holds for all $N \geq 1$, while the existence of solutions in Theorems 1.1 and 1.3 are for $2 \leq N \leq 5$ (the existence also holds for $N = 1$). See Theorem 6.4 for more on the subtlety of the spatial dimension N.
2. If an eigenvalue λ_k of (6.1) with $k \geq 2$ is also simple with eigenfunction φ_k, and all eigenvalues of (6.2) with φ_1 replaced by φ_k are also simple, then similar bifurcation result as in Theorem 6.2 part 3 can also be established near $(\alpha, v, w) = (-\lambda_k, 0, \mu_n \varphi_k)$.

Theorem 6.4 *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. Let Σ_n be the connected component of $\overline{\Sigma}$ containing $(-\lambda_1, 0, \mu_n\varphi_1)$ in Theorem 6.2, and let $proj_\alpha \Sigma_n$ be the projection of Σ_n into α-axis.*

1. *For $N \geq 6$ and Ω is a star-shaped domain, $proj_\alpha \Sigma_n \subset [-\lambda_1, 0)$ for each $n \in \mathbb{N}$.*
2. *For $N = 2, 3$, for each $n \in \mathbb{N}$, either $proj_\alpha \Sigma_n = [-\lambda_1, \infty)$ or $\Sigma_n = \Sigma_m$ for some $m \neq n$. Moreover $proj_\alpha \Sigma_1 = [-\lambda_1, \infty)$.*
3. *For $N = 1$ and $\Omega = (-L, L)$, for each $n \in \mathbb{N}$, $proj_\alpha \Sigma_n = [-\lambda_1, \infty)$ and $\Sigma_n \cap \Sigma_m = \emptyset$ for any $m \neq n$. Hence for any $\alpha > -\lambda_1$ and $n \in \mathbb{N}$, (1.14) has at least one pair of non-trivial solutions $(\pm v(\alpha, \cdot), w(\alpha, \cdot))$ such that $v(\alpha, \cdot)$ changes sign exactly $n - 1$ times, and $w(\alpha, \cdot) > 0$ on $(-L, L)$.*

Proof For any dimension $N \geq 1$, $proj_\alpha \Sigma_n \subset [-\lambda_1, \infty)$ from Theorem 6.2 part 2, as $\Sigma_n \cap \{\alpha = -\lambda_1\} = \{(-\lambda_1, 0, \mu_n\varphi_1)\}$ and Theorem 6.2 part 3 shows the local structure of Σ_n near the bifurcation point $(-\lambda_1, 0, \mu_n\varphi_1)$. For any non-trivial solution (v, w) of (1.14) with $\alpha > -\lambda_1$, we must have $w = (1/2)(-\Delta + \alpha)^{-1}(v^2) > 0$.

1. Without loss of generality, we assume that $0 \in \Omega$ and Ω is star-shaped with respect to the origin. The Pohozaev identity for (1.14) is
$$\frac{1}{2}\int_{\partial\Omega} \left(|\nabla v|^2 + |\nabla w|^2\right)(x \cdot \nu) dS + \frac{1}{2}\int_{\partial\Omega} v^2 w (x \cdot \nu) dS$$
$$= -\int_{\Omega} (v^2 + \alpha w^2)\, dx + \frac{6 - N}{4}\int_{\Omega} v^2 w\, dx, \tag{6.6}$$
where ν denotes the unit outward normal to $\partial\Omega$. Since $w > 0$ for any non-trivial solution (v, w) of (1.14) and $x \cdot \nu \geq 0$ for any $x \in \partial\Omega$, then (1.14) has only the trivial solution for $\alpha \geq 0$ and $N \geq 6$. Therefore all the branches Σ_n in Theorem 6.2 only exist for $\alpha \in (-\lambda_1, 0)$ when $N \geq 6$.
2. For $N = 1, 2, 3$, the a priori estimates in Lemma 4.6 still hold for a bounded domain Ω and $\alpha \in [-\lambda_1 + \delta, \infty)$ for some $\delta > 0$. If Σ_n is not connected to another bifurcation point $(-\lambda_1, 0, \mu_m\varphi_1)$, then Σ_n is unbounded in $(-\lambda_1, \infty) \times X^2$. The a priori estimates for $\alpha \in [-\lambda_1 + \delta, \infty)$ implies that there is no bifurcation from infinity at any $-\lambda_1 + \delta \leq \alpha < \infty$. Suppose that there is a sequence (α^k, v^k, w^k) of solutions to (1.14) satisfying $\alpha^k \to -\lambda_1$, and $||(v^k, w^k)||_{X^2} = \infty$ as $k \to \infty$, then Σ_n and Σ_* are two distinct continua of solutions of (1.14) near $(\alpha, (v, w)) = (-\lambda_1, \infty)$, which is a contradiction to the bifurcation from infinity result in [40, Corollary 1.8]. Thus the projection of Σ_n into X^2 is bounded for all $\alpha \geq -\lambda_1$, and we must have $proj_\alpha \Sigma_n = [-\lambda_1, \infty)$.
For Σ_1, we let $\Sigma_1 = \Sigma_1^+ \cup \Sigma_1^- \cup \{(-\lambda_1, 0, \mu_1\varphi_1)\}$, where Σ_1^+ is the connected component of $\Sigma_1 \backslash \{(-\lambda_1, 0, \mu_1\varphi_1)\}$ which contains the positive solutions near bifurcation point, and $\Sigma_1^- = \{(\alpha, -v, w) : (\alpha, v, w) \in \Sigma_1^+\}$. It is shown in the proof of Theorem 6.2 that Σ_1 is either unbounded or Σ_1 contains another $(-\lambda_1, 0, \mu_k\varphi_1)$. But from the maximum principle, any solution on Σ_1^+ is positive, hence Σ_1^+ cannot connect to $(-\lambda_1, 0, \mu_k\varphi_1)$,

as near that bifurcation point the v-component of solutions of (5.2) is not positive. Thus Σ_1^+ is unbounded and $proj_\alpha \Sigma_1 = [-\lambda_1, \infty)$.

3. For $N = 1$, we assume that $\Omega = (-L, L)$ and we follow a classical approach given in [38]. Define $E = \{u \in C^1[-\pi, \pi] : u(\pm\pi) = 0\}$. Let S_n^+ denote the set of $u \in E$ such that u has exactly $n-1$ simple zeros in $(-\pi, \pi)$, all zeros of u in $[-\pi, \pi]$ are simple, and u is positive in a small neighborhood $(-\pi, \pi + \delta)$ of $x = -\pi$. Set $S_n^- = -S_n^+$ and $S_n = S_n^+ \cup S_n^-$. We claim that if (α, v, w) is a non-trivial solution of (5.2) on $\Omega = (-\pi, \pi)$ with $\alpha > -\lambda_1$, then $(v, w) \in S_n \times S_1^+$ where $n-1$ is the number of zeros of v in $(-\pi, \pi)$. We have shown that $w > 0$ in $(-\pi, \pi)$ whenever $\alpha > -\lambda_1$. Suppose that $x = x_0 \in [-\pi, \pi]$ is a zero of $v(x)$, then $v'(x_0) \neq 0$ otherwise the uniqueness of solution to a second order ordinary differential equation implies that $v(x) \equiv 0$, which contradicts with the assumption that $v(x)$ is non-trivial. Hence $(v, w) \in S_n \times S_1^+$.

Now near the bifurcation point, each Σ_n can be decomposed into $\Sigma_n = \Sigma_n^+ \cup \Sigma_n^- \cup \{(-\lambda_1, 0, \mu_n\varphi_1)\}$, where $\Sigma_n^+ \subset S_n^+$ and $\Sigma_1^- = \{(\alpha, -v, w) : (\alpha, v, w) \in \Sigma_1^+\}$. Now the argument above (see also [38]) implies that the entire connected component $\Sigma_n^+ \subset S_n^+$, and in particular $\Sigma_n \cap \Sigma_m = \emptyset$. In part 2, we have proved that $proj_\alpha \Sigma_n = [-\lambda_1, \infty)$ in case $\Sigma_n \cap \Sigma_m = \emptyset$. This completes the proof. □

The existence of solutions with precise nodal structure in part 3 of Theorem 6.4 can also be proved for $N = 2, 3$ when Ω is the unit ball. Such results in general do not hold for systems of equations but only for scalar equations, and other related result was shown in [3,50] for the Schrödinger system (1.7) with $K = 2$ and $\mu_1 = \mu_2$, also on a interval of ball domain. Also when n is an even number, the solution $(v, w) \in \Sigma_n$ satisfies $v(-x) = v(x)$ and $w(-x) = w(x)$ which is again inherited from the eigenfunction. Thus for $N = 1$ and bounded interval, both types of multi-pulse solutions mentioned in the introduction are proved in Theorem 6.4.

7 Appendix: proof of Lemma 4.6

Proof of Lemma 4.6 For $m \in \mathbb{N}$ and $\beta > 1$, define

$$A_m = \{x \in \mathbb{R}^N : |\widetilde{v}_\alpha(x)|^{\beta-1} \leq m\}, \ B_m = \mathbb{R}^N \backslash A_m,$$

and

$$\varphi_m(x) = \begin{cases} \widetilde{v}_\alpha(x)|\widetilde{v}_\alpha(x)|^{2(\beta-1)}, & x \in A_m, \\ m^2 \widetilde{v}_\alpha(x), & x \in B_m. \end{cases} \tag{7.1}$$

One can observe that $\varphi_m \in H^1(\mathbb{R}^N)$, $|\varphi_m| \leq |\widetilde{v}_\alpha|^{2\beta-1}$ pointwisely, and

$$\nabla\varphi_m = (2\beta - 1)|\widetilde{v}_\alpha|^{2(\beta-1)}\nabla\widetilde{v}_\alpha, \ \text{ in } A_m, \ \nabla\varphi_m = m^2\nabla\widetilde{v}_\alpha, \ \text{ in } B_m. \tag{7.2}$$

Using φ_m as a test function in (4.17), we obtain

$$\int_{\mathbb{R}^N} (\nabla\widetilde{v}_\alpha\nabla\varphi_m + \widetilde{v}_\alpha\varphi_m) \, dx = \int_{\mathbb{R}^N} \widetilde{v}_\alpha w_\alpha \varphi_m \, dx. \tag{7.3}$$

From (7.2), we have

$$\int_{\mathbb{R}^N} \nabla\widetilde{v}_\alpha\nabla\varphi_m \, dx = (2\beta - 1)\int_{A_m} |\widetilde{v}_\alpha|^{2(\beta-1)}|\nabla\widetilde{v}_\alpha|^2 \, dx + m^2 \int_{B_m} |\nabla\widetilde{v}_\alpha|^2 \, dx. \tag{7.4}$$

Similarly we define

$$\psi_m(x) = \begin{cases} \widetilde{v}_\alpha(x)|\widetilde{v}_\alpha(x)|^{\beta-1}, & x \in A_m, \\ m\widetilde{v}_\alpha(x), & x \in B_m. \end{cases} \tag{7.5}$$

Then $\psi_m^2 = \widetilde{v}_\alpha \varphi_m$ and

$$\nabla\psi_m = \beta|\widetilde{v}_\alpha|^{\beta-1}\nabla\widetilde{v}_\alpha \text{ in } A_m, \ \nabla\psi_m = m\nabla\widetilde{v}_\alpha \text{ in } B_m. \tag{7.6}$$

Hence we obtain

$$\int_{\mathbb{R}^N} |\nabla\psi_m|^2\,dx = \beta^2\int_{A_m} |\widetilde{v}_\alpha|^{2(\beta-1)}|\nabla\widetilde{v}_\alpha|^2\,dx + m^2\int_{B_m}|\nabla\widetilde{v}_\alpha|^2\,dx. \tag{7.7}$$

It follows from (7.1), (7.4), (7.5) and (7.7) that

$$\begin{aligned}\int_{\mathbb{R}^N} \left(|\nabla\psi_m|^2 + \psi_m^2\right)\,dx - \int_{\mathbb{R}^N}\left(\nabla\widetilde{v}_\alpha\nabla\varphi_m + \widetilde{v}_\alpha\varphi_m\right)\,dx \\ = (\beta-1)^2\int_{A_m}|\widetilde{v}_\alpha|^{2(\beta-1)}|\nabla\widetilde{v}_\alpha|^2\,dx.\end{aligned} \tag{7.8}$$

From (7.4), we have

$$\begin{aligned}(2\beta-1)\int_{A_m}|\widetilde{v}_\alpha|^{2(\beta-1)}|\nabla\widetilde{v}_\alpha|^2\,dx &\le \int_{\mathbb{R}^N}\nabla\widetilde{v}_\alpha\nabla\varphi_m\,dx \\ &\le \int_{\mathbb{R}^N}\left(\nabla\widetilde{v}_\alpha\nabla\varphi_m + \widetilde{v}_\alpha\varphi_m\right)\,dx,\end{aligned} \tag{7.9}$$

and consequently from (7.3), (7.8) and (7.9), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}\left(|\nabla\psi_m|^2 + \psi_m^2\right)\,dx &\le \left(\frac{(\beta-1)^2}{2\beta-1}+1\right)\int_{\mathbb{R}^N}\left(\nabla\widetilde{v}_\alpha\nabla\varphi_m + \widetilde{v}_\alpha\varphi_m\right)\,dx \\ &\le \beta^2\int_{\mathbb{R}^N}\widetilde{v}_\alpha w_\alpha\varphi_m\,dx.\end{aligned} \tag{7.10}$$

For $N=3$, let S be the best constant of the Sobolev embedding of $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$. That is, for every $u \in D^{1,2}(\mathbb{R}^N)$,

$$|u|_{2^*}^2 \le S\int_{\mathbb{R}^N}|\nabla u|^2\,dx, \tag{7.11}$$

For $N = 1$ or 2, we define 2^* to be some positive constant larger than 4 and define S to be the best constant of the Sobolev embedding of $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$. Then for $N = 1, 2$ or 3, from (7.10), we have

$$\left(\int_{A_m}|\psi_m|^{2^*}\,dx\right)^{2/2^*} \le \left(\int_{\mathbb{R}^N}|\psi_m|^{2^*}\,dx\right)^{2/2^*} \le S\beta^2\int_{\mathbb{R}^N}\widetilde{v}_\alpha w_\alpha\varphi_m\,dx \tag{7.12}$$

Since $|\psi_m| = |\widetilde{v}_\alpha|^\beta$ in A_m and $\varphi_m \le |\widetilde{v}_\alpha|^{2\beta-1}$ in $\mathbb{R}^N$, from (7.12) we have

$$\left(\int_{A_m}|\widetilde{v}_\alpha|^{2^*\beta}\,dx\right)^{2/2^*} \le S\beta^2\int_{\mathbb{R}^N}\widetilde{v}_\alpha^{2\beta}w_\alpha\,dx. \tag{7.13}$$

Let $m \to \infty$ in (7.13). Then from the Monotone Convergence Theorem, we obtain

$$\left(\int_{\mathbb{R}^N} |\widetilde{v}_\alpha|^{2^*\beta}\,dx\right)^{2/2^*} \leq S\beta^2 \int_{\mathbb{R}^N} \widetilde{v}_\alpha^{2\beta} w_\alpha\,dx. \tag{7.14}$$

By the Hölder inequality, we have

$$|\widetilde{v}_\alpha|_{2^*\beta} \leq (S\beta^2)^{1/(2\beta)} |\widetilde{v}_\alpha|_{4\beta} |w_\alpha|_2^{\frac{1}{2\beta}} \tag{7.15}$$

As the estimate (7.15) holds for any $\beta > 1$, we can obtain the desired conclusion by the iterations of the estimate (7.15). In fact, let $\sigma = 2^*/4$, then $\sigma > 1$ since $N \leq 3$. Taking $\beta = \sigma$ in (7.15), we have

$$|\widetilde{v}_\alpha|_{2^*\sigma} \leq S^{\frac{1}{2\sigma}} \sigma^{\frac{1}{\sigma}} |\widetilde{v}_\alpha|_{2^*} |w_\alpha|_2^{\frac{1}{2\sigma}}. \tag{7.16}$$

Taking $\beta = \sigma^2$ in (7.15), we have

$$|\widetilde{v}_\alpha|_{2^*\sigma^2} \leq S^{\frac{1}{2\sigma^2}} \sigma^{\frac{1}{\sigma^2}} |\widetilde{v}_\alpha|_{2^*\sigma} |w_\alpha|_2^{\frac{1}{2\sigma^2}}. \tag{7.17}$$

Substituting (7.16) into (7.17), we find

$$|\widetilde{v}_\alpha|_{2^*\sigma^2} \leq S^{\frac{1}{2\sigma^2}+\frac{1}{2\sigma}} \sigma^{\frac{1}{\sigma^2}+\frac{1}{\sigma}} |\widetilde{v}_\alpha|_{2^*} |w_\alpha|_2^{\frac{1}{2\sigma^2}+\frac{1}{2\sigma}}. \tag{7.18}$$

By induction, taking $\beta = \sigma^j$, $j = 1, 2, \ldots$, yields

$$|\widetilde{v}_\alpha|_{2^*\sigma^j} \leq S^{\sum_{i=1}^{j} \frac{1}{2\sigma^i}} \sigma^{\sum_{i=1}^{j} \frac{1}{\sigma^i}} |\widetilde{v}_\alpha|_{2^*} |w_\alpha|_2^{\sum_{i=1}^{j} \frac{1}{2\sigma^i}}.$$

Taking $j \to \infty$, the conclusion in (4.25) follows from that $|\widetilde{v}_\alpha|_{2^*}$ and $|w_\alpha|_2$ are bounded from Lemma 4.4, and the fact that the series $\sum_{i=1}^{\infty} \sigma^{-i}$ is convergent. □

Remark 7.1 The Moser iteration arguments given in the proof of Lemma 4.6 is also useful for the case of $N = 4, 5$ in the following sense.

1. For $N = 4, 5$, one can show that there exists $\delta > 0$ independent of α such that $|\widetilde{v}_\alpha|_\infty \geq \delta$ and $|w_\alpha|_\infty \geq \delta$. In fact, by using (7.14) (again with $\beta > 1$) and the Hölder inequality with 2^* and $\widetilde{2}^*$ defined in (2.1), we have
$$|\widetilde{v}_\alpha|_{2^*\beta} \leq (CS\beta^2)^{\frac{1}{2\beta}} |\widetilde{v}_\alpha|_{2\widetilde{2}^*\beta} |w_\alpha|_{2^*}^{\frac{1}{2\beta}} \leq (CS\beta^2)^{\frac{1}{2\beta}} |\widetilde{v}_\alpha|_{2\widetilde{2}^*\beta} |\widetilde{v}_\alpha|_\infty^{\frac{1}{2\beta}}. \tag{7.19}$$
Let $\sigma = \frac{2^*}{2\widetilde{2}^*} = \frac{N+2}{2(N-2)}$, then $\sigma > 1$ since $N < 6$. Taking $\beta = \sigma^j$ in (7.19), we have
$$|\widetilde{v}_\alpha|_{2^*\sigma^j} \leq (CS)^{\sum_{i=1}^{j} \frac{1}{2\sigma^i}} \sigma^{\sum_{i=1}^{j} \frac{1}{\sigma^i}} |\widetilde{v}_\alpha|_{2^*} |\widetilde{v}_\alpha|_\infty^{\sum_{i=1}^{j} \frac{1}{2\sigma^i}}.$$
Let $j \to \infty$, we find that $|\widetilde{v}_\alpha|_\infty$ is bounded from below, since $\sum_{i=1}^{\infty} \frac{1}{2\sigma^i} = \frac{1}{2(\sigma-1)} > 1$ when $N = 4, 5$.
2. For $N = 4, 5$, it follows from Step 3 in the proof of Theorem 1.3 that
$$|\widetilde{v}_\alpha|_\infty \to \infty \text{ and } |w_\alpha|_\infty \to \infty, \quad \text{as } \alpha \to \infty.$$
Here, as a comparison, we obtain an estimate from the Moser iteration
$$|\widetilde{v}_\alpha|_\infty \leq C\alpha^{\frac{(N-2)^2}{4(6-N)}}, \text{ and } |w_\alpha|_\infty \leq C\alpha^{\frac{N-2}{6-N}}. \tag{7.20}$$

In fact, by using (7.14) and Lemma 4.5 part 3, we have

$$|\widetilde{v}_\alpha|_{2^*\beta} \leq (C_3 S\beta^2)^{\frac{1}{2\beta}} (\alpha^{\frac{N-2}{4}})^{\frac{1}{2\beta}} |\widetilde{v}_\alpha|_\infty^{\frac{1}{2\beta}} |\widetilde{v}_\alpha|_{2\beta}.$$

Taking $\sigma = 2^*/2$, we can get (7.20) by the iteration method.

Acknowledgments The authors would like to thank an anonymous reviewer for helpful suggestions and comments. This work was done when Zhao and Zhao visited Department of Mathematics, College of William and Mary in 2013–2014, and they would like to thank the College for hospitality.

References

1. Ambrosetti, A., Colorado, E.: Standing waves of some coupled nonlinear Schrödinger equations. J. Lond. Math. Soc. **75**(1), 67–82 (2007)
2. Ambrosetti, A., Ruiz, D.: Multiple bound states for the Schrödinger-Poisson problem. Commun. Contemp. Math. **10**(3), 391–404 (2008)
3. Bartsch, T., Dancer, N., Wang, Z.-Q.: A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system. Calc. Var. Partial Differ. Equ. **37**(3–4), 345–361 (2010)
4. Bartsch, T., Wang, Z.-Q., Wei, J.-C.: Bound states for a coupled Schrödinger system. J. Fixed Point Theory Appl. **2**(2), 353–367 (2007)
5. Bates, P.W., Shi, J.-P.: Existence and instability of spike layer solutions to singular perturbation problems. J. Funct. Anal. **196**(2), 211–264 (2002)
6. Benci, V., Fortunato, D.: An eigenvalue problem for the Schrödinger-Maxwell equations. Topol. Methods Nonlinear Anal. **11**(2), 283–293 (1998)
7. Berestycki, H., Lions, P.-L.: Nonlinear scalar field equations. I. Existence of a ground state. Arch. Ration. Mech. Anal. **82**(4), 313–345 (1983)
8. Brezis, H., Lieb, E.: Minimum action solutions of some vector field equations. Commun. Math. Phys. **96**(1), 97–113 (1984)
9. Brown, K.J., Lin, S.S.: On the existence of positive eigenfunctions for an eigenvalue problem with indefinite weight function. J. Math. Anal. Appl. **75**(1), 112–120 (1980)
10. Buryak, A.V., Di Trapani, P., Skryabin, D.V., Trillo, S.: Optical solitons due to quadratic nonlinearities: from basic physics to futuristic applications. Phys. Rep. **370**(2), 63–235 (2002)
11. Buryak, A.V., Kivshar, Y.S.: Solitons due to second harmonic generation. Phys. Lett. A **197**(5–6), 407–412 (1995)
12. Busca, J., Sirakov, B.: Symmetry results for semilinear elliptic systems in the whole space. J. Differ. Equ. **163**(1), 41–56 (2000)
13. Cazenave, T.: Semilinear Schrödinger Equations, vol. 10 of Courant Lecture Notes in Mathematics. New York University Courant Institute of Mathematical Sciences, New York (2003)
14. Chen, C.-C., Lin, C.-S.: Uniqueness of the ground state solutions of $\Delta u + f(u) = 0$ in $\mathbf{R}^n$, $n \geq 3$. Commun. Partial Differ. Equ. **16**(8–9), 1549–1572 (1991)
15. Chen, Z.-J., Lin, C.-S., Zou, W.-M.: Multiple sign-changing and semi-nodal solutions for coupled Schrödinger equations. J. Differ. Equ. **255**(11), 4289–4311 (2013)
16. Chen, Z.-J., Zou, W.-M.: Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent. Arch. Ration. Mech. Anal. **205**(2), 515–551 (2012)
17. Crandall, M.G., Rabinowitz, P.H.: Bifurcation from simple eigenvalues. J. Funct. Anal. **8**, 321–340 (1971)
18. Dancer, E.N., Wei, J.-C.: Spike solutions in coupled nonlinear Schrödinger equations with attractive interaction. Trans. Am. Math. Soc. **361**(3), 1189–1208 (2009)
19. Dancer, E.N., Wei, J.-C., Weth, T.: A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system. Ann. Inst. H. Poincaré Anal. Non Linéaire **27**(3), 953–969 (2010)
20. de Figueiredo, D.G., Lopes, O.: Solitary waves for some nonlinear Schrödinger systems. Ann. Inst. H. Poincaré Anal. Non Linéaire **25**(1), 149–161 (2008)
21. Ding, W.-Y., Ni, W.-M.: On the existence of positive entire solutions of a semilinear elliptic equation. Arch. Ration. Mech. Anal. **91**(4), 283–308 (1986)
22. Han, J., Huh, H., Seok, J.: Chern-Simons limit of the standing wave solutions for the Schrödinger equations coupled with a neutral scalar field. J. Funct. Anal. **266**(1), 318–342 (2014)
23. Hayashi, N., Ozawa, T., Tanaka, K.: On a system of nonlinear Schrödinger equations with quadratic interaction. Ann. Inst. H. Poincaré Anal. Non Linéaire **30**(4), 661–690 (2013)

24. Kwong, M.-K.: Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbf{R}^n$. Arch. Ration. Mech. Anal. **105**(3), 243–266 (1989)
25. Kwong, M.-K., Zhang, L.-Q.: Uniqueness of the positive solution of $\Delta u + f(u) = 0$ in an annulus. Differ. Integr. Equ. **4**(3), 583–599 (1991)
26. Lieb, E.H.: Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. Stud. Appl. Math. **57**(2), 93–105 (1976/77)
27. Lieb, E.H., Loss, M.: Analysis, vol. 14 of Graduate Studies in Mathematics, 2nd edn. American Mathematical Society, Providence, RI (2001)
28. Lin, T.-C., Wei, J.-C.: Ground state of N coupled nonlinear Schrödinger equations in $\mathbf{R}^n$, $n \leq 3$. Commun. Math. Phys. **255**(3), 629–653 (2005)
29. Lions, P.-L.: The Choquard equation and related questions. Nonlinear Anal. **4**(6), 1063–1072 (1980)
30. Lions, P.-L.: The concentration-compactness principle in the calculus of variations. The locally compact case. I. Ann. Inst. H. Poincaré Anal. Non Linéaire **1**(2), 109–145 (1984)
31. Lions, P.-L.: The concentration-compactness principle in the calculus of variations. The locally compact case. II. Ann. Inst. H. Poincaré Anal. Non Linéaire **1**(4), 223–283 (1984)
32. Liu, P., Shi, J.-P., Wang, Y.-W.: A double saddle-node bifurcation theorem. Commun. Pure Appl. Anal. **12**(6), 2923–2933 (2013)
33. Liu, Z.-L., Wang, Z.-Q.: Multiple bound states of nonlinear Schrödinger systems. Commun. Math. Phys. **282**(3), 721–731 (2008)
34. Lopes, O.: Uniqueness of a symmetric positive solution to an ODE system. Electron. J. Differ. Equ. **8**, 162 (2009)
35. Ma, L., Zhao, L.: Classification of positive solitary solutions of the nonlinear Choquard equation. Arch. Ration. Mech. Anal. **195**(2), 455–467 (2010)
36. Moroz, V., Van Schaftingen, J.: Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics. J. Funct. Anal. **265**(2), 153–184 (2013)
37. Noris, B., Tavares, H., Terracini, S., Verzini, G.: Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition. Commun. Pure Appl. Math. **63**(3), 267–302 (2010)
38. Rabinowitz, P.H.: Nonlinear Sturm-Liouville problems for second order ordinary differential equations. Commun. Pure Appl. Math. **23**, 939–961 (1970)
39. Rabinowitz, P.H.: Some global results for nonlinear eigenvalue problems. J. Funct. Anal. **7**, 487–513 (1971)
40. Rabinowitz, P.H.: On bifurcation from infinity. J. Differ. Equ. **14**, 462–475 (1973)
41. Rabinowitz, P.H.: Minimax methods in critical point theory with applications to differential equations, vol. 65 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI (1986)
42. Ruiz, D.: The Schrödinger-Poisson equation under the effect of a nonlinear local term. J. Funct. Anal. **237**(2), 655–674 (2006)
43. Sato, Y., Wang, Z.-Q.: On the multiple existence of semi-positive solutions for a nonlinear Schrödinger system. Ann. Inst. H. Poincaré Anal. Non Linéaire **30**(1), 1–22 (2013)
44. Shi, J.-P., Wang, X.-F.: On global bifurcation for quasilinear elliptic systems on bounded domains. J. Differ. Equ. **246**(7), 2788–2812 (2009)
45. Sirakov, B.: Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$. Commun. Math. Phys. **271**(1), 199–221 (2007)
46. Strauss, W.A.: Existence of solitary waves in higher dimensions. Commun. Math. Phys. **55**(2), 149–162 (1977)
47. Sulem, C., Sulem, P.-L.: The Nonlinear Schrödinger Equation: Self-focusing and wave collapse, vol. 139 of Applied Mathematical Sciences. Springer-Verlag, New York (1999)
48. Terracini, S., Verzini, G.: Multipulse phases in k-mixtures of Bose-Einstein condensates. Arch. Ration. Mech. Anal. **194**(3), 717–741 (2009)
49. Tian, R.-S., Wang, Z.-Q.: Multiple solitary wave solutions of nonlinear Schrödinger systems. Topol. Methods Nonlinear Anal. **37**(2), 203–223 (2011)
50. Tian, R.-S., Wang, Z.-Q.: Bifurcation results on positive solutions of an indefinite nonlinear elliptic system. Discrete Contin. Dyn. Syst. **33**(1), 335–344 (2013)
51. Wei, J.-C., Weth, T.: Radial solutions and phase separation in a system of two coupled Schrödinger equations. Arch. Ration. Mech. Anal. **190**(1), 83–106 (2008)
52. Wei, J.-C., Yao, W.: Uniqueness of positive solutions to some coupled nonlinear Schrödinger equations. Commun. Pure Appl. Anal. **11**(3), 1003–1011 (2012)
53. Willem, M.: Minimax theorems. Progress in Nonlinear Differential Equations and their Applications, vol. 24. Birkhäuser Boston Inc., Boston (1996)

54. Yew, A.C.: Multipulses of nonlinearly coupled Schrödinger equations. J. Differ. Equ. **173**(1), 92–137 (2001)
55. Yew, A.C., Champneys, A.R., McKenna, P.J.: Multiple solitary waves due to second-harmonic generation in quadratic media. J. Nonlinear Sci. **9**(1), 33–52 (1999)
56. Zhao, L.-G., Zhao, F.-K.: On the existence of solutions for the Schrödinger-Poisson equations. J. Math. Anal. Appl. **346**(1), 155–169 (2008)
57. Zhao, Y.-H., Wang, Y.-W., Shi, J.-P.: Steady states and dynamics of an autocatalytic chemical reaction model with decay. J. Differ. Equ. **253**(2), 533–552 (2012)