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Uniqueness of the positive solution for a non-cooperative model of nuclear reactors

Jun Zhou [a], Junping Shi [b,*]

[a] School of Mathematics and Statistics, Southwest University, Chongqing, 400715, PR China

[b] Department of Mathematics, College of William and Mary, Williamsburg, VA, 23187-8795, USA

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ABSTRACT

We prove the uniqueness of the positive solution for a non-cooperative reaction–diffusion model of nuclear reactors, by converting the system to an equivalent cooperative one.

1. Introduction

In this work we consider the uniqueness of the positive solution for the following non-cooperative reaction–diffusion steady state model of nuclear reactors:

$$\begin{cases} -\Delta u = au - buv, & x \in \Omega, \\ -\Delta v = cu - duv - ev, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded connected domain in $\mathbb{R}^n$ ($n \geq 1$), with a smooth boundary $\partial\Omega$, and the parameters a, b, c, d and e take positive values. A solution (u, v) of (1.1) is called a positive one if both u and v are positive in Ω.

The problem (1.1) has been studied in [1,2,5–7]. Here we recall some of their results concerning the existence and uniqueness of a positive solution (see [1, Theorems 3.1 and 3.2], [5, Theorems 1.1 and 1.2], [6, Theorems 1.3 and 1.4], and [7, Theorems 1.2 and 1.3]):

Theorem 1.1. *Suppose that a, b, c, d, e are positive constants, and let ρ_1 be the principal eigenvalue of $-\Delta$ under a homogeneous Dirichlet boundary condition. Then the following results hold.*

(A) *(See [5].) Problem (1.1) admits a positive solution if and only if $\rho_1 < a < \rho_1 + bc/d$.*

(B) *(See [5,6].) There is an unbounded continuum Σ of positive solutions of (1.1) emanating from $(a, u, v) = (\rho_1, 0, 0)$ such that the projection of Σ to the a-axis is $(\rho_1, \rho_1 + bc/d)$, and for any sequence of solutions (a_n, u_n, v_n) such that $\lim_{n\to\infty} a_n = \rho_1 + bc/d$, we have*

$$\limsup_{n\to\infty} v_n(x) = \frac{c}{d}, \qquad \lim_{n\to\infty} u_n(x) = \infty.$$

Here the limit of u_n as $n \to \infty$ is taken uniformly in compact subsets of Ω.

* Corresponding author. Tel.: +1 757 2212030.
E-mail addresses: jzhouwm@gmail.com (J. Zhou), jxshix@wm.edu (J. Shi).

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(C) *(See [5].) The positive solution of (1.1) is unique when* $n = 1$ *and* $\rho_1 < a < \rho_1 + bc/d$.
(D) *(See [6,7].) The positive solution of (1.1) is unique for any* $n \geq 1$ *if* $\rho_1 < a \leq bc/d - e$ *and* $0 \leq a^2 - (bc/d - e)a + 2e^2$.
(E) *(See [1].) There exists a positive constant* $\varepsilon \in (0, bc/d)$ *such that the positive solution of (1.1) is unique for any* $n \geq 1$ *if* $\rho_1 < a \leq \rho_1 + \varepsilon$.
(F) *(See [1].) Assume that* $bc/d - e > \rho_1$. *Then there exists a positive constant* $\epsilon \in (0, bc/d - e - \rho_1] \cap (0, \rho_1 + e]$ *such that the positive solution of (1.1) is unique for any* $n \geq 1$ *if* $bc/d - e - \epsilon < a < bc/d - e + \epsilon$.

Using the result developed in [4, Theorem 2.3], we continue to study the uniqueness of positive solutions for (1.1). Our new uniqueness result for (1.1) is:

Theorem 1.2. *Suppose that* a, b, c, d, e *are positive constants, and let* ρ_1 *be the principal eigenvalue of* $-\Delta$ *under a homogeneous Dirichlet boundary condition. Then Eq. (1.1) has a unique positive solution for any bounded smooth domain* $\Omega \subset \mathbb{R}^n$ *with any* $n \geq 1$ *if* $bc/d - e < a < \rho_1 + bc/d$. *Furthermore, the unique positive solution is linearly stable.*

Our result improves the uniqueness result given in Theorem 1.1(F) for the higher dimensional domain case as we prove the uniqueness of the positive solution for all possible values of $a > bc/d - e$. The uniqueness results in Theorem 1.1(F) are all for $a \in (\rho_1, bc/d - e)$. Combining our results with Theorem 1.1(D)–(F), the only region of a for which the uniqueness is still not known is an interval $(\rho_1 + \varepsilon, bc/d - e - \epsilon)$. We also remark that for the corresponding Neumann boundary problem of (1.1), it is known that the unique constant steady state solution is globally asymptotically stable [3].

2. Proof of Theorem 1.2

In order to prove the uniqueness of the positive solution to (1.1), we cite a result from [4, Theorem 2.3]:

Lemma 2.1. *Let* Ω *be a bounded smooth domain in* $\mathbb{R}^n$ *for* $n \geq 1$, *and let* f *and* g *be smooth real-valued functions defined on* $[0, +\infty) \times [0, +\infty)$. *Suppose that* (U, V) *is a positive solution of*

$$\begin{cases} \Delta u + f(u, v) = 0, & x \in \Omega, \\ \Delta v + g(u, v) = 0, & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

and for any $x \in \Omega$, *the following relations are satisfied:*

$$\begin{aligned} & f_v(U(x), V(x)) \geq 0, \qquad g_u(U(x), V(x)) \geq 0, \\ & f(U(x), V(x)) \geq f_u(U(x), V(x))U(x) + f_v(U(x), V(x))V(x), \\ & g(U(x), V(x)) \geq g_u(U(x), V(x))U(x) + g_v(U(x), V(x))V(x). \end{aligned} \tag{2.2}$$

If at least one of the last two inequalities in (2.2) is strict, then (U, V) *is linearly stable in the sense that any eigenvalue* μ *of*

$$\begin{cases} -\Delta\xi = f_u(U, V)\xi + f_v(U, V)\eta + \mu\xi, & x \in \Omega, \\ -\Delta\eta = g_u(U, V)\xi + g_v(U, V)\eta + \mu\eta, & x \in \Omega, \\ \xi(x) = \eta(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

has positive real part. In particular, (2.3) has a positive principal eigenvalue μ_1 *with a positive eigenfunction* (ξ_1, η_1).

We remark that the original result in Lemma 2.1 requires the relations in (2.2) to hold for all $(u, v) \in \mathbb{R}^2_+$. But the proof there can be used to prove the version here; hence we omit the proof.

Proof of Theorem 1.2. The existence of a positive solution of (1.1) follows from Theorem 1.1(A), so we only prove the uniqueness. Let (u, v) be a positive solution of (1.1). Set $w = u - bv/d$; then (w, u) satisfies

$$\begin{cases} -\Delta w + ew = \left(a + e - \dfrac{bc}{d}\right)u, & x \in \Omega, \\ -\Delta u = u(a - du + dw), & x \in \Omega, \\ w = u = 0, & x \in \partial\Omega. \end{cases} \tag{2.4}$$

Let $f(w, u) := -ew + (a + e - bc/d)u$ and $g(w, u) := dwu + au - du^2$; then (w, u) satisfies

$$\begin{cases} \Delta w + f(w, u) = 0, & x \in \Omega, \\ \Delta u + g(w, u) = 0, & x \in \Omega, \\ w = u = 0, & x \in \partial\Omega. \end{cases} \tag{2.5}$$

Since u is positive and $a+e-bc/d>0$, then $w>0$ as $(-\Delta+e)^{-1}$ maps a positive function to a positive one. It is easy to calculate that for $u, v>0$,

$$\begin{aligned} &f_u(w,u)=a+e-bc/d>0, \qquad g_w(w,u)=du>0,\\ &f(w,u)-f_w(w,u)w-f_u(w,u)u=0,\\ &g(w,u)-g_w(w,u)w-g_u(w,u)u=du(u-w)=buv>0. \end{aligned}$$

Note that the last inequality does not hold for all $(w,u)\in\mathbb{R}^2_+$, but it holds for a positive solution $(w(x),u(x))$ of (2.5) satisfying $u(x)-w(x)=bv(x)/d>0$. Therefore the relations in (2.2) are satisfied; then any positive solution (w,u) of (2.5) is linearly stable by Lemma 2.1.

On the other hand, the existence of a positive solution (u,v) of (1.1) implies the existence of a positive solution (w,u) of (2.5). Because of the stability of (w,u), for any positive solution (a^0, w^0_a, u^0_a) of (2.5) with $bc/d-e<a_0<\rho_1+bc/d$, there is a unique positive solution (a, w_a, u_a) of (2.5) for a near a^0, and all these solutions are on a smooth curve by the implicit function theorem. This curve can be extended to $a\in(bc/d-e,\rho_1+bc/d)$ as (w_a,u_a) are uniformly bounded for any compact subinterval of $(bc/d-e,\rho_1+bc/d)$ (see [5, Lemma 2.4]). This shows the existence of a curve of positive solutions of (2.5): $\Gamma=\{(a,w_a,u_a): a\in(bc/d-e,\rho_1+bc/d)\}$. It is clear that $w_a\to 0$ as $a\to(bc/d-e)^+$, and u_a must approach a non-negative solution $\hat{u}$ of

$$-\Delta u=u(bc/d-e-du), \quad x\in\Omega,\ u=0,\ x\in\partial\Omega. \tag{2.6}$$

The only non-negative solutions of (2.6) are $u=0$ and a unique positive solution θ. Since we assume that $bc/d-e>\rho_1$, then $\hat{u}\neq 0$ since $a=\rho_1$ is the unique bifurcation point for the trivial solution $(u,v)=(0,0)$ of (1.1). Thus $\hat{u}=\theta$. This also shows that such a curve Γ must be unique by applying the implicit function theorem near $(a,w,u)=(bc/d-e,0,\theta)$. Hence the system (2.5) has a unique positive solution (w_a,u_a) for any $a\in(bc/d-e,\rho_1+bc/d)$. This also implies the uniqueness of the positive solution (u,v) of (1.1), since for a positive solution (u,v) of (1.1), $(w,u)=(u-bv/d,u)$ must be a positive solution of (2.5) from the assumption that $a>bc/d-e$.

For the linear stability, we have proved that any positive solution (w,u) of (2.5) is linearly stable by Lemma 2.1; thus any eigenvalue of the linearized equation

$$\begin{cases} -\Delta\varphi=f_w(w,u)\varphi+f_u(w,u)\psi+\mu\varphi, & x\in\Omega,\\ -\Delta\psi=g_w(w,u)\varphi+g_u(w,u)\psi+\mu\psi, & x\in\Omega,\\ \varphi=\psi=0, & x\in\partial\Omega. \end{cases} \tag{2.7}$$

has a positive real part. The linearized eigenvalue problem of (1.1) is

$$\begin{cases} -\Delta\psi=(a-bv)\psi-bu\phi+\mu\psi, & x\in\Omega,\\ -\Delta\phi=(c-dv)\psi-(du+e)\phi+\mu\phi, & x\in\Omega,\\ \psi=\phi=0, & x\in\partial\Omega, \end{cases} \tag{2.8}$$

and it is easy to see that μ is an eigenvalue of (2.8) if and only if μ is an eigenvalue of (2.7) via the change of variables $\varphi=\psi-(b/d)\phi$. Hence the unique positive solution (u,v) is also linearly stable with respect to (1.1). This completes the proof. □

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