Chapter 5
Absolute Stability and Conditional Stability in General Delayed Differential Equations

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Introduction

Delay differential equations are a class of mathematical models describing various natural and engineered phenomena with delayed feedbacks in the system. Mathematical theory of delay differential equations or functional-differential equations have been developed in the second half of twentieth century to study mathematical questions from models of population biology, biochemical reactions, neural conduction, and other applications [4, 6, 10, 17, 20].

A basic delay differential equation was proposed by renowned biologist George Evelyn Hutchinson in 1948 (see [8]):

$$\frac{du(t)}{dt} = ru(t)\left(1 - u(t-\tau)\right), \tag{1}$$

where $u(t)$ is the population as a function of time t, r is growth rate per capita parameter, and the system carrying capacity is assumed to be rescaled to 1. When $\tau = 0$, the Eq. (1) is reduced to the classical logistic equation, and it is well-known that the equilibrium $u = 1$ is globally asymptotically stable for all positive initial values. On the other hand, when τ is larger, then $u = 1$ becomes unstable, and there exists a periodic orbit of (1) which attracts all positive initial values except $u = 1$. To illustrate the cause of instability, we linearize the Eq. (1) at $u = 1$ to obtain

$$v'(t) = -rv(t-\tau). \tag{2}$$

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If an exponential function $v(t) = \exp(\lambda t)$ is a solution of (2), then the exponent λ satisfies a characteristic equation in form

$$\lambda + re^{-\lambda\tau} = 0. \tag{3}$$

While the exponent λ in the Eq. (3) cannot be explicitly solved, one can observe that $\lambda = 0$ is not a root of (3), and also the root λ of (3) varies continuously with respect to parameters r and τ. Since when $\tau = 0$, the only root of (3) is $\lambda = -r < 0$, then (3) can only have a root with positive real part if $\lambda = \omega i$ is a root of (3) for some (r,τ). Thus one can assume the "neutral stability" condition $\lambda = \omega i$ for some $\omega > 0$ (as $\lambda = -\omega i$ is also a root), which implies

$$\omega i + re^{-\omega\tau i} = 0$$

and

$$\cos(\omega\tau) = 0, \;\; r\sin(\omega\tau) = \omega. \tag{4}$$

Solving (4) we obtain that only when

$$\tau_n = \frac{(2n+1)\pi}{2r}, \;\; n \in \mathbb{N} \cup \{0\}, \tag{5}$$

the neutral stability condition holds with $\omega = r$. This simple example demonstrates that an equilibrium in delay differential equation can lose the stability with a larger delay value $\tau > 0$. In this case, we call the equilibrium $u = 1$ conditionally stable for the delay differential equation (1).

In general, for a delay differential equation with k different delays and variable $x \in \mathbb{R}^n$:

$$\dot{x}(t) = f(x(t), x(t-\tau_1), \cdots, x(t-\tau_k)), \tag{6}$$

A steady state $x = x_*$ of system (6) is said to be *absolutely stable* (i.e., asymptotically stable independent of the delays) if it is locally asymptotically stable for all delays $\tau_j \geq 0$ $(1 \leq j \leq k)$, and $x = x_*$ is said to be *conditionally stable* (i.e., asymptotically stable depending on the delays) if it is locally asymptotically stable for τ_j $(1 \leq j \leq k)$ in some intervals, but not necessarily for all delays (see [13]).

A variation of (1) can demonstrate the absolute stability of an equilibrium. Consider

$$\frac{du}{dt} = ru(t)[1 - au(t) - bu(t-\tau)]. \tag{7}$$

Here a and b represent the portions of instantaneous and delayed dependence of the growth rate on the population, respectively, and we assume that $a, b \in (0,1)$ and $a + b = 1$ (see [14]). Then $u_* = 1$ is an equilibrium. Following [14], we use the same procedure as above, then the linearized equation is now:

$$v'(t) = -arv(t) - brv(t-\tau), \tag{8}$$

and the characteristic equation becomes

$$\lambda + ar + bre^{-\lambda\tau} = 0. \tag{9}$$

By substituting the neutral stability condition $\lambda = \omega i$ into (9) and separating the real and imaginary parts, we obtain

$$\cos(\omega\tau) = -\frac{a}{b}, \;\; \sin(\omega\tau) = \frac{\omega}{br}. \tag{10}$$

If $a < b$, then one can find that the neutral stability condition $\lambda = \omega i$ can be achieved when $\tau = \tau_n$ as defined by

$$\tau_n = \frac{1}{r\sqrt{b^2 - a^2}} \left(\arccos\left(-\frac{a}{b}\right) + 2n\pi \right), \tag{11}$$

with

$$\omega = r\sqrt{b^2 - a^2}.$$

In this case, similar to (1), the equilibrium $u_* = 1$ is conditionally stable. However, if $a \geq b$, then the neutral stability condition cannot be achieved for any $\tau > 0$; hence it is absolutely stable, that is, the equilibrium $u_* = 1$ is locally asymptotically stable for any $\tau \geq 0$. Indeed one can prove that $u_* = 1$ is globally asymptotically stable by using a Lyapunov function argument (see [9, 11, 14]).

Biologically the phenomenon described above has the following meaning: if the instantaneous feedback of the population dominates the delayed feedback, then the system has a globally asymptotically stable equilibrium; but if the delayed feedback is more dominant, then the equilibrium is conditionally stable, and it loses the stability for a larger value of delay. It is the aim of this notes to show that this phenomenon occurs for a wider class of delayed differential equations, including some systems from biology or physics. Some recent results by the author and his collaborators in this direction will be reviewed in section "Main Results", while the proof of these results can be found in references given below. In section "Concluding Remarks" some concluding remarks and open questions will be given.

Main Results

Scalar Equations

First we state a result for scalar equation which generalizes the example of instantaneous and delayed feedback given in the Introduction. Consider a general delayed differential equation:

$$\frac{du}{dt} = f(u(t), u(t-\tau)). \tag{12}$$

Here $f = f(u, w)$ is a smooth function, and we assume that $u = u_*$ is an equilibrium. Then the linearization of (12) at $u = u_*$ is

$$v'(t) = f_u(u_*,u_*)v(t) + f_w(u_*,u_*)v(t-\tau), \tag{13}$$

where $f_u(u_*,u_*)$ and $f_w(u_*,u_*)$ are the partial derivatives of f with respect to the variables u and w, respectively. In the following when there is no confusion, we will simply write f_u and f_w, with the understanding of evaluation at (u_*,u_*). Then the corresponding characteristic equation is

$$\lambda - f_u - f_w e^{-\lambda\tau} = 0. \tag{14}$$

We assume that when $\tau = 0$, the equilibrium $u = u_*$ is stable; hence the following condition is satisfied:

$$f_u(u_*,u_*) + f_w(u_*,u_*) < 0. \tag{15}$$

Substituting the neutral stability condition $\lambda = \omega i$ into (14), we get

$$\cos(\omega\tau) = -\frac{f_u}{f_w}, \;\; \sin(\omega\tau) = -\frac{\omega}{f_w}. \tag{16}$$

Squaring each equation in (16) and taking the sum, we obtain

$$\omega^2 = f_w^2 - f_u^2. \tag{17}$$

By using the well-known stability result, we obtain the following general criterion.

Theorem 1. *Suppose that $u = u_*$ is an equilibrium of (12), and (15) is satisfied.*

1. *If $|f_u(u_*,u_*)| \geq |f_w(u_*,u_*)|$ (or equivalently $f_u(u_*,u_*) \leq f_w(u_*,u_*)$), then the neutral stability condition cannot be achieved for any $\tau \geq 0$. Hence u_* is absolutely stable.*
2. *If $|f_u(u_*,u_*)| < |f_w(u_*,u_*)|$ (or equivalently $f_u(u_*,u_*) > f_w(u_*,u_*)$), then $u = u_*$ is locally asymptotically stable when $0 \leq \tau < \tau_0$, and it is unstable when $\tau > \tau_0$, where*

$$\tau_0 = \frac{1}{\sqrt{f_w^2(u_*,u_*) - f_u^2(u_*,u_*)}} \arccos\left(-\frac{f_u(u_*,u_*)}{f_w(u_*,u_*)}\right). \tag{18}$$

Moreover the characteristic equation (14) has a pair of purely imaginary root $\lambda = \pm\omega i$ for $\omega > 0$ if and only if $|f_u(u_*,u_*)| < |f_w(u_*,u_*)|$, $\tau = \tau_n$ which is defined by

$$\tau_n = \frac{1}{\sqrt{f_w^2(u_*,u_*) - f_u^2(u_*,u_*)}} \arccos\left(-\frac{f_u(u_*,u_*)}{f_w(u_*,u_*)} + 2n\pi\right) \tag{19}$$

for $n \in \mathbb{N} \cup \{0\}$, and

$$\omega = \sqrt{f_w^2(u_*,u_*) - f_u^2(u_*,u_*)}. \tag{20}$$

An obvious example of Theorem 1 is the instantaneous and delayed feedback given in the Introduction in which $f_u < 0$ and $f_w < 0$. Note that Theorem 1 can also be applied to the case (i) $f_u > 0$ and $f_w < 0$ (conditionally stable), and (ii) $f_u < 0$ and $f_w > 0$ (absolutely stable).

Planar Systems with One Transcendental Term

It is common that in a system of differential equations, there are delayed feedbacks on one of the variables. A general form of such equations can be written as

$$\begin{cases} u_t = f(u,v,u_\tau), & t>0, \\ v_t = g(u,v,u_\tau), & t>0, \\ u(t) = \phi_1(t), & t \in [-\tau, 0], \\ v(0) = \phi_2, \end{cases} \tag{21}$$

where $u = u(t)$, $v = v(t)$, and $u_\tau = u(t-\tau)$. The functions $f(u,v,w)$ and $g(u,v,w)$ are continuously differentiable in $\mathbb{R}^3$. We assume that there exist $u^*, v^* \in \mathbb{R}$ such that

$$f(u^*,v^*,u^*) = 0, \quad g(u^*,v^*,u^*) = 0.$$

Then (u^*,v^*) is a constant equilibrium of system (21). Linearizing system (21) at (u^*,v^*), we obtain

$$\frac{d}{dt}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix} + \begin{pmatrix} f_w\phi(t-\tau) \\ g_w\phi(t-\tau)\end{pmatrix}. \tag{22}$$

And the characteristic equation can be derived from

$$Det\begin{pmatrix} \lambda - f_u - f_w e^{-\lambda\tau} & -f_v \\ -g_u - g_w e^{-\lambda\tau} & \lambda - g_v \end{pmatrix} = 0, \tag{23}$$

and it is in a form

$$\lambda^2 + a\lambda + b + (c\lambda + d)e^{-\lambda\tau} = 0, \tag{24}$$

where

$$a = -(f_u + g_v), \;\; b = f_u g_v - f_v g_u, \;\; c = -f_w, \;\; and \;\; d = f_w g_v - f_v g_w. \tag{25}$$

Note that if we define

$$L_1 = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}, \quad L_2 = \begin{pmatrix} f_w & 0 \\ g_w & 0 \end{pmatrix}, \tag{26}$$

then

$$\begin{aligned} &a = -Tr(L_1), \;\; b = Det(L_1), \;\; c = -Tr(L_2), \\ &b + d = Det(L_1 + L_2), \;\; b - d = Det(L_1 - L_2). \end{aligned} \tag{27}$$

Similar to before, we substitute the neutral stability condition $\lambda = \omega i$ into (24), then we obtain

$$-\omega^2 + a\omega i + b + (c\omega i + d)e^{-i\omega\tau} = 0, \tag{28}$$

or equivalently

$$
\begin{aligned}
-d\cos(\omega\tau)+c\omega\sin(\omega\tau)&=b-\omega^2,\\
-c\omega\cos(\omega\tau)-d\sin(\omega\tau)&=a\omega.
\end{aligned}
\tag{29}
$$

Squaring each equation in (29) and taking the sum, we obtain an equation of ω^2 in the form

$$
\omega^4-(c^2-a^2+2b)\omega^2+(b^2-d^2)=0. \tag{30}
$$

The existence of a positive root ω^2 to (30) determines the stability of equilibrium (u_*,v_*). Again, we assume that the equilibrium (u_*,v_*) is stable when $\tau=0$; hence the following condition is satisfied (partial derivatives are evaluated at (u_*,v_*,u_*)):

$$
\begin{aligned}
a+c&=-Tr(L_1+L_2)=-(f_u+f_w+g_v)>0,\\
b+d&=Det(L_1+L_2)=(f_u+f_w)g_v-(g_u+g_w)f_v>0.
\end{aligned}
\tag{31}
$$

We first state a result for the characteristic equation (24), which was proved in Ruan [13] (see also references therein for earlier results).

Theorem 2. *Suppose that $a,b,c,d\in\mathbb{R}$ satisfy*

$$
a+c>0,\;\; b+d>0. \tag{32}
$$

1. *If (i) $c^2-a^2+2b<0$ and $b-d>0$ or (ii) $(c^2-a^2+2b)^2-4(b^2-d^2)<0$ is satisfied, then all roots of (24) have negative real parts for any $\tau\ge 0$.*
2. *If (iii) $b-d<0$ or (iv) $c^2-a^2+2b>0$, $b-d>0$, and $(c^2-a^2+2b)^2-4(b^2-d^2)\ge 0$ are satisfied, then (24) has purely imaginary roots $\pm\omega i$ if and only if (30) has a positive root ω_+ or ω_- where*

$$
\omega=\omega_\pm=\sqrt{\frac{c^2-a^2+2b\pm\sqrt{(c^2-a^2+2b)^2-4(b^2-d^2)}}{2}}, \tag{33}
$$

and for $n\in\mathbb{N}\cup\{0\}$,

$$
\tau=\tau_n=\frac{1}{\omega_\pm}\left(\arccos\left(\frac{(d-ac)\omega_\pm^2-bd}{d^2+c^2\omega_\pm^2}\right)+2n\pi\right). \tag{34}
$$

Applying Theorem 2 to the stability of equilibrium of (21), we have

Theorem 3. *Suppose that $(u,v)=(u_*,v_*)$ is an equilibrium of (21), and (31) is satisfied.*

1. *If the matrices L_1 and L_2 satisfy either*

   (i) $Det(L_1)+Tr(L_1+L_2)Tr(L_1-L_2)<0$ and $Det(L_1-L_2)>0$ or
   (ii) $[Det(L_1)+Tr(L_1+L_2)Tr(L_1-L_2)]^2-4Det(L_1+L_2)Det(L_1+L_2)<0$,

   then the neutral stability condition cannot be achieved for any $\tau\ge 0$. Hence (u_*,v_*) is absolutely stable.

2. If the matrices L_1 and L_2 satisfy either

(iii) $Det(L_1 - L_2) < 0$ *or*
(iv) $Det(L_1 - L_2) > 0$ *and*
$Det(L_1) + Tr(L_1 + L_2)Tr(L_1 - L_2) > 2\sqrt{Det(L_1 + L_2)Det(L_1 + L_2)},$

then there exists $\tau_0 > 0$ *such that* (u_*, v_*) *is locally asymptotically stable when* $0 \le \tau < \tau_0$*, and it is unstable when* $\tau > \tau_0$.

In the second case of Theorem 3, the critical value τ_0 and $(\omega_\pm, \tau_n)$ can all be calculated from the formulas in Theorem 2, and we omit the long formulas due to their long expression. We remark that results in Theorem 3 again demonstrate the phenomenon that if the instantaneous feedback dominates the delayed one, then the equilibrium is absolutely stable; but if the delayed feedback is more dominant, then it is conditionally stable. This is best seen in the scenario (iii) in Theorem 3 as $Det(L_1 - L_2)$ provides a measure of the difference of the two feedbacks.

There are numerous examples from applications in which the above absolute or conditional stability can be determined. Here we show two examples. First one is a Rosenzweig–MacArthur predator–prey model with a delay effect (see [2]):

$$\begin{cases} u'(t) = u(t)\left(1 - \dfrac{u(t)}{k}\right) - \dfrac{mu(t)v(t)}{u(t)+1}, & t > 0, \\ v'(t) = -rv(t) + \dfrac{mu(t-\tau)v(t)}{u(t-\tau)+1}, & t > 0, \\ u(t) = u_0(t) \ge 0, v(t) = v_0(t) \ge 0, & t \in [-\tau, 0], \end{cases} \tag{35}$$

From well-known results (see, e.g., [21]), (35) has a unique positive equilibrium (β, v_β) where $\beta = \dfrac{r}{m-r}$ and $v_\beta = \dfrac{(K-\beta)(1+\beta)}{Km}$. To consider the stability of (β, v_β), we find that

$$L_1 = \begin{pmatrix} \dfrac{\beta(k-1-2\beta)}{k(1+\beta)} & -r \\ 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 \\ \dfrac{(k-\beta)}{k(\beta+1)} & 0 \end{pmatrix}. \tag{36}$$

Hence the characteristic equation is in form (24) with

$$a = -\frac{\beta(k-1-2\beta)}{k(1+\beta)}, \quad b = c = 0, \quad d = \frac{r(k-\beta)}{k(\beta+1)}. \tag{37}$$

Then for $(k-1)/2 < \beta < k$, $a + c > 0$ and $b + d > 0$; hence (31) is satisfied; it is also obvious that $b < d$. Therefore case (iii) in Theorem 3 occurs, and the coexistence equilibrium (β, v_β) is conditionally stable.

The second example is a Leslie–Gower predator–prey system with delay effect [3].

$$\begin{cases} u'(t) = u(t)(p - \alpha u(t) - \beta v(t-\tau_1)), & t > 0, \\ v'(t) = \mu v(t)\left(1 - \dfrac{v(t)}{u(t-\tau_2)}\right), & t > 0, \\ u(t) = u_0(t) \geq 0, & t \in [-\tau_2, 0]. \\ v(t) = v_0(t) \geq 0, & t \in [-\tau_1, 0]. \end{cases} \tag{38}$$

A unique positive equilibrium of (38) is $(u_*, v_*) = \left(\dfrac{p}{\alpha+\beta}, \dfrac{p}{\alpha+\beta}\right)$. Note that (38) is not in the form of (21), but the characteristic equation is still (24) with

$$a = \frac{\alpha p}{\alpha+\beta} + \mu, \;\; b = \frac{\mu\alpha p}{\alpha+\beta}, \;\; c = 0, \;\; d = \frac{\mu\beta p}{\alpha+\beta}, \quad \text{and} \quad \tau = \tau_1 + \tau_2. \tag{39}$$

Thus $a + c > 0$ and $b + d > 0$; hence (31) is satisfied. If $\alpha > \beta$, then

$$\begin{aligned} b - d &= \frac{\mu(\alpha-\beta)p}{\alpha+\beta} > 0, \\ c^2 - a^2 + 2b &= -\left(\frac{\alpha p}{\alpha+\beta}\right)^2 - \mu^2 < 0. \end{aligned} \tag{40}$$

Thus case (i) in Theorem 2 is applicable, and (u_*, v_*) is absolutely stable. Indeed, in [3], it is proved that (u_*, v_*) is globally asymptotically stable for any $\tau_1 \geq 0, \; \tau_2 \geq 0$. On the other hand, if $\alpha < \beta$, then $b - d < 0$, then again case (iii) in Theorem 2 is applicable. Hence there exists $\tau_0 > 0$ such that (u_*, v_*) is stable for $\tau_1 + \tau_2 < \tau_0$, and it is unstable for $\tau_1 + \tau_2 > \tau_0$.

We remark that for many planar systems not in the form of (21), one can still use Theorem 2 to consider the stability of equilibrium in such systems, as long as the characteristic equation is still (24). For example, planar systems in form of

$$\begin{cases} \dot{u}(t) = f(u(t), v(t-\tau_1)), \\ \dot{v}(t) = g(u(t-\tau_2), v(t)), \end{cases} \tag{41}$$

and planar systems in form of

$$\begin{cases} \dot{u}(t) = f(u(t), u(t)) \pm k_1 g(u(t-\tau), v(t-\tau)), \\ \dot{v}(t) = h(u(t), v(t)) \pm k_2 g(u(t-\tau), v(t-\tau)). \end{cases} \tag{42}$$

Notice that Eq. (41) includes the case of Kolmogorov-type predator–prey systems with two delays [13, 15], and Eq. (42) includes the cases of competitive, mutualistic, and predator–prey models with symmetric delayed interaction terms. Yet another example is the second-order delayed feedback system in form

$$u'' + au' + bu = F(u(t-\tau)). \tag{43}$$

A discussion of this system by comparing the delayed feedback and the instantaneous one is given in Smith [17, Sect. 6.4].

General Planar Systems with One Delay

For a general planar system

$$\begin{cases} \dot{x}(t) = f(x(t), y(t), x(t-\tau), y(t-\tau)), \\ \dot{y}(t) = g(x(t), y(t), x(t-\tau), y(t-\tau)), \end{cases} \tag{44}$$

the corresponding characteristic equation is in a form

$$\lambda^2 + a\lambda + b + (c\lambda + d)e^{-\lambda\tau} + he^{-2\lambda\tau} = 0. \tag{45}$$

Here $\tau > 0$ and $a, b, c, d, h \in \mathbb{R}$. Notice that (45) has an additional transcendental term $he^{-2\lambda\tau}$ compared with (24). The characteristic equation (45) was considered recently in [1]. Here we will briefly describe the results in [1] and refer all the details and proofs to [1].

If $\pm i\omega$, $(\omega > 0)$, is a pair of roots of (45), then we have

$$-\omega^2 + a\omega i + b + (c\omega i + d)e^{-i\omega\tau} + he^{-2i\omega\tau} = 0. \tag{46}$$

If $\frac{\omega\tau}{2} \neq \frac{\pi}{2} + j\pi$, $j \in \mathbb{Z}$, then let $\theta = \tan\frac{\omega\tau}{2}$, and we have $e^{-i\omega\tau} = \frac{1-i\theta}{1+i\theta}$. Separating the real and imaginary parts, we obtain that θ satisfies

$$\begin{cases} (\omega^2 - b + d - h)\theta^2 - 2a\omega\theta = \omega^2 - b - d - h, \\ (c\omega - a\omega)\theta^2 + (-2\omega^2 + 2b - 2h)\theta = -(a\omega + c\omega). \end{cases} \tag{47}$$

Denote

$$M_1 = \begin{pmatrix} \omega^2 - b + d - h & -2a\omega \\ (c-a)\omega & -2\omega^2 + 2b - 2h \end{pmatrix},$$

$$M_2 = \begin{pmatrix} \omega^2 - b - d - h & -2a\omega \\ -(c+a)\omega & -2\omega^2 + 2b - 2h \end{pmatrix},$$

and

$$M_3 = \begin{pmatrix} \omega^2 - b + d - h & \omega^2 - b - d - h \\ (c-a)\omega & -(c+a)\omega \end{pmatrix}.$$

And define

$$D(\omega) = det(M_1), \; E(\omega) = det(M_2), \quad \text{and} \quad F(\omega) = det(M_3). \tag{48}$$

If $D(\omega) \neq 0$, then we can solve from (47) that

$$\theta^2 = \frac{E(\omega)}{D(\omega)}, \quad \theta = \frac{F(\omega)}{D(\omega)}, \tag{49}$$

and from Eq. (49), we have that ω satisfies

$$D(\omega)E(\omega) = F(\omega)^2, \tag{50}$$

which is a polynomial equation for ω with degree 8:

$$\omega^8 + s_1\omega^6 + s_2\omega^4 + s_3\omega^2 + s_4 = 0, \tag{51}$$

where

$$\begin{aligned} s_1 &= 2a^2 - 4b - c^2, \\ s_2 &= 6b^2 - 2h^2 - 4ba^2 - d^2 + a^4 - a^2c^2 + 2c^2b + 2hc^2, \\ s_3 &= 2d^2b - a^2d^2 - 4b^3 + 2b^2a^2 - c^2b^2 - 2bc^2h \\ &\quad + 4acdh - 2d^2h + 4bh^2 - 2h^2a^2 - c^2h^2, \\ s_4 &= b^4 - d^2b^2 - 2b^2h^2 + 2bd^2h - d^2h^2 + h^4 = (b-h)^2[-d^2 + (b+h)^2], \end{aligned} \tag{52}$$

and ω^2 is a positive root of

$$z^4 + s_1z^3 + s_2z^2 + s_3z + s_4 = 0. \tag{53}$$

The following lemma gives the algorithm of solving the critical delay values for purely imaginary roots of (45).

Lemma 4. *If (53) has a positive root ω_N^2 ($\omega_N > 0$) and $D(\omega_N) \neq 0$, then Eq. (47) has a unique real root $\theta_N = \dfrac{F(\omega_N)}{D(\omega_N)}$ when $\omega = \omega_N$. Hence Eq. (45) has a pair of purely imaginary roots $\pm i\omega_N$ when*

$$\tau = \tau_N^j = \frac{2\arctan\theta_N + 2j\pi}{\omega_N}, \quad j \in \mathbb{Z}. \tag{54}$$

The nondegeneracy condition $D(\omega_N) \neq 0$ can be verified in certain situations, and the tranversality condition for the roots moving across the imaginary axis can also be formulated for (45). The analysis of the quartic polynomial (53) is more complicated than (30), and a complete solution would be cumbersome to present. In [1], several different ways of solving the characteristic equation were presented. Here we only state one of them:

Theorem 5. *Suppose that $a, b, c, d, h \in \mathbb{R}$ satisfy*

(i) $c \neq 0$ and $h \neq 0$.
(ii) $b \neq h$ and $d^2 > (b+h)^2$.
(iii) $b + h \leq \dfrac{ad}{c}$ or $\left(\dfrac{d}{c}\left(2h - \dfrac{ad}{c}\right) - a\left(b + h - \dfrac{ad}{c}\right)\right) \cdot (a - c) \neq 0$.

Recall that $D(\omega)$ and $F(\omega)$ are defined as in (48). Then

1. *The quartic equation (53) has a positive root ω_N^2 for some $\omega_N > 0$ satisfying $D(\omega_N) \neq 0$.*
2. *Let*

$$\theta_N = \frac{F(\omega_N)}{D(\omega_N)} \text{ and } \tau = \tau_N^j = \frac{2\arctan\theta_N + 2j\pi}{\omega_N},$$

where $j \in \mathbb{Z}$. Then the characteristic equation (45) has a pair of purely imaginary eigenvalues $\pm i\omega_N$ when $\tau = \tau_N^j$.

Moreover if $a,b,c,d,h \in \mathbb{R}$ also satisfy

(iv) $a+c > 0$ and $b+d+h > 0$,

then there exists $\tau_* > 0$ such that when $\tau \in [0, \tau_*)$, all the roots of Eq. (45) have negative real parts; if a nondegeneracy condition holds, then when $\tau = \tau_*$, all the roots of Eq. (45) have nonpositive real parts, but Eq. (45) has at least one pair of simple purely imaginary roots $\pm i\omega_0$, and for $\tau \in (\tau_*, \tau_* + \varepsilon)$ with some small $\varepsilon > 0$, Eq. (45) has at least one pair of conjugate complex roots with positive real parts.

We refer to [1] for the detail of the nondegeneracy condition. We remark that all the conditions (i), (ii), and (iii) except $d^2 > (b+h)^2$ hold for all parameter values except a zero measure set. Combining with the condition (iv), we have the following observation for the appearance of roots of Eq. (45) with positive real parts for $\tau > 0$.

Corollary 6. *Define a subset in the parameter space*

$$P = \{(a,b,c,d,h) \in \mathbb{R}^5 : a+c > 0,\ b+d+h > 0,\ b-d+h < 0\}. \tag{55}$$

Then for almost every $(a,b,c,d,h) \in P$, there exists $\tau_* > 0$ such that when $\tau \in [0, \tau_*)$, all the roots of Eq. (45) have negative real parts; when $\tau = \tau_*$, Eq. (45) has at least one pair of simple purely imaginary roots $\pm i\omega_*$, and for $\tau \in (\tau_*, \tau_* + \varepsilon)$ with some small $\varepsilon > 0$, Eq. (45) has at least one pair of conjugate complex roots with positive real parts.

Now we apply these results to (44). We assume that the functions $f(u,v,w,z)$ and $g(u,v,w,z)$ are continuously differentiable in $\mathbb{R}^4$, and there exist $u^*, v^* \in \mathbb{R}$ such that

$$f(u^*,v^*,u^*,v^*) = 0, \quad g(u^*,v^*,u^*,v^*) = 0.$$

The linearized equation is

$$\frac{d}{dt}\begin{pmatrix}\phi(t)\\ \psi(t)\end{pmatrix} = L_1\begin{pmatrix}\phi(t)\\ \psi(t)\end{pmatrix} + L_2\begin{pmatrix}\phi(t-\tau)\\ \psi(t-\tau)\end{pmatrix}, \tag{56}$$

where

$$L_1 = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}, \quad L_2 = \begin{pmatrix} f_w & f_z \\ g_w & g_z \end{pmatrix}, \tag{57}$$

Then the characteristic equation at (u_*, v_*) is

$$Det \begin{pmatrix} \lambda - f_u - f_w e^{-\lambda\tau} & -f_v - f_z e^{-\lambda\tau} \\ -g_u - g_w e^{-\lambda\tau} & \lambda - g_v - g_z e^{-\lambda\tau} \end{pmatrix} = 0, \tag{58}$$

which becomes (45) with

$$\begin{aligned} a =& - (f_u + g_v),\ \ b = f_u g_v - f_v g_u,\ \ c = -(f_w + g_z), \\ d =& (f_u g_z - f_z g_u) + (f_w g_v - f_v g_w),\ \ h = f_w g_z - f_z g_w, \end{aligned} \tag{59}$$

or equivalently

$$\begin{aligned} a &= -Tr(L_1),\ \ b = Det(L_1),\ \ c = -Tr(L_2), \\ d &= \frac{1}{2}\left[Det(L_1 + L_2) - Det(L_1 - L_2)\right],\ \ h = Det(L_2). \end{aligned} \tag{60}$$

Then we can state a general delay-induced instability result based on Theorem 5:

Theorem 7. *Suppose that $f, g \in C^1(\mathbb{R}^4)$, and (u^*, v^*) is an equilibrium of (44). Let L_1 and L_2 be the Jacobian matrices defined as in (57). Assume that*

$$Tr(L_2) \neq 0,\ \ Tr(L_2) \neq Tr(L_1),\ \ Det(L_2) \neq 0,\ \ Det(L_2) \neq Det(L_1), \tag{61}$$

and for a, b, c, d, h defined in (59), we have

$$b + h \le \frac{ad}{c} \quad or \quad \frac{d}{c}\left(2h - \frac{ad}{c}\right) - a\left(b + h - \frac{ad}{c}\right) \neq 0. \tag{62}$$

If L_1 and L_2 satisfy

$$Tr(L_1 + L_2) < 0,\ \ Det(L_1 + L_2) > 0, \ \ and \ \ Det(L_1 - L_2) < 0, \tag{63}$$

then there exists $\tau_0 > 0$, the equilibrium (u^*, v^*) is stable for (44) when $0 \le \tau < \tau_0$, but it is unstable when $\tau \in (\tau_0, \tau_0 + \varepsilon)$ for $\varepsilon > 0$ and small.

Similarly Corollary 6 implies the following observation:

Corollary 8. *Suppose that f, g, (u^*, v^*), L_1 and L_2 are same as in Theorem 7. Let $M_{2\times 2}$ be the set of all real-valued 2×2 matrices, and let $\mathscr{M}_1$ be a subset of $(M_{2\times 2})^2$ consisting of all matrix pairs (L_1, L_2) satisfying (63). Then for almost every $(L_1, L_2) \in \mathscr{M}_1$, the conclusions in Theorem 7 hold.*

We remark that the results in [1] are mainly about under what conditions, conditional stability is achieved. Only in a very special case, we find a condition for absolute stability. Hence more general condition on a, b, c, d, h for absolute stability is still largely open.

We apply the result above to another Leslie–Gower predator–prey system with delays:

$$\begin{cases} u'(t) = u(t)(p - \alpha u(t) - \beta v(t-\tau)), & t > 0, \\ v'(t) = \mu v(t)\left(1 - \dfrac{v(t-\tau)}{u(t-\tau)}\right), & t > 0, \end{cases} \tag{64}$$

where p, α, β, and μ are positive parameters, and $\tau \geq 0$ is the delay. System (64) has a unique positive equilibrium

$$(u^*, v^*) = \left(\frac{p}{\alpha+\beta}, \frac{p}{\alpha+\beta}\right), \tag{65}$$

and the Jacobian matrices at (u^*, v^*) are

$$L_1 = \begin{pmatrix} -\dfrac{\alpha p}{\alpha+\beta} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad L_2 = \begin{pmatrix} 0 & -\dfrac{\beta p}{\alpha+\beta} \\ \mu & -\mu \end{pmatrix}.$$

Hence the characteristic equation of system (64) is in the same form as (45) with

$$a = \frac{\alpha p}{\alpha+\beta}, \quad b = 0, \quad c = \mu, \quad d = \frac{\mu \alpha p}{\alpha+\beta}, \text{ and } \quad h = \frac{\mu \beta p}{\alpha+\beta}. \tag{66}$$

Since $a+c>0$ and $b+d+h>0$ hold for any parameter $\alpha, \beta, p, \mu > 0$, then (u^*, v^*) is always locally asymptotically stable when $\tau = 0$. If $\alpha > \beta$, then $b-d+h<0$, and one can apply Theorem 7 to show that there exists a $\tau_0 > 0$, such that (u_*, v_*) is locally asymptotically stable when $0 \leq \tau < \tau_0$, and it is unstable when $\tau \in (\tau_*, \tau_* + \varepsilon)$ for small $\varepsilon > 0$.

Concluding Remarks

For the simplicity of presentation, we only state our results for delayed differential equations without spatial variables. The results in section "Main Results" also hold for the stability of a constant equilibrium of reaction-diffusion systems with Neumann boundary condition; see details in [1] and also [2, 3] for the examples in Section "Planar Systems with One Transcendental Term". For reaction-diffusion systems, the interaction between diffusion and delay can also produce more complex spatiotemporal pattern formation; see [1, 5, 16]. On the other hand, for the reaction-diffusion equation with Dirichlet boundary condition, the positive equilibrium is spatially nonhomogenous, and the corresponding characteristic equation is also nonhomogenous. Thus the stability analysis for Dirichlet boundary PDE models is much more involved. For the instantaneous and delayed feedback model (7), the corresponding PDE model is

$$\begin{cases} u_t(x,t) = d\Delta_x u(x,t) + ru(x,t)(1 - au(x,t) - bu(x,t-\tau)), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{67}$$

It is known that when $a \geq b$ and $r > d\lambda_1$, then the unique positive equilibrium $u_r(x)$ is globally asymptotically stable for any $\tau \geq 0$ (see [7, 12]); on the other hand, when $a < b$, and assume $r > d\lambda_1$ but $r - d\lambda_1$ is small, then there is a $\tau_0(r) > 0$ satisfying $\lim_{r \to d\lambda_1} (r - d\lambda_1)\tau_0(r) = \frac{1}{r\sqrt{b^2 - a^2}} \arccos\left(-\frac{a}{b}\right)$ such that the unique positive equilibrium $u_r(x)$ is stable when $\tau < \tau_0(r)$, and it is unstable when $\tau > \tau_0(r)$ (see [18]).

For a planar system with two variables and two equations, we have shown here that a general stability/instability criterion can be formulated in terms of Jacobian matrices at the equilibrium point. This delay-induced instability can be compared to the Turing's diffusion-induced instability for planar reaction-diffusion systems [19]. See [1] for more in that direction. It would be interesting to extend such notion for systems with three or more variables. Another interesting question is to prove the stability or instability for distributed delay instead of discrete delays.

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