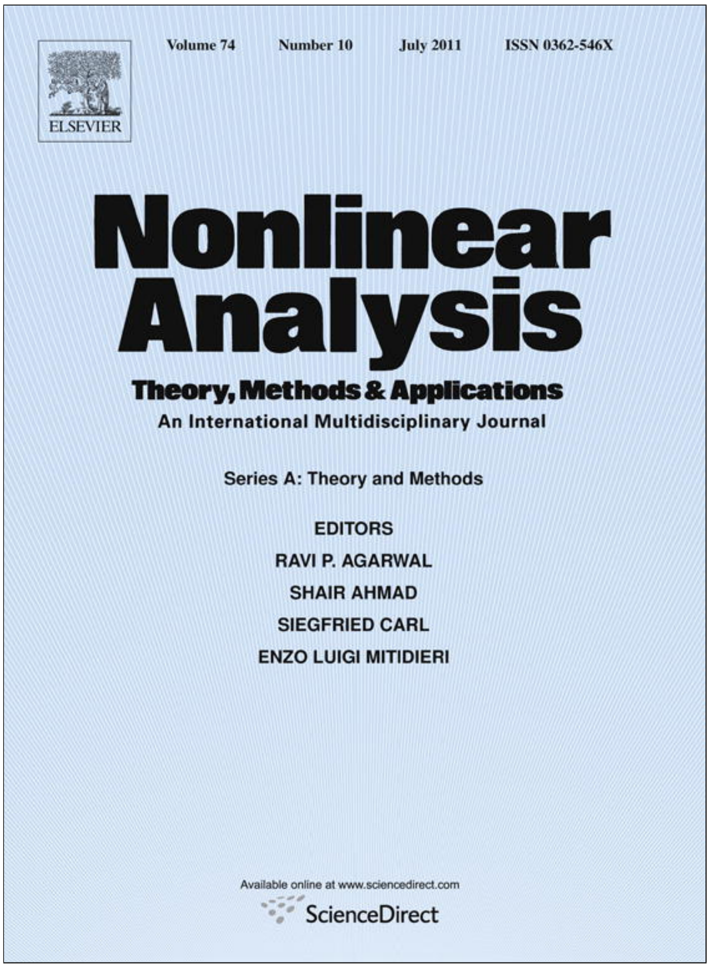
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Stability of impulsive stochastic differential delay systems and its application to impulsive stochastic neural networks

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ABSTRACT

This paper is concerned with the stability of n-dimensional stochastic differential delay systems with nonlinear impulsive effects. First, the equivalent relation between the solution of the n-dimensional stochastic differential delay system with nonlinear impulsive effects and that of a corresponding n-dimensional stochastic differential delay system without impulsive effects is established. Then, some stability criteria for the n-dimensional stochastic differential delay systems with nonlinear impulsive effects are obtained. Finally, the stability criteria are applied to uncertain impulsive stochastic neural networks with time-varying delay. The results show that, this convenient and efficient method will provide a new approach to study the stability of impulsive stochastic neural networks. Some examples are also discussed to illustrate the effectiveness of our theoretical results.

1. Introduction

Impulsive systems arise naturally in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time. Such processes are often investigated in various fields, such as economics, physics, population dynamics, engineering, biology, etc. For instance, in implementation of electronic networks in which the state is subject to instantaneous perturbations and experiences abrupt change at certain moments, which may be caused by switching phenomenon, frequency change or other sudden noise, does exhibit impulsive effects. Besides the impulsive effects, time delays are frequently encountered in real world, which can cause instability and oscillations in the system. For example, time delays can be caused by the finite switching speed of amplifier circuits in neural networks or deliberately introduced to achieve tasks that deal with motion-related problems such as moving image processing. Recently, impulsive delay systems have been used with great success in a variety of applications, and a large number of important results have been reported (see [1–5] and the references therein).

Yan and Zhao [1] considered the following linear impulsive delay differential equation:

$$\begin{cases} y'(t) + \sum_{i=1}^{n} p_i(t) y(t-\tau_i(t)) = 0, & t \neq t_k \\ y(t_k^+) - y(t_k) = b_k y(t_k), & k = 1, 2, \ldots. \end{cases} \tag{1.1}$$

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They showed the oscillation and the stability results of Eq. (1.1) by transforming (1.1) into

$$x'(t)+\sum_{i=1}^{n}p_i(t)\prod_{t-\tau_i(t)\le t_k<t}(1+b_k)^{-1}x(t-\tau_i(t))=0. \tag{1.2}$$

Li and Huo [2] considered the following delay differential system with impulses:

$$\begin{cases}\dfrac{y(t)}{\mathrm{d}t}=y(t)F(t,y(t-\tau_1(t)),\ldots,y(t-\tau_n(t))), & t\neq t_k\\ y(t_k^+)-y(t_k)=b_ky(t_k), & k=1,2,\ldots.\end{cases} \tag{1.3}$$

They showed the existence and global attractivity of positive periodic solutions results of Eq. (1.3) by transforming (1.3) into

$$z'(t)=z(t)G(t,z(t-\tau_1(t)),\ldots,z(t-\tau_n(t))) \tag{1.4}$$

where

$$G(t,z(t-\tau_1(t)),\ldots,z(t-\tau_n(t)))=F\left(t,\prod_{\sigma\le t_k<t-\tau_1(t)}(1+b_k)z(t-\tau_1(t)),\ldots,\prod_{\sigma\le t_k<t-\tau_n(t)}(1+b_k)z(t-\tau_n(t))\right).$$

Yan et al. [3] considered the following scalar impulsive delay differential equation:

$$\begin{cases}y'(t)+a(t)y(t)+F(t,y(\cdot))=0, & t\neq t_k\\ y(\tau_k^+)-y(\tau_k)=I_k(y(\tau_k)), & k=1,2,\ldots.\end{cases} \tag{1.5}$$

They showed the stability results of Eq. (1.5) by transforming (1.5) into

$$x'(t)+a(t)x(t)+\prod_{t_0\le\tau_k<t}J_k(y(\tau_k))F(t,y(\cdot))=0 \tag{1.6}$$

where $J_k(u)=\frac{u}{u+I_k(u)}$.

Alzabut et al. [4] considered the following impulsive delay logarithmic population model:

$$\begin{cases}y'(t)=-a_1(t)y(t)-a_2(t)y(t-\tau(t))+\lambda(t), & t\neq t_k\\ y(t_k^+)-y(t_k)=b_ky(t_k), & k=1,2,\ldots.\end{cases} \tag{1.7}$$

They showed the existence of a globally attractive periodic solution results of Eq. (1.7) by transforming (1.7) into

$$z'(t)=-a_1(t)z(t)-a_2(t)z(t-\tau(t))+\gamma(t) \tag{1.8}$$

where $\gamma(t)=\prod_{0<t_k<t}(1+b_k)^{-1}\lambda(t)$.

In practice, a real system is usually affected by external random perturbations, i.e., the system has stochastic effects which may lead to complex dynamic behaviors such as oscillation, divergence, chaos, instability or other poor performance. For instance, stochastic effects can result from stochastic failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, etc. Hence, qualitative analysis of systems with stochastic effects has become a significant topic in both theoretical research and practical applications (see [6–11] and the references therein).

Zhao et al. [8] considered the following linear stochastic differential delay systems under impulsive control:

$$\begin{cases}\mathrm{d}y(t)+\displaystyle\sum_{i=1}^{n}p_i(t)y(t-\tau_i(t))\mathrm{d}t+\sum_{i=1}^{n}q_i(t)y(t-\tau_i(t))\mathrm{d}w(t)=0, & t\neq t_k\\ y(t_k^+)-y(t_k)=b_ky(t_k), & t=t_k,\,k\in N.\end{cases} \tag{1.9}$$

They showed the stability results of Eq. (1.9) by transforming (1.9) into

$$\mathrm{d}x(t)+\sum_{i=1}^{n}p_i(t)\prod_{t-\tau_i(t)\le t_k<t}(1+b_k)^{-1}x(t-\tau_i(t))\mathrm{d}t+\sum_{i=1}^{n}q_i(t)\prod_{t-\tau_i(t)\le t_k<t}(1+b_k)^{-1}x(t-\tau_i(t))\mathrm{d}w(t)=0. \tag{1.10}$$

Li and Sun [9] considered the following stochastic differential delay systems under impulsive control:

$$\begin{cases}\mathrm{d}y(t)=y(t)F_1(t,y(t),y(t-\tau_1(t)),\ldots,y(t-\tau_n(t)))\mathrm{d}t\\ \qquad +y(t)F_2(t,y(t),y(t-\tau_1(t)),\ldots,y(t-\tau_n(t)))\mathrm{d}w(t), & t\neq t_k\\ y(t_k^+)-y(t_k)=b_ky(t_k), & t=t_k,\,k\in N.\end{cases} \tag{1.11}$$

They showed the stability results of Eq. (1.11) by transforming (1.11) into

$$\begin{aligned}\mathrm{d}x(t)&=x(t)G_1(t,x(t),x(t-\tau_1(t)),\ldots,x(t-\tau_n(t)))\mathrm{d}t\\ &\quad +x(t)G_2(t,x(t),x(t-\tau_1(t)),\ldots,x(t-\tau_n(t)))\mathrm{d}w(t)\end{aligned} \tag{1.12}$$

where for $i = 1, 2$,

$$G_i(t, x(t), x(t-\tau_1(t)), \dots, x(t-\tau_n(t))) \\ = F_i\left(t, \prod_{\sigma \le t_k < t}(1+b_k)x(t), \prod_{\sigma \le t_k < t-\tau_1(t)}(1+b_k)x(t-\tau_1(t)), \dots, \prod_{\sigma \le t_k < t-\tau_n(t)}(1+b_k)x(t-\tau_n(t))\right).$$

Li et al. [10] considered the following 1-dimensional stochastic delay differential equations with nonlinear impulsive effects:

$$\begin{cases} \mathrm{d}y(t) = \{-a_1(t)y(t) - a_2(t)y(t-\tau(t))\}\mathrm{d}t + \{-b_1(t)y(t) - b_2(t)y(t-\tau(t))\}\mathrm{d}w(t), & t \ne t_k \\ y(t_k^+) - y(t_k) = I_k(y(t_k)), & t = t_k,\ k \in N. \end{cases} \tag{1.13}$$

They showed the stability results of Eq. (1.13) by transforming (1.13) into

$$\mathrm{d}x(t) = \{-a_1(t)x(t) - \hat{a}_2(t)x(t-\tau(t))\}\mathrm{d}t + \{-b_1(t)x(t) - \hat{b}_2(t)x(t-\tau(t))\}\mathrm{d}w(t), \quad t \ge 0 \tag{1.14}$$

where $\hat{a}_2(t) = \prod_{t-\tau(t) \le t_k < t} J_k(y(t_k))a_2(t)$, $\hat{b}_2(t) = \prod_{t-\tau(t) \le t_k < t} J_k(y(t_k))b_2(t)$, and $J_k(u) = \frac{u}{u+I_k(u)}$.

From the reported results listed above, we can see that some researchers have qualitatively analyzed the impulsive systems by establishing the equivalence between the solution of the impulsive systems and that of a corresponding system without impulses, here we call it '*equivalent method*' below. In this way, the existing results for systems without impulses may possibly be used to qualitatively analyze more complicate systems—impulsive systems. Moreover, particularly for stochastic systems, the traditional methods to study stochastic differential delay system (such as Itô's formula) cannot be effectively used in impulsive stochastic differential delay system, since it is difficult to deal with when integrating intervals contain impulses, and we can overcome this difficulty by using '*equivalent method*'.

However, to the best of our knowledge, the existing results related to '*equivalent method*' are classified into two categories: n-dimensional system with linear impulses ($b_k y_{t_k}$), and 1-dimensional system with nonlinear impulses ($I_k(y_{t_k})$). Note that no results related to '*equivalent method*' of n-dimensional system with nonlinear impulses have been reported. Thus, it is important and significant to establish the '*equivalent method*' in an n-dimensional system with nonlinear impulses.

Stability is one of the most fundamental concepts both in theoretical research and practical applications. Therefore, in this paper, we aim to establish the '*equivalent method*' and study the stability of the n-dimensional stochastic differential delay system with nonlinear impulses as follows:

$$\begin{cases} \mathrm{d}y_i(t) = F_i(t, y_1(t), \dots, y_n(t), y_1(t-\tau(t)), \dots, y_n(t-\tau(t)))\mathrm{d}t \\ \qquad + G_i(t, y_1(t), \dots, y_n(t), y_1(t-\tau(t)), \dots, y_n(t-\tau(t)))\mathrm{d}w_i(t), & t \ne t_k \\ y_i(t_k^+) - y_i(t_k) = I_{ki}(y_1(t_k), \dots, y_n(t_k)), & t = t_k, k \in N \end{cases}$$

for $i = 1, 2, \dots, n$, or

$$\begin{cases} \mathrm{d}y(t) = F(t, y(t), y(t-\tau(t)))\mathrm{d}t + G(t, y(t), y(t-\tau(t)))\mathrm{d}w(t), & t \ne t_k \\ y(t_k^+) - y(t_k) = I_k(y(t_k)), & t = t_k,\ k \in N. \end{cases}$$

In addition, various classes of neural networks have been active research topics in the past few year, due to its important applications, such as pattern recognition, associative memory and combinatorial optimization, etc. Stability is one of the main properties of neural networks, which is a crucial feature in the design of neural networks. Therefore, the stability analysis of neural networks has received a great deal of interest (see [12–37] and the references therein). As artificial electronic system, neural networks are often subject to impulsive perturbation which can affect dynamical behaviors. Therefore, impulsive effects should also be taken into account. Recently, many interesting results on impulsive effects to the stability of neural networks have been reported (see [14–21] and the references therein). In this paper, the derived stability results of the impulsive stochastic differential delay system will be applied to uncertain impulsive stochastic neural networks with time-varying delay. It can be easily seen that, a convenient and efficient method, that is, our stability results will provide a new approach to study the stability of impulsive stochastic neural networks. It is worth noting that there are only very few results on impulsive stochastic neural networks (see [35–37]).

This paper is organized as follows: In Section 2, we briefly recall some basic notations. In Section 3, we first establish the equivalence between the solution of an n-dimensional impulsive stochastic differential delay system and that of a corresponding n-dimensional stochastic differential delay system without impulses. Then, we give several sufficient conditions ensuring various stabilities of n-dimensional stochastic differential delay systems with and without impulses. In Section 4, the stability criteria are applied to uncertain impulsive stochastic neural networks with time-varying delay. Finally, some concluding remarks are given in Section 5.

2. Preliminaries

To begin with, we introduce several notations and recall some basic definitions which are used throughout the paper.

Let $\{\Omega, F, \{F_t\}_{t\ge 0}, P\}$ be a complete probability space with a filtration $\{F_t\}_{t\ge 0}$ satisfying the usual conditions (i.e. right continuous and F_0 containing all P-null sets). Let $w(t) = (w_1(t), \dots, w_n(t))^T$ be an n-dimensional standard Brownian motion defined on the probability space. Let $|\cdot|$ denote the Euclidean norm in R^n. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^TA)}$. Let $PC([-\tau,0],R^n)$ denote the family of all piecewise right continuous functions $\phi : [-\tau,0] \to R^n$ with the norm $\|\phi\| = \sup_{-\tau\le s\le 0}|\phi(s)|$. Let $PC^b_{F_0}([-\tau,0],R^n) = \{\phi \in PC([-\tau,0],R^n) : \phi$ is bounded F_0-measurable, and $E\|\phi(s)\|^p < \infty\}$, $PC^b_{F_0}(\delta) = \{\phi \in PC^b_{F_0}([-\tau,0],R^n) : E\|\phi(s)\|^p \le \delta\}$, where E denotes the expectation of the stochastic process.

Consider the n-dimensional stochastic differential delay system with nonlinear impulsive effects as follows:

$$\begin{cases} \mathrm{d}y_i(t) = F_i(t, y_1(t), \ldots, y_n(t), y_1(t-\tau(t)), \ldots, y_n(t-\tau(t)))\mathrm{d}t \\ \qquad + G_i(t, y_1(t), \ldots, y_n(t), y_1(t-\tau(t)), \ldots, y_n(t-\tau(t)))\mathrm{d}w_i(t), \quad t \ne t_k \\ y_i(t_k^+) - y_i(t_k) = I_{ki}(y_1(t_k), \ldots, y_n(t_k)), \quad t = t_k, k \in N \end{cases} \tag{2.1}$$

for $i = 1, 2, \ldots, n$, with the initial condition:

$$y_i(t) = \phi_i(t), \quad t \in [-\tau, 0], \ \tau = \sup_{t\ge 0}\tau(t). \tag{2.2}$$

Let $y(t) = (y_1(t), \ldots, y_n(t))^T$, $y(t-\tau(t)) = (y_1(t-\tau(t)), \ldots, y_n(t-\tau(t)))^T$, $F = (F_1, \ldots, F_n)^T$, $G = (G_1, \ldots, G_n)^T$, $I_k = (I_{k1}, \ldots, I_{kn})^T$, $k \in N$, $\phi = (\phi_1, \ldots, \phi_n)^T$, then system (2.1) with initial condition (2.2) becomes

$$\begin{cases} \mathrm{d}y(t) = F(t, y(t), y(t-\tau(t)))\mathrm{d}t + G(t, y(t), y(t-\tau(t)))\mathrm{d}w(t), \quad t \ne t_k \\ y(t_k^+) - y(t_k) = I_k(y(t_k)), \quad t = t_k, \ k \in N \\ y(t) = \phi(t), \quad t \in [-\tau, 0]. \end{cases} \tag{2.3}$$

For system (2.1) and (2.2) (or (2.3)), we assume the following hypotheses:

(H1) $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ are fixed impulsive points such that $t_k \to \infty$ for $k \to \infty$;
(H2) $F, G : R_+ \times R^n \times R^n \to R^n$ satisfy $F(t,0,0) \equiv 0$ and $G(t,0,0) \equiv 0$. Meanwhile, $\tau(t) : R_+ \to [0,\tau]$. Here, $R_+ := [0,\infty)$;
(H3) $I_k : R^n \to R^n$ satisfies $I_k(0) \equiv 0$, $k \in N$;
(H4) $\phi(t) \in PC^b_{F_0}([-\tau,0],R^n)$.

Definition 2.1. A function $y(t) = (y_1(t), \ldots, y_n(t))^T$ defined on $[-\tau,\infty)$ is said to be a solution of the impulsive stochastic differential delay system (2.1) with initial condition (2.2) if

(i) $y(t)$ is absolutely continuous on $(0, t_1]$ and each interval $(t_k, t_{k+1}]$, $k \in N$.
(ii) For any t_k, $k \in N$, $y(t_k^+) = \lim_{t\to t_k^+} y(t)$ and $y(t_k^-) = \lim_{t\to t_k^-} y(t)$ exist and $y(t_k^-) = y(t)$.
(iii) $y(t)$ satisfies Eq. (2.1) for almost everywhere in $[0,+\infty) \setminus \{t_k\}$ and the impulsive conditions at each $t = t_k$, $k \in N$.

From hypotheses (H1)–(H4) and Definition 2.1, one knows that $y(t) \equiv 0$ is a solution of the impulsive stochastic differential delay system (2.1) and (2.2). It is said to be a zero solution of (2.1) and (2.2). Throughout this paper, except for the zero solution of (2.1), we also suppose that for any solution $y(t) = (y_1(t), \ldots, y_n(t))^T$ of (2.1), $y_i(t_k) \ne 0$ and $I_{ki}(y_1(t_k), \ldots, y_n(t_k)) \ne -y_i(t_k)$, $i = 1, 2, \ldots, n$, $k \in N$. Moreover, we always assume that a product equals unity if the number of factors is zero.

Definition 2.2. The zero solution of the impulsive stochastic differential delay system (2.1) with initial condition (2.2) is said to be

(i) p-stable, if for any $\varepsilon > 0$, there is a $\delta > 0$ such that the initial function $\phi \in PC^b_{F_0}(\delta)$ implies $E|y(t)|^p < \varepsilon$ for $t > 0$. Especially, when $p = 1$, it is said to be stable.
(ii) Exponentially p-stable ($p \ge 2$), if there is a pair of positive constants λ and K such that for any initial condition $\phi \in PC^b_{F_0}([-\tau,0],R^n)$,

$$E|y(t)|^p \le K\|\phi\|^p \mathrm{e}^{-\lambda t}, \quad t \ge 0.$$

Here λ is called the exponential convergence rate. Especially, when $p = 2$, it is said to be exponentially stable in mean square.
(iii) Asymptotically stable, if it is stable and there exists a $\sigma > 0$ such that the initial function $\phi \in PC^b_{F_0}(\sigma)$ implies $\lim_{t\to\infty} E|y(t)| = 0$.

We define functions,

$$\mathcal{J}_{ki}(u_1, \ldots, u_n) = \frac{u_i}{u_i + I_{ki}(u_1, \ldots, u_n)},$$

where $u_1, \ldots, u_n \in R$, $i = 1, 2, \ldots, n$, $k \in N$.

For simplicity, we give the following notation,

$$J_{ki} := \mathcal{J}_{ki}(y_1(t_k), \ldots, y_n(t_k)) = \frac{y_i(t_k)}{y_i(t_k) + I_{ki}(y_1(t_k), \ldots, y_n(t_k))},$$

where $y_1(t_k), \ldots, y_n(t_k) \in R$, $i = 1, 2, \ldots, n$, $k \in N$.

Under conditions (H2) and (H4), we consider the following n-dimensional stochastic differential delay system without impulsive effects,

$$\begin{aligned} dx_i(t) = & \prod_{0\le t_k<t} J_{ki}\widetilde{F}_i(t, x_1(t), \cdots, x_n(t), x_1(t-\tau(t)), \ldots, x_n(t-\tau(t)))dt \\ & + \prod_{0\le t_k<t} J_{ki}\widetilde{G}_i(t, x_1(t), \ldots, x_n(t), x_1(t-\tau(t)), \ldots, x_n(t-\tau(t)))dw_i(t), \end{aligned} \tag{2.4}$$

for $i = 1, 2, \ldots, n$, where

$$\begin{aligned} & \widetilde{F}_i(t, x_1(t), x_2(t), \ldots, x_n(t), x_1(t-\tau(t)), \ldots, x_n(t-\tau(t))) \\ & \quad = F_i(t, \prod_{0\le t_k<t} J_{k1}^{-1}x_1(t), \prod_{0\le t_k<t} J_{k2}^{-1}x_2(t), \ldots, \prod_{0\le t_k<t} J_{kn}^{-1}x_n(t), \\ & \quad \prod_{0\le t_k<t-\tau(t)} J_{k1}^{-1}x_1(t-\tau(t)), \ldots, \prod_{0\le t_k<t-\tau(t)} J_{kn}^{-1}x_n(t-\tau(t))), \quad i = 1, 2, \ldots, n \end{aligned}$$

and

$$\begin{aligned} & \widetilde{G}_i(t, x_1(t), x_2(t), \ldots, x_n(t), x_1(t-\tau(t)), \ldots, x_n(t-\tau(t))) \\ & \quad = G_i\Bigg(t, \prod_{0\le t_k<t} J_{k1}^{-1}x_1(t), \prod_{0\le t_k<t} J_{k2}^{-1}x_2(t), \ldots, \prod_{0\le t_k<t} J_{kn}^{-1}x_n(t), \\ & \qquad \prod_{0\le t_k<t-\tau(t)} J_{k1}^{-1}x_1(t-\tau(t)), \ldots, \prod_{0\le t_k<t-\tau(t)} J_{kn}^{-1}x_n(t-\tau(t))\Bigg), \quad i = 1, 2, \ldots, n \end{aligned}$$

with the initial function

$$x_i(t) = \phi_i(t), \quad t \in [-\tau, 0], \ \tau = \sup_{t\ge 0} \tau(t). \tag{2.5}$$

Let $x(t) = (x_1(t), \ldots, x_n(t))^T$, $x(t-\tau(t)) = (x_1(t-\tau(t)), \ldots, x_n(t-\tau(t)))^T$, $\widetilde{F} = (\widetilde{F}_1, \ldots, \widetilde{F}_n)^T$, $\widetilde{G} = (\widetilde{G}_1, \ldots, \widetilde{G}_n)^T$ and $J_k = \mathrm{diag}(J_{k1}, J_{k2}, \ldots, J_{kn})$, then system (2.4) with initial condition (2.5) becomes

$$\begin{cases} dx(t) = \prod_{0\le t_k<t} J_k\widetilde{F}(t, x(t), x(t-\tau(t)))dt + \prod_{0\le t_k<t} J_k\widetilde{G}(t, x(t), x(t-\tau(t)))dw(t), & t \ge 0 \\ x(t) = \phi(t), & t \in [-\tau, 0] \end{cases} \tag{2.6}$$

where

$$\widetilde{F}(t, x(t), x(t-\tau(t))) = F\left(t, \prod_{0\le t_k<t} J_k^{-1}x(t), \prod_{0\le t_k<t-\tau(t)} J_k^{-1}x(t-\tau(t))\right),$$

$$\widetilde{G}(t, x(t), x(t-\tau(t))) = G\left(t, \prod_{0\le t_k<t} J_k^{-1}x(t), \prod_{0\le t_k<t-\tau(t)} J_k^{-1}x(t-\tau(t))\right).$$

By a solution of (2.4) and (2.5) (or (2.6)) we mean an absolutely continuous function $x(t) = (x_1(t), \ldots, x_n(t))^T$ on $[-\tau, +\infty)$ satisfying (2.4) almost everywhere for $t \ge 0$ and satisfies (2.5). Similarly to Definition 2.2, various stabilities of the solution of system (2.4) can be defined.

Remark 2.1. The results proposed later can be easily extended to investigate the stability for the following impulsive stochastic differential delay system:

$$\begin{cases} dy_i(t) = F_i(t, y_1(t), \ldots, y_n(t), y_1(t-\tau_1(t)), \ldots, y_n(t-\tau_1(t)), y_1(t-\tau_2(t)), \ldots, \\ \quad y_n(t-\tau_2(t)), \ldots, y_1(t-\tau_m(t)), \ldots, y_n(t-\tau_m(t)))dt \\ \quad + G_i(t, y_1(t), \ldots, y_n(t), y_1(t-\tau_1(t)), \ldots, y_n(t-\tau_1(t)), y_1(t-\tau_2(t)), \ldots, \\ \quad y_n(t-\tau_2(t)), \ldots, y_1(t-\tau_m(t)), \ldots, y_n(t-\tau_m(t)))dw_i(t), \quad t \ne t_k \\ y_i(t_k^+) - y_i(t_k) = I_{ki}(y_1(t_k), \ldots, y_n(t_k)), \quad t = t_k, k \in N \end{cases}$$

for $i = 1, 2 \cdots, n$, or

$$\begin{cases} dy(t) = F(t, y(t), y(t-\tau_1(t)), \ldots, y(t-\tau_m(t)))dt \\ \quad + G(t, y(t), y(t-\tau_1(t)), \ldots, y(t-\tau_m(t)))dw(t), \quad t \ne t_k \\ y(t_k^+) - y(t_k) = I_k(y(t_k)), \quad t = t_k, \ k \in N \end{cases}$$

where $y(t-\tau_i(t)) = (y_1(t-\tau_i(t)), \ldots, y_n(t-\tau_i(t)))^T$, $i = 1, 2, \ldots, m$. For the sake of understandability and simplicity, here we only discuss the stability problem of system (2.1) or (2.3).

3. Stability criteria

In this section, first we establish a fundamental lemma that enables us to reduce the stabilities of n-dimensional stochastic differential delay system with nonlinear impulses (2.1) to the corresponding problems of an n-dimensional stochastic differential delay system without impulses (2.4).

Lemma 3.1. *Assume that* (H1)–(H4) *hold, then*

(i) *if* $x(t) = (x_1(t), \ldots, x_n(t))^T$ *is a solution of* (2.4), *then* $y(t) = (y_1(t), \ldots, y_n(t))^T$ *is a solution of* (2.1) *on* $[-\tau, +\infty)$, *where* $y_i(t) = \prod_{0\le t_k<t} J_{ki}^{-1} x_i(t),\ i = 1, 2, \ldots, n$.

(ii) *if* $y(t) = (y_1(t), \ldots, y_n(t))^T$ *is a solution of* (2.1), *then* $x(t) = (x_1(t), \ldots, x_n(t))^T$ *is a solution of* (2.4) *on* $[-\tau, +\infty)$, *where* $x_i(t) = \prod_{0\le t_k<t} J_{ki} y_i(t),\ i = 1, 2, \ldots, n$.

Proof. First we prove (i). Let $x(t) = (x_1(t), \ldots, x_n(t))^T$ be a possible solution of system (2.4), it is easy to see that $y_i(t) = \prod_{0\le t_k<t} J_{ki}^{-1} x_i(t),\ i = 1, 2, \ldots, n$, is absolutely continuous on each interval $(t_k,\ t_{k+1}) \in [0, \infty)$, $k \in N$, and for any $t \ne t_k$,

$$\begin{aligned}
\mathrm{d}y_i(t) &= d\left\{\prod_{0\le t_k<t} J_{ki}^{-1} x_i(t)\right\} \\
&= \prod_{0\le t_k<t} J_{ki}^{-1} dx_i(t) \\
&= \widetilde{F}_i(t, x_1(t), \ldots, x_n(t), x_1(t-\tau(t)), \ldots, x_n(t-\tau(t)))\mathrm{d}t \\
&\quad + \widetilde{G}_i(t, x_1(t), \ldots, x_n(t), x_1(t-\tau(t)), \ldots, x_n(t-\tau(t)))\mathrm{d}w_i(t) \\
&= F_i\left(t, \prod_{0\le t_k<t} J_{k1}^{-1} x_1(t), \ldots, \prod_{0\le t_k<t} J_{kn}^{-1} x_n(t), \prod_{0\le t_k<t-\tau(t)} J_{k1}^{-1} x_1(t-\tau(t)), \ldots, \prod_{0\le t_k<t-\tau(t)} J_{kn}^{-1} x_n(t-\tau(t))\right)\mathrm{d}t \\
&\quad + G_i\left(t, \prod_{0\le t_k<t} J_{k1}^{-1} x_1(t), \ldots, \prod_{0\le t_k<t} J_{kn}^{-1} x_n(t), \prod_{0\le t_k<t-\tau(t)} J_{k1}^{-1} x_1(t-\tau(t)), \ldots, \prod_{0\le t_k<t-\tau(t)} J_{kn}^{-1} x_n(t-\tau(t))\right)\mathrm{d}w_i(t) \\
&= F_i(t, y_1(t), \ldots, y_n(t), y_1(t-\tau(t)), \ldots, y_n(t-\tau(t)))\mathrm{d}t \\
&\quad + G_i(t, y_1(t), \ldots, y_n(t), y_1(t-\tau(t)), \ldots, y_n(t-\tau(t)))\mathrm{d}w_i(t)
\end{aligned}$$

Thus, $y(t) = (y_1(t), \ldots, y_n(t))^T$ satisfies the impulsive controlled stochastic differential delay system (2.1) for almost everywhere in $[0, +\infty) \setminus t_k$.

On the other hand, for $i = 1, 2, \ldots, n$, and every k, $k \in N$ and t_k situated in $[0, +\infty)$,

$$\begin{aligned}
y_i(t_k^+) &= \lim_{t\to t_k^+} \prod_{0\le t_j<t} J_{ji}^{-1} x_i(t) \\
&= \prod_{0\le t_j\le t_k} J_{ji}^{-1} x_i(t_k^+) \\
&= J_{ki}^{-1} \prod_{0\le t_j<t_k} J_{ji}^{-1} x_i(t_k) \\
&= J_{ki}^{-1} y_i(t_k) \\
&= \frac{y_i(t_k) + I_{ki}(y_1(t_k), \ldots, y_n(t_k))}{y_i(t_k)} y_i(t_k) \\
&= y_i(t_k) + I_{ki}(y_1(t_k), \ldots, y_n(t_k))
\end{aligned}$$

and

$$\begin{aligned}
y_i(t_k^-) &= \lim_{t\to t_k^-} \prod_{0\le t_j<t} J_{ji}^{-1} x_i(t) \\
&= \prod_{0\le t_j<t_k} J_{ji}^{-1} x_i(t_k^-) \\
&= \prod_{0\le t_j<t_k} J_{ji}^{-1} x_i(t_k) \\
&= y_i(t_k).
\end{aligned}$$

Therefore, we arrive at a conclusion that $y(t) = (y_1(t), \dots, y_n(t))^T$ is the solution of (2.1) corresponding to initial condition (2.2). In fact, if $x(t) = (x_1(t), \dots, x_n(t))^T$ is the solution of (2.4) with initial condition (2.5), then $y_i(t) = \prod_{0\le t_k<t} J_{ki}^{-1} x_i(t) = x_i(t) = \phi_i(t)$ on $[-\tau, 0]$, $i = 1, 2, \dots, n$.

Next we prove (ii). Since $y(t) = (y_1(t), \dots, y_n(t))^T$ is a solution of (2.1), $y_i(t)$, $i = 1, 2, \dots, n$, is absolutely continuous on each interval $(t_k, t_{k+1}) \in [0, +\infty)$, $k \in N$. Therefore, $x_i(t) = \prod_{0\le t_k<t} J_{ki} y_i(t)$, $i = 1, 2, \dots, n$ is absolutely continuous on $(t_k, t_{k+1}) \in [0, +\infty)$. What is more, it follows that, for $i = 1, 2, \dots, n$, and every k, $k \in N$ and t_k situated in $[0, +\infty)$,

$$\begin{aligned}
x_i(t_k^+) &= \lim_{t\to t_k^+} \prod_{0\le t_j<t} J_{ji} y_i(t) \\
&= \prod_{0\le t_j\le t_k} J_{ji} y_i(t_k^+) \\
&= \prod_{0\le t_j<t_k} J_{ji} \cdot J_{ki} y_i(t_k^+) \\
&= \prod_{0\le t_j<t_k} J_{ji} \cdot \frac{y_i(t_k)(y_i(t_k) + I_{ki}(y_1(t_k), \dots, y_n(t_k)))}{y_i(t_k) + I_{ki}(y_1(t_k), \dots, y_n(t_k))} \\
&= \prod_{0\le t_j<t_k} J_{ji} y_i(t_k) \\
&= x_i(t_k)
\end{aligned}$$

and

$$\begin{aligned}
x_i(t_k^-) &= \lim_{t\to t_k^-} \prod_{0\le t_j<t} J_{ji} y_i(t) \\
&= \prod_{0\le t_j<t_k} J_{ji} y_i(t_k^-) \\
&= x_i(t_k)
\end{aligned}$$

which implies that $x(t) = (x_1(t), \dots, x_n(t))^T$ is continuous and easy to prove absolutely continuous on $[0, +\infty)$. Now, similar to the proof in case (i), we can easily check that $x(t) = (x_1(t), \dots, x_n(t))^T$, where $x_i(t) = \prod_{0\le t_k<t} J_{ki} y_i(t)$, $i = 1, 2, \dots, n$ is the solution of (2.4) on $[0, +\infty)$ corresponding to the initial condition (2.5). This completes the proof. □

Based on the previous certificate process, we can establish the equivalent relation between the solution of the n-dimensional stochastic differential delay system with nonlinear impulsive effects (2.3) and the solution of a corresponding n-dimensional stochastic differential delay system without impulsive effects (2.6) easily by the following lemma.

Lemma 3.2. *Assume that* (H1)–(H4) *hold, then*

(i) *if $x(t)$ is a solution of* (2.6), *then $y(t) = \prod_{0\le t_k<t} J_k^{-1} x(t)$ is a solution of* (2.3) *on $[-\tau, +\infty)$.*
(ii) *if $y(t)$ is a solution of* (2.3), *then $x(t) = \prod_{0\le t_k<t} J_k y(t)$ is a solution of* (2.6) *on $[-\tau, +\infty)$.*

Remark 3.1. Lemmas 3.1 and 3.2 give the equivalent relation between the solution of an n-dimensional stochastic differential delay system with nonlinear impulsive effects and the solution of a corresponding n-dimensional stochastic differential delay system without impulses. Based on this '*equivalent method*', the existence and uniqueness of an n-dimensional stochastic differential delay system with nonlinear impulsive effects (2.1) (or (2.3)) can be derived by a new way, that is, any conditions that ensure the existence and uniqueness of an n-dimensional stochastic differential delay system (2.4) (or (2.6)) will also ensure the existence and uniqueness of an n-dimensional stochastic differential delay system with nonlinear impulsive effects (2.1) (or (2.3)).

By applying Lemma 3.1 we can provide the following stability theory for the stability analysis of the solution of n-dimensional stochastic differential delay system with nonlinear impulsive effects (2.1).

Theorem 3.1. *Assume that* (H1)–(H4) *hold, and there exists a positive constant M such that for any $t > 0$,*

$$\left| \prod_{0\le t_k<t} J_{ki}^{-1} \right| \le M, \quad i = 1, 2, \dots, n. \tag{3.1}$$

(i) If the zero solution of (2.4) *is p-stable, then the zero solution of* (2.1) *is also p-stable.*

(i) *If the zero solution of* (2.4) *is exponentially p-stable ($p \ge 2$), then the zero solution of* (2.1) *is also exponentially p-stable. Moreover they have the same exponential convergence rate.*
(ii) *If the zero solution of* (2.4) *is asymptotically stable, then the zero solution of* (2.1) *is also asymptotically stable.*

Proof. We prove (i) only, and (ii), (iii) can be proved similarly and proofs will be omitted here. Let $x(t) = (x_1(t), \ldots, x_n(t))^T$ and $y(t) = (y_1(t), \ldots, y_n(t))^T$ be the possible solutions of system (2.1) and (2.4) corresponding to initial conditions (2.2) and (2.5).

From the hypotheses that the zero solution of (2.4) is p-stable, we find that, for any $\varepsilon > 0$, there is a scalar $\delta > 0$ such that the initial condition $\phi \in PC^b_{F_0}(\delta)$ implies

$$E|x(t)|^p < \frac{\varepsilon}{M^p}.$$

In view of Lemma 3.1, $y(t) = (y_1(t), \ldots, y_n(t))^T$ is a solution of (2.1) on $[-\tau, +\infty)$, where $y_i(t) = \prod_{0\le t_k<t} J_{ki}^{-1} x_i(t)$, $i = 1, 2, \ldots, n$. Furthermore, it is easy to see that,

$$\begin{aligned}
E|y(t)|^p &= E|(y_1(t), \ldots, y_n(t))^T|^p \\
&= E\left|\left(\prod_{0\le t_k<t} J_{k1}^{-1}x_1(t), \ldots, \prod_{0\le t_k<t} J_{kn}^{-1}x_n(t)\right)^T\right|^p \\
&= E\left(\prod_{0\le t_k<t} J_{k1}^{-2}x_1^2(t) + \cdots + \prod_{0\le t_k<t} J_{kn}^{-2}x_n^2(t)\right)^{\frac{p}{2}} \\
&\le E(M^p(x_1^2(t) + \cdots + x_n^2(t))^{\frac{p}{2}}) \\
&= M^p E|x(t)|^p \\
&< M^p \frac{\varepsilon}{M^p} = \varepsilon
\end{aligned}$$

which implies that the zero solution of (2.1) is also p-stable. This completes the proof. □

Example 3.1. Consider the nonlinear stochastic differential delay system under impulsive control as follows:

$$\begin{cases} dy(t) = y(t)(-a - by^2(t) + \cos ty(t-\tau) + \sin ty^2(t-\tau))dt + y(t)y(t-\tau)dw(t), & t \ne t_k \\ y(t_k^+) - y(t_k) = \dfrac{y(t_k)^{k+1}}{(1+y(t_k)^2)^k}, & t = t_k, k \in N \end{cases} \tag{3.2}$$

then

$$J_k = \frac{y(t_k)}{y(t_k) + \frac{y(t_k)^{k+1}}{(1+y(t_k)^2)^k}} = \frac{1}{1 + \frac{y(t_k)^k}{(1+y(t_k)^2)^k}}, \quad k = 1, 2, \ldots$$

and we can see that there exists a positive constant M such that, for any $t > 0$,

$$1 \le \prod_{0\le t_k<t} J_k^{-1} = \prod_{0\le t_k<t}\left(1 + \frac{y(t_k)^k}{(1+y(t_k)^2)^k}\right) \le \prod_{k=1}^{\infty}\left(1 + \frac{1}{2^k}\right) \le M \tag{3.3}$$

Equivalently, we consider the following nonlinear stochastic differential delay system without impulses:

$$\begin{aligned}
dx(t) &= x(t)\Bigg\{-a - b\prod_{0\le t_k<t} J_k^{-2}x^2(t) + \prod_{0\le t_k<t-\tau} J_k^{-1}\cos tx(t-\tau) \\
&\quad + \prod_{0\le t_k<t-\tau} J_k^{-2}\sin tx^2(t-\tau)\Bigg\}dt + \prod_{0\le t_k<t-\tau} J_k^{-1}x(t)x(t-\tau)dw(t)
\end{aligned} \tag{3.4}$$

If we choose $V(x, t) = x^2$, one has

$$\begin{aligned}
\mathcal{L}V &= 2x^2(t)\Bigg\{-a - b\prod_{0\le t_k<t} J_k^{-2}x^2(t) + \prod_{0\le t_k<t-\tau} J_k^{-1}\cos tx(t-\tau) \\
&\quad + \prod_{0\le t_k<t-\tau} J_k^{-2}\sin tx^2(t-\tau)\Bigg\} + \prod_{0\le t_k<t-\tau} J_k^{-2}x^2(t)x^2(t-\tau) \\
&\le 2x^2(t)\{-a - bx^2(t) + Mx(t-\tau) + M^2x^2(t-\tau)\} + M^2x^2(t)x^2(t-\tau) \\
&\le -2ax^2(t) + Mx^2(t-\tau) - (2b - M - \frac{3}{2}M^2)x^4(t) + \frac{3}{2}M^2x^4(t-\tau)
\end{aligned}$$

Here, $\mathcal{L}$ denotes the well-known $\mathcal{L}$-operator given by Itô's formula.

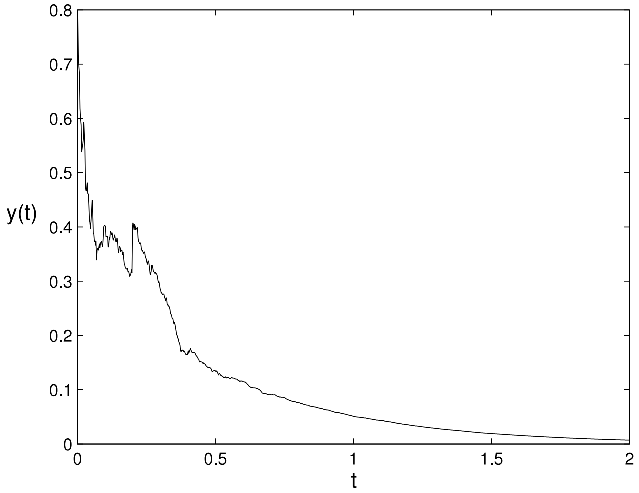

Fig. 1. Impulsive stabilization of $y(t)$ in Eq. (3.2) with $a = 2$, $b = 17$.

Then due to Theorem 3.4 in Ref. [7], we can deduce a conclusion that when $a > \frac{M}{2}$ and $b > \frac{3M^2}{2} + \frac{M}{2}$, the zero solution of the nonlinear stochastic differential delay system (3.4) is exponentially stable in mean square. Take into account inequality (3.3), by Theorem 3.1, we obtain that the zero solution of impulsive controlled nonlinear stochastic differential delay system (3.2) is also exponentially stable in mean square, shown in Fig. 1.

Similarly we can establish the following stability analysis of the nonlinear stochastic differential equation with time-varying delay (2.4).

Theorem 3.2. *Assume that* (H1)–(H4) *hold, and there exists a positive constant L such that, for any $t > 0$,*

$$\left| \prod_{0 \le t_k < t} J_{ki} \right| \le L, \quad i = 1, 2, \ldots, n. \tag{3.5}$$

(i) *If the zero solution of* (2.1) *is p-stable, then the zero solution of* (2.4) *is also p-stable.*
(ii) *If the zero solution of* (2.1) *is exponentially p-stable ($p \ge 2$), then the zero solution of* (2.4) *is also exponentially p-stable. Moreover they have the same exponential convergence rate.*
(iii) *If the zero solution of* (2.1) *is asymptotically stable, then the zero solution of* (2.4) *is also asymptotically stable.*

It would be of interest to observe that by combining Theorem 3.1 with Theorem 3.2 one easily obtains the following result.

Corollary 3.1. *Assume that* (H1)–(H4) *hold, and inequalities* (3.1) *and* (3.5) *are satisfied, then*

(i) *The zero solution of* (2.1) *is p-stable if and only if the zero solution of* (2.4) *is p-stable.*
(ii) *The zero solution of* (2.1) *is exponentially p-stable ($p \ge 2$), if and only if the zero solution of* (2.4) *is exponentially p-stable, and they have the same exponential convergence rate.*
(iii) *The zero solution of* (2.1) *is asymptotically stable if and only if the zero solution of* (2.4) *is asymptotically stable.*

Similar to Theorems 3.1 and 3.2 and Corollary 3.1, we have the following stability results between systems (2.3) and (2.6).

Theorem 3.3. *Assume that* (H1)–(H4) *hold,*

(i) *While the condition in Theorem 3.1 (the inequality (3.1)) is replaced by:*

$$\left| \prod_{0 \le t_k < t} J_k^{-1} \right| \le M, \tag{3.6}$$

a stability result similar to Theorem 3.1 holds between systems (2.3) and (2.6).

(ii) *While the condition in Theorem 3.2 (the inequality (3.5)) is replaced by:*

$$\left| \prod_{0 \le t_k < t} J_k \right| \le L, \tag{3.7}$$

a stability result similar to Theorem 3.2 holds between systems (2.3) and (2.6).

(iii) *While the condition in Corollary 3.1 (the inequalities (3.1) and (3.5)) is replaced by the inequalities (3.6) and (3.7), a stability result similar to Corollary 3.1 holds between systems (2.3) and (2.6).*

Proof. We prove (i) only, and others can be proved similarly and proofs will be omitted here. For any $t > 0$, note that

$$\begin{aligned}\left|\prod_{0\le t_k<t} J_k^{-1}\right| \le \quad M &\Longleftrightarrow \sqrt{\operatorname{trace}\left(\prod_{0\le t_k<t} J_k^{-T}\prod_{0\le t_k<t} J_k^{-1}\right)} \le M\\ &\Longleftrightarrow \sqrt{\prod_{0\le t_k<t} J_{k1}^{-2}+\prod_{0\le t_k<t} J_{k2}^{-2}+\cdots+\prod_{0\le t_k<t} J_{kn}^{-2}} \le M\\ &\Longrightarrow \left|\prod_{0\le t_k<t} J_{ki}^{-1}\right| \le M,\quad i=1,2,\ldots,n\end{aligned}$$

which implies that Theorem 3.1 holds. This completes the proof. □

Remark 3.2. The '*equivalent method*' offers a feasible way to the researchers to qualitatively analyze impulsive systems by the relative results and approaches of systems without impulses. Compared to traditional approaches, '*equivalent method*' may be more feasible, flexible and effective. For example, Itô's formula, a traditional method to study stochastic differential delay system, cannot be effectively used in impulsive stochastic differential delay system, since it is difficult to deal with when integrating intervals contain impulses, and we can overcome this difficulty by using '*equivalent method*'.

Remark 3.3. Novel stability criteria are established in Theorems 3.1 and 3.2, Corollary 3.1 and Theorem 3.3 of the n-dimensional stochastic differential delay system with and without nonlinear impulsive effects. Based on the '*equivalent method*', the stability criteria depend on the impulsive operator conditions. For instance, the inequality (3.1) in Theorem 3.1, the inequality (3.5) in Theorem 3.2, the inequalities (3.6) and (3.7) in Theorem 3.3. Obviously, the impulsive operator conditions are the possible further improvement within the 'equivalent method', even equivalent conditions for the stability study of the two systems, namely n-dimensional stochastic differential delay systems with and without impulsive effects, can be expected in future. It is worth pointing out that, as will be shown later in Remark 4.1, many of the existing research concerning stability problems of impulsive systems satisfy our requirements nicely.

4. Application to impulsive stochastic neural networks

Recently, various neural networks have been active research topics due to its practical importance and successful application (see [12–37]). In practice, because of the existence of modeling inaccuracies and changes in environment of the model, network parameter uncertainties can be often encountered in a neural system. Therefore, the stability analysis for uncertain neural networks emerges as a research topic of primary significance.

Wang et al. [23], Zhang et al. [26] have investigated the following neural network with parameter uncertainties and stochastic perturbations which is described as follows,

$$\mathrm{d}y(t) = [A(t)y(t) + W_1(t)\sigma(y(t-\tau(t)))]\mathrm{d}t + [C(t)y(t) + D(t)y(t-\tau(t))]\mathrm{d}w(t).$$

Huang and Feng [24] have shown the stability results of a more complex uncertain stochastic neural network with time-varying delays as follows,

$$\mathrm{d}y(t) = [A(t)y(t) + W_0(t)\sigma(y(t)) + W_1(t)\sigma(y(t-\tau(t)))]\mathrm{d}t + [C(t)y(t) + D(t)y(t-\tau(t))]\mathrm{d}w(t).$$

What is more, Rakkiyappan et al. [27] have studied the asymptotic stability of the following stochastic delayed recurrent neural networks with the nonlinear activation function g,

$$\mathrm{d}y(t) = [-Cy(t) + A(t)\sigma(y(t)) + B(t)\sigma(y(t-\tau(t)))]\mathrm{d}t + g(t, y(t), y(t-\tau(t)))\mathrm{d}w(t).$$

It is easy to see that the papers list above are the special cases of the following uncertain stochastic neural network with time-varying delays,

$$\mathrm{d}y(t) = [A(t)y(t) + W_0(t)\sigma(y(t)) + W_1(t)\sigma(y(t-\tau(t)))]\mathrm{d}t + g(t, y(t), y(t-\tau(t)))\mathrm{d}w(t),$$

which has been investigated in Refs. Chen and Lu [28], Wu et al. [29], Yu et al. [30], Kwon et al. [31].

However, the impulsive effect was not taken into account in the above literature and few results of impulsive stochastic neural networks have been reported. Based on this consideration, in this section, we apply our stability results above to analyze the stability of impulsive uncertain stochastic neural networks with time-varying delay.

Consider the following uncertain impulsive stochastic neural network with time-varying delay described by:

$$\begin{cases}\mathrm{d}y(t) = [A(t)y(t) + W_0(t)\sigma(y(t)) + W_1(t)\sigma(y(t-\tau(t)))]\mathrm{d}t + g(t, y(t), y(t-\tau(t)))\mathrm{d}w(t), & t \ne t_k\\ y(t_k^+) - y(t_k) = I_k(y(t_k)), \quad t = t_k,\ k \in N\end{cases} \tag{4.1}$$

where $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T \in R^n$ is the state vector associated with the n neuron; $\sigma(y(\cdot)) = (\sigma_1(y_1(\cdot)), \sigma_2(y_2(\cdot)), \ldots, \sigma_n(y_n(\cdot)))^T \in R^n$ is the activation function, and each function σ_i satisfies $\sigma_i(0) = 0$; $g(t, 0, 0) = 0$

and $g(t,x,y)$ is locally Lipschitz continuous and satisfies the linear growth condition as well. The matrices $A(t) = A + \Delta A(t)$, $W_0(t) = W_0 + \Delta W_0(t)$, $W_1(t) = W_1 + \Delta W_1(t)$, where A is a diagonal matrix representing self-feedback term, and W_0, W_1 are the connection weight matrix and the delayed connection weight matrix, respectively. $\Delta A(t), \Delta W_0(t), \Delta W_1(t)$ represent the parameter uncertainties in (4.1) while $\tau(t)$ is the time-varying delay, impulsive operator I_k satisfies $I_k(0) = 0$ as well.

Let $y(t,\phi)$ be the state trajectory of impulsive stochastic neural network (4.1) with the initial function $y(t) = \phi(t)$ in $PC^b_{F_0}([-\tau, 0], R^n)$. As a standing hypothesis, we assume that for any $\phi = (\phi_1, \dots, \phi_n)^T \in PC^b_{F_0}([-\tau, 0], R^n)$, there exists a unique solution of system (4.1) which is denoted by $y(t)$ if no confusion arises. Obviously, system (4.1) admits an equilibrium solution $y(t) = 0$.

Equivalently, we consider the following uncertain stochastic neural network with time-varying delay without impulses:

$$\begin{aligned} dx(t) = {}& \left[A(t)x(t) + \prod_{0\le t_k<t} J_k W_0(t)\sigma\left(\prod_{0\le t_k<t} J_k^{-1}x(t)\right) + \prod_{0\le t_k<t} J_k W_1(t)\sigma\left(\prod_{0\le t_k<t-\tau(t)} J_k^{-1}x(t-\tau(t))\right)\right]dt \\ & + \prod_{0\le t_k<t} J_k g\left(t, \prod_{0\le t_k<t} J_k^{-1}x(t), \prod_{0\le t_k<t-\tau(t)} J_k^{-1}x(t-\tau(t))\right)dw(t). \end{aligned} \tag{4.2}$$

We are now in a position to apply our stability criteria to the stability analysis of impulsive stochastic uncertain neural networks, that is, the stability of impulsive stochastic uncertain neural networks can be judged by the stability of a corresponding stochastic uncertain neural network without impulses.

Example 4.1. We consider the following stochastic two-neuron uncertain neural networks with nonlinear impulsive effects:

$$\begin{cases} dy(t) = [W_0 y(t) + W_1\sigma(y(t-\tau(t)))]dt + [H_0 y(t) + H_1 y(t-\tau(t))]dw(t), \quad t \ne t_k \\ y_1(t_k^+) - y_1(t_k) = \dfrac{1}{2^k}y_1(t_k) + \dfrac{1}{k^2}y_1(t_k)\sin^2(y_2(t_k)) \\ y_2(t_k^+) - y_2(t_k) = \dfrac{1}{3^k}y_2(t_k) + \dfrac{1}{k^2}y_2(t_k)\cos^2(y_1(t_k)) \quad t = t_k,\ k \in N. \end{cases} \tag{4.3}$$

Now, suppose $\sigma(y(t-\tau(t))) = y(t-\tau(t))$, $\tau(t) = \tau > 0$ is a constant and the matrices W_0, W_1, H_0, H_1 are as follows:

$$W_0 = \begin{pmatrix} -9 & 0 \\ 0 & -9 \end{pmatrix}, \qquad W_1 = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}, \qquad H_0 = \begin{pmatrix} 0.75 & 0.5 \\ 0.25 & 0.25 \end{pmatrix}, \qquad H_1 = \begin{pmatrix} 0.25 & 0.75 \\ 0.25 & 0.25 \end{pmatrix}$$

Equivalently, we consider the following stochastic two-neuron uncertain networks without impulsive effects:

$$dx(t) = \left[W_0 x(t) + W_1 \prod_{t-\tau\le t_k<t} J_k\sigma(x(t-\tau))\right]dt + \left[H_0 x(t) + H_1\prod_{t-\tau\le t_k<t} J_k x(t-\tau)\right]dw(t), \quad t \ne t_k \tag{4.4}$$

where $J_k = \begin{pmatrix} J_{k1} & 0 \\ 0 & J_{k2} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{1+\frac{1}{2^k}+\frac{1}{k^2}\sin^2(y_2(t_k))} & 0 \\ 0 & \dfrac{1}{1+\frac{1}{3^k}+\frac{1}{k^2}\cos^2(y_1(t_k))} \end{pmatrix}$.

Let $A_0 = W_0, A_{0m} = 0, A_1 = \frac{2}{3}W_1, A_{1m} = \frac{1}{3}W_1, B_0 = H_0, B_{0m} = 0, B_1 = \frac{2}{3}H_1, B_{1m} = \frac{1}{3}H_1$ in Theorem 2 of [6], then we can conclude that the stochastic two-neuron uncertain networks without impulsive effects (4.4) is exponentially stable in mean square.

Moreover, note that there exists a positive constant L such that, for any $t > 0$,

$$\begin{aligned} \left|\prod_{0\le t_k<t} J_k^{-1}\right| &= \left|\prod_{0\le t_k<t}\begin{pmatrix} 1+\dfrac{1}{2^k}+\dfrac{1}{k^2}\sin^2(y_2(t_k)) & 0 \\ 0 & 1+\dfrac{1}{3^k}+\dfrac{1}{k^2}\cos^2(y_1(t_k)) \end{pmatrix}\right| \\ &< \left|\prod_{0\le t_k<t}\begin{pmatrix} 1+\dfrac{1}{2^k}+\dfrac{1}{k^2} & 0 \\ 0 & 1+\dfrac{1}{3^k}+\dfrac{1}{k^2} \end{pmatrix}\right| \\ &= \prod_{0\le t_k<t}\left[\left(1+\frac{1}{2^k}+\frac{1}{k^2}\right)^2 + \left(1+\frac{1}{3^k}+\frac{1}{k^2}\right)^2\right]^{\frac{1}{2}} \\ &\le L. \end{aligned}$$

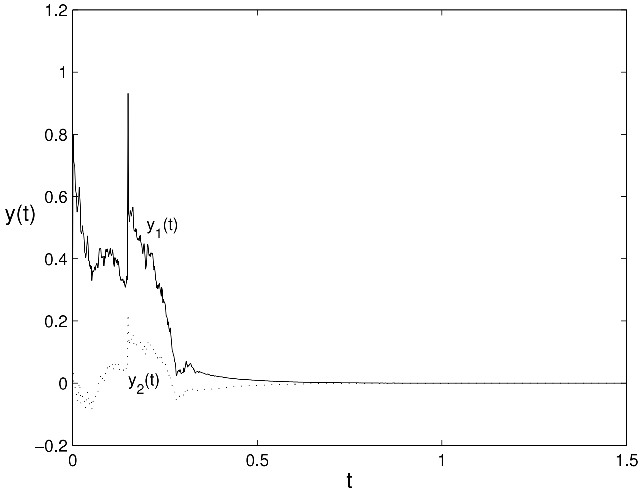

Fig. 2. Impulsive stabilization of $y(t)$ in Eq. (4.3).

Then, by Theorem 3.3(i), we can deduce a conclusion that stochastic two-neuron uncertain networks with nonlinear impulsive effects (4.3) is also exponentially stable in mean square, showed in Fig. 2.

Remark 4.1. In many works on the stability analysis of impulsive neural networks (eg. [14–19]), the impulsive operator I_k satisfies the following condition, for $i = 1, 2, \ldots, n,\ k \in N$,

$$I_{ik}(x_i(t_k^-)) = -\delta_{ik}(x_i(t_k^-) - x_i^*) \quad 0 < \delta_{ik} < 2, \tag{4.5}$$

From the definition of J_{ki}, we have $|J_{ki}^{-1}| = |1 - \delta_{ik}| < 1, \quad i = 1, 2, \ldots, n$, then $|\prod_{0 \le t_k < t} J_{ki}^{-1}| \le 1, \quad i = 1, 2, \ldots, n$, that is clear. Note that, in Theorem 3.1, the condition that ensures the stability of the stochastic differential delay system without impulses that can be used to judge the stability of the impulsive stochastic differential delay system is inequality (3.1), $|\prod_{0 \le t_k < t} J_{ki}^{-1}| \le M, \quad i = 1, 2, \ldots, n$, which is less conservative than inequality (4.5) that is required in many previous works. Therefore, it can be easily seen that our stability results will provide a new and convenient, efficient approach to study the stability of impulsive stochastic neural networks.

5. Conclusion

In this paper, we have investigated the stability of an n-dimensional stochastic differential delay system with nonlinear impulsive effects. First, the equivalent relation between the solution of the n-dimensional stochastic differential delay system with nonlinear impulsive effects and that of a corresponding n-dimensional stochastic differential delay system without impulsive effects is established, namely, *equivalent method*. It should be noticed that, to the best of our knowledge, there is no effective *equivalent method* of n-dimensional system with nonlinear impulsive effects has been established. Thus, no results of an n-dimensional system with nonlinear impulsive effects related to *equivalent method* have been reported. In contrast, many results of 1-dimensional system with nonlinear impulsive effects (eg. [1,2,4,8,9]) or n-dimensional system with linear impulsive effects (eg. [3,10]) related to *equivalent method* have been reported. In our future work, we could further consider stability, existence and attractivity of periodic solutions of impulsive stochastic differential delay systems by using '*equivalent method*'. Second, by using *equivalent method*, some stability criteria for the n-dimensional stochastic differential delay systems with and without impulsive effects are obtained. Finally, the stability criteria are applied to uncertain impulsive stochastic neural networks with time-varying delay. The results show that this convenient and efficient method will provide a new approach to study the stability of impulsive stochastic neural networks. It is worthwhile to note that there are only very few results on impulsive stochastic neural networks. Some examples are also presented to illustrate the effectiveness of our results.

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