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Periodic solutions of a logistic type population model with harvesting ☆

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ABSTRACT

We consider a bifurcation problem arising from population biology

$$\frac{du(t)}{dt} = f\big(u(t)\big) - \varepsilon h(t),$$

where $f(u)$ is a logistic type growth rate function, $\varepsilon \geqslant 0$, $h(t)$ is a continuous function of period T such that $\int_0^T h(t)\,dt > 0$. We prove that there exists an $\varepsilon_0 > 0$ such that the equation has exactly two T-periodic solutions when $0 < \varepsilon < \varepsilon_0$, exactly one T-periodic solution when $\varepsilon = \varepsilon_0$, and no T-periodic solution when $\varepsilon > \varepsilon_0$.

1. Introduction

When a population grows at a density-dependent growth rate $f(u)$ and it is harvested with a seasonal harvesting rate $h(t)$ with period T, the population can be described by a differential equation (see for example, [3,4,8])

$$\frac{du(t)}{dt} = f\big(u(t)\big) - h(t). \tag{1.1}$$

Here we assume that the non-linear function f is a logistic type function which satisfies

(f1) $f \in C^2(\mathbf{R})$, $f(0) = 0$, $f'(0) > 0$, $f(u) > 0$ for $u \in (0, M)$, $f(M) = 0$ and $f'(M) < 0$;
(f2) $f''(u) < 0$ for $u \in \mathbf{R}$.

Some typical examples of $f(u)$ are $f(u) = au - bu^p$, where $a, b > 0$, $p \geqslant 2$, see [10,12,17,20]. When $h(t)$ is a constant h, then it is easy to know that there is a threshold (maximum sustainable yield) $h_* > 0$ such that when $h > h_*$, (1.1) has no equilibrium and the population is destined to extinction, and when $h < h_*$, there are exactly two positive equilibria. When the seasonal effect on the harvesting is considered ($h(t)$ periodic), then one expects that periodic solutions play similar role as equilibria in the constant case, and the question is: how many periodic solutions does (1.1) have?

Here we assume that the total yield over one season (period) is positive, that is $\int_0^T h(t)\,dt > 0$. Thus we allow $h(t)$ to be negative, that is stocking instead of harvesting, but the total effort is still harmful to the population. Without loss of

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generality, we can normalize $h(t)$ so that $\int_0^T h(t)\,dt = T$, and we rewrite (1.1) to be

$$\frac{du(t)}{dt} = f\big(u(t)\big) - \varepsilon h(t), \tag{1.2}$$

where $\varepsilon \in \mathbf{R}$ measures the harvesting strength. Our result is

Theorem 1.1. *Suppose that f satisfies* (f1) *and* (f2). *Let $h(t)$ be a continuous function of period T such that $\int_0^T h(t)\,dt = T$. Then there exists an $\varepsilon_0 > 0$ such that* (1.2) *has exactly two T-periodic solutions when $\varepsilon < \varepsilon_0$, exactly one T-periodic solution when $\varepsilon = \varepsilon_0$, and no T-periodic solution when $\varepsilon > \varepsilon_0$.*

Thus the dynamics of (1.2) is qualitatively similar to the autonomous equation with constant $h(t)$, with two equilibria replaced by two periodic solutions. One can also define ε_0 as the maximum sustainable yield in this case.

It is known that (1.2) has at most two periodic solutions due to the concavity of f, see Pliss [18], Lazer and Sànchez [13], Mawhin [15], and Korman and Ouyang [11]. The turning point (fold) structure is also studied in [11,15], as well as McKean and Scovel [16]. But the main result in [11,16] is for a more general problem, and the result is abstract in describing the singular points. The result in [15] assumes that f depends on t, but $h(t)$ is assumed to be a constant or strictly positive (see [15, Remark 2]). Our result here is more specific in term of harvesting model, and it is more general than the one in [15] since we only assume that $\int_0^T h(t)\,dt > 0$. Our proof uses some ingredients in previous approach, but also some more recent bifurcation theory. A different approach was given in Benardete, Noonburg and Pollina [3], based on Poincaré map and dynamical systems arguments, and they proved a special case of Theorem 1.1 when $f(x) = Rx(1-x)$ and $h(t) = 1 + \alpha\sin(2\pi t)$. Other recent discussions can be found in [2,5–7], for example.

We give preliminaries in Section 2, and we prove the main result in Section 3. Some discussions, numerical examples and conjectures are given in Section 4. An earlier version of Theorem 1.1 appeared in Problem Section of Electronic Journal of Differential Equations in 2006 [19].

2. Preliminaries

To prove the theorem we recall the following result based on the implicit function theorem:

Lemma 2.1. *Consider*

$$x' = f(\varepsilon, t, x), \tag{2.1}$$

where $f \in C^1(\mathbf{R} \times \mathbf{R} \times \mathbf{R}^n, \mathbf{R}^n)$, and $x \in \mathbf{R}^n$. We suppose that $f(\varepsilon, t+T, x) = f(\varepsilon, t, x)$ for all $(\varepsilon, t, x) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n$, and for $\varepsilon = 0$, (2.1) *has a T-periodic solution $y = y(t)$. Let $z(\varepsilon, t, \xi)$ be the solution of the initial value problem*:

$$z' = f(\varepsilon, t, z), \quad t > 0, \qquad z(0) = \xi, \tag{2.2}$$

and let $A(\varepsilon, t, \xi) = \partial z(\varepsilon, t, \xi)/\partial\xi$. Suppose that $\lambda = 1$ is not an eigenvalue of $A(0, T, y(0))$. Then there exists a $\delta > 0$ such that for $|\varepsilon| < \delta$, there exists a C^1 function $\xi(\varepsilon)$ such that $\xi(0) = y(0)$, and (2.1) *has a unique T-periodic solution $y_\varepsilon(t)$ with $y_\varepsilon(0) = \xi(\varepsilon)$.*

Proof. This lemma is well known, see for example, [1]. For the sake of completeness, we include the proof here. Notice that a T-periodic solution satisfies $z(\varepsilon, T, \xi) = \xi$. Define $F : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ by $F(\varepsilon, \xi) = z(\varepsilon, T, \xi) - \xi$. Then F is continuously differentiable, $F(0, y(0)) = 0$, $F_\xi(0, y(0)) = A(0, T, y(0)) - I$. Since $\lambda = 1$ is not an eigenvalue of $A(0, T, y(0))$, then $F_\xi(0, y(0))$ is invertible, and the claimed result follows from the implicit function theorem. □

We also recall a well-known result for concave non-linearity. A particular case of Lemma 2.2 was known in Pliss [18], and the current version is due to Mawhin [15] (see also Korman and Ouyang [11]).

Lemma 2.2.

$$x' = f(t, x), \tag{2.3}$$

where $f(t+T, x) = f(t, x)$ and $f_{xx}(t, x) < 0$ for all $(t, x) \in \mathbf{R} \times \mathbf{R}^n$. Then (2.3) *has at most two T-periodic solutions.*

We also recall the following well-known bifurcation theorem in [9] and a new bifurcation theorem of the authors [14].

Theorem 2.3 *(Saddle-node bifurcation theorem of Crandall and Rabinowitz [9]). Suppose that X and Y are Banach spaces. Let $(\lambda_0, u_0) \in \mathbf{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood V of (λ_0, u_0) into Y. Suppose that*

(F1) *dim* $N(F_u(\lambda_0, u_0)) = \operatorname{codim} R(F_u(\lambda_0, u_0)) = 1$, *and* $N(F_u(\lambda_0, u_0)) = \operatorname{span}\{w_0\}$, *and*
(F2) $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$.

Then

1. *If* Z *is a complement of* $\operatorname{span}\{w_0\}$ *in* X, *then the solutions of* $F(\lambda, u) = 0$ *near* (λ_0, u_0) *form a curve* $(\lambda(s), u(s)) = (\lambda(s), u_0 + sw_0 + z(s))$, *where* $s \mapsto (\lambda(s), z(s)) \in \mathbf{R} \times Z$ *is a continuously differentiable function near* $s = 0$ *and* $\lambda(0) = \lambda_0$, $\lambda'(0) = 0$, $z(0) = z'(0) = 0$.
2. *Suppose that* F *is* C^2 *in* u, *then*

$$\lambda''(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle}{\langle l, F_\lambda(\lambda_0, u_0)\rangle}, \tag{2.4}$$

where $N(F_u)$ *and* $R(F_u)$ *are the null space and the range space of linear operator* F_u *and there exists* $l \in Y^*$ *such that* $R(F_u(\lambda_0, u_0)) = \{h \in Y\colon \langle h, l\rangle = 0\}$.

Theorem 2.4. *(See [14].) Suppose that* X *and* Y *are Banach spaces. Let* $F : \mathbf{R} \times X \to Y$ *be a* C^2 *mapping. Suppose that* $F(\lambda_0, u_0) = 0$, F *satisfies* (F1) *and* $F_\lambda(\lambda_0, u_0) = 0$. *We assume that the matrix*

$$H \equiv \begin{pmatrix} \langle l, F_{\lambda\lambda}(\lambda_0, u_0)\rangle & \langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle \\ \langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle & \langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle \end{pmatrix} \tag{2.5}$$

is non-degenerate, i.e., $\det(H) \neq 0$.

1. *If* H *is definite, i.e.* $\det(H) > 0$, *then the solution set of* $F(\lambda, u) = 0$ *near* $(\lambda, u) = (\lambda_0, u_0)$ *is* $\{(\lambda_0, u_0)\}$.
2. *If* H *is indefinite, i.e.* $\det(H) < 0$, *then the solution set of* $F(\lambda, u) = 0$ *near* $(\lambda, u) = (\lambda_0, u_0)$ *is the union of two intersecting* C^1 *curves, and the two curves are in form of* $(\lambda_i(s), u_i(s)) = (\lambda_0 + \mu_i s + s\theta_i(s), u_0 + \eta_i s w_0 + s v_i(s))$, $i = 1, 2$, *where* $s \in (-\delta, \delta)$ *for some* $\delta > 0$, (μ_1, η_1) *and* (μ_2, η_2) *are non-zero linear independent solutions of the equation*

$$\big\langle l, F_{\lambda\lambda}(\lambda_0, u_0)\big\rangle\mu^2 + 2\big\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\big\rangle\mu\eta + \big\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\big\rangle\eta^2 = 0. \tag{2.6}$$

3. Proof of main result

Proof of Theorem 1.1. When $\varepsilon = 0$, (1.2) has exactly two equilibrium solutions, $u_0(t) = 0$ and $v_0(t) = M$, which are also the only T-periodic solutions, as all other solutions are monotonic. We use Lemma 2.1 to show that for $\varepsilon > 0$ small, there are exactly two periodic solutions $u_\varepsilon(t)$ and $v_\varepsilon(t)$ which are perturbations of $u_0(t)$ and $v_0(t)$ respectively. When $y(0) = y_0$, $A(0, t, y_0)$ satisfies the equation:

$$A' = f_x(0, t, y_0)A, \qquad A(0, 0, y_0) = 1. \tag{3.1}$$

At u_0, (3.1) becomes $A' = f'(0)A$ with $A(0) = 1$, hence $A(t) = e^{f'(0)t}$ and $A(T) = e^{f'(0)T} \neq 1$ since $f'(0) > 0$; similarly at v_0, (3.1) becomes $A' = f'(M)A$ with $A(0) = 1$, hence $A(t) = e^{f'(M)t}$ and $A(T) = e^{f'(M)T} \neq 1$ since $f'(M) < 0$. Hence from Lemma 2.1, the T-periodic solutions of (1.2) near u_0 and v_0 can be represented by curves $(\varepsilon, u_\varepsilon(t))$ and $(\varepsilon, v_\varepsilon(t))$ respectively.

The argument of the implicit function theorem can be used repeatedly at a T-periodic solution $u(t)$ as long as the solution $A(t)$ of linearized equation:

$$A' = f'\big(u(t)\big)A, \qquad A(0, 0, y_0) = 1, \tag{3.2}$$

satisfies $A(T) \neq 1$. We show that, there exists $\varepsilon_0 > 0$ such that when $\varepsilon > \varepsilon_0$, (1.2) has no T-periodic solutions. From our assumption, there exist a unique $u_1 \in (0, M)$ such that $f(u_1) = \max_{u \in \mathbf{R}^+} f(u)$, $f'(u_1) = 0$ and $\bar{h} = T^{-1}\int_0^T h(t)\,dt > 0$. If $u(t)$ is a T-periodic solution, then

$$0 = u(T) - u(0) = \int_0^T u'(t)\,dt = \int_0^T \big[f(u) - \varepsilon h(t)\big]\,dt$$

$$= \int_0^T \big[\big(f(u) - f(u_1)\big) + f(u_1) - \varepsilon\bar{h} - \varepsilon\big(h(t) - \bar{h}\big)\big]\,dt < 0,$$

if $\varepsilon > \varepsilon_0 \equiv f(u_1)/\bar{h}$. Hence the curves containing $(\varepsilon, u_\varepsilon)$ and $(\varepsilon, v_\varepsilon)$ cannot be continued indefinitely. A degenerate solution $(\varepsilon_*, u_*(t))$ must occur. At such a degenerate solution, $A(T) = 1$. Indeed $A(t) = \exp(\int_0^t f'(u_*(s))\,ds)$, thus u_* satisfies $\int_0^T f'(u_*(s))\,ds = 0$.

Let $(\varepsilon_*, u_*(t))$ be the first degenerate solution when we increase ε along the branch $\{(\varepsilon, u_\varepsilon)\}$. From definition, $F_\xi(\varepsilon_*, u_*)[\tau] = (\exp(\int_0^T f'(u_*(s))\,ds))\tau - \tau = 0$, then the null space of $F_\xi(\varepsilon_*, u_*) : \mathbf{R} \to \mathbf{R}$ is one-dimensional and the range of $F_\xi(\varepsilon_*, u_*)$ is co-dimension one. Hence the condition (F1) defined in Section 2 is satisfied. We claim that $F_\varepsilon(\varepsilon_*, u_*) \notin R(F_\xi(\varepsilon_*, u_*))$. It is sufficient to show that $F_\varepsilon(\varepsilon_*, u_*) \neq 0$. Indeed $F_\varepsilon(\varepsilon_*, u_*) = \partial z(\varepsilon_*, T, u_*(0))/\partial\varepsilon$, and $\partial z(\varepsilon_*, t, u_*(0))/\partial\varepsilon$ satisfies

$$B' = f'\big(u_*(t)\big)B - h(t), \quad t > 0, \qquad B(0) = 0. \tag{3.3}$$

Solving (3.3), one obtains $B(t) = -A(t)\int_0^t [A(s)]^{-1}h(s)\,ds$.

If $F_\varepsilon(\varepsilon_*, u_*) = B(T) = -A(T)\int_0^T [A(s)]^{-1}h(s)\,ds = 0$, then the conditions in Theorem 2.4 except the one about matrix H are satisfied. For the matrix H, $F_{\xi\xi}(\varepsilon_*, u_*) = C(T)$ can be evaluated by the equation:

$$C' = f'\big(u_*(t)\big)C + f''\big(u_*(t)\big)A^2, \qquad C(0) = 0, \tag{3.4}$$

where $A(t) = \exp(\int_0^t f'(u_*(s))\,ds)$. Then from (f2), $C(t) = f''(u_*)A(t)\int_0^t A(s)\,ds < 0$ hence $C(T) < 0$; $F_{\varepsilon\xi}(\varepsilon_*, u_*) = D(T)$ can be evaluated by the equation:

$$D' = f'\big(u_*(t)\big)D + f''\big(u_*(t)\big)AB, \qquad D(0) = 0, \tag{3.5}$$

then $D(t) = f''(u_*(t))A(t)\int_0^t B(s)\,ds$; and $F_{\varepsilon\varepsilon}(\varepsilon_*, u_*) = E(T)$ can be evaluated by the equation:

$$E' = f'\big(u_*(t)\big)E + f''\big(u_*(t)\big)B^2, \qquad E(0) = 0, \tag{3.6}$$

then $E(t) = f''(u_*(t))A(t)\int_0^t A(s)^{-1}B(s)^2\,ds$ and $E(T) < 0$. Hence the matrix H is

$$H = H(\varepsilon_*, u_*) = \begin{pmatrix} E(T) & D(T) \\ D(T) & C(T) \end{pmatrix}, \tag{3.7}$$

and from Cauchy–Schwarz inequality we have

$$\det H = E(T)C(T) - D^2(T) \tag{3.8}$$

$$= \big(f''\big(u_*(T)\big)A(T)\big)^2\left[\int_0^T A(s)\,ds \int_0^T A(s)^{-1}B(s)^2\,ds - \left(\int_0^T B(s)\,ds\right)^2\right] > 0. \tag{3.9}$$

Thus we can apply Theorem 2.4, the solution set of $F(\varepsilon, \xi) = 0$ near $(\varepsilon, \xi) = (\varepsilon_*, u_*)$ is the singleton $\{(\varepsilon_*, u_*)\}$, which contradicts with the fact that (ε_*, u_*) is a limit point of the curve $\{(\varepsilon, u_\varepsilon)\}$ from left. Hence $F_\varepsilon(\varepsilon_*, u_*) = B(T) = -A(T)\int_0^T [A(s)]^{-1}h(s)\,ds \neq 0$ and $F_\varepsilon(\varepsilon_*, u_*) \notin R(F_\xi(\varepsilon_*, u_*))$, and Theorem 2.3 is applicable.

Near a degenerate solution (ε_*, u_*), the T-periodic solutions of (1.2) form a curve $(\varepsilon(s), u(s))$ such that $\varepsilon(0) = \varepsilon_*$, $\varepsilon'(0) = 0$,

$$\varepsilon''(0) = -\frac{F_{\xi\xi}(\varepsilon_*, u_*)}{F_\varepsilon(\varepsilon_*, u_*)}, \tag{3.10}$$

and $u(s) = u_* + s + o(|s|)$. From the last paragraph, $F_{\xi\xi}(\varepsilon_*, u_*) = C(T) < 0$. If $B(T) > 0$, then $\varepsilon''(0) > 0$ and $\varepsilon(s)$ is parabola-like opening to the right, which contradicts again with the fact that (ε_*, u_*) is a limit point of the curve $\{(\varepsilon, u_\varepsilon)\}$ from left. Therefore we must have $B(T) = -A(T)\int_0^T [A(s)]^{-1}h(s)\,ds < 0$ and $\varepsilon''(0) < 0$.

Now we continue the curve $\{(\varepsilon, u_\varepsilon)\}$ of periodic solutions turned back from (ε_*, u_*) with decreasing ε. If we reach another degenerate solution (ε^*, u^*) at some $\varepsilon^* \in (0, \varepsilon_*)$, then similar to arguments above, we can apply Theorem 2.3 at (ε^*, u^*) and we must have $\varepsilon''(0) > 0$. But that would imply (1.2) has at least three periodic solutions for $\varepsilon \in (\varepsilon^*, \varepsilon^* + \delta)$ with small $\delta > 0$, which contradicts with Lemma 2.2. Hence there is no other degenerate solution when we continue the lower branch from (ε_*, u_*) with decreasing ε until $\varepsilon = 0$. But at $\varepsilon = 0$, the periodic solutions are equilibria $u = 0$ and $u = M$, since the lower branch cannot be identical to the upper branch which is from $u = M$, then the lower branch must connect to $u = 0$. Hence the lower branch is identical to the branch $\{(\varepsilon, v_\varepsilon)\}$ continuing from $(\varepsilon, u) = (0, 0)$.

So far we have shown that (1.2) has at least two periodic solutions when $\varepsilon \in (0, \varepsilon_*)$, then from Lemma 2.2, (1.2) has exactly two periodic solutions for $\varepsilon \in (0, \varepsilon_*)$. Suppose that for some $\varepsilon > \varepsilon_*$, (1.2) has a periodic solution (ε, u), then the same arguments above can be used to show that (ε, u) belongs to another curve which has exactly one degenerate solution, and the curve can be extended to $\varepsilon = 0$. But there are only two equilibria when $\varepsilon = 0$, hence this curve must be identical

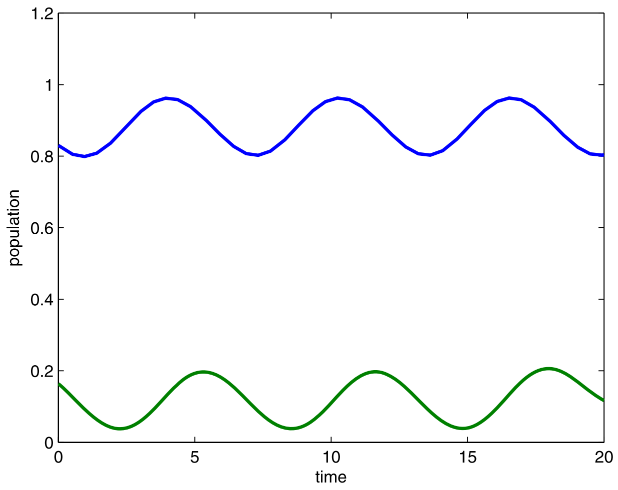

Fig. 1. Two periodic solutions for $\varepsilon < \varepsilon_0$. Here $h(t) = 1 + \cos(t)$. Left: $\varepsilon = 0.1$. Right: $\varepsilon = 0.22$.

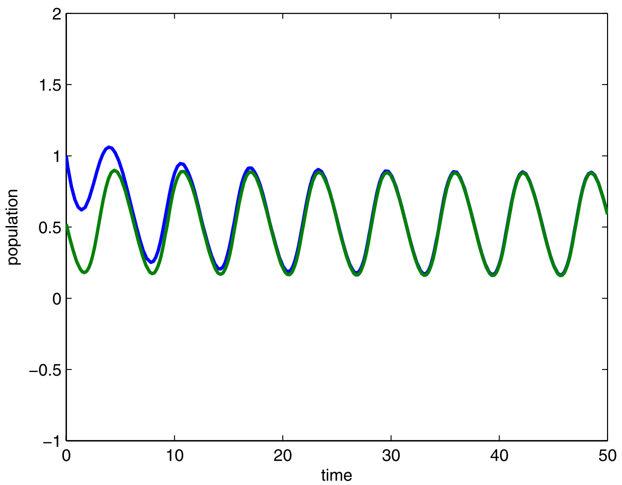

Fig. 2. From sustainable to collapse. Here $h(t) = 1 + 2\cos(t)$. Left: $\varepsilon = 0.185$. Right: $\varepsilon = 0.187$.

to the one in the last paragraph. This shows that all periodic solutions are on the curve connecting $(\varepsilon, u) = (0, 0)$ and $(\varepsilon, u) = (0, M)$, and it completes the proof. $\square$

4. Discussion

1. In the proof, we give an upper bound for the threshold value ε_*, that is $\varepsilon_* < \max f(u)/\bar{h} \equiv \varepsilon_\sharp$. Here let us assume that $f(u) = u(1-u)$ for simplicity. If $h(t) \equiv 1$, then $\varepsilon_* = 0.25 = \varepsilon_\sharp$. But if $h(t)$ is not a constant, then $\varepsilon_* < \varepsilon_\sharp$. For example, for $h(t) = 1 + \cos(t)$, one can numerically find $\varepsilon_* \approx 0.226 < 0.25 = \varepsilon_\sharp$ (see Fig. 1), and for $h(t) = 1 + 2\cos(t)$, the numerical value of ε_* is about $0.186 < \varepsilon_\sharp$ (see Fig. 2). We conjecture that when the variation of the periodic harvesting function $h(t)$ increases, the maximum sustainable yield decreases.
2. If $h(t)$ is assumed to be positive, then the result in Theorem 1.1 has been obtained in [15,19], and the proof would be easier. Our assumption that $\int_0^T h(t)\,dt > 0$ is sufficient for the fold type bifurcation diagram, but not optimal. For example, if $h(t) = \cos(t)$ (so $\int_0^T h(t)\,dt = 0$), one still can find the existence of a threshold $\varepsilon_* \approx 0.77$ (see Fig. 3).
3. Our result can be extended to periodic $h(t)$ with less smoothness condition such as $h \in L^1$.

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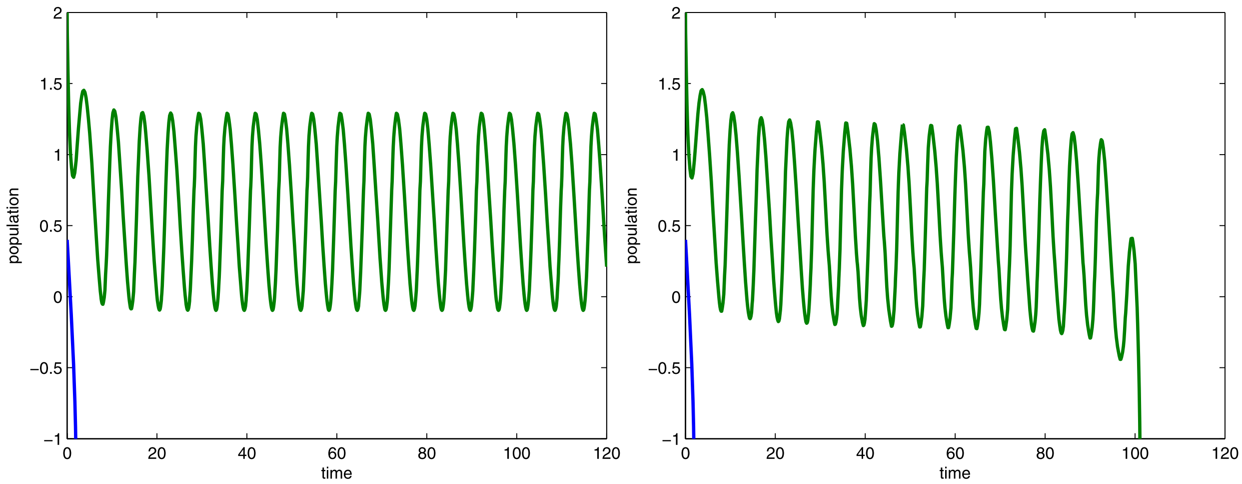

Fig. 3. Threshold for zero-average harvesting. Here $h(t) = \cos(t)$. Left: $\varepsilon = 0.77$. Right: $\varepsilon = 0.78$.

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