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Results in Mathematics

Existence and Multiplicity of Positive Solutions to a Quasilinear Elliptic Equation with Strong Allee Effect Growth Rate

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Abstract. In this paper we consider a p-Laplacian equation with strong Allee effect growth rate and Dirichlet boundary condition

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda f(x,u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \qquad (P_\lambda)$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ for $N \geq 1$, $p > 1$, and λ is a positive parameter. By using variational methods and a suitable truncation technique, we prove that problem (P_λ) has at least two positive solutions for large parameter and it has no positive solutions for small parameter. In addition, a nonexistence result is investigated.

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Keywords. p-Laplacain, Allee effect, positive solutions, multiplicity.

1. Introduction

Consider a boundary value problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda f(x,u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \qquad (P_\lambda)$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ for $N \geq 1$, $p > 1$, and λ is a positive parameter.

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The existence, nonexistence and/or multiplicity of positive solutions to problem (P_λ) have been studied extensively in the literature; see, for example, [1,2,5,6,8,9,11–15,17–25] and the references therein. In many previous studies, the nonlinear function $f(x,s)/s^{p-1}$ was assumed to be nonincreasing in s, and under this condition, it can be shown that problem (P_λ) has at most one positive solution (see, e.g., [1,6,8,9,22]). On the other hand, the uniqueness of positive solution no longer holds if $f(x,s)/s^{p-1}$ is not nonincreasing in s. In [12,24,25], it was shown that problem (P_λ) has at least two positive solutions for sufficiently large λ if the nonlinearity $f(x,s)$ is homogeneous, i.e., $f(x,s) = f(s)$, and it satisfies that $f(0) = f(\alpha) = 0$ for some $\alpha > 0$, $\lim_{s\to 0+} f(s)/s^{p-1} = 0$, $f > 0$ in $(0,\alpha)$ and $f < 0$ in (α,∞). In [14], it was shown that problem (P_λ) has at least two positive solutions for sufficiently large λ if $p > 2$ and homogeneous nonlinearity $f(x,u) = f(u)$ satisfies that $f(0) = 0$, $\lim_{s\to 0+} f(s)/s^{p-1} = -m < 0$, and there are precisely two numbers $0 < \rho_1 < \rho_2$ such that $f(\rho_1) = f(\rho_2) = 0$, $f < 0$ in $(0,\rho_1)$, $f > 0$ in (ρ_1,ρ_2) and $\int_0^{\rho_2} f(s)ds > 0$.

Throughout this paper, we assume that $f(x,u)$ satisfies

($f1$) $f \in C(\overline{\Omega} \times [0,\infty), \mathbb{R})$;

($f2$) There exist $b(x), c(x) \in C(\overline{\Omega})$ such that $0 < b(x) < c(x) \le M$ for some constant $M > 0$, and $f(x,0) = f(x,b(x)) = f(x,c(x)) = 0$ for all $x \in \overline{\Omega}$;

($f3$) For all $x \in \overline{\Omega}$, $f(x,s) < 0$ for any $s \in (0,b(x)) \cup (c(x),\infty)$, and $f(x,s) > 0$ for any $s \in (b(x),c(x))$;

($f4$) There exists $N_1 > 0$ such that $f(x,s) \ge -N_1 s^{p-1}$ for all $x \in \overline{\Omega}$ and $0 \le s \le M$;

($f5$) There exists an open ball B_1 of Ω such that $c(x) \in C^1(\overline{B}_1)$ and

$$F(x,c(x)) > 0, \quad x \in B_1,$$

where

$$F(x,s) = \int_0^s f(x,\tau)d\tau \quad \text{for } (x,s) \in \Omega \times [0,\infty).$$

Using variational methods and a suitable truncation technique, we show that problem (P_λ) has at least two positive solutions for sufficiently large λ and problem (P_λ) has no positive solutions for small λ when inhomogeneous nonlinearity $f(x,u)$ satisfies ($f1$)-($f5$). Our result extends the result of [17] for $p = 2$ to $p > 1$, and it also extends the result of [14] for $p > 2$ and homogeneous nonlinearity to $p > 1$ and inhomogeneous nonlinearity.

2. Preliminaries

In this section we will establish some basic setups and preliminary results concerning the p-Laplacian problems (see, e.g., [7]). Consider a boundary value problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + g(x,u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{Q}$$

where $p > 1$ and suppose $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and it satisfies the growth condition:

$$|g(x,s)| \leq C|s|^{q-1} + b(x) \quad \text{for } (x,s) \in \Omega \times \mathbb{R}, \tag{2.1}$$

where $C \geq 0$ is constant, $q > 1$, $b \in L^{q'}(\Omega)$, $1/q + 1/q' = 1$.

A function $u \in W_0^{1,p}(\Omega)$ is said to be a solution of problem (Q) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \int_\Omega g(x,u) v dx \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

Define $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\Phi(u) = \int_\Omega G(x, u(x)) dx \quad \text{for } u \in W_0^{1,p}(\Omega),$$

where $G : \Omega \times \mathbb{R} \to \mathbb{R}$ is defined by

$$G(x,s) = \int_0^s g(x,\tau) d\tau \quad \text{for } (x,s) \in \Omega \times \mathbb{R}.$$

Then Φ is continuously Fréchet differentiable on $W_0^{1,p}(\Omega)$ if $g(x,s)$ satisfies (2.1) with $q \in (1, p^*)$, where

$$p^* = \begin{cases} \frac{Np}{N-p}, & p < N \\ \infty, & p \geq N, \end{cases}$$

and

$$\Phi'(u)\phi = \int_\Omega g(x,u)\phi dx \quad \text{for } u, \phi \in W_0^{1,p}(\Omega).$$

Consequently, the functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \Phi(u) \quad \text{for } u \in W_0^{1,p}(\Omega)$$

is continuously Fréchet differentiable on $W_0^{1,p}(\Omega)$ if $g(x,s)$ satisfies (2.1) with $q \in (1, p^*)$, and

$$I'(u)\phi = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi dx - \int_\Omega g(x,u)\phi dx \quad \text{for all } \phi \in W_0^{1,p}(\Omega).$$

Thus critical points of I are solutions of (Q).

We define $u \in W^{1,p}(\Omega)$ to be a sub-solution to problem (Q) if $u \leq 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi dx - \int_\Omega g(x,u)\phi dx \leq 0 \quad \text{for all } \phi \in W_0^{1,p}(\Omega),\ \phi \geq 0.$$

Similarly, $u \in W^{1,p}(\Omega)$ is a super-solution to problem (Q) if in the above the reverse inequalities hold.

Finally we recall the following existence result based on super-subsolution method: ([10, Theorem 4.11]).

Theorem 2.1. *Assume that $g(x, s)$ satisfies (2.1) with $q = p$, and assume that $\rho \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a sub-solution and $\psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a super-solution to (Q) such that $\rho \le \psi$. Then (Q) has a minimal solution u_* and a maximal solution u^* in the order interval $[\rho, \psi]$ such that any solution u of (Q) in $[\rho, \psi]$ satisfies $u_* \le u \le u^*$.*

3. Main Results

Consider the following truncation of the nonlinearity of $f(x, s)$:

$$\hat{f}(x, s) := \begin{cases} 0, & (x, s) \in \overline{\Omega} \times (-\infty, 0], \\ f(x, s), & (x, u) \in \overline{\Omega} \times (0, M], \\ f(x, M), & (x, s) \in \overline{\Omega} \times (M, \infty). \end{cases}$$

Then for each $q \in (1, p^*)$, there exists a constant $C(q) > 0$ such that

$$\hat{f}(x, s) \le C(q)|s|^{q-1}, \quad (x, s) \in \Omega \times \mathbb{R}. \tag{3.1}$$

Define the functional $\hat{I}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\hat{I}_\lambda(u) := \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx - \lambda \int_\Omega \hat{F}(x, u(x))dx, \quad u \in W_0^{1,p}(\Omega),$$

where $\hat{F}(x, s) = \int_0^s \hat{f}(x, \tau)d\tau,\ (x, s) \in \Omega \times \mathbb{R}$. Since $\hat{f}(x, s)$ is bounded, the functional $\hat{I}_\lambda$ is continuously Fréchet differentiable on $W_0^{1,p}(\Omega)$, and it is also weakly lower-semicontinuous and coercive on $W_0^{1,p}(\Omega)$. Moreover, the functional $\hat{I}_\lambda$ satisfies the Palais-Smale condition. Indeed, let $\{u_n\}$ be any sequence in $W_0^{1,p}(\Omega)$ such that $\{\hat{I}_\lambda(u_n)\}$ is bounded and $\hat{I}'_\lambda(u_n) \to 0$ as $n \to \infty$. Then it follows from the boundedness of $\hat{F}$ that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. By Lemma 2 on page 363 of [7], the sequence $\{u_n\}$ has a convergent subsequence, and thus the functional $\hat{I}_\lambda$ satisfies the Palais-Smale condition.

Let u be any critical point of $\hat{I}_\lambda$. Then $0 \le u(x) \le M$ by the same argument as in the proof of [25, Propsition 2.1]. Here we use the facts that $\hat{f}(x, s) = 0$ for all $(x, s) \in \Omega \times (-\infty, 0]$ and $\hat{f}(x, s) \le 0$ for all $(x, s) \in \Omega \times [M, \infty)$, and M is the constant in the condition $(f2)$. Thus u is a nonnegative bounded solution of (P_λ), and $u \in C_0^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, 1)$ by Lieberman's regularity result [16, Theorem 1]. It follows from the maximum principle due to Vázquez [26, Theorem 5] that $u > 0$ in Ω and $\partial u/\partial \nu < 0$ on $\partial\Omega$ if $u \not\equiv 0$ in Ω.

Theorem 3.1. *Let $p > 1$ and suppose that $f(x, s)$ satisfies $(f1) - f(5)$. Then problem (P_λ) has at least two positive solutions for sufficiently large λ, and it* *has no positive solutions for* $\lambda \le \lambda_1/C(p)$*. Here* $C(p)$ *is the constant in* (3.1) *with* $q = p$*, and* $\lambda_1 > 0$ *is the principal eigenvalue of the problem*

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases}$$

with the associated eigenfunction $\phi_1 > 0$ *in* Ω*.*

Proof. Since $\hat{I}_\lambda$ is weakly lower-semicontinuous and coercive on $W_0^{1,p}(\Omega)$, $\hat{I}_\lambda$ has a global minimizer $u_1 \in W_0^{1,p}(\Omega)$. We will show that for large λ, there exists $v \in W_0^{1,p}(\Omega)$ such that $\hat{I}_\lambda(v) < 0 = \hat{I}_\lambda(0)$. Then $u_1 \not\equiv 0$, and thus u_1 is a positive solution of problem (P_λ) for large λ.

For small $\epsilon > 0$, we define

$$v_\epsilon(x) := \begin{cases} 0, & x \in \Omega \setminus B_1^\epsilon \\ c_\epsilon(x), & x \in B_1^\epsilon \setminus \overline{B}_1 \\ c(x), & x \in B_1, \end{cases}$$

where $B_1^\epsilon := \{x \in \Omega : dist(x, B_1) \le \epsilon\}$, B_1 is the open set in $(f4)$, $c(x)$ is the function in $(f2)$ and $c_\epsilon(x)$ is an appropriate function such that $0 \le v_\epsilon(x) \le c(x),\ x \in \Omega$ and $v_\epsilon \in C_0^1(\overline{\Omega})$. Then $\hat{F}(x, v_\epsilon(x)) = F(x, v_\epsilon(x)),\ x \in \Omega$ and

$$\begin{aligned} \hat{I}_\lambda(v_\epsilon) &= \frac{1}{p}\int_\Omega |\nabla v_\epsilon(x)|^p dx - \lambda \int_\Omega F(x, v_\epsilon(x)) dx \\ &= \frac{1}{p}\int_\Omega |\nabla v_\epsilon(x)|^p dx - \lambda \int_{B_1} F(x, c(x)) dx - \lambda \int_{B_1^\epsilon \setminus \overline{B}_1} F(x, c_\epsilon(x)) dx \\ &\le \frac{1}{p}\int_\Omega |\nabla v_\epsilon(x)|^p dx - \lambda \int_{B_1} F(x, c(x)) dx + \lambda A |B_1^\epsilon \setminus \overline{B}_1|, \end{aligned}$$

where $A = \max_{0 \le u \le M} |F(x, u)|$. Since $F(x, c(x)) > 0$ is continuous in B_1, there exist an open subset B_0 with $\overline{B}_0 \subseteq B_1$ and a constant $\delta_0 > 0$ such that $|B_0| > 0$ and $F(x, c(x)) \ge \delta_0$ for $x \in B_0$. Choose a sufficiently small $\epsilon_0 > 0$ so that

$$\delta_0 |B_0| - A|B_1^{\epsilon_0} \setminus \overline{B}_1| > \frac{\delta_0 |B_0|}{2}.$$

Then

$$\begin{aligned} \hat{I}_\lambda(v_{\epsilon_0}) &\le \frac{1}{p}\int_\Omega |\nabla v_{\epsilon_0}(x)|^p dx - \lambda \int_{B_0} F(x, c(x)) dx + \lambda A |B_1^{\epsilon_0} \setminus \overline{B}_1| \\ &\le \frac{1}{p}\int_\Omega |\nabla v_{\epsilon_0}(x)|^p dx - \lambda(\delta |B_0| - A|B_1^{\epsilon_0} \setminus \overline{B}_1|) \\ &\le \frac{1}{p}\int_\Omega |\nabla v_{\epsilon_0}(x)|^p dx - \lambda \frac{\delta_0 |B_0|}{2}, \end{aligned}$$

which implies that $\hat{I}_\lambda(v_{\epsilon_0}) < 0$ for sufficiently large λ. Consequently (P_λ) has a positive solution u_1 satisfying $\hat{I}_\lambda(u_1) = \inf \hat{I}_\lambda(u) < 0$ for all large λ.

Fix $q^* \in (p, p^*)$. Then it follows from (3.1) that

$$\hat{F}(x,s) \le \frac{C(q^*)}{q^*}|s|^{q^*}, \quad (x,s) \in \Omega \times \mathbb{R},$$

and for $u \in W_0^{1,p}(\Omega)$, by the Sobolev inequality,

$$\begin{aligned} \hat{I}_\lambda(u) &\ge \frac{1}{p}\|u\|^p_{W_0^{1,p}} - \frac{\lambda C(q^*)}{q^*}\int_\Omega |u(x)|^{q^*}dx \\ &\ge \frac{1}{p}\left[1 - \lambda C_1 \|u\|^{q^*-p}_{W_0^{1,p}}\right]\|u\|^p_{W_0^{1,p}} \end{aligned}$$

for some constant $C_1 > 0$. Thus for each $\lambda > 0$, there exists $\rho > 0$ such that $\hat{I}_\lambda(u) > 0 = \hat{I}_\lambda(0)$ if $0 < \|u\|_{W_0^{1,p}} \le \rho$. Fix $\lambda > 0$ such that $\hat{I}_\lambda(u_1) < 0$. It follows from Mountain pass lemma that $\hat{I}_\lambda$ has another critical point u_2 such that

$$\hat{I}_\lambda(u_2) > 0 > \hat{I}_\lambda(u_1),$$

and thus problem (P_λ) has another positive solution u_2 for all large λ.

Finally we show that problem (P_λ) has no positive solution for all $\lambda \le \lambda_1/C(p)$. Assume on the contrary that there exists a positive solution u_λ of problem (P_λ) such that $\lambda \le \lambda_1/C(p)$. Let $u = u_\lambda$, $v = \phi_1$, $A = B = 1$, $a(x) = \lambda f(x, u_\lambda(x))/u_\lambda(x)^{p-1}$, $b(x) = \lambda_1$ in [3, Theorem 1]. Then $a(x) \le b(x)$ in Ω, and

$$\int_\Omega L(u_\lambda, \phi_1)dx \le 0$$

since $\phi_1 > 0$ in Ω. Here

$$L(u_\lambda, \phi_1) := |\nabla u_\lambda|^p - p\left(\frac{u_\lambda}{\phi_1}\right)^{p-1}|\phi_1|^{p-2}\nabla\phi_1\nabla u_\lambda + (p-1)\left(\frac{u_\lambda}{\phi_1}\right)^p|\nabla\phi_1|^p.$$

On the other hand, $L(u_\lambda, \phi_1) \ge 0$ by Picone's identity (see, e.g., [4, Theorem 1.1]). Thus $L(u_\lambda, \phi_1) = 0$, for a.e. in Ω, which implies $u_\lambda = k\phi_1$ for some constant k, and one can easily proceed a contradiction. □

As an example of Theorem 3.1, we consider the following inhomogeneous cubic nonlinearity case:

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u^{p-1}(u - b(x))(c(x) - u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3.2}$$

where $b(x), c(x) \in C(\overline{\Omega})$ such that $0 < b(x) < c(x)$ for any $x \in \overline{\Omega}$.

It is easy to verify that $f(x,s) = s^{p-1}(s - b(x))(c(x) - s)$ satisfies $(f1) - (f4)$. Moreover f satisfies $(f5)$ if there exists an open ball $B_1 \subseteq \Omega$ such that $c(x) \in C^1(\overline{B}_1)$ and

$$0 < \left(1 + \frac{2}{p}\right) b(x) < c(x) \ \text{ in } B_1.$$

Then by Theorem 3.1, problem (3.2) has at least two positive solutions for large λ, and no positive solutions for small λ.

Finally we give the nonexistence of the positive solutions of problem (P_λ) when $(f5)$ does not hold. We define $\overline{f}(s) := \max_{x\in\overline{\Omega}} f(x,s)$, $b_* := \min_{x\in\overline{\Omega}} b(x)$ and $c^* := \max_{x\in\overline{\Omega}} c(x)$. Then $\overline{f}$ is a continuous function on $[0,\infty)$, and it satisfies that $\overline{f}(0) = \overline{f}(b_*) = \overline{f}(c^*) = 0$, $\overline{f}(s) < 0$ for any $s \in (0, b_*) \cup (c^*, \infty)$, and $\overline{f}(s) > 0$ for any $s \in (b_*, c^*)$ when f satisfies $(f1) - (f3)$.

Theorem 3.2. *Let $p > 1$ and suppose that $f(x,s)$ satisfies $(f1) - f(3)$. If $\int_0^{c^*} \overline{f}(s)ds < 0$, then problem (P_λ) has no positive solutions for any $\lambda > 0$.*

Proof. Assume on the contrary that there exists a positive solution (λ, u_λ) of problem (P_λ). Then u_λ is a sub-solution of

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda \overline{f}(u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3.3}$$

and c^* is a super-solution of problem (3.3). Since $\|u_\lambda\|_\infty \in (b_*, c^*]$, problem (3.3) has a positive solution u^* such that $u_\lambda \le u^* \le c^*$ in view of Theorem 2.1. On the other hand, it follows from Loc and Schmitt [18, Remark 2] that $\int_0^{c^*} \overline{f}(s)ds \ge 0$, which contradicts the hypothesis $\int_0^{c^*} \overline{f}(s)ds < 0$. □

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