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Hopf Bifurcation in a Diffusive Logistic Equation with Mixed Delayed and Instantaneous Density Dependence

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Abstract A diffusive logistic equation with mixed delayed and instantaneous density dependence and Dirichlet boundary condition is considered. The stability of the unique positive steady state solution and the occurrence of Hopf bifurcation from this positive steady state solution are obtained by a detailed analysis of the characteristic equation. The direction of the Hopf bifurcation and the stability of the bifurcating periodic orbits are derived by the center manifold theory and normal form method. In particular, the global continuation of the Hopf bifurcation branches are investigated with a careful estimate of the bounds and periods of the periodic orbits, and the existence of multiple periodic orbits are shown.

Keywords Reaction-diffusion equation · Logistic equation · Delayed and instantaneous density dependence · Stability · Local Hopf bifurcation · Global Hopf bifurcation

Mathematics Subject Classification (2010) 35K57 · 35R10 · 35B32 · 35B10 · 92D25 · 92D40

1 Introduction

In a density-dependent population model, the growth rate of a population relies on the population size. However it is unrealistic that the newborns have an immediate impact on the

Y. Su · J. Wei
Department of Mathematics, Harbin Institute of Technology, Harbin, 150001 Heilongjiang, People's Republic of China

Present Address:
Y. Su
Department of Applied Mathematics, University of Western Ontario, London, ON N6A 5B7, Canada

J. Shi (✉)
Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795, USA
e-mail: jxshix@wm.edu

population growth, and such impact can only be felt after the newborns become mature adults. This effect can be achieved by introducing a time-delay in the growth rate per capita. In 1948, renowned ecologist Hutchinson [16] proposed a time delayed logistic population model

$$\frac{du}{dt} = ru(t)[1 - bu(t-\tau)], \tag{1.1}$$

where r is the maximum growth rate per capita, $1/b$ is the carrying capacity, and τ is the time delay due to the maturation. The Hutchinson model (1.1) is considered as the milestone in population ecology which first embodied the time delay effect [25,26,34]. In general, a large delay will destabilize a positive equilibrium in a population model like (1.1) and cause oscillations (e.g. see [15,25]). Indeed it can be shown that there exists a critical value $\tau^0 = \pi/(2r)$ such that the positive equilibrium $u^* = 1/b$ of (1.1) loses the stability when $\tau > \tau^0$, and an oscillatory pattern (a periodic orbit) emerges as the dominant dynamical behavior.

On the other hand, the absence of the instantaneous density dependence in (1.1) could also make the prediction of population inaccurate. Thus a more reasonable and more realistic time delayed model would depend on an average over past populations [24,25]. A simplified average can be taken between the present population at time t and a fixed past time $t - \tau$, which results in a modified Hutchinson's equation (see [13,35,38]):

$$\frac{du}{dt} = ru(t)[1 - au(t) - bu(t-\tau)]. \tag{1.2}$$

Here the meaning of r and τ are same as in (1.1), and the parameters a and b represent the portions of instantaneous and delayed dependence of the growth rate respectively, and system (1.2) has a carrying capacity $u_* = 1/(a+b)$.

If the instantaneous dependence is dominant, i.e. $a > b$, it has been shown that the unique positive equilibrium u_* is globally asymptotically stable, see [5,6,22,28,35,36,38]. On the other hand, if the delayed dependence is more dominant, i.e. $a < b$, then it has been shown that [13,35] there exists a critical value $\tau_0 > 0$ given by

$$\tau_0 = \frac{a+b}{r\sqrt{b^2 - a^2}} \arccos\left(-\frac{a}{b}\right),$$

such that the positive equilibrium u_* is locally asymptotically stable when $\tau \in [0, \tau_0)$ and is unstable when $\tau > \tau_0$. Moreover a Hopf bifurcation occurs at the positive equilibrium when τ passes through τ_0.

In this paper, we consider the following logistic type reaction-diffusion population model with mixed delayed and instantaneous density dependence:

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} = d\frac{\partial u^2(x,t)}{\partial x^2} + ru(x,t)[1 - au(x,t) - bu(x,t-\tau)],\ x \in (0,\pi),\ t > 0, \\ &u(0,t) = u(\pi,t) = 0,\ t \geq 0, \end{aligned} \tag{1.3}$$

where $u(x,t)$ is the population density at location x and time t, and $x \in (0,\pi)$ which is the spatial domain; Dirichlet boundary condition is imposed so that the exterior environment is hostile; $d > 0$ is the diffusion coefficient, and parameters a, b, τ are as in (1.2). For convenience we normalize the time delay by a time-scaling $\hat{u}(x,t) = u(x, t\tau)$, and drop the hat for the simplicity of notation, then (1.3) becomes

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} = d\tau\frac{\partial u^2(x,t)}{\partial x^2} + r\tau u(x,t)[1 - au(x,t) - bu(x,t-1)],\ x \in (0,\pi),\ t > 0, \\ &u(0,t) = u(\pi,t) = 0,\ t \geq 0, \end{aligned} \tag{1.4}$$

We consider Eq. (1.4) with the following initial condition

$$u(x,s)=\eta(x,s), \qquad x\in(0,\pi),\ t\in[-1,0], \tag{1.5}$$

where $\eta\in\mathcal{C}:=C([-1,0];Y)$ and $Y=L^2\big((0,\pi)\big)$.

It is well-known that (1.4) admits no positive steady state solution when $r\le d$, and it possesses a unique positive steady state solution u_r when $r>d$ [17,37]. The local stability of steady states has been studied by Green and Stech [14] and Parrot [32], while the global stability was studied by Huang [18] and Pao [31]. In Huang [18] and Pao [31], it was proved that when $r\le d$, the zero solution is the global attractor of all nonnegative solutions to (1.4) for any $\tau\ge 0$; and when $r>d$ and $a>b$ (instantaneous dominant case), the unique positive steady state solution u_r is globally attractive for all nonnegative solutions to (1.4) for any $\tau\ge 0$. Dynamical system approach was used in [18] and upper-lower solution method was used in [31] for the proof of global stability. On the other hand, Friesecke [12] proved that for small delay, the dynamics of (1.4)–(1.5) is the same as that of (1.4)–(1.5) without delay.

In [42], we studied the following general delayed diffusive population model:

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t}&=d\frac{\partial u^2(x,t)}{\partial x^2}+\lambda u(x,t)f(u(x,t-\tau)), \qquad x\in(0,\ell),\ t>0,\\ u(0,t)&=u(\ell,t)=0, \qquad t\ge 0,\end{aligned} \tag{1.6}$$

where f is a smooth decreasing function and $f(0)>0$. We proved that when $\lambda>d\pi^2/\ell^2$, the model has a unique positive steady state solution u_λ, and for a fixed λ satisfying $0<\lambda-d\pi^2/\ell^2\ll 1$, there exists a sequence of the delay values $\{\tau_n\}_{n=0}^{\infty}$ so that a forward Hopf bifurcation occurs at each $\tau=\tau_n$ from the positive steady state u_λ. For (1.6), the stability of the bifurcating periodic solutions were studied by Yan and Li [46]. The result in [42] generalized earlier result of Busenberg and Huang [2], in which a diffusive Hutchinson equation with Dirichlet boundary condition was considered.

In this paper we consider the stability and associated Hopf bifurcations of system (1.4) in the delayed dominant case of $a<b$, and we show that the dynamics of (1.4) when $a<b$ is similar to (1.6). That is, a large delay will destabilize the positive steady state and causes oscillatory patterns. Our main results can be summarized as follows: assume that $r>d$ and $0<r-d\ll 1$, then

(i) (1.4) has a unique positive steady state solution $u_r(x)$.

(ii) If $a<b$, then there exists a constant $\tau_0=\tau_0(r)$ satisfying

$$\lim_{r\to d^+}(r-d)\tau_0(r)=\frac{a+b}{\sqrt{b^2-a^2}}\arccos\left(\frac{-a}{b}\right), \tag{1.7}$$

such that for (1.4), u_r is locally asymptotically stable when $\tau\in[0,\tau_0)$, whereas it is unstable when $\tau\in(\tau_0,\infty)$. Moreover, there exist a sequence of values $\{\tau_n(r)\}_{n=0}^{\infty}$, which satisfies

$$\lim_{r\to d^+}(r-d)\tau_n(r)=\frac{a+b}{\sqrt{b^2-a^2}}\left[\arccos\left(\frac{-a}{b}\right)+2n\pi\right], \tag{1.8}$$

such that for (1.4), a forward Hopf bifurcation occurs at each $\tau=\tau_n(r)(n=0,1,2,\cdots)$ from $u=u_r$, and the bifurcating periodic solutions are orbitally asymptotically stable on the center manifold.

(iii) Assuming the conditions in (ii), then (1.4) has at least one periodic orbit for any $\tau>\tau_1$, and (1.4) has at least two distinct periodic orbits when $0<\tau-\tau_n\ll 1$, for $n=2,3,\cdots$.

The particular equation considered here is a canonical example for considering the combined effect of delay and diffusion which has interested many authors. For the local stability and Hopf bifurcation around the positive steady state solution of (1.4), we analyze the characteristic equation using the approach in Busenberg and Huang [2], which has been utilized in many other studies of stability of non-constant steady state solution [1,42,46]. The first part of analysis here is conducted under a similar framework as in [2,42], but the analysis here is more difficult with the presence of both delayed and instantaneous effect on the growth rate. For the normal form calculation on the center manifold in Sect. 4, we adopt the framework in [8,9] to handle the complicated computation, which is different from the ones in [2,42]. In Sect. 5 we combine the upper and lower solutions method and the global Hopf bifurcation theorem in [44] to obtain global continuation of branch of periodic solutions bifurcating from the local Hopf bifurcation, which has not been obtained for reaction-diffusion equation with delay effect.

The diffusive logistic equation with mixed delayed and instantaneous density dependence (1.4) (including the case without instantaneous effect) with Neumann boundary condition has also been considered. The local stability and Hopf bifurcations from the constant steady state solution were studied in [27,29,47], and global stability for this case has been proved in [12,20,21,31]. Similar analysis for a constant steady state solution in a Dirichlet boundary value problem has also been investigated [41]. It is recognized that the stability and bifurcation analysis for a non-constant steady state solution (which is natural for Dirichlet boundary condition) is more difficult than the one for a constant steady state solution (which is natural for Neumann boundary condition) [2,7,18,40,42], as the spatial profile of the non-constant steady state solution is usually not known, which makes the characteristic equation analysis much harder. Analysis in [2,42] has also been extended to a diffusive logistic equation with nonlocal delay effect [3].

The rest of this paper is organized as follows. In Sect. 2, the eigenvalue problem of the associated characteristic equation is investigated. In Sect. 3, the stability of the steady state solutions and the occurrence of the Hopf bifurcations are considered. The direction of the Hopf bifurcations and the stability of bifurcating periodic solutions on center manifold are established in Sect. 4. In Sect. 5, the global continuation of the branch of periodic orbits from Hopf bifurcations is studied. Finally some numerical simulations motivated by our theoretical studies are presented in Sect. 6.

Throughout the paper, we use standard notation L^2, H^k, H_0^k for the real-valued Sobolev spaces based on L^2 spaces, and the underlying spatial domain is always the interval $(0, \pi)$. Moreover we denote $X = H^2 \bigcap H_0^1$, $Y = L^2$ and, for any real-valued vector space Z, we also denote the complexification of Z to be $Z_{\mathbb{C}} := Z \oplus iZ = \{x_1 + ix_2 | \, x_1, x_2 \in Z\}$. For the complex-valued Hilbert space $Y_{\mathbb{C}}$, we use the standard inner product $\langle u, v\rangle = \int_0^\pi u(x)\overline{v}(x)dx$. We also define by $\mathscr{D}(L)$, $\mathscr{N}(L)$, and $\mathscr{R}(L)$, the domain, the null space and the range space of a linear operator L, and define by $\text{Span}\{A\}$ the space spanned by all the elements in A. For a nonlinear mapping F, we denote by $D_u F$ the Fréchet derivative with respect to variable(s) u. In the remaining part of this paper, we will always assume that $a < b$ unless specified otherwise.

2 Eigenvalue Problems

In this section we first study the existence and properties of the positive steady state solutions of (1.4), which satisfy the following boundary value problem

$$
\begin{aligned}
& d\frac{d^2u(x)}{dx^2} + ru(x)[1-(a+b)u(x)] = 0, \quad x \in (0,\pi), \\
& u(0) = u(\pi) = 0.
\end{aligned} \tag{2.1}
$$

It is well known that

$$Y = \mathscr{N}(dD^2 + d) \oplus \mathscr{R}(dD^2 + d),$$

where

$$D^2 = \frac{\partial^2}{\partial x^2}, \quad \mathscr{N}(dD^2 + d) = \text{Span}\{\sin(\cdot)\}$$

and

$$\mathscr{R}(dD^2 + d) = \left\{ y \in Y : \langle \sin(\cdot), y \rangle = \int_0^\pi \sin(x) y(x) dx = 0 \right\}.$$

Now we give a result on the existence of positive steady state as follows.

Theorem 2.1 *There exist $r^* > d$ and a continuously differentiable mapping $r \mapsto (\xi_r, \alpha_r)$ from $[d, r^*]$ to $(X \cap \mathscr{R}(dD^2 + d)) \times \mathbb{R}^+$ such that (1.4) has a positive steady state solution (solution of (2.1)) given by*

$$u_r(x) = \alpha_r(r-d)[\sin(x) + (r-d)\xi_r(x)], \qquad r \in [d, r^*]. \tag{2.2}$$

Moreover,

$$\alpha_d = \frac{\int_0^\pi \sin^2 x dx}{d(a+b)\int_0^\pi \sin^3 x dx}$$

and $\xi_d \in X$ is the unique solution of the equation

$$(dD^2 + d)\xi + [1 - d(a+b)\alpha_d \sin(\cdot)]\sin(\cdot) = 0, \qquad \langle \sin(\cdot), \xi \rangle = 0.$$

Proof Since $dD^2 + d$ is bijective from $X \cap \mathscr{R}(dD^2 + d)$ to $\mathscr{R}(dD^2 + d)$ we know that ξ_d is well-defined. Let $m : X \times \mathbb{R} \times \mathbb{R} \to Y \times \mathbb{R}$ be defined as

$$
\begin{aligned}
m(\xi, \alpha, r) = \Big(& (dD^2 + d)\xi + \sin(\cdot) + (r-d)\xi \\
& - r(a+b)\alpha_r[\sin(\cdot) + (r-d)\xi]^2, \langle \sin(\cdot), \xi \rangle \Big).
\end{aligned}
$$

Using the definition of ξ_d, we have that

$$m(\xi_d, \alpha_d, d) = (dD^2 + d)\xi_d + [1 - d(a+b)\alpha_d \sin(\cdot)]\sin(\cdot), \ \langle \sin(\cdot), \xi_d \rangle\big) = 0,$$

and

$$D_{(\xi,\alpha)} m(\xi_d, \alpha_d, d)(\eta, \epsilon) = \big((dD^2 + d)\eta - d(a+b)\epsilon \sin^2(\cdot), \ \langle \sin(\cdot), \eta \rangle\big).$$

From $\sin^2(\cdot) \notin \mathscr{R}(dD^2 + d)$, it follows that $D_{(\xi,\alpha)} m(\xi_d, \alpha_d, d)$ is bijective from $X \times \mathbb{R}$ to $Y \times \mathbb{R}$. Therefore, the implicit function theorem implies that there exist $r^* > d$ and a continuously differentiable mapping $r \mapsto (\xi_r, \alpha_r) \in X \times \mathbb{R}^+$ such that

$$m(\xi_r, \alpha_r, r) = 0, \quad r \in [d, r^*].$$

An easy calculation shows that $\alpha_r(r-d)[\sin(\cdot) + (r-d)\xi_r]$ solves (2.1). □

In the remaining part of this paper, we will always assume $r \in [d, r^*]$ unless otherwise specified, and $0 < r^* - d \ll 1$. But the value of r^* may change from one place to another when further perturbation arguments are used.

The linearization of (1.4)–(1.5) at u_r is given by

$$\begin{aligned} &\frac{\partial v(x,t)}{\partial t} = d\tau \frac{\partial^2 v(x,t)}{\partial x^2} + r\tau[1-(2a+b)u_r]v(x,t) - rb\tau u_r v(x,t-1),\ t>0,\\ &v(0,t) = v(\pi,t) = 0,\ t \ge 0,\\ &v(x,t) = \eta(x,t),\ (x,t) \in [0,\pi]\times[-1,0], \end{aligned} \tag{2.3}$$

where $\eta \in \mathcal{C}$.

We introduce the operator $A(r) : \mathscr{D}(A(r)) \to Y_{\mathbb{C}}$ defined by

$$A(r) = dD^2 + r - r(2a+b)u_r, \tag{2.4}$$

with domain

$$\mathscr{D}(A(r)) = \{y \in Y_{\mathbb{C}} :\ \dot{y},\ \ddot{y} \in Y_{\mathbb{C}},\ y(0) = y(\pi) = 0\} = X_{\mathbb{C}},$$

and set $v(t) = v(\cdot, t),\ \eta(t) = \eta(\cdot, t)$. Then (2.3) can be rewritten as

$$\begin{aligned} &\frac{dv(t)}{dt} = \tau A(r)v(t) - \tau bru_r v(t-1), \quad t>0,\\ &v(t) = \eta(t), \quad t \in [-1,0], \quad \eta \in \mathcal{C}, \end{aligned} \tag{2.5}$$

with $A(r)$ an infinitesimal generator of a compact C_0-semigroup [33]. From [43] (or [44]), the semigroup induced by the solutions of (2.5) has the infinitesimal generator $\mathcal{A}_\tau(r)$ given by

$$\begin{aligned} &\mathcal{A}_\tau(r)\phi = \dot{\phi},\\ &\mathscr{D}(\mathcal{A}_\tau(r)) = \{\phi \in \mathcal{C}_{\mathbb{C}} \cap \mathcal{C}^1_{\mathbb{C}} :\ \phi(0) \in X_{\mathbb{C}},\ \dot{\phi}(0) = \tau A(r)\phi(0) - br\tau u_r \phi(-1)\}, \end{aligned}$$

where $\mathcal{C}^1_{\mathbb{C}} = C^1([-1,0]; Y_{\mathbb{C}})$. The spectral set $\sigma(\mathcal{A}_\tau(r)) = \big\{\lambda\tau \in \mathbb{C} :\ \Delta(r,\lambda,\tau)y = 0,\ \text{for some } y \in X_{\mathbb{C}} \setminus \{0\}\big\}$, where

$$\Delta(r,\lambda,\tau) = A(r) - bru_r e^{-\lambda\tau} - \lambda.$$

The eigenvalues of $\mathcal{A}_\tau(r)$ depend continuously on τ (see e.g. [4]).

It is clear that $\mathcal{A}_\tau(r)$ has a purely imaginary eigenvalue $\lambda\tau = i\nu\tau$ ($\nu \ne 0$) for some $\tau > 0$ if and only if

$$[A(r) - bru_r e^{-i\theta} - i\nu]y = 0, \quad y(\ne 0) \in X_{\mathbb{C}} \tag{2.6}$$

is solvable for some value of $\nu > 0$ and $\theta \in [0, 2\pi)$.

One can see that if we find a pair of (ν, θ) such that (2.6) has a non-zero solution y, then

$$\Delta(r, i\nu, \tau_n)y = 0, \quad \tau_n = \frac{\theta + 2n\pi}{\nu}, \quad n = 0, 1, 2, \cdots.$$

Next we shall show that, for $r \in (d, r^*]$, there is a unique pair (ν, θ) which solves (2.6). Now we give two lemmas which will be used to conclude our assertion.

Lemma 2.2 *If* $z \in X_{\mathbb{C}}$ *and* $\langle \sin(\cdot), z\rangle = 0$, *then* $|\langle (dD^2 + d)z, z\rangle| \ge 3d\|z\|^2_{Y_{\mathbb{C}}}$.

This is exactly the Lemma 2.3 of [2] and we omit its proof here.

Lemma 2.3 *For $r \in (d, r^*]$, if (ν, θ, y) solves (2.6) with $y(\neq 0) \in X_{\mathbb{C}}, \nu > 0$ and $\theta \in [0, 2\pi)$, then $\dfrac{\nu}{r-d}$ is uniformly bounded for $r \in (d, r^*]$.*

Proof Noting that

$$\langle [A(r) - bru_r e^{-i\theta} - i\nu]y, y\rangle = 0,$$

and also $A(r)$ is self-adjoint, then separating the real and imaginary parts of the above equality, we obtain

$$\nu\langle y, y\rangle = \langle br \sin\theta u_r y, y\rangle.$$

Hence

$$\frac{\nu}{r-d} = \frac{br\alpha_r|\sin\theta|\langle[\sin(\cdot) + (r-d)\xi_r]y, y\rangle}{\|y\|^2_{Y_{\mathbb{C}}}}.$$

It follows that there is a constant $M > 0$ such that

$$\frac{\nu}{r-d} \le M[1 + (r-d)\|\xi_r\|_\infty], \quad r \in (d, r^*].$$

The boundedness of $\nu/(r-d)$ follows from the continuity of $r \mapsto (\|\xi_r\|_\infty, \alpha_r)$. □

Now, for $r \in (d, r^*]$, suppose that (ν, θ, y) is a solution of (2.6) with $y(\neq 0) \in X_{\mathbb{C}}$. We normalize y so it can be represented as

$$\begin{aligned} y = \beta\sin(\cdot) + (r-d)z, \quad \langle \sin(\cdot), z\rangle = 0, \quad \beta \ge 0, \\ \|y\|^2_{Y_{\mathbb{C}}} = \beta^2\|\sin(\cdot)\|^2_{Y_{\mathbb{C}}} + (r-d)^2\|z\|^2_{Y_{\mathbb{C}}} = \|\sin(\cdot)\|^2_{Y_{\mathbb{C}}}. \end{aligned} \tag{2.7}$$

Substituting (2.2), (2.7) and $\nu = (r-d)h$ into (2.6), we obtain an equivalent system to (2.6):

$$\begin{aligned} g_1(z, \beta, h, \theta, r) :=&(dD^2 + d)z + [\beta\sin(\cdot) + (r-d)z] \\ &\cdot \left(1 - [r(2a+b)\alpha_r + br\alpha_r e^{-i\theta}][\sin(\cdot) + (r-d)\xi_r] - ih\right) = 0, \\ g_2(z) :=&\mathrm{Re}\langle \sin(\cdot), z\rangle = 0, \\ g_3(z) :=&\mathrm{Im}\langle \sin(\cdot), z\rangle = 0, \\ g_4(z, \beta, r) :=&(\beta^2 - 1)\|\sin(\cdot)\|^2_{Y_{\mathbb{C}}} + (r-d)^2\|z\|^2_{Y_{\mathbb{C}}} = 0. \end{aligned} \tag{2.8}$$

We define $G : X_{\mathbb{C}} \times \mathbb{R}^3 \times \mathbb{R} \mapsto Y_{\mathbb{C}} \times \mathbb{R}^3$ by $G = (g_1, g_2, g_3, g_4)$ and define (recall that $a < b$)

$$z_d = \left(1 - \frac{\sqrt{b^2 - a^2}}{a+b}i\right)\xi_d, \quad \beta_d = 1, \quad h_d = \frac{\sqrt{b^2 - a^2}}{a+b}, \quad \theta_d = \arccos\left(-\frac{a}{b}\right), \tag{2.9}$$

with ξ_d defined as in Theorem 2.1. An easy calculation shows that

$$G(z_d, \beta_d, h_d, \theta_d, d) = 0.$$

Now we are in the position to give the main theorem of this section.

Theorem 2.4 *There exists a continuously differentiable mapping $r \mapsto (z_r, \beta_r, h_r, \theta_r)$ from $[d, r^*]$ to $X_{\mathbb{C}} \times \mathbb{R}^3$ such that $G(z_r, \beta_r, h_r, \theta_r, r) = 0$. Moreover, if $r \in (d, r^*]$, then the solution for $G = 0$ is unique for given r, that is, if $(z^r, \beta^r, h^r, \theta^r, r)$ solves the equation $G = 0$ with $h^r > 0$, and $\theta^r \in [0, 2\pi)$, then $(z^r, \beta^r, h^r, \theta^r) = (z_r, \beta_r, h_r, \theta_r)$.*

Proof Let $T = (T_1, T_2, T_3, T_4) : X_{\mathbb{C}} \times \mathbb{R}^3 \mapsto Y_{\mathbb{C}} \times \mathbb{R}^3$ be defined by

$$T = D_{(z,\beta,h,\theta)} G(z_d, \beta_d, h_d, \theta_d, d).$$

Thus, we have

$$\begin{aligned} T_1(\chi, \kappa, \epsilon, \vartheta) =& (dD^2 + d)\chi - i\epsilon \sin(\cdot) + id\vartheta\alpha_d \left(-\frac{a}{b} - i\frac{\sqrt{b^2 - a^2}}{b} \right) \sin^2(\cdot) \\ &+ \kappa \left(1 - \frac{\sqrt{b^2 - a^2}}{a + b} i \right) \sin(\cdot) \left[1 + d(a + b)\alpha_d \sin(\cdot) \right], \end{aligned}$$

$$T_2(\chi) = \operatorname{Re}\langle \sin(\cdot), \chi \rangle, \quad T_3(\chi) = \operatorname{Im}\langle \sin(\cdot), \chi \rangle, \quad T_4(\kappa) = 2\kappa \| \sin(\cdot) \|^2_{Y_{\mathbb{C}}}.$$

It is routine to verify that T is bijective from $X_{\mathbb{C}} \times \mathbb{R}^3$ to $Y_{\mathbb{C}} \times \mathbb{R}^3$. It follows from the implicit function theorem that there exists a continuously differentiable mapping $r \mapsto (z_r, \beta_r, h_r, \theta_r)$ from $[d, r^*]$ (with a smaller r^*) to $X_{\mathbb{C}} \times \mathbb{R}^3$ such that $G(z_r, \beta_r, h_r, \theta_r, r) = 0$. Hence the existence is proved, and it remains to prove the uniqueness. By virtue of the uniqueness of the implicit function theorem, now we only need to show that if $G(z^r, \beta^r, h^r, \theta^r, r) = 0$, $h^r > 0$ and $\theta^r \in [0, 2\pi)$, then

$$(z^r, \beta^r, h^r, \theta^r) \to (z_d, \beta_d, h_d, \theta_d)$$

as $r \to d$ in the norm of $X_{\mathbb{C}} \times \mathbb{R}^3$. From the definitions of $(z^r, \beta^r, h^r, \theta^r)$, it is easy to see that $\{h^r\}$, $\{\beta^r\}$ and $\{\theta^r\}$ are bounded. From Lemma 2.2 and the first equation of Eq. (2.8) we have

$$\| z^r \|^2_{Y_{\mathbb{C}}} \le \frac{1}{3d} |\langle \varrho(h^r, \theta^r, r)[\beta^r \sin(\cdot) + (r - d)z^r], z^r \rangle|,$$

where

$$\varrho(h^r, \theta^r, r) = 1 - [r(2a + b)\alpha_r + br\alpha_r e^{-i\theta^r}][\sin(\cdot) + (r - d)\xi_r] - ih^r.$$

The boundedness of $\{h^r\}, \{\alpha_r\}$ and $\{\xi_r\}$ yield that there is $M > 0$ such that $\|\varrho(h^r, \theta^r, r)\|_\infty \le 3dM$, for $r \in [d, r^*]$. Thus we have

$$\| z^r \|^2_{Y_{\mathbb{C}}} \le M|\beta^r| \cdot \| \sin(\cdot) \|_{Y_{\mathbb{C}}} \| z^r \|_{Y_{\mathbb{C}}} + M(r - d) \| z^r \|^2_{Y_{\mathbb{C}}}.$$

Without loss of generality, assume $M(r^* - d) < 1/2$, then

$$\| z^r \|_{Y_{\mathbb{C}}} \le 2M|\beta^r| \cdot \| \sin(\cdot) \|_{Y_{\mathbb{C}}}, \quad r \in [d, r^*].$$

Hence $\{z^r\}$ is bounded in $Y_{\mathbb{C}}$. On the other hand, $(dD^2 + d) : X_{\mathbb{C}} \cap \mathscr{R}_{\mathbb{C}}(dD^2 + d) \to Y_{\mathbb{C}} \cap \mathscr{R}_{\mathbb{C}}(dD^2 + d)$ has a bounded inverse, by applying $(dD^2 + d)^{-1}$ on $g_1(z^r, \beta^r, h^r, \theta^r, r) = 0$ one sees that $\{z^r\}$ is also bounded in $X_{\mathbb{C}}$, and hence $\{(z^r, \beta^r, h^r, \theta^r) : r \in (d, r^*]\}$ is precompact in $Y_{\mathbb{C}} \cap \mathbb{R}^3$. Therefore, there is a subsequence $\{(z^n, \beta^n, h^n, \theta^n)\}$ such that

$$(z^n, \beta^n, h^n, \theta^n) \to (z^d, \beta^d, h^d, \theta^d), \quad r \to d \text{ as } n \to \infty,$$

by taking the limit of the equation $G(z^n, \beta^n, h^n, \theta^n, r^n) = 0$ as $n \to \infty$. We claim that $G(z, \beta, h, \theta, d) = 0$ has a unique solution given by $(z, \beta, h, \theta) = (z_d, \beta_d, h_d, \theta_d)$ defined in (2.9), thus $(z^d, \beta^d, h^d, \theta^d) = (z_d, \beta_d, h_d, \theta_d)$. In fact, we take the limit in equation $G(z^n, \beta^n, h^n, \theta^n, r^n) = 0$ as $n \to \infty$ to obtain

$$\begin{aligned} &(dD^2 + d)z^d + \beta^d \sin(\cdot)\left(1 - [d(2a+b)\alpha_d + bd\alpha_d e^{-i\theta^d}]\sin(\cdot) - ih^d\right) = 0, \\ &\langle \sin(\cdot), z^d \rangle = 0, \quad \left((\beta^d)^2 - 1\right)\|\sin(\cdot)\|^2_{Y_{\mathbb{C}}} = 0. \end{aligned} \tag{2.10}$$

It follows that $\beta = 1$. Multiplying the first equation of (2.10) by $\sin(\cdot)$ and integrating it from 0 to π, and separating the real and imaginary parts, we obtain that

$$\begin{cases} \int_0^\pi \left(1 - [d(2a+b)\alpha_d + bd\alpha_d \cos\theta^d]\sin x\right)\sin^2 x dx = 0, \\ \int_0^\pi \left(h^d - bd\alpha_d \sin\theta^d\right)\sin^2 x dx = 0. \end{cases}$$

Noting that $\alpha_d = \dfrac{\int_0^\pi \sin^2 x dx}{d(a+b)\int_0^\pi \sin^3 x dx}$ and $h^d > 0$, it follows that $\theta^d = \arccos\left(-\dfrac{a}{b}\right) = \theta_d$ and $h^d = \dfrac{\sqrt{b^2 - a^2}}{a+b} = h_d$. Therefore, (2.10) yields that

$$(dD^2 + d)z^d + \left(1 - \frac{\sqrt{b^2-a^2}}{a+b}i\right)\left[\sin(\cdot) - d(a+b)\alpha_d \sin^2(\cdot)\right] = 0, \quad \langle \sin(\cdot), z^d \rangle = 0.$$

From the uniqueness of the solution of this equation in $X_{\mathbb{C}}$, we have $z^d = z_d$. Hence, $(z^r, \beta^r, h^r, \theta^r) \to (z_d, \beta_d, h_d, \theta_d)$ as $r \to d$ in the norm of $Y_{\mathbb{C}} \times \mathbb{R}^3$. In addition, $(dD^2 + d)^{-1}$ is a continuous linear operator from $\mathscr{R}_{\mathbb{C}}(dD^2 + d)$ into $X_{\mathbb{C}} \cap \mathscr{R}_{\mathbb{C}}(dD^2 + d)$, we get the convergence in $X_{\mathbb{C}} \times \mathbb{R}^3$, which follows that $(z^r, \beta^r, h^r, \theta^r) = (z_r, \beta_r, h_r, \theta_r)$. □

Corollary 2.5 *For $r \in (d, r^*]$, the eigenvalue problem*

$$\Delta(r, i\nu, \tau)y = 0, \quad \nu \geq 0, \quad \tau > 0, \quad y(\neq 0) \in X_{\mathbb{C}}$$

has a nontrivial solution, or equivalently, $i\nu\tau \in \sigma(\mathcal{A}_\tau(r))$ if and only if

$$\nu = \nu_r = (r-d)h_r, \quad \tau = \tau_n = \frac{\theta_r + 2n\pi}{\nu_r}, \quad n = 0, 1, 2, \cdots \tag{2.11}$$

and

$$y = cy_r, \quad y_r = \beta_r \sin(\cdot) + (r-d)z_r,$$

where c is a nonzero constant, and $z_r, \beta_r, h_r, \theta_r$ are defined as in Theorem 2.4.

Remark 2.6 Combining Theorem 2.4, Eq. (2.9) and Corollary 2.5, we can obtain the estimate of Hopf bifurcation values given in (1.7) and (1.8).

3 Stability of Steady State Solutions

In this section we study the stability of non-constant steady state solution u_r of Eq.(1.3) with a fixed $r \in (d, r^*]$, and the time delay τ is considered as a parameter.

We recall the following facts:

Lemma 3.1 *Let $r \in (d, r^*]$.*

1. *If* $\tau \geq 0$, *then* 0 *is not an eigenvalue of* $\mathcal{A}_\tau(r)$*;*
2. *If* $\tau = 0$, *then all eigenvalues of* $\mathcal{A}_\tau(r)$ *have negative real parts.*

Part 1 can be proved by using So and Yang [40, Lemma 4.1] and it is very similar to [42, Lemma 3.2], and part 2 is essentially same as [40, Theorem 4.2], hence we omit their proof here.

We now show that $\lambda\tau_n = i\nu\tau_n$ is a simple eigenvalue of $\mathcal{A}_{\tau_n}$ for $n = 0, 1, 2, \cdots$. For this purpose, we first give the following lemma.

Lemma 3.2 *For fixed* $r \in (d, r^*]$,

$$S_n(r) := \int_0^\pi [1 - br\tau_n e^{-i\theta_r} u_r] y_r^2(x)dx \neq 0, \quad n = 0, 1, 2, \cdots .$$

Proof From the expressions of u_r, y_r, τ_n, and the fact that $\theta_r \to \arccos(-\frac{a}{b})$ as $r \to d$, it is easy to obtain

$$S_n(r) \to \left[1 + \left(\frac{a}{\sqrt{b^2 - a^2}} + i\right)\left(\arccos\left(-\frac{a}{b}\right) + 2n\pi\right)\right] \int_0^\pi \sin^2 x dx, \quad \text{as } r \to d \tag{3.1}$$

It follows that $S_n(r) \neq 0$ for $r \in (d, r^*]$ and for all τ_n, $n = 0, 1, 2, \cdots$. □

Theorem 3.3 *For each fixed* $r \in (d, r^*]$, $\lambda\tau_n = i\nu_r\tau_n$ *is a simple eigenvalue of* $\mathcal{A}_{\tau_n}$ *for* $n = 0, 1, 2, \cdots$.

Proof From Corollary 2.5 we have $\mathscr{N}[\mathcal{A}_{\tau_n}(r) - i\nu_r\tau_n] = \text{Span}\{e^{i\nu_r\tau_n \cdot} y_r\}$. Suppose that for some $\phi \in \mathscr{D}(\mathcal{A}_{\tau_n}(r)) \cap \mathscr{D}([A_{\tau_n}(r)]^2)$, we have

$$[\mathcal{A}_{\tau_n}(r) - i\nu_r\tau_n]^2\phi = 0.$$

This implies that

$$[\mathcal{A}_{\tau_n}(r) - i\nu_r\tau_n]\phi \in \mathscr{N}[\mathcal{A}_{\tau_n}(r) - i\nu_r\tau_n] = \text{Span}\{e^{i\nu_r\tau_n \cdot} y_r\}.$$

So there is a constant c such that

$$[\mathcal{A}_{\tau_n}(r) - i\nu_r\tau_n]\phi = ce^{i\nu_r\tau_n \cdot} y_r.$$

Hence

$$\begin{aligned} \dot{\phi}(\theta) &= i\nu_r\tau_n\phi(\theta) + ce^{i\nu_r\tau_n\theta} y_r, \quad \theta \in [-1, 0], \\ \dot{\phi}(0) &= A(r)\phi(0) - br\tau_n u_r \phi(-1). \end{aligned} \tag{3.2}$$

The first equation of (3.2) yields

$$\begin{aligned} \phi(\theta) &= \phi(0)e^{i\nu_r\tau_n\theta} + c\theta e^{i\nu_r\tau_n\theta} y_r, \\ \dot{\phi}(0) &= i\nu_r\tau_n\phi(0) + cy_r. \end{aligned} \tag{3.3}$$

From (3.2) and (3.3) we have

$$\begin{aligned} \Delta(r, i\nu, \tau_n)\phi(0) &= [A(r) - br\tau_n u_r e^{-i\theta_r} - i\nu_r\tau_n]\phi(0) \\ &= c(1 - br\tau_n u_r e^{-i\theta_r}) y_r. \end{aligned}$$

Hence

$$0 = \int_0^\pi \phi(0)[\Delta(r, i\nu, \tau_n)y_r]dx$$
$$= \int_0^\pi y_r[\Delta(r, i\nu, \tau_n)\phi(0)]dx$$
$$= c\int_0^\pi (1 - br\tau_n u_r e^{-i\theta_r})y_r^2 dx.$$

As a consequence of Lemma 3.2 we have $c = 0$, which leads to that $\phi \in \mathscr{N}[\mathcal{A}_{\tau_n}(r) - i\nu_r\tau_n]$. By induction we obtain

$$\mathscr{N}[\mathcal{A}_{\tau_n}(r) - i\nu_r\tau_n]^j = \mathscr{N}[\mathcal{A}_{\tau_n}(r) - i\nu_r\tau_n] \quad j = 1, 2, 3, \cdots, \quad n = 0, 1, 2, \cdots.$$

Therefore, $\lambda\tau_n = i\nu_r\tau_n$ is a simple eigenvalue of $\mathcal{A}_{\tau_n}$ for $n = 0, 1, 2, \cdots$. □

Since $\lambda\tau_n = i\nu_r\tau_n$ is a simple eigenvalue of $\mathcal{A}_{\tau_n}$, by using the implicit function theorem it is not difficult to show that there are a neighborhood $O_n \times D_n \times H_n \subset \mathbb{R} \times \mathbb{C} \times X_{\mathbb{C}}$ of $(\tau_n, i\nu_r\tau_n, y_r)$ and a continuously differential function $(\lambda, y) : O_n \to D_n \times H_n$ such that for each $\tau \in O_n$, the only eigenvalue of $\mathcal{A}_\tau(r)$ in D_n is $\lambda(\tau)$, and

$$\lambda(\tau_n) = i\nu_r\tau_n, \quad y(\tau_n) = y_r, \quad ||y(\tau)||_{Y_{\mathbb{C}}} = \|\sin(\cdot)\|_{Y_{\mathbb{C}}},$$
$$\Delta(r, \lambda(\tau)/\tau, \tau) = [A(r) - br\tau u_r e^{-\lambda(\tau)} - \lambda(\tau)]y(\tau) = 0, \quad \tau \in O_n. \tag{3.4}$$

We show that $\lambda(\tau)$ moves across the imaginary axis at $\tau = \tau_n$ transversally.

Theorem 3.4 *For $r \in (d, r^*]$, we have*

$$\mathrm{Re}\left\{\frac{d\lambda(\tau_n)}{d\tau}\right\} > 0, \quad n = 0, 1, 2, \cdots.$$

Proof Differentiating (3.4) with respect to τ at $\tau = \tau_n$, we have

$$\frac{d\lambda(\tau_n)}{d\tau}[-1 + bru_r\tau_n e^{-i\theta_r}]y_r + \Delta(r, i\nu_r, \tau_n)\frac{dy(\tau_n)}{d\tau} + i\nu_r y_r = 0.$$

Multiplying the equation by y_r and integrating on $(0, \pi)$, we obtain

$$\frac{d\lambda(\tau_n)}{d\tau} = \frac{i\nu_r\int_0^\pi y_r^2 dx}{\int_0^\pi [1 - br\tau_n u_r e^{-i\theta_r}]y_r^2 dx}$$
$$= \frac{1}{|S_n(r)|^2}\left(i\nu_r\left|\int_0^\pi y_r^2 dx\right|^2 - ibr\nu_r\tau_n e^{i\theta_r}\int_0^\pi y_r^2 dx\overline{\int_0^\pi u_r y_r^2 dx}\right). \tag{3.5}$$

Noting that

$$\int_0^\pi y_r^2(x)dx = \left|\int_0^\pi y_r^2(x)dx\right|e^{i\rho_r},$$

where $\rho_r = \mathrm{Arg}\big(\int_0^\pi y_r^2(x)dx\big)$, $-\pi < \rho_r \le \pi$. Then from (3.5) it follows that

$$\mathrm{Re}\left\{\frac{d\lambda(\tau_n)}{d\tau}\right\} = \frac{br\tau_n\nu_r(r-d)}{|S_n(r)|^2}\left|\int_0^\pi y_r^2 dx\right|\mathrm{Re}\left\{-ie^{i(\theta_r+\rho_r)}\overline{\int_0^\pi \frac{u_r y_r^2}{r-d}dx}\right\}.$$

Hence, by

$$\mathrm{Re}\left\{-ie^{i(\theta_r+\rho_r)}\int_0^\pi \frac{u_r y_r^2}{r-d}\right\} \to \frac{\sqrt{b^2-a^2}}{b}\frac{\int_0^\pi \sin^2 x dx}{d(a+b)} > 0 \quad \text{as } r \to d,$$

we have $\mathrm{Re}\left\{\frac{d\lambda(\tau_n)}{d\tau}\right\} > 0$ when $r \in (d, r^*]$. □

We summarize the stability properties of steady state solutions and associated bifurcations of Eq. (1.4) as follows:

Theorem 3.5 *Suppose that $a, b, d, r > 0$ and $\tau \geq 0$.*

1. *If $0 < r \leq d$, then for all $a,\ b > 0$, each solution $u(x, t)$ with nonnegative initial value of the problems (1.4)–(1.5) satisfies $||u(\cdot, t)||_Y \to 0$ as $t \to \infty$ for any $\tau \geq 0$.*
2. *If $r > d$, then the trivial steady state $u = 0$ is unstable, and Eq. (1.4) has a unique positive steady state solution u_r.*
3. *There exists $r^* > d$ such that for $r \in (d, r^*]$, the infinitesimal generator $\mathcal{A}_\tau(r)$ associated with u_r has exactly $2(n+1)$ eigenvalues with positive real part when $\tau \in (\tau_n, \tau_{n+1}]$, $n = 0, 1, 2, \cdots$. In particular, for $r \in (d, r^*]$, the positive steady state solution u_r of Eq. (1.4) is locally asymptotically stable when $\tau \in [0, \tau_0)$ and is unstable when $\tau \in (\tau_0, \infty)$, and a Hopf bifurcation occurs at each $\tau = \tau_n$ and $u = u_r$.*

Proof Part 1 is similar to [42, Theorem 6.2] and we omit the proof. Part 2 is well-known, see for example [17,37]. The eigenvalue distribution in part 3 follows directly from Lemma 3.1 and Theorem 3.4, and the stability/instability follows from Lemma 3.1, Corollary 2.5 and the eigenvalue distribution. □

4 Hopf Bifurcation

We have shown in Theorem 3.5 the existence of Hopf bifurcations for the problem (1.4)–(1.5) occurring around the positive steady state solution u_r and at $\tau = \tau_n$ with τ as bifurcation parameter. In this section, the detailed local Hopf bifurcation analysis at $\tau = \tau_n$ is carried out by using the normal form method described in [8].

We first transform the steady state to the origin via a translation $U(t) = u(\cdot, t) - u_r(\cdot)$ with $u(x, t)$ satisfying (1.4), and introduce a new bifurcating parameter $\alpha = \tau - \tau_n$, then (1.4) is transformed into

$$\frac{dU(t)}{dt} = \tau_n A(r)U(t) - \tau_n rbu_r U(t-1) + F(U_t, \alpha), \tag{4.1}$$

where $A(r)$ is defined in (2.4), $U_t \in \mathcal{C}$, and for $\phi \in \mathcal{C}$, F is defined as

$$F(\phi, \alpha) = \alpha A(r)\phi(0) - \alpha rbu_r\phi(-1) - ar(\tau_n + \alpha)\phi^2(0) - br(\tau_n + \alpha)\phi(0)\phi(-1).$$

For the linearized equation of (4.1):

$$\frac{dv(t)}{dt} = \tau_n A(r)v(t) - \tau_n rbu_r v(t-1), \tag{4.2}$$

following [9], we introduce a formal duality $\langle\langle\cdot,\cdot\rangle\rangle$, which is a bilinear form defined in $\mathcal{C}^*_{\mathbb{C}} \times \mathcal{C}_{\mathbb{C}}$ where $\mathcal{C}^* := C([0, 1]; Y_{\mathbb{C}})$, and it is defined by

$$\langle\langle\psi, \phi\rangle\rangle = \langle\psi(0), \phi(0)\rangle^* - \int_{-1}^0 \langle\psi(s+1), br\tau_n u_r\phi(s)\rangle^* ds, \quad \text{for } \phi \in \mathcal{C}_{\mathbb{C}},\ \psi \in \mathcal{C}^*_{\mathbb{C}},$$

with $\langle \cdot, \cdot \rangle^*$ the natural duality in $Y_{\mathbb{C}}$ when it is considered as a Banach space, that is $\langle u, v\rangle^* = \int_0^\pi u(x)v(x)dx$ for $u, v \in Y_{\mathbb{C}}$. In the following we will also use $\langle\langle \cdot, \cdot \rangle\rangle$ and $\langle \cdot, \cdot \rangle^*$ for vectors or matrices, which should be understood as matrix multiplication with entry multiplication given by these dualities.

Recall that $\mathcal{A}_{\tau_n}(r)$ is the infinitesimal generator of (4.2). We denote by $\Lambda = \Lambda_{r,n}$ the set of pure imaginary eigenvalues of the $\mathcal{A}_{\tau_n}(r)$. It is clear that

$$\Lambda = \{i\nu_r\tau_n, -i\nu_r\tau_n\}.$$

We also introduce P as the generalized eigenspace associated with Λ. From Sects. 2 and 3, it is clear that

$$P = \text{Span}\{\Phi\}, \text{ where } \Phi(\theta) = (\phi_1(\theta), \phi_2(\theta)) = (y_r e^{i\nu_r\tau_n\theta}, \overline{y_r} e^{-i\nu_r\tau_n\theta}), \text{ for } \theta \in [-1, 0].$$

From the formal duality theory in [10], the phase space $\mathcal{C}_{\mathbb{C}}$ can be decomposed as $\mathcal{C}_{\mathbb{C}} = P \oplus Q$, in which

$$Q = \{\phi \in \mathcal{C}_{\mathbb{C}} : \langle\langle \psi, \phi \rangle\rangle = 0, \text{ for all } \psi \in P^*\},$$

where P^* is the generalized eigenspace of the adjoint equation of (4.2) associated with Λ. By an easy computation we have that

$$P^* = \text{Span}\{\Psi\}, \text{ where } \Psi(s) = \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{S_n} y_r e^{-i\nu_r\tau_n s} \\ \dfrac{1}{\overline{S_n}} \overline{y_r} e^{i\nu_r\tau_n s} \end{pmatrix}, \text{ for } s \in [0, 1].$$

Here $S_n = S_n(r)$ is defined in Lemma 3.2 and $\langle\langle \Psi, \Phi \rangle\rangle = I$, where $I \in \mathbb{R}^{2\times 2}$ is the identity matrix.

In order to obtain the normal form, one needs to consider an enlarged phase space:

$$BC = \{\psi : [-1, 0] \to Y_{\mathbb{C}} : \psi \text{ is continuous on} [-1, 0) \\ \text{with a possible jump discontinuity at} 0\},$$

with the sup norm. From [8], BC can be decomposed by Λ as $BC = P \oplus \mathcal{N}(\pi)$, where π is a continuous projection from BC onto P, which is defined by

$$\pi(\phi + X_0 y) = \Phi\big(\langle\langle \Psi, \phi \rangle\rangle + \langle \Psi(0), y\rangle^*\big), \ \phi \in \mathcal{C}_{\mathbb{C}}, \ y \in Y_{\mathbb{C}},$$

where

$$X_0(\theta) = \begin{cases} 0, & -1 \le \theta < 0, \\ I, & \theta = 0. \end{cases}$$

We define an extension $\mathcal{A}_n$ of $\mathcal{A}_{\tau_n}$ by

$$\mathcal{A}_n v = \dot{v} + X_0[\tau_n A(r)v(0) - \tau_n r b u_r v(-1) - \dot{v}(0)]$$

for $v \in \mathcal{C}_0^1 := \{\phi \in \mathcal{C}_{\mathbb{C}} | \ \dot{\phi} \in \mathcal{C}_{\mathbb{C}}, \phi(0) \in X_{\mathbb{C}}\}$. Then we can state the following result from [8].

Lemma 4.1 *In* BC *decomposed by* Λ, (4.1) *can be written as*

$$\begin{cases} \dfrac{dz}{dt} = B_n z(t) + \langle \Psi(0), F(\Phi z(t) + y(t), \alpha)\rangle^*, \\ \dfrac{dy}{dt} = \mathcal{A}_n^1 y(t) + (I - \pi)X_0 F(\Phi z(t) + y(t), \alpha), \end{cases} \tag{4.3}$$

where $z(t) \in \mathbb{C}^2$, $y(t) \in Q_0^1 := Q \cap \mathcal{C}_0^1$, $B_n = \begin{pmatrix} i\nu_r\tau_n & 0 \\ 0 & -i\nu_r\tau_n \end{pmatrix}$ *and* $\mathcal{A}_n^1$ *is defined by* $\mathcal{A}_n^1 : Q_0^1 \to \mathcal{N}(\pi)$, $\mathcal{A}_n^1 v = \mathcal{A}_n v$ *for* $v \in Q_0^1$.

In order to give the main result, we now write F as a Taylor polynomial in form

$$F(U_t, \alpha) = \frac{1}{2!}F_2(U_t, \alpha) + \frac{1}{3!}F_3(U_t, \alpha),$$

where F_2, F_3 are the second and third Fréchet derivatives of F at $(0, 0)$ respectively, that is

$$F_2(U_t, \alpha) = 2\alpha[A(r)U(t) - rbu_r U(t-1)] - 2ar\tau_n U^2(t) - 2br\tau_n U(t)U(t-1),$$
$$F_3(U_t, \alpha) = -6ar\alpha U^2(t) - 6br\alpha U(t)U(t-1).$$

Then (4.3) can be rewritten as

$$\begin{cases} \dfrac{dz}{dt} = B_n z + \dfrac{1}{2} f_2^1(z, y, \alpha) + \dfrac{1}{3!} f_3^1(z, y, \alpha), \\ \dfrac{dy}{dt} = \mathcal{A}_n^1 y + \dfrac{1}{2} f_2^2(z, y, \alpha) + \dfrac{1}{3!} f_3^2(z, y, \alpha), \end{cases}$$

with $f_j := (f_j^1, f_j^2)$, $j = 2, 3$, defined by

$$f_j^1(z, y, \alpha) = \langle \Psi(0), F_j(\Phi z + y, \alpha)\rangle^*, \quad f_j^2(z, y, \alpha) = (I - \pi)X_0 F_j(\Phi z + y, \alpha).$$

Following the approach in [9], we prepare the following Lemmas 4.2 and 4.3.

Lemma 4.2 *For fixed* $r \in (d, r^*]$, *there exist* $A_1, A_2 \in \mathbb{C}$ *such that the normal form of the flow of Eq.* (4.1) *on the center manifold near* $\alpha = 0$ *is given by*

$$\frac{dz}{dt} = B_n z + \begin{pmatrix} A_1 z_1 \alpha \\ \bar{A}_1 z_2 \alpha \end{pmatrix} + \begin{pmatrix} A_2 z_1^2 z_2 \\ \bar{A}_2 z_1 z_2^2 \end{pmatrix} + O(\alpha^2 |z| + |(\alpha, z)|^4) \tag{4.4}$$

where $z(t) = (z_1(t), z_2(t))^T$, *and* $\bar{A}_i$ *is the complex conjugation of* A_i *(depending on* r *and* n*) for* $i = 1, 2$.

Proof By the procedure in [8], the normal form of (4.1) can be obtained by a recursive process of changes of variables. At each step, using the transformation

$$(z, y) = (\hat{z}, \hat{y}) + \frac{1}{j}\left(U_j^1(\hat{z}, \alpha), U_j^2(\hat{z}, \alpha)\right), \quad j = 2, 3, \cdots$$

with $U_j^i(\hat{z}, \alpha)$, $i = 1, 2$, are homogeneous polynomials of degree j in $(\hat{z}, \alpha)$.

Ultimately Eq. (4.3) can be transformed to

$$\begin{cases} \dfrac{dz}{dt} = B_n z + \displaystyle\sum_{j \geq 2} \frac{1}{j!} g_j^1(z, y, \alpha), \\ \dfrac{dy}{dt} = \mathcal{A}_n^1 y + \displaystyle\sum_{j \geq 2} \frac{1}{j!} g_j^2(z, y, \alpha), \end{cases}$$

with $g_j := (g_j^1, g_j^2)$ are homogeneous polynomials of degree j in (z, y, α), defined by

$$g_j = \tilde{f}_j - M_j U_j,$$

where $\tilde{f}_j := (\tilde{f}_j^1, \tilde{f}_j^2)$ are the terms of order j in (z, y, α) obtained after the $(j-1)$-th transformation and linear operators $M_j := (M_j^1, M_j^2)$ are defined by

$$
\begin{aligned}
(M_j^1 p)(z,\alpha) &= D_z p(z,\alpha) B_n z - B_n p(z,\alpha),\\
(M_j^2 p)(z,\alpha) &= D_z p(z,\alpha) B_n z - \mathcal{A}_n^1 p(z,\alpha),
\end{aligned} \tag{4.5}
$$

where $p(z,\alpha)$ is a homogeneous polynomials of degree j in (z,α).

An easy calculation yields that

$$M_j^1(z^q \alpha^l e_k) = i\nu_r \tau_n \big(q_1 - q_2 + (-1)^k\big) z^q \alpha^l e_k, \quad j = 2, 3, \cdots, \quad k = 1, 2, \tag{4.6}$$

with $z^q = z_1^{q_1} z_2^{q_2}$, $q_1, q_2, l \in \mathbb{N}_0$, $l + q_1 + q_2 = j$, and $\{e_1, e_2\}$ is the canonical basis for $\mathbb{C}^2$.

Therefore,

$$
\begin{aligned}
\mathscr{N}(M_2^1) &= \text{Span}\left\{ \begin{pmatrix} z_1\alpha \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2\alpha \end{pmatrix} \right\},\\
\mathscr{N}(M_3^1) &= \text{Span}\left\{ \begin{pmatrix} z_1^2 z_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1 z_2^2 \end{pmatrix}, \begin{pmatrix} z_1\alpha^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2\alpha^2 \end{pmatrix} \right\}.
\end{aligned} \tag{4.7}
$$

From [8],

$$g_j^1(z, 0, \alpha) = \text{Proj}_{\mathscr{N}(M_j^1)} \tilde{f}_j^1(z, 0, \alpha).$$

It follows that the normal form up to third order of the flow of Eq. (4.1) on the center manifold near $\alpha = 0$ is given by (4.4), with the coefficients can be found to be complex conjugates. □

Using Lemma 4.2 and well-known calculations, we can state the following Lemma.

Lemma 4.3 *For each fixed $r \in (d, r^*)$ and each fixed $n \in \mathbb{N}_0$, (4.1) has a 2-dimensional local center manifold of the origin at $\tau = \tau_n$, on which the flow is given by an ODE written in normal form and in polar coordinates (ρ, ξ) as*

$$
\begin{aligned}
\dot{\rho} &= K_1(\tau - \tau_n)\rho + K_2\rho^3 + O\big((\tau - \tau_n)^2\rho + |(\tau - \tau_n, \rho)|^4\big),\\
\dot{\xi} &= -i\nu_r\tau_n + O\big(|(\tau - \tau_n, \rho)|\big),
\end{aligned} \tag{4.8}
$$

where $K_1 = \text{Re}\{A_1\}$, $K_2 = \text{Re}\{A_2\}$.

Next we calculate the signs of $\text{Re}\{A_1\}$ and $\text{Re}\{A_2\}$ which determine the direction of Hopf bifurcation and the stability of bifurcating periodic orbits.

Proposition 4.4 *Let $r \in (d, r^*]$ and $n = 0, 1, 2, \cdots$, and let A_1 be defined as in (4.4). Then* $\text{Re}\{A_1\} > 0$.

Proof From the definition of f_2^1, we have

$$
\begin{aligned}
&\frac{1}{2} f_2^1(z, y, \alpha)\\
&\quad = \alpha\langle \Psi_0, A(r)\big(\Phi_0 z + y_0\big) - rbu_r\big(\Phi_{-1}z + y_{-1}\big)\rangle^*\\
&\qquad - ar\tau_n\langle \Psi_0, \big(\Phi_0 z + y_0\big)^2\rangle^* - br\tau_n\langle \Psi_0, \big(\Phi_0 z + y_0\big)\big(\Phi_{-1}z + y_{-1}\big)\rangle^*,
\end{aligned} \tag{4.9}
$$

where $\Psi_0 = \Psi(0)$, $\Phi_\theta = \Phi(\theta)$ and $y_\theta = y(\theta)$, $\theta = 0, -1$. Note that A_1 is the coefficient of the term $\begin{pmatrix} x_1\alpha \\ 0 \end{pmatrix}$ in $\frac{1}{2} f_2^1(z, 0, \alpha) = \frac{1}{2} \tilde{f}_2^1(z, 0, \alpha)$. Then from Corollary 2.5, we obtain that

$$A_1 = \frac{1}{S_n} \langle y_r, A(r)y_r - rbu_r e^{-i\theta_r} y_r \rangle^* \\ = \frac{i\nu_r}{S_n} \langle y_r, y_r \rangle^* = \frac{i\nu_r \overline{S_n} \int_0^\pi y_r^2 dx}{|S_n|^2}.$$

From (3.1) and $y_r \to \sin x$ as $r \to d$, we have

$$\mathrm{Re} \left\{ i\overline{S_n} \int_0^\pi y_r^2 dx \right\} \to (\theta_d + 2n\pi) \left(\int_0^\pi \sin^2 x dx \right)^2, \quad \text{as } r \to d.$$

Therefore $\mathrm{Re}\{A_1\} > 0$. □

From Lemma 4.2, A_2 is the coefficient of the term $\begin{pmatrix} z_1^2 z_2 \\ 0 \end{pmatrix}$ in $\frac{1}{3!} \tilde{f}_3^1(z, 0, 0)$. Then following [8], we write

$$\begin{aligned} \tilde{f}_3^1(z,0,0) =& f_3^1(z,0,0) + \frac{3}{2}\big[(D_z f_2^1)(z,0,0)U_2^1(z,0) \\ & - (D_z U_2^1)(z,0) g_2^1(z,0,0) + (D_y f_2^1)(z,0,0) U_2^2(z,0)\big], \\ =& \frac{3}{2}\left[(D_z f_2^1)(z,0,0)U_2^1(z,0) + (D_y f_2^1)(z,0,0)U_2^2(z,0)\right], \end{aligned} \tag{4.10}$$

since $f_3^1(z, 0, 0) = 0$ and $g_2^1(z, 0, 0) = 0$. Thus

$$A_2 = \frac{3}{2} [C_1 + C_2],$$

where C_1 and C_2 are the coefficients of $\begin{pmatrix} z_1^2 z_2 \\ 0 \end{pmatrix}$ contributed from $(D_z f_2^1)(z, 0, 0)U_2^1(z, 0)$ and $(D_y f_2^1)(z, 0, 0)U_2^2(z, 0)$ respectively. We calculate C_1 and C_2 separately in the following lemma.

Lemma 4.5 *Let y_r and θ_r be defined as in Theorem 2.4, and let C_1 and C_2 be defined as above. We define a matrix*

$$\mathrm{R} = \begin{pmatrix} (a + be^{-i\theta_r}) y_r^2 & (a + be^{-i\theta_r})|y_r|^2 \\ (a + be^{i\theta_r})|y_r|^2 & (a + be^{i\theta_r}) \overline{y_r}^2 \end{pmatrix} := \begin{pmatrix} \mathrm{R}_{11} & \mathrm{R}_{12} \\ \mathrm{R}_{21} & \mathrm{R}_{22} \end{pmatrix}.$$

Then

$$\begin{aligned} C_1 =& \frac{4r^2\tau_n}{i\nu_r}\Bigg(-\frac{1}{S_n^2}\langle y_r, \mathrm{R}_{11}\rangle^* \langle y_r, \mathrm{R}_{21} + \mathrm{R}_{12}\rangle^* + \frac{2}{3|S_n|^2}\langle y_r, \mathrm{R}_{22}\rangle^* \langle \overline{y_r}, \mathrm{R}_{11}\rangle^* \\ & + \frac{1}{|S_n|^2}\langle y_r, \mathrm{R}_{21} + \mathrm{R}_{12}\rangle^* \langle \overline{y_r}, \mathrm{R}_{21} + \mathrm{R}_{12}\rangle^* \Bigg), \end{aligned} \tag{4.11}$$

and we have

$$\lim_{r \to d^+} (r - d)^2 \mathrm{Re}\{C_1(r)\} = 0. \tag{4.12}$$

Proof We define $\mathrm{H}(z) = z^{\mathrm{T}}\mathrm{R}z$ to be the quadratic norm $\mathrm{H}(z) = \mathrm{R}_{11}z_1^2 + (\mathrm{R}_{12}+\mathrm{R}_{21})z_1z_2 + \mathrm{R}_{22}z_2^2$. We observe that

$$\mathrm{Proj}_{\mathscr{R}(M_2^1)} f_2^1(z,0,0) = -2r\tau_n\langle \Psi_0, z^{\mathrm{T}}\mathrm{R}z\rangle^*,$$

and from (4.6),

$$M_2^1(z_1^2, z_1z_2, z_2^2)e_1 = i\nu_r\tau_n(z_1^2, -z_1z_2, -3z_2^2)e_1,$$
$$M_2^1(z_1^2, z_1z_2, z_2^2)e_2 = i\nu_r\tau_n(3z_1^2, z_1z_2, -z_2^2)e_2.$$

Hence,

$$\begin{aligned} U_2^1(z,\alpha) &= (M_2^1)^{-1}(\mathrm{Proj}_{\mathscr{R}(M_2^1)} f_2^1(z,0,\alpha)) \\ &= -\frac{2r}{i\nu_r}\begin{pmatrix} \langle \frac{y_r}{S_n}, \mathrm{R}_{11}z_1^2 - (\mathrm{R}_{12}+\mathrm{R}_{21})z_1z_2 + \frac{1}{3}\mathrm{R}_{22}z_2^2\rangle^* \\ \langle \frac{\overline{y_r}}{\overline{S_n}}, \frac{1}{3}\mathrm{R}_{11}z_1^2 + (\mathrm{R}_{12}+\mathrm{R}_{21})z_1z_2 - \mathrm{R}_{22}z_2^2\rangle^* \end{pmatrix}. \end{aligned}$$

On the other hand,

$$D_z f_2^1(z,0,0) = -2r\tau_n\langle \Psi_0, \nabla(z^{\mathrm{T}}\mathrm{R}z)\rangle^*,$$

where

$$\begin{aligned} \nabla(z^{\mathrm{T}}\mathrm{R}z) &= \left(\frac{\partial}{\partial z_1}(z^{\mathrm{T}}\mathrm{R}z), \frac{\partial}{\partial z_2}(z^{\mathrm{T}}\mathrm{R}z)\right) \\ &= (2\mathrm{R}_{11}z_1 + (\mathrm{R}_{12}+\mathrm{R}_{21})z_2, (\mathrm{R}_{12}+\mathrm{R}_{21})z_1 + 2\mathrm{R}_{22}z_2). \end{aligned}$$

Hence

$$\begin{aligned} &D_z f_2^1(z,0,0) \\ &= -2r\tau_n\begin{pmatrix} \langle \frac{y_r}{S_n}, 2\mathrm{R}_{11}z_1 + (\mathrm{R}_{12}+\mathrm{R}_{21})z_2\rangle^* & \langle \frac{y_r}{S_n}, (\mathrm{R}_{12}+\mathrm{R}_{21})z_1, +2\mathrm{R}_{22}z_2\rangle^* \\ \langle \frac{\overline{y_r}}{\overline{S_n}}, 2\mathrm{R}_{11}z_1 + (\mathrm{R}_{12}+\mathrm{R}_{21})z_2\rangle^* & \langle \frac{\overline{y_r}}{\overline{S_n}}, (\mathrm{R}_{12}+\mathrm{R}_{21})z_1, +2\mathrm{R}_{22}z_2\rangle^* \end{pmatrix}. \end{aligned}$$

Then we can obtain (4.11) from the expressions of $U_2^1(z,\alpha)$ and $D_z f_2^1(z,0,0)$.

We rewrite $C_1(r)$ as the following:

$$\begin{aligned} C_1 = &\frac{4r^2\tau_n}{i\nu_r}\Bigg(-\frac{2}{S_n^2}\langle y_r, (a+be^{-i\theta_r})y_r^2\rangle^*\langle y_r, (a+b\mathrm{Re}\{e^{i\theta_r}\})|y_r|^2\rangle^* \\ &+\frac{2}{3|S_n|^2}|\langle y_r, (a+be^{i\theta_r})\overline{y_r}^2\rangle^*|^2 + \frac{4}{|S_n|^2}|\langle y_r, (a+b\mathrm{Re}\{e^{i\theta_r}\})|y_r|^2\rangle^*|^2\Bigg). \end{aligned}$$

Then

$$\mathrm{Re}\{C_1\} = \mathrm{Re}\left\{\frac{4r^2\tau_n}{i\nu_r}\left(-\frac{2}{S_n^2}\langle y_r, (a+be^{-i\theta_r})y_r^2\rangle^*\langle y_r, (a+b\mathrm{Re}\{e^{i\theta_r}\})|y_r|^2\rangle^*\right.\right\}.$$

It follows that

$$\begin{aligned} \lim_{r\to d^+}(r-d)^2\mathrm{Re}\{C_1\} = &-\mathrm{Re}\left\{\frac{8d^2(\theta_d+2n\pi)}{i(h_d)^2(S_n^d)^2}(a+be^{-i\theta_d})(a+b\cos\theta_d)\left(\int_0^\pi \sin^3 x dx\right)^2\right\} \\ =&0, \end{aligned}$$

since $a + b\cos\theta_d = 0$, here $S_n^d = \lim_{r\to d^+} S_n(r)$. □

From [8], the $U_2^2(z,0)$ can be uniquely determined by $M_2^2 U_2^2(z,0) = f_2^2(z,0,0)$, which is equivalent to

$$D_z U_2^2(z,0)B_n z - \mathcal{A}_n^1\big((U_2^2(z,0)\big) = (I-\pi)X_0 F_2(\Phi z, 0).$$

Define

$$U_2^2(z,0) := p(z)(\theta) := p_{20}(\theta)z_1^2 + p_{11}(\theta)z_1 z_2 + p_{02}(\theta)z_2^2, \tag{4.13}$$

here $p_{20},\ p_{11},\ p_{02} \in Q_0^1$. From (4.9),

$$f_2^1(z,y,0) = -2r\tau_n\langle\Psi_0, \big(\Phi_0 z + y_0\big)\big(a\Phi_0 z + ay_0 + b\Phi_{-1}z + by_{-1}\big)\rangle^*.$$

Then for $W \in Q_0^1$,

$$\begin{aligned} D_y f_2^1(z,y,0)W =& -2r\tau_n\langle\Psi_0, W_0\big(\Phi_0 z + y_0\big)\big(a\Phi_0 z + ay_0 + b\Phi_{-1}z + by_{-1}\big)\rangle^* \\ & -2r\tau_n\langle\Psi_0, \big(\Phi_0 z + y_0\big)\big(aW_0 + bW_{-1}\big)\rangle^*, \end{aligned}$$

where $W_\theta = W(\theta),\ \theta = 0,\ -1$. Hence

$$D_y f_2^1(z,y,0)U_2^2(z,0) = -2r\tau_n\langle\Psi_0, \big(2a\Phi_0 z p(z)(0) + b\Phi_{-1}z p(z)(0) + b\Phi_0 z p(z)(-1)\big)\rangle^*.$$

Now the coefficient of $z_1^2 z_2$ in $2a\Phi_0 z p(z)(0) + b\Phi_{-1} z p(z)(0) + b\Phi_0 z p(z)(-1)$ is

$$(2a + be^{-i\theta_r})y_r p_{11}(0) + (2a + be^{-i\theta_r})\overline{y_r}p_{20}(0) + by_r p_{11}(-1) + b\overline{y_r}p_{20}(-1),$$

Hence

$$\begin{aligned} C_2 = -\frac{2r\tau_n}{S_n}\Big\{ & \langle y_r, [bp_{11}(-1) + (2a + be^{-i\theta_r})p_{11}(0)]y_r\rangle^* \\ & + \langle y_r, [bp_{20}(-1) + (2a + be^{i\theta_r})p_{20}(0)]\overline{y_r}\rangle^* \Big\}. \end{aligned} \tag{4.14}$$

Now, we need to determine $p_{20}(0),\ p_{20}(-1),\ p_{11}(0)$ and $p_{11}(-1)$.

By using the definitions of $\mathcal{A}_n^1$ and F_2, we have

$$\begin{cases} \dot{p}(z)(\theta) - D_z p(z)(\theta)B_n z = -2r\tau_n\Phi\langle\Psi_0, z^T \mathrm{R} z\rangle^*, \\ \dot{p}(z)(0) - \tau_n A(r)p(z)(0) + rb\tau_n u_r p(z)(-1) = -2r\tau_n z^{\mathrm{T}}\mathrm{R}z. \end{cases} \tag{4.15}$$

Substituting (4.13) into the first equation of (4.15) we obtain

$$\begin{cases} \dot{p}_{20}(\theta) - 2i\nu_r\tau_n p_{20}(\theta) = \tau_n k_r^1 e^{i\nu_r\tau_n\theta} + \tau_n k_r^2 e^{-i\nu_r\tau_n\theta} \\ \dot{p}_{11}(\theta) = \tau_n k_r^3 e^{i\nu_r\tau_n\theta} + \tau_n k_r^4 e^{-i\nu_r\tau_n\theta}, \end{cases} \tag{4.16}$$

where

$$\begin{aligned} k_r^1 &= -2r\frac{\langle y_r, \mathrm{R}_{11}\rangle^* y_r}{S_n},\ k_r^2 = -2r\frac{\langle\overline{y_r}, \mathrm{R}_{11}\rangle^*\overline{y_r}}{\overline{S_n}}, \\ k_r^3 &= -2r\frac{\langle y_r, \mathrm{R}_{12} + \mathrm{R}_{21}\rangle^* y_r}{S_n},\ k_r^4 = -2r\frac{\langle\overline{y_r}, \mathrm{R}_{12} + \mathrm{R}_{21}\rangle^*\overline{y_r}}{\overline{S_n}}. \end{aligned}$$

Solving the equations in (4.16), we obtain that, for $-1 \le \theta < 0$,

$$\begin{cases} p_{20}(\theta) = e^{2i\nu_r\tau_n\theta}p_{20}(0) + \dfrac{k_r^1 i}{\nu_r}(e^{i\nu_r\tau_n\theta} - e^{2i\nu_r\tau_n\theta}) + \dfrac{k_r^2 i}{3\nu_r}(e^{-i\nu_r\tau_n\theta} - e^{2i\nu_r\tau_n\theta}), \\ p_{11}(\theta) = p_{11}(0) - \dfrac{k_r^3 i}{\nu_r}(e^{i\nu_r\tau_n\theta} - 1) + \dfrac{k_r^4 i}{\nu_r}(e^{-i\nu_r\tau_n\theta} - 1). \end{cases} \tag{4.17}$$

Using the second equation of (4.15) we get

$$\begin{cases} \dot{p}_{20}(0) - \tau_n A(r) p_{20}(0) + br\tau_n u_r p_{20}(-1) = -2r\tau_n \mathrm{R}_{11}, \\ \dot{p}_{11}(0) - \tau_n A(r) p_{11}(0) + br\tau_n u_r p_{11}(-1)] = -2r\tau_n (\mathrm{R}_{12} + \mathrm{R}_{11}). \end{cases} \tag{4.18}$$

However, using (4.16) we have

$$\begin{cases} \dot{p}_{20}(0) = 2i\nu_r \tau_n p_{20}(0) + \tau_n k_r^1 + \tau_n k_r^2, \\ \dot{p}_{11}(0) = \tau_n k_r^3 + \tau_n k_r^4. \end{cases} \tag{4.19}$$

Combining (4.18) and (4.19) we have the following equations of $p_{20}(0)$ and $p_{11}(0)$:

$$\begin{cases} A_{20}(r) p_{20}(0) = \dfrac{bru_r i}{\nu_r}[k_r^1(e^{-i\theta_r} - e^{-2i\theta_r}) + \dfrac{k_r^2}{3}(e^{i\theta_r} - e^{-2i\theta_r})] + k_r^1 + k_r^2 + 2r\mathrm{R}_{11}, \\ A_{11}(r) p_{11}(0) = -\dfrac{bru_r i}{\nu_r}[k_r^3(e^{-i\theta_r} - 1) - k_r^4(e^{i\theta_r} - 1)] + k_r^3 + k_r^4 + 2r(\mathrm{R}_{12} + \mathrm{R}_{21}), \end{cases}$$

with

$$\begin{cases} A_{20}(r) = A(r) - 2i\nu_r - rbu_r e^{-2i\theta_r}, \\ A_{11}(r) = A(r) - bru_r. \end{cases}$$

Let $q_{20}(\theta) = \nu_r p_{20}(\theta)$ and $q_{11}(\theta) = \nu_r p_{11}(\theta)$, then from (4.17),

$$\begin{cases} q_{20}(\theta) = e^{2i\nu_r\tau_n\theta} q_{20}(0) + k_r^1 i(e^{i\nu_r\tau_n\theta} - e^{2i\nu_r\tau_n\theta}) + \dfrac{k_r^2 i}{3}(e^{-i\nu_r\tau_n\theta} - e^{2i\nu_r\tau_n\theta}), \\ q_{11}(\theta) = q_{11}(0) - k_r^3 i(e^{i\nu_r\tau_n\theta} - 1) + k_r^4 i(e^{-i\nu_r\tau_n\theta} - 1). \end{cases} \tag{4.20}$$

We decompose $q_*(0)$ as $q_*(0) = s_*^r \sin(\cdot) + z_*^r$, where $s_*^r \in \mathbb{C}$, $z_*^r \in Q_0^1$ satisfy $\int_0^\pi q_*(x)\sin x dx = 0$ for $* = 20, 11$. Since $\lim_{r\to d^+} ||A_*(r) - (dD^2 + d)|| = 0$, and $\lim_{r\to d^+} \nu_r \rho_*(r) = 0$ uniformly for $x \in [0, \pi]$, then $\lim_{r\to d^+} ||z_*^r|| = 0$. Define $q_*^d(\theta) = \lim_{r\to d^+} q_*^r(\theta)$, $s_*^d = \lim_{r\to d^+} s_*^r$ and $k_d^j = \lim_{r\to d^+} k_d^j$ for $* = 20, 11$ and $j = 1, 2, 3, 4$. Then,

$$\begin{cases} q_*^d(0) = s_*^d \sin x, \\ q_{20}^d(\theta) = s_{20}^d e^{2i\theta_d\theta}\sin x + k_d^1 i(e^{i\theta_d\theta} - e^{2i\theta_d\theta}) + \dfrac{k_d^2 i}{3}(e^{-i\theta_d\theta} - e^{2i\theta_d\theta}), \\ q_{11}^d(\theta) = s_{11}^d \sin x - k_d^3 i(e^{i\theta_d\theta} - 1) + k_d^4 i(e^{-i\theta_d\theta} - 1). \end{cases} \tag{4.21}$$

Now from

$$\langle\langle \Psi, q_*(\theta)\rangle\rangle = \langle \Psi_0, q_*(0)\rangle^* - br\tau_n \int_{-1}^0 \langle \Psi_{s+1}, u_r q_*(s)\rangle^* ds = 0,$$

taking limit as $r \to d^+$, and using (4.21), we obtain

$$\begin{aligned} s_{20}^d = &\frac{bi(\theta_d + 2n\pi)e^{-i\theta_d}}{\mu_d(a+b)h_d \int_0^\pi \sin^3 x dx}\left[\left(1 - \frac{1 - e^{-i\theta_d}}{i\theta_d}\right)\int_0^\pi k_d^1 \sin^2 x dx\right. \\ &\left. - \frac{3 + e^{2i\theta_d} - 2e^{-i\theta_d}}{6i\theta_d}\int_0^\pi k_d^2 \sin^2 x dx\right], \end{aligned} \tag{4.22}$$

where

$$\mu_d = 1 - \frac{b(\theta_d + 2n\pi)}{i(a+b)h_d\theta_d}(e^{-i\theta_d} - e^{-2i\theta_d}). \tag{4.23}$$

By using

$$\begin{pmatrix} k_d^1 & k_d^2 \\ k_d^3 & k_d^4 \end{pmatrix} = -2d \sin x \int_0^\pi \sin^3 x dx \begin{pmatrix} \dfrac{a + be^{-i\theta_d}}{S_n^d} & \dfrac{a + be^{-i\theta_d}}{\overline{S_n^d}} \\ \dfrac{2(a + b\cos\theta_d)}{S_n^d} & \dfrac{2(a + b\cos\theta_d)}{\overline{S_n^d}} \end{pmatrix}, \tag{4.24}$$

and noting that $a + b\cos\theta_d = 0$ so that $k_d^3 = k_d^4 = 0$, hence $s_{11}^d = 0$ and $q_{11}^d(\theta) \equiv 0$ for $\theta \in [-1, 0]$. Using (4.14), we obtain that

$$\begin{aligned}
\lim_{r\to d^+} (r-d)^2 C_2 = &- \frac{2d(\theta_d + 2n\pi)}{S_n^d h_d^2} \int_0^\pi \sin^3 x dx \Big[(2a + be^{i\theta_d} + be^{-2i\theta_d}) s_{20}^d \\
&- 2bdi(a + be^{-i\theta_d}) \int_0^\pi \sin^3 x dx \Big(\frac{e^{-i\theta_d} - e^{-2i\theta_d}}{S_n^d} + \frac{e^{i\theta_d} - e^{-2i\theta_d}}{3\overline{S_n^d}} \Big) \Big] \\
= &\frac{4bd^2 i(\theta_d + 2n\pi)(a + be^{-i\theta_d})}{S_n^d h_d^2 \mu_d} \Big(\int_0^\pi \sin^3 x dx \Big)^2 \\
&\cdot \Big\{ \frac{(\theta_d + 2n\pi)(2a + be^{i\theta_d} + be^{-2i\theta_d})e^{-i\theta_d}}{(a+b)h_d\mu_d} \Big[\frac{1}{S_n^d} \Big(1 - \frac{1 - e^{-i\theta_d}}{i\theta_d}\Big) \\
&- \frac{1}{\overline{S_n^d}} \frac{3 + e^{2i\theta_d} - 2e^{-i\theta_d}}{6i\theta_d} \Big] + \frac{e^{-i\theta_d} - e^{-2i\theta_d}}{S_n^d} + \frac{e^{i\theta_d} - e^{-2i\theta_d}}{3\overline{S_n^d}} \Big\} \\
= &\frac{4bd^2(\theta_d + 2n\pi)(a+b)^2\sqrt{b^2 - a^2}(\int_0^\pi \sin^3 x dx)^2}{b^2 - a^2} I(n, a, b),
\end{aligned}$$

where

$$\begin{aligned}
I(n, a, b) = &\frac{(\theta_d + 2n\pi)(2a + be^{i\theta_d} + be^{-2i\theta_d})e^{-i\theta_d}}{(a+b)h_d\mu_d} \Big[\frac{1}{(S_n^d)^2} \Big(1 - \frac{1 - e^{-i\theta_d}}{i\theta_d}\Big) \\
&- \frac{1}{|S_n^d|^2} \frac{3 + e^{2i\theta_d} - 2e^{-i\theta_d}}{6i\theta_d} \Big] + \frac{e^{-i\theta_d} - e^{-2i\theta_d}}{(S_n^d)^2} + \frac{e^{i\theta_d} - e^{-2i\theta_d}}{3|S_n^d|^2}.
\end{aligned} \tag{4.25}$$

From Lemmas 4.3 and 4.5, the determination of the direction of the Hopf bifurcations and the stability of the bifurcating periodic solutions for (1.4) is calculating the sign of the quantity $\mathrm{Re}\{I(n, a, b)\}$. Without loss of generality, we can assume that $a + b = 1$, it follows that $b \in (0.5, 1)$. Notice that if we introduce a transformation $v(x, t) = (a + b)u(x, t)$, then v satisfies the following equation

$$\frac{\partial v(x,t)}{\partial t} = d\tau \frac{\partial v^2(x,t)}{\partial x^2} + r\tau v(x,t) \Big[1 - \frac{a}{a+b} v(x,t) - \frac{b}{a+b} v(x, t-1) \Big], \tag{4.26}$$

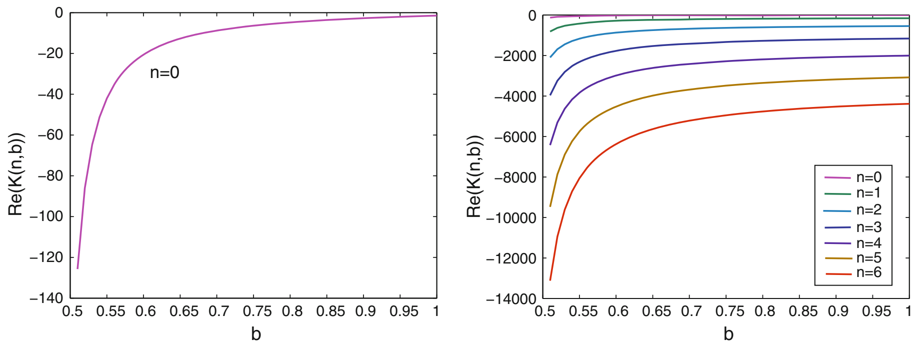

Fig. 1 Graph of functions $\mathrm{Re}\{K(n,b)\}$. *Left* $n=0$. *Right* $0 \le n \le 6$

which is same as (1.4) ignoring a scalar factor. Note that we can rewrite $I(n,a,b) = I(n,b)$ as follows:

$$\begin{aligned} I(n,b) =& \frac{(\int_0^\pi \sin^2 x)^2}{|S_n^d|^4} \left\{ \frac{(\theta_d + 2n\pi)(2a + be^{i\theta_d} + be^{-2i\theta_d})e^{-i\theta_d}}{(a+b)h_d\mu_d} \right. \\ & \left[\frac{(\overline{S}_n^d)^2}{(\int_0^\pi \sin^2 x)^2}\left(1 - \frac{1-e^{-i\theta_d}}{i\theta_d}\right) - \frac{|S_n^d|^2}{(\int_0^\pi \sin^2 x)^2} \frac{3 + e^{2i\theta_d} - 2e^{-i\theta_d}}{6i\theta_d} \right] \\ & \left. + \frac{(\overline{S}_n^d)^2}{(\int_0^\pi \sin^2 x)^2}(e^{-i\theta_d} - e^{-2i\theta_d}) + \frac{|S_n^d|^2}{(\int_0^\pi \sin^2 x)^2} \frac{e^{i\theta_d} - e^{-2i\theta_d}}{3} \right\} \\ :=& \frac{(\int_0^\pi \sin^2 x)^2}{|S_n^d|^4} K(n,b), \end{aligned} \tag{4.27}$$

where $K(n,b)$ is a function of variables $b \in (0.5, 1)$ and $n = 0, 1, 2, \cdots$.

It is difficult to determine the sign of $\mathrm{Re}\{K(n,b)\}$ analytically due to its complicated form, but for $b \in (0.5, 1)$ and each fixed $n = 0, 1, 2, \cdots$, the graph of function $\mathrm{Re}\{K(n,b)\}$ can be plotted (see Fig. 1), and it follows that $\mathrm{Re}\{C_2\} < 0$ for any $b \in (0.5, 1)$ and $n = 0, 1, 2, \cdots$. This implies that the constant $K_2 = \mathrm{Re}\{A_2\} < 0$ from Lemma 4.5. Hence the direction of each Hopf bifurcation at $\tau = \tau_n$ is forward (periodic orbits exist for $\tau \in (\tau_n, \tau_n + \epsilon)$, and the bifurcating periodic orbits are locally orbitally stable from Lemma 4.3 and the well-known formulas for Hopf bifurcations [44].

Summarizing above results, we obtain the following theorem about the direction of local Hopf bifurcation and the stability of the bifurcating periodic solutions.

Theorem 4.6 *For each fixed* $r \in (d, r^*)$, (1.4) *undergoes a Hopf bifurcation at* $u = u_r$ *when* $\tau = \tau_n$ $(n = 0, 1, 2, \cdots)$. *Moreover, all bifurcating periodic solutions are locally orbitally asymptotically stable on the center manifold near* $\tau = \tau_n$ *and* $u = u_r$ *and the direction of bifurcations are forward. In particular, the bifurcating periodic solutions from the first bifurcation value* $\tau = \tau_0$ *are orbitally asymptotically stable in the entire phase space.*

5 Global Existence of Periodic Solutions

In this section, we study the global continuation of periodic solutions bifurcating from the point (u_r, τ_n), $n = 0, 1, 2, \cdots$ for Eq. (1.4) using global Hopf bifurcation theorem given by

Wu [44,45]. Throughout this section, we closely follow the notations in [44]. To state the global Hopf bifurcation theorem, we define that

(i) $E = C(S^1; X)$ is a real isometric Banach representation of the group $G = S^1 := \{z \in \mathbb{C} : |z| = 1\}$;
(ii) Let $E^G := \{x \in E : gx = x \text{ for all} g \in G\}$. Then $E^G = X$, and E has an isotypical direct sum decomposition $E = E^G \bigoplus_{k=1}^{\infty} E_k$ where $E_k = \{e^{ikt}x : x \in X\}$ for $k \geq 1$.

Then from [44], Eq. (1.4) can be casted into an integral equation which is continuously differentiable, completely continuous, and G-invariant.

We fix $r \in (d, r^*)$ and $n \in \mathbb{N} \bigcup \{0\}$. Recall that u_r is the unique positive steady state solution of (1.4). From Lemma 3.1, for any $\tau \geq 0$, 0 is not an eigenvalue of $\mathcal{A}_\tau(r)$, hence the assumption **(H1)** in [44, Sect. 6.5] is satisfied. When $\tau = \tau_n$, $\mathcal{A}_\tau(r)$ has a unique pair of purely imaginary eigenvalues $\pm i\nu_r\tau_n$, hence the assumption **(H2)** in [44, Sect. 6.5] is satisfied. We choose sufficiently small $\epsilon_0, \varsigma_0 > 0$, and we define the local steady state manifold

$$M_r = \{(u_r, \tau, \omega) : |\tau - \tau_n| < \epsilon_0, |\omega - \nu_r\tau_n| < \varsigma_0\} \subset E^G \times \mathbb{R} \times \mathbb{R}_+.$$

Then for $(\tau, \omega) \in [\tau_n - \epsilon_0, \tau_n + \epsilon_0] \times [\nu_r\tau_n - \varsigma_0, \nu_r\tau_n + \varsigma_0]$, $i\omega$ is an eigenvalue of $\mathcal{A}_\tau(r)$ if and only if $\tau = \tau_n$ and $\omega = \nu_r\tau_n$ from our results in Sect. 2. This verifies the assumption **(H3)** in [44, Sect. 6.5]. From [44, Lemma 6.5.3], we conclude that $(u_r, \tau_n, \nu_r\tau_n)$ is an isolated singular point in M_r.

Let $\mu_k(u_r, \tau_n, \nu_r\tau_n)$ $(k = 1, 2, \cdots)$ be the generalized crossing number defined in [44, Sect. 6.5]. Then from Theorem 3.4, if $\lambda(\tau) = \alpha(\tau) \pm i\beta(\tau)$ are the eigenvalues of $\mathcal{A}_\tau(r)$ satisfying $\lambda(\tau_n) = \pm i\nu_r\tau_n$, then $\alpha'(\tau_n) > 0$. This implies that $\mu_1(u_r, \tau_n, \nu_r\tau_n) = 1$. Hence one obtains the local topological Hopf bifurcation for Eq. (1.4) at $\tau = \tau_n$.

Next we consider the global nature of the Hopf bifurcation. Let S be the closure of the set

$$\{(z, \tau, \omega) \in E \times \mathbb{R} \times \mathbb{R}_+ : u(\cdot, t) = z(\cdot, \omega t) \text{ is a nontrivial } 2\pi/\omega \text{ periodic solution of (1.4)}\}.$$

Then from the local bifurcation theorem, $(u_r, \tau_n, \nu_r\tau_n) \in S$. We also define the complete steady state manifold:

$$M_r^* = \{(u_r, \tau) : \tau \in \mathbb{R}\} \subset E^G \times \mathbb{R}.$$

Let $\mathfrak{C}_n = \mathfrak{C}(u_r, \tau_n, \nu_r\tau_n)$ denote the connected component of S for which $(u_r, \tau_n, \nu_r\tau_n)$ belongs to. Then the global Hopf bifurcation theorem of (1.4) can be adapted from [44, Theorem 6.5.5]:

Theorem 5.1 *Let S, M_r^*, and $\mathfrak{C}_n$ be defined as above. Then for each $n \in \mathbb{N} \bigcup \{0\}$, $\mathfrak{C}_n$ is unbounded, i.e.,*

$$\sup \left\{ \max_{t \in \mathbb{R}} |z(t)| + |\tau| + \omega + \omega^{-1} : (z, \tau, \omega) \in \mathfrak{C}_n \right\} = \infty. \tag{5.1}$$

Proof From Theorem 6.5.5 in [44], one of the following holds:

(i) $\mathfrak{C}_n$ is unbounded, i.e. (5.1) holds; or
(ii) $\mathfrak{C}_n \cap (M_r^* \times \mathbb{R}_+)$ is finite and for all $k \geq 1$, one has the equality

$$\sum_{(u_r, \tau_j, \nu_r\tau_j) \in \mathfrak{C}_n \cap (M_r^* \times \mathbb{R}_+)} \mu_k(u_r, \tau_j, \nu_r\tau_j) = 0, \tag{5.2}$$

where μ_k is the k-th generalized crossing number. However from Theorem 3.4, if $\lambda(\tau) = \alpha(\tau) \pm i\beta(\tau)$ are the eigenvalues of $\mathcal{A}_\tau(r)$ satisfying $\lambda(\tau_n) = \pm i\nu_r\tau_n$, then $\alpha'(\tau_n) > 0$. This implies that $\mu_1(u_r, \tau_n, \nu_r\tau_n) = 1$. Hence if case (ii) occurs, then the sum $\sum \mu_1(u_r, \tau_j, \nu_r\tau_j) = p > 0$, where p is the number of elements in $\mathfrak{C}_n \cap (M_r^* \times \mathbb{R}_+)$. That is a contradiction to (5.2) when $k = 1$. Therefore the second alternative could not happen, and $\mathfrak{C}_n$ is unbounded. □

For the further structure of each $\mathfrak{C}_n$, we prove the following properties of solutions of Eq. (1.4):

Lemma 5.2 *Suppose that $r > d$, and $u(x,t)$ is the solution of* (1.4)–(1.5) *with $\eta(x,t) \geq 0$ for $t \in [-1, 0]$, $x \in (0, \pi)$, then $u(\cdot, t)$ exists for all $t \in (0, \infty)$, and there exists $T_\eta > 0$ so that*

$$0 \leq u(x,t) \leq \frac{1}{a}, \quad t > T_\eta. \tag{5.3}$$

Moreover, for $t \in (0, \infty)$ and $x \in (0, \pi)$, $u(x,t) > 0$ if $\eta(x, 0) \geq (\not\equiv) 0$, and $u(x,t) \equiv 0$ if $\eta(x, 0) \equiv 0$.

Proof From [44, Chap. 2], (1.4)–(1.5) has a local solution $u(x,t)$. We choose $K \geq \max\{1/a, \max |\eta(t,x)|\}$, and let $v(x,t)$ be the unique solution of

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial t} = d\tau \dfrac{\partial v^2(x,t)}{\partial x^2} + r\tau v(x,t)[1 - av(x,t)], \ x \in (0,\pi), \ t > 0, \\ v(0,t) = v(\pi,t) = 0, \ t \geq 0, \\ v(x,0) = K. \end{cases} \tag{5.4}$$

It is well-known that $v(x,t)$ exists and $v(x,t) > 0$ for $t \in (0,\infty)$ and $x \in (0,\pi)$, and $\lim_{t\to\infty} v(x,t) = v_a(x)$, which is the unique positive steady state solution of (5.4) (see [17]). Note that we assume that $r > d$ here so $v_a(x)$ exists. Then $\underline{u}(x,t) = 0$ and $\overline{u}(x,t) = v(x,t)$ are a pair of upper and lower solutions of (1.4)–(1.5) as [31, Definition 2.2]. Then from the comparison principle (see [31, Theorem 2.1]), $u(x,t)$ exists and $0 = \underline{u}(x,t) \leq u(x,t) \leq \overline{u}(x,t) = v(x,t)$ for all $t \in (0,\infty)$. Since $0 < v_a(x) < 1/a$ for $0 < x < \pi$, then (5.3) holds for $t > T_\eta$. The last assertion follows from strong maximum principle of parabolic equations. □

Secondly we claim that Eq. (1.4) has no positive periodic orbit for small $\tau > 0$.

Lemma 5.3 *Suppose that $d < r < r^*$, then Eq.* (1.4) *has no positive nontrivial periodic orbit for small $\tau > 0$.*

Proof From [12] (see also [44, Sect. 10.2]), for small enough $\tau > 0$, the unique positive steady state solution u_r is globally asymptotically stable for all positive initial values for Eq. (1.4). Thus the conclusion of this lemma holds. □

Next we prove a lemma about the nonexistence of positive periodic orbits of (1.4) with certain periods:

Lemma 5.4 *Assume that $b > a > 0, r > d > 0$ and $\tau > 0$. Suppose that $u(x,t)$ is a nontrivial T-periodic solution of* (1.4) *satisfying $u(x,t) > 0$ for $t \in \mathbb{R}$ and $x \in (0,\pi)$, and $T > 0$. Then $T \neq 1/m$ for $m \in \mathbb{N}$.*

Proof We prove that (1.4) has no nontrivial positive 1-periodic solution. Indeed a nontrivial 1-periodic solution of (1.4) is also a nontrivial periodic solution of the following diffusive logistic equation:

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial t} = d\tau \dfrac{\partial v^2(x,t)}{\partial x^2} + r\tau v(x,t)[1-(a+b)v(x,t)], \ x\in(0,\pi), \ t>0, \\ v(0,t)=v(\pi,t)=0, \ t\geq 0. \end{cases} \tag{5.5}$$

It is well-known that (5.5) has no nonconstant periodic solution as a gradient system, and the unique positive steady state solution is globally asymptotically stable with respect to all non-negative initial values ($\not\equiv 0$) (see [17,31,37]). Therefore (1.4) has no nonconstant 1-periodic solution. Since any nontrivial $1/m$-periodic solution is also a nontrivial 1-periodic solution, then there is no nontrivial positive $1/m$-periodic solution of (1.4) as well. □

The nonexistence of nontrivial positive $1/m$-periodic solution implies the following important estimate for the periods of periodic orbits on $\mathfrak{C}_n$:

Corollary 5.5 *Suppose that* $(z,\tau,\omega)\in\mathfrak{C}_n$ *for* $n\in\mathbb{N}\bigcup\{0\}$. *Then* $1/(n+1)<\omega<1/n$ *if* $n\geq 1$, *and* $\omega>1$ *if* $n=0$.

Proof From (1.8) and (2.9), we know that

$$\lim_{r\to d^+} \nu_r\tau_n(r) = \arccos\left(\frac{-a}{b}\right)+2n\pi.$$

Hence near the bifurcation point $(u_r,\tau_n,\nu_r\tau_n)$, the period ω of periodic orbit is close to $2\pi/(\nu_r\tau_n)$ which satisfies

$$\frac{2}{2n+1}=\frac{2\pi}{\pi+2n\pi}<\frac{2\pi}{\tau_n\nu_r}<\frac{2\pi}{\pi/2+2n\pi}=\frac{4}{4n+1}. \tag{5.6}$$

When $n\geq 1$, (5.6) implies that

$$\frac{1}{n+1}<\frac{2\pi}{\tau_n\nu_r}<\frac{1}{n}. \tag{5.7}$$

From Lemma 5.4 and the continuity of ω on $\mathfrak{C}_n$, $1/(n+1)<\omega<1/n$ for any $(z,\tau,\omega)\in\mathfrak{C}_n$. Similarly for $n=0$, we have $\omega>1$ for any $(z,\tau,\omega)\in\mathfrak{C}_0$. □

Next we show that the strong maximum principle also implies that every periodic orbit on $\mathfrak{C}_n$ must be strictly positive.

Lemma 5.6 *Suppose that* $(z,\tau,\omega)\in\mathfrak{C}_n$ *for* $n\in\mathbb{N}\bigcup\{0\}$, *and let* $u(x,t)$ *be a* ω-*periodic solution of* (1.4) *with delay* τ *which is a representation of* z. *Then* $u(x,t)>0$ *for* $t\in\mathbb{R}$ *and* $x\in(0,\pi)$.

Proof Since $(u_r,\tau_n,\nu_r\tau_n)\in\mathfrak{C}_n$, then near the bifurcation point $(u_r,\tau_n,\nu_r\tau_n)$, any $(z,\tau,\omega)\in\mathfrak{C}_n$ satisfies $u(x,t)>0$ for $t\in\mathbb{R}$ and $x\in(0,\pi)$, where $u(x,t)$ is an ω-periodic solution of (1.4) with delay τ and u is a representation of z. Suppose that the assertion does not hold for all $(z,\tau,\omega)\in\mathfrak{C}_n$. Then there exists a $(z^*,\tau^*,\omega^*)\in\mathfrak{C}_n$ such that if $u^*(x,t)$ is an ω^*-periodic solution of (1.4) with delay τ^* and u^* is a representation of z^*, and either (i) $u^*(x^*,t^*)=0$ for some $x^*\in(0,\pi)$ and $t^*\in\mathbb{R}$, or (ii) $u^*(x,t^*)>0$ for all $x\in(0,\pi)$ but $u^*_x(x^*,t^*)=0$ for $x^*=0$ or π. For case (i), the strong maximum principle of parabolic equations [30,39] implies that $u^*(x,t)\equiv 0$; and for case (ii), the Hopf boundary lemma of parabolic equations [11,39] (especially Corollary 9.14 of [39]) implies that $u^*(x,t)\equiv 0$.

From Lemma 5.3, $\tau^* > 0$, and from Corollary 5.5, $\omega^* > 0$. This would make $\tau = \tau^*$ a Hopf bifurcation point for periodic orbits with period near ω^* from the steady state solution $u = 0$. Hence the linearized Eq. (2.3) at $u = 0$ has a pair of purely imaginary eigenvalues $\pm\omega^* i$. But this is impossible since the linearization at $u = 0$ does not any eigenvalues with nonzero imaginary part. That is a contradiction, and hence the assertion of the lemma holds. □

Now we are in the position to state a structure theorem about $\mathfrak{C}_n$ and the multiplicity of periodic orbits of (1.4).

Theorem 5.7 *Let $\mathfrak{C}_n$ be the connected component of S for which $(u_r, \tau_n, \nu_r\tau_n)$ belongs to, where $n = 0, 1, 2, \cdots$.*

1. *For any $n, m \in \mathbb{N}\bigcup\{0\}, n \neq m, \mathfrak{C}_n \bigcap \mathfrak{C}_m = \emptyset$;*
2. *For each $n \in \mathbb{N}$, the projection of $\mathfrak{C}_n$ to the τ-component is unbounded, and indeed, define $Proj_\tau\mathfrak{C}_n := \{\tau : (z, \tau, \omega) \in \mathfrak{C}_n\}$, then $Proj_\tau\mathfrak{C}_n \supseteq (\tau_n, \infty)$;*
3. *For any $\tilde{\tau} > \tau_n$, $S_{\tilde{\tau}} := \{(z, \tilde{\tau}, \omega) \in S\}$ contains at least n elements $(z_i, \tilde{\tau}, \omega_i)$, $(1 \leq i \leq n)$, and $(z_i, \tilde{\tau}, \omega_i) \in S \bigcap \mathfrak{C}_i$.*

Proof From Corollary 5.5, the ranges of period ω on $\mathfrak{C}_n$ and $\mathfrak{C}_m$ are disjoint, hence $\mathfrak{C}_n \bigcap \mathfrak{C}_m = \emptyset$ for any $n, m \in \mathbb{N}\bigcup\{0\}, n \neq m$. From Theorem 5.1, each $\mathfrak{C}_n$ is unbounded in the sense of (5.1). For $n \in \mathbb{N}\bigcup\{0\}$, $\max_{t\in\mathbb{R}} |z(t)|$ is uniformly bounded for all $(z, \tau, \omega) \in \mathfrak{C}_n$ from Lemma 5.2. From Corollary 5.5, for $n \in \mathbb{N}$, $\omega + \omega^{-1}$ is also uniformly bounded for all $(z, \tau, \omega) \in \mathfrak{C}_n$. Thus the projection of $\mathfrak{C}_n$ to τ component must be unbounded from (5.1). From Lemma 5.3, $\mathrm{Proj}_\tau\mathfrak{C}_n$ does not extend to small $\tau > 0$, hence it must extend to $\tau = \infty$. Therefore $\mathrm{Proj}_\tau\mathfrak{C}_n \supseteq (\tau_n, \infty)$. Since $\mathrm{Proj}_\tau\mathfrak{C}_i \supseteq (\tau_i, \infty)$ for any $i \geq 1$, then $S_{\tilde{\tau}} \bigcap \mathfrak{C}_i \neq \emptyset$ for $1 \leq i \leq n$ if $\tilde{\tau} > \tau_n$. Thus $S_{\tilde{\tau}}$ contains at least n elements for any $\tilde{\tau} > \tau_n$. □

We now has the following existence and multiplicity result for the periodic orbits of Eq. (1.4):

Theorem 5.8 *For each fixed $r \in (d, r^*)$, (1.4) has at least one periodic orbit when $\tau > \tau_1$, and (1.4) has at least two distinct periodic orbits when $\tau \in (\tau_n, \tau_n + \epsilon)$ for $n \geq 2$ and some $\epsilon > 0$.*

Proof Since $\mathrm{Proj}_\tau\mathfrak{C}_1 \supseteq (\tau_1, \infty)$, then (1.4) has at least one periodic orbit when $\tau > \tau_1$. We denote by $(z_1(\tau), \tau, \omega_1(\tau))$ an element of $\mathfrak{C}_1$ with delay value $\tau > \tau_1$. When $\tau > \tau_n$, $S_\tau \bigcap \mathfrak{C}_n \neq \emptyset$, hence we denote by $(z_n(\tau), \tau, \omega_n(\tau))$ an element of $\mathfrak{C}_n$ with delay value $\tau > \tau_n$.

We call (z_1, τ, ω_1) and $(z_2, \tau, \omega_2) \in S$ to be geometrically identical, if there exist representation $u_i(x, t)$ of z_i $(i = 1, 2)$ such that $u_1(x, t) \equiv u_2(x, t)$ for $t \in \mathbb{R}$ and $x \in (0, \pi)$, otherwise we call (z_1, τ, ω_1), $(z_2, \tau, \omega_2) \in S$ to be geometrically distinctive. It is evident that if (z_1, τ, ω_1) and $(z_2, \tau, \omega_2) \in S$ are geometrically identical, then $\omega_1 = k\omega_2$ or$\omega_2 = k\omega_1$ for some $k \in \mathbb{N}$. When $\tau \to \tau_n^+$ with $n \geq 2$, $(z_1(\tau), \tau, \omega_1(\tau))$ and $(z_n(\tau), \tau, \omega_n(\tau))$ are two geometrically distinctive periodic orbits since $z_n(\tau) \to u_r$ and $z_1(\tau) \not\to u_r$ as $\tau \to \tau_n^+$ from the uniqueness of bifurcating periodic orbits near $u = u_r$ and $\tau = \tau_n$. Hence (1.4) has at least two distinct periodic orbits when $\tau \in (\tau_n, \tau_n + \epsilon)$ for $n \geq 2$ and some $\epsilon > 0$. □

We remark that in general, for the global Hopf bifurcation theorem, it is hard to assert the existence of n geometrically distinctive periodic orbits when $\tau > \tau_n$, although one can obtain

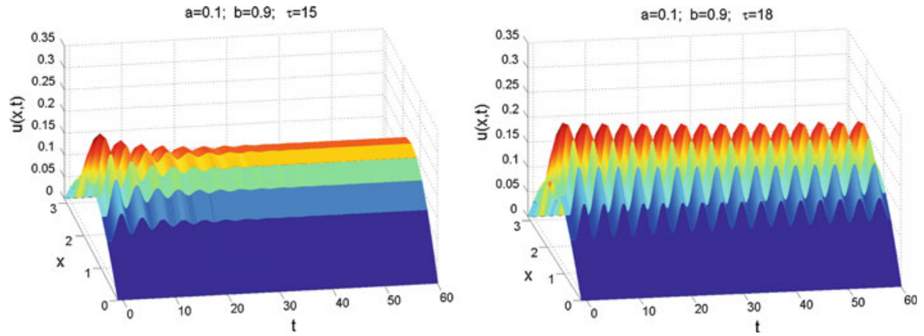

Fig. 2 *Left* $\tau = 15$, the solution tends to a positive steady state. *Right* $\tau = 18$, the solution converges to a time-periodic solution with small amplitude

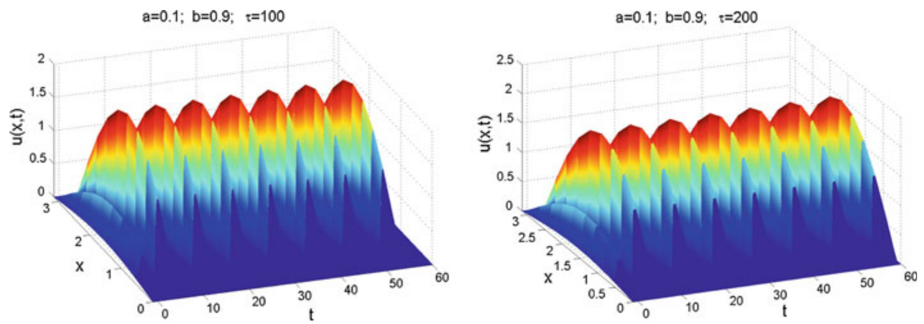

Fig. 3 *Left* $\tau = 100$. *Right* $\tau = 200$. In both cases, the solution converges to a time-periodic solution with larger amplitude

n different elements on the different connected components $\mathfrak{C}_m$ $(1 \leq m \leq n)$ of the set of periodic orbits as in Theorem 5.7. In general we cannot exclude the possibility that (z_1, τ, ω_1) and $(z_2, \tau, \omega_2) \in S$, but they are geometrically identical. It is an interesting question to study the nodal properties of these bifurcating periodic orbits similar to the steady state bifurcation case.

6 Numerical Simulations and Discussion

In this section, we present some numerical simulations to demonstrate the analytic results in previous sections. As an example we consider (1.4)–(1.5) with $d = 0.5$ and $r = 0.6$, and in the following simulations, we always use the initial condition $\eta(x, t) = 0.15 \sin x, \ t \in [-1, 0]$. In Figs. 2 and 3 and 4, left we use $a = 0.1, \ b = 0.9$. For this set of parameter, $\tau_0 \approx 16.9$ and the period of bifurcating orbits is near $T \approx 3.74$ from Corollary 2.5. We know that the positive steady state solution u_r is locally asymptotically stable when $\tau < \tau_0$, which is shown in Fig. 2-left with $\tau = 15$. One can see that the maximum of the steady state in Fig. 2-left is close to $\dfrac{(r-d)\int_0^\pi \sin^2 x dx}{d(a+b)\int_0^\pi \sin^3 x dx} \approx 0.23$. A Hopf bifurcation occurs when τ crosses τ_0, the positive steady state u_r loses its stability and the bifurcating periodic solutions state is stable

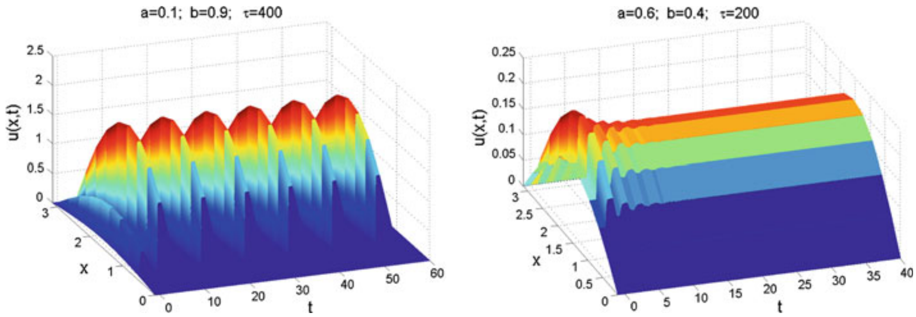

Fig. 4 *Left* $\tau = 400$, the solution converges to a time-periodic solution with large amplitude which is less than $1/a$. *Right* $a = 0.6,\ b = 0.4$ and $\tau = 200$, the solution converges to a steady state solution

as shown in Fig. 2-right, and one can see that period of the periodic solution in Fig. 2-left is close to 3.74.

As τ increases, the solutions still converge to a time periodic solution, see Figs. 3 and 4-left, and the period of the periodic solutions increases slowly. But after $\tau = 200$, the amplitude of periodic solutions almost does not increase anymore, and it is smaller than $1/a = 10$, see Figs. 3-right and 4-left. This partly verifies the uniform boundedness of all solutions proved in Lemma 5.2. Finally in Fig. 4-right we show that when $a > b$ (we use $a = 0.6,\ b = 0.4$), the steady state u_r is asymptotically stable even with a large delay τ.

Our analytical results prove the local Hopf bifurcation at τ_n for $n \geq 1$ can be extended to $\tau = \infty$, but such an assertion cannot be made for $n = 0$ as the periods of periodic orbits on $\mathfrak{C}_0$ is not bounded. The numerical simulations here suggest that, for any $\tau > \tau_0$ there is a stable periodic orbit, which provides evidence for the global continuation of local Hopf bifurcation from τ_0.

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