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Diffusion-driven instability and bifurcation in the Lengyel–Epstein system[☆]

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Abstract

Lengyel–Epstein reaction–diffusion system of the CIMA reaction is considered. We derive the precise conditions on the parameters so that the spatial homogenous equilibrium solution and the spatial homogenous periodic solution become Turing unstable or diffusively unstable. We also perform a detailed Hopf bifurcation analysis to both the ODE and PDE models, and derive conditions for determining the bifurcation direction and the stability of the bifurcating periodic solution.

Keywords: Lengyel–Epstein system; The CIMA reaction; Turing diffusion-driven instability; Hopf bifurcation

1. Introduction

One of the most fundamental problems in theoretical biology is to explain the mechanisms by which patterns and forms are created in the living world. In his seminal paper *The Chemical Basis of Morphogenesis*, Turing [13] showed that a system of coupled reaction–diffusion equations can be used to describe patterns and forms in biological systems. Turing's theory shows that diffusion could destabilize an otherwise stable equilibrium of the reaction–diffusion system and lead to nonuniform spatial patterns. This kind of instability is usually called *Turing instability* or *diffusion-driven instability*.

Over the years, Turing's idea has attracted the attention of a great number of investigators and was successfully developed on the theoretical backgrounds. Not only it has been studied in biological and chemical fields, some investigations range as far as economics, semiconductor physics, and star formation (see [4]). However, the search for Turing patterns in real chemical or biological systems turned out to be difficult. Finally in early 1990s, working with the chlorite–iodide–malonic acid or so-called CIMA reaction, De Kepper et al. [3] discovered the formation of stationary three-dimensional (but almost two-dimensional) structures with characteristic wavelengths of 0.2 mm, which is the first experimental evidence to the Turing patterns nearly 40 years after the publication of [13]. The fact is that

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there are five reactants involved in the CIMA reaction which make the mathematical discussion of the system more complicated. Lengyel and Epstein [7,8] were able to reduce the original system to a two-dimensional one, which we call *Lengyel–Epstein model*. A more detailed historical account of the development of CIMA reaction model and experiments can be found in Epstein and Pojman [4].

We assume that the reactor Ω is a bounded domain in $\mathbf{R}^n$, with a smooth boundary $\partial\Omega$. Let $u=u(x,t)$ and $v=v(x,t)$ denote the chemical concentrations of the activator iodide (I^-) and the inhibitor chlorite (ClO_2^-), respectively, at time $t>0$ and a point $x \in \Omega$. The Lengyel–Epstein model is in form of

$$\frac{\partial u}{\partial t} = \Delta u + a - u - \frac{4uv}{1+u^2},$$
$$\frac{\partial v}{\partial t} = \sigma\left[c\Delta v + b\left(u - \frac{uv}{1+u^2}\right)\right], \tag{1.1}$$

where a and b are parameters related to the feed concentrations; c is the ratio of the diffusion coefficients; $\sigma>0$ is a rescaling parameter depending on the concentration of the starch, enlarging the effective diffusion ratio to σc. In laboratory conditions, a sample of parameters is taken in the range $0<a<35$, $0<b<8$, $c=1.5$ and $\sigma=8$. We shall assume accordingly that all constants a, b, c, and σ are positive.

In [10], Ni and Tang studied both existence and nonexistence for the steady states of the reaction–diffusion system (1.1) subject to the initial condition:

$$u(x,0)=u_0(x)>0, \quad v(x,0)=v_0(x)>0, \quad x\in\Omega, \quad u_0, v_0 \in C^2(\Omega)\cap C^0(\overline{\Omega}); \tag{1.2}$$

and the no-flux boundary condition:

$$\frac{\partial u}{\partial \vec{n}} = \frac{\partial v}{\partial \vec{n}} = 0, \quad x\in\partial\Omega, \quad t>0, \tag{1.3}$$

where $\vec{n}$ is the unit outer normal to $\partial\Omega$. They obtain the *a priori* bound of solutions to the system (1.1)–(1.3), nonexistence of nonconstant steady states for small effective diffusion rate, and existence of nonconstant steady states for large effective diffusion rate. These results partially verify the diffusion-driven instability of Turing for the CIMA reaction system. In [6], Jang et al. further considered the global bifurcation structure of the set of the nonconstant steady states in the one-dimensional case and clarified the limiting behavior of the steady states by using a shadow system approach.

It has been observed that Eq. (1.1) possesses a spatially homogeneous periodic solution for some parameter ranges, and the interaction of the Hopf and Turing bifurcations could be the driving force of more complicated spatiotemporal phenomena for Eq. (1.1) (see [11]). The purpose of this paper is to study the stability of the periodic orbit as a spatial homogeneous solution of the reaction–diffusion Lengyel–Epstein model, and it is shown that the spatially homogenous periodic solution becomes unstable if parameters related to diffusion coefficients are properly chosen. Notice that Rovinsky and Menzinger [11] also considered the parameter ranges of Hopf and Turing instability, as well as bifurcation directions. But our analysis are more complete and rigorous, and the stability of bifurcating periodic solutions are considered. Ruan [12] investigated the stability of equilibrium and bifurcating periodic solutions of Gierer–Meinhardt system.

The rest of the paper is organized as follows. In Section 2, we study the asymptotical behavior of the equilibrium of the local system (the ODE model) and show that for the local system Hopf bifurcation occurs; we consider the diffusion-driven instability of the equilibrium solution in Section 3; in Section 4, we analyze the stability of the bifurcating (spatial homogeneous) periodic solution through the Hopf bifurcation when the spatial domain is a finite interval. In Section 5, we illustrate our results with numerical simulations; in Section 6, we end our investigation with concluding remarks.

2. Analysis of the local system

For the reaction–diffusion Lengyel–Epstein system (1.1), the local system is an ordinary differential equation in form of

$$\frac{du}{dt} = a - u - \frac{4uv}{1+u^2} := F(u,v),$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \sigma b\left(u - \frac{uv}{1+u^2}\right) := G(u,v). \tag{2.1}$$

The system (2.1) has a unique equilibrium point $(u^*, v^*) = (\alpha, 1+\alpha^2)$, where $\alpha = a/5$. The Jacobian matrix of the system of (2.1) at (u^*, v^*) is

$$J := \begin{pmatrix} \dfrac{3\alpha^2-5}{1+\alpha^2} & -\dfrac{4\alpha}{1+\alpha^2} \\ \dfrac{2\sigma\alpha^2 b}{1+\alpha^2} & -\dfrac{\sigma\alpha b}{1+\alpha^2} \end{pmatrix}.$$

The characteristic equation is given by $\lambda^2 - \lambda T + D = 0$, where

$$T := \operatorname{tr} J = \frac{3\alpha^2 - 5 - \sigma\alpha b}{1+\alpha^2}, \quad D := \det\ J = \frac{5\sigma\alpha b}{1+\alpha^2}.$$

Note that the system (2.1) is an activator–inhibitor system under the condition

(H_1) $\quad 3\alpha^2 - 5 > 0,$

since $F_u(u^*, v^*) > 0$, $G_v(u^*, v^*) < 0$, $F_v(u^*, v^*) < 0$ and $G_u(u^*, v^*) > 0$ (see discussion of activator–inhibitor systems in [9]). It is clear that if

$$0 < 3\alpha^2 - 5 < \sigma\alpha b$$

holds, then the equilibrium (u^*, v^*) of system (2.1) is locally asymptotically stable.

Next we analyze the Hopf bifurcation occurring at (u^*, v^*) by choosing b as the bifurcation parameter. Denote

$$b_0 := \frac{3\alpha^2-5}{\sigma\alpha}.$$

Then when $b = b_0$, the Jacobian matrix J has a pair of imaginary eigenvalues $\lambda = \pm\mathrm{i}\sqrt{5\sigma\alpha b_0/(1+\alpha^2)}$. Let $\lambda = \beta(b) \pm \mathrm{i}\omega(b)$ be the roots of $\lambda^2 - \lambda T + D = 0$, then

$$\beta(b) = \frac{3\alpha^2 - 5 - \sigma\alpha b}{2(1+\alpha^2)}, \quad \omega(b) = \frac{1}{2}\sqrt{\frac{20\sigma\alpha b}{1+\alpha^2} - \left(\frac{3\alpha^2-5-\sigma\alpha b}{1+\alpha^2}\right)^2},$$

and

$$\beta'(b)|_{b=b_0} = -\frac{\sigma\alpha}{2(1+\alpha^2)} < 0.$$

By the *Poincaré–Andronov–Hopf Bifurcation Theorem* (for example [14] Theorem 3.1.3), we know that system (2.1) undergoes a Hopf bifurcation at (u^*, v^*) when $b = b_0$. However, the detailed nature of the Hopf bifurcation needs further analysis of the normal form of the system. To that end we translate the equilibrium (u^*, v^*) to the origin by the translation $\widetilde{u} = u - u^*$, $\widetilde{v} = v - v^*$. For the sake of convenience, we still denote $\widetilde{u}$ and $\widetilde{v}$ by u and v, respectively. Thus, the local system (2.1) becomes

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= 4\alpha - u - \frac{4(u+\alpha)(v+1+\alpha^2)}{1+(u+\alpha)^2}, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= \sigma b\left[u + \alpha - \frac{(u+\alpha)(v+1+\alpha^2)}{1+(u+\alpha)^2}\right]. \end{aligned} \tag{2.2}$$

Rewrite system (2.2) to

$$\begin{pmatrix} \dfrac{\mathrm{d}u}{\mathrm{d}t} \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} \end{pmatrix} = J\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v,b) \\ g(u,v,b) \end{pmatrix}, \tag{2.3}$$

where

$$f(u,v,b) := \frac{4\alpha(3-\alpha^2)}{(1+\alpha^2)^2}u^2 + \frac{4(\alpha^2-1)}{(1+\alpha^2)^2}uv + \frac{4(\alpha^4-6\alpha^2+1)}{(1+\alpha^2)^3}u^3 + \frac{4\alpha(3-\alpha^2)}{(1+\alpha^2)^3}u^2v + \mathcal{O}(|u|^4, |u|^3|v|),$$

$$g(u,v,b) := \frac{\sigma b}{4} f(u,v,b).$$

Set matrix

$$P := \begin{pmatrix} 1 & 0 \\ N & M \end{pmatrix},$$

where

$$M = \frac{(1+\alpha^2)\sqrt{4D-T^2}}{8\alpha} \quad \text{and} \quad N = \frac{3\alpha^2 - 5 + \sigma\alpha b}{8\alpha}.$$

Clearly,

$$P^{-1} = \begin{pmatrix} 1 & 0 \\ -\dfrac{N}{M} & \dfrac{1}{M} \end{pmatrix},$$

and when $b = b_0$,

$$N_0 := N|_{b=b_0} = \frac{1}{4}\sigma b_0, \quad M_0 := M|_{b=b_0} = \frac{\sqrt{5(1+\alpha^2)(3\alpha^2-5)}}{4\alpha}, \quad \omega(b_0) = \sqrt{\frac{5(3\alpha^2-5)}{1+\alpha^2}}.$$

By the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix},$$

system (2.3) becomes

$$\begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = J(b) \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} F^1(x,y,b) \\ F^2(x,y,b) \end{pmatrix}. \tag{2.4}$$

Here

$$J(b) := \begin{pmatrix} \beta(b) & -\omega(b) \\ \omega(b) & \beta(b) \end{pmatrix},$$

$$F^1(x,y,b) := A_{20}x^2 + A_{11}xy + A_{21}x^2y + A_{30}x^3 + \mathcal{O}(|x|^4, |x|^3|y|),$$

where,

$$A_{20} := \frac{4\alpha(3-\alpha^2)}{(1+\alpha^2)^2}, \quad A_{11} := \frac{4M(\alpha^2-1)}{(1+\alpha^2)^2},$$

$$A_{21} := \frac{4M\alpha(3-\alpha^2)}{(1+\alpha^2)^3}, \quad A_{30} := \frac{4\alpha(3-\alpha^2)}{(1+\alpha^2)^2} + \frac{4N(\alpha^2-1)}{(1+\alpha^2)^2},$$

and

$$F^2(x,y,b) := \left(-\frac{N}{M} + \frac{1}{4M\sigma b}\right) F^1(x,y,b).$$

Rewrite (2.4) in the following polar coordinates form:

$$\dot{r} = \beta(b)r + a(b)r^3 + \cdots,$$
$$\dot{\theta} = \omega(b) + c(b)r^2 + \cdots, \tag{2.5}$$

then the Taylor expansion of (2.5) at $b = b_0$ yields

$$\dot{r} = \beta'(b_0)(b - b_0)r + a(b_0)r^3 + \mathcal{O}\left((b - b_0)^2 r, (b - b_0)r^3, r^5\right),$$
$$\dot{\theta} = \omega(b_0) + \omega'(b_0)(b - b_0) + c(b_0)r^2 + \mathcal{O}\left((b - b_0)^2, (b - b_0)r^2, r^4\right). \tag{2.6}$$

In order to determine the stability of the periodic solution, we need to calculate the sign of the coefficient $a(b_0)$, which is given by

$$a(b_0) := \frac{1}{16}\left[F^1_{xxx} + F^1_{xyy} + F^2_{xxy} + F^2_{yyy}\right] + \frac{1}{16\omega(b_0)}\left[F^1_{xy}(F^1_{xx} + F^1_{yy}) - F^2_{xy}(F^2_{xx} + F^2_{yy}) - F^1_{xx}F^2_{xx} + F^1_{yy}F^2_{yy}\right],$$

where all partial derivatives are evaluated at the bifurcation point, i.e., $(x, y, b) = (0, 0, b_0)$.

Since the highest order of y in both $F^1(x, y, b)$ and $F^2(x, y, b)$ is less than 2, we have that

$$F^1_{xyy}(0, 0, b_0) = F^2_{yyy}(0, 0, b_0) = F^1_{yy}(0, 0, b_0) = F^2_{yy}(0, 0, b_0) \equiv 0.$$

On the other hand, it is easy to calculate that

$$F^2_{xxy}(0, 0, b_0) = F^2_{xx}(0, 0, b_0) = F^2_{xy}(0, 0, b_0) = 0.$$

Thus,

$$a(b_0) = \frac{1}{16}F^1_{xxx}(0, 0, b_0) + \frac{1}{16\omega(b_0)}F^1_{xy}(0, 0, b_0)F^1_{xx}(0, 0, b_0).$$

By tedious but simple calculations, we can obtain

$$a(b_0) = \frac{2\alpha^4 - 27\alpha^2 - 5}{2\alpha^2(1 + \alpha^2)},$$

which implies that $a(b_0) < 0$ if and only if $0 < \alpha^2 < (27 + \sqrt{769})/4$.

Now from *Poincaré–Andronov–Hopf Bifurcation Theorem*, $\beta'(b_0) < 0$ and the above calculation of $a(b_0)$, we summarize our results as follows.

Theorem 2.1. *Suppose that* $\sigma, \alpha > 0$ *so that* (H_1) *is satisfied, and let* $b_0 = (3\alpha^2 - 5)/(\sigma\alpha)$.

1. *The equilibrium* (u^*, v^*) *of system* (2.1) *is locally asymptotically stable when* $b > b_0$, *and unstable when* $b < b_0$;
2. *The system* (2.1) *undergoes a Hopf bifurcation at* (u^*, v^*) *when* $b = b_0$; *the direction of the Hopf bifurcation is subcritical and the bifurcating periodic solutions are orbitally asymptotically stable if*

$$(H_2) \quad \frac{5}{3} < \alpha^2 < \frac{27 + \sqrt{769}}{4};$$

and the direction of the Hopf bifurcation is supercritical and the bifurcating periodic solutions are unstable if

$$(H_2') \quad \alpha^2 > \frac{27 + \sqrt{769}}{4}.$$

The discussion above shows that the local system (2.1) has periodic solutions arising from Hopf bifurcation. In the following we employ the Poincaré–Bendixson theorem to verify that the system (2.1) has periodic solutions.

Theorem 2.2. *Suppose that* $\sigma, \alpha > 0$ *so that* (H$_1$) *is satisfied, and* $b < (3\alpha^2 - 5)/(\sigma\alpha)$. *Then system* (2.1) *has at least one stable periodic solution satisfying* $0 < u(t) < a$ *and* $0 < v(t) \leqslant 1 + \varepsilon$ for some $\varepsilon > \frac{\alpha^2}{25}$.

Proof. Set $l_1 = \{(u, v) : 0 \leqslant u \leqslant a, v = 0\}$, $l_2 = \{(u, v) : u = a, 0 \leqslant v \leqslant 1 + \varepsilon\}$, $l_3 = \{(u, v) : 0 \leqslant u \leqslant a, v = 1 + \varepsilon\}$, and $l_4 = \{(u, v) : u = 0, 0 \leqslant v \leqslant 1 + \varepsilon\}$, where $\varepsilon > \alpha^2/25$ is a constant. Let $\mathscr{C}$ denote the Jordan curve consisting of the line segments l_1, l_2, l_3 and l_4, and $\mathscr{D}$ denote the interior of $\mathscr{C}$. Then we have that $(\mathrm{d}v/\mathrm{d}t)|_{(u,v)\in l_1} > 0$, $(\mathrm{d}u/\mathrm{d}t)|_{(u,v)\in l_2} < 0$, $(\mathrm{d}v/\mathrm{d}t)|_{(u,v)\in l_3} < 0$ and $(\mathrm{d}u/\mathrm{d}t)|_{(u,v)\in l_4} > 0$. This implies that the trajectories starting at the boundary of $\mathscr{D}$ point inwards. Meanwhile, the unique positive equilibrium (u^*, v^*) is in the domain $\mathscr{D}$, and is unstable. Hence, there exists at least a stable periodic solution which belongs to $\mathscr{D}$ from the Poincaré–Bendixson theorem. This completes the proof. □

3. Diffusion-driven instability of the equilibrium solution

In this part, we will derive conditions for the diffusion-driven instability with respect to the equilibrium solution, the spatially homogenous solution of the reaction–diffusion Lengyel–Epstein system. Such diffusion-driven instability for the equilibrium solution (u^*, v^*) has been investigated in [10], and here we derive only the special case when $\Omega = (0, \pi)$ for completeness of our analysis.

Consider the following system with the no-flux boundary condition in a one-dimensional "cube" $(0, \pi)$:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a - u - \frac{4uv}{1+u^2},\\
&\frac{\partial v}{\partial t} = \sigma\left[c\frac{\partial^2 v}{\partial x^2} + b\left(u - \frac{uv}{1+u^2}\right)\right],\\
&u_x(0,t) = u_x(\pi,t) = 0,\\
&v_x(0,t) = v_x(\pi,t) = 0.
\end{aligned} \tag{3.1}$$

It is well known that the operator $u \to -u_{xx}$ with the above no-flux boundary condition has eigenvalues and eigenfunctions as follows:

$$\mu_0 = 0, \quad \phi_0(x) = \sqrt{\frac{1}{\pi}}, \quad \mu_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}}\cos(kx), \quad k = 1, 2, 3, \ldots .$$

The linearized system of Eq. (3.1) at $(\alpha, 1 + \alpha^2)$ has the form:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = L\begin{pmatrix} u \\ v \end{pmatrix} := D\begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix} + J\begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.2}$$

where

$$D := \begin{pmatrix} 1 & 0 \\ 0 & \sigma c \end{pmatrix}, \quad J := \begin{pmatrix} \dfrac{3\alpha^2 - 5}{1+\alpha^2} & -\dfrac{4\alpha}{1+\alpha^2} \\ \dfrac{2\sigma\alpha^2 b}{1+\alpha^2} & -\dfrac{\sigma\alpha b}{1+\alpha^2} \end{pmatrix},$$

with domain

$$\{(u, v) \in H^2[(0,\pi)] \times H^2[(0,\pi)] : \ u_x(0,t) = u_x(\pi,t) = 0, v_x(0,t) = v_x(\pi,t) = 0\},$$

where the $H^2[(0, \pi)]$ is the standard Sobolev space.

From the standard linear operator theory (or Theorem 1 of [1]), it is known that if all the eigenvalues of the operator L have negative real parts, then (u^*, v^*) is asymptotically stable, and if some eigenvalues have positive real parts, the (u^*, v^*) is unstable.

We consider the following characteristic equation of the operator L:

$$L\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\mu\begin{pmatrix}\phi\\ \psi\end{pmatrix}.$$

Let $(\phi(x), \psi(x))^{\mathrm{T}}$ be an eigenfunction of L corresponding to the eigenvalue μ, and let

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\sum_{k=0}^{\infty}\begin{pmatrix}a_k\\ b_k\end{pmatrix}\cos kx, \tag{3.3}$$

where a_k and b_k are coefficients, we obtain that

$$-k^2 D\sum_{k=0}^{\infty}\begin{pmatrix}a_k\\ b_k\end{pmatrix}\cos kx + J\sum_{k=0}^{\infty}\begin{pmatrix}a_k\\ b_k\end{pmatrix}\cos kx=\mu\sum_{k=0}^{\infty}\begin{pmatrix}a_k\\ b_k\end{pmatrix}\cos kx. \tag{3.4}$$

Hence,

$$(J-k^2D)\begin{pmatrix}a_k\\ b_k\end{pmatrix}=\mu\begin{pmatrix}a_k\\ b_k\end{pmatrix}\quad (k=0,1,2,\ldots). \tag{3.5}$$

Denote

$$J_k := J-k^2D=\begin{pmatrix}\dfrac{3\alpha^2-5}{1+\alpha^2}-k^2 & -\dfrac{4\alpha}{1+\alpha^2}\\ \dfrac{2\sigma\alpha^2 b}{1+\alpha^2} & -\dfrac{\sigma\alpha b}{1+\alpha^2}-k^2\sigma c\end{pmatrix}\quad (k=0,1,2,\ldots).$$

It follows from this, that the eigenvalues of L are given by the eigenvalues of J_k for $k=0,1,2,\ldots$. The characteristic equation of J_k is

$$\mu^2-\mu T_k+D_k=0,\quad k=0,1,2,\ldots, \tag{3.6}$$

where

$$T_k := \operatorname{tr} J_k=-k^2(1+\sigma c)+\frac{3\alpha^2-5-\sigma\alpha b}{1+\alpha^2},$$

$$D_k := \det J_k=\sigma c k^4+\sigma\left(\frac{\alpha b}{1+\alpha^2}-\frac{(3\alpha^2-5)c}{1+\alpha^2}\right)k^2+\frac{5\sigma\alpha b}{1+\alpha^2}.$$

By analyzing the distribution of the roots of Eq. (3.6), we can obtain the following conclusions.

Theorem 3.1. *Suppose that $b>b_0:=(3\alpha^2-5)/\sigma\alpha$ so that (u^*, v^*) is a locally asymptotically stable equilibrium for* (2.1). *Then (u^*, v^*) is an unstable equilibrium solution of* (3.1) *if*

$$(\mathrm{H}_3)\quad \alpha^2>3 \quad\text{and}\quad c>\frac{3\alpha b}{\alpha^2-3};$$

and (u^*, v^*) is a locally asymptotically stable equilibrium solution of (3.1) *if*

$$(\mathrm{H}_4)\quad \frac{5}{3}<\alpha^2\leqslant 3,$$

or

$$(\mathrm{H}_5)\quad \alpha^2>3 \quad\text{and}\quad 0<c<\frac{3\alpha b}{\alpha^2-3}.$$

Proof. For convenience, we rewrite D_k as

$$D_k=\sigma c k^2\left(k^2-\frac{3\alpha^2-5}{1+\alpha^2}\right)+\frac{\sigma\alpha b}{1+\alpha^2}k^2+\frac{5\sigma\alpha b}{1+\alpha^2}.$$

Clearly, $D_1 < 0$ follows from (H_3). This implies that Eq. (3.6) has at least one root with positive real part. Hence (u^*, v^*) is an unstable equilibrium solution of (3.1).

We have $D_{k+1} > D_k$ for $k \geqslant 0$. (H_4) implies that $1 - (3\alpha^2 - 5)/(1 + \alpha^2) \geqslant 0$, and hence $D_1 > 0$. Thus $D_k > 0$ for $k \geqslant 1$. Meanwhile, we know that $T_{k+1} < T_k$ for $k \geqslant 0$ from the definition of T_k. This and $T_0 < 0$ leads to that all the roots of Eq. (3.6) have negative real parts. Therefore, (u^*, v^*) is a locally asymptotically stable equilibrium solution of (3.1). Similarly, we can obtain that (H_5) ensure that all the roots of Eq. (3.6) have negative real parts, and hence the conclusion follows. This completes the proof. □

The result here is relatively simpler than that in [10] since we choose the spatial domain to be the interval $(0, \pi)$. For a general interval $(0, L)$, similar results can be obtained but the wave number k be other than 1. For more general analysis on a general smooth domain in $\mathbf{R}^n$, we refer to [10].

4. Stability of spatial homogeneous periodic solution: bounded spatial domain

A periodic solution $\phi(t)$ of (2.1) is also a (spatially homogeneous) periodic solution of (3.1). Thus, (3.1) also possess any periodic solution as (2.1), including the ones from Hopf bifurcation in Theorem 2.1. A Hopf bifurcation analysis (see [5,2]) can also be performed for the partial differential equation (3.1) at the same bifurcation point, but from the local uniqueness of periodic solutions near Hopf bifurcation point, only spatial homogeneous periodic solutions exist near $b = b_0 = (3\alpha^2 - 5)/\sigma\alpha$. However, the stability of these periodic solutions with respect to (3.1) could be different from that for (2.1). First if $\phi(t)$ is an unstable periodic solution of (2.1), then it is clearly also unstable for (3.1); second if (u^*, v^*) is an unstable equilibrium solution of (3.1) but stable for (2.1), then the nearby bifurcating periodic solutions through Hopf bifurcation are also unstable. The latter case illustrates the interaction of Hopf instability and Turing instability. For fixed $b > b_0$, (u^*, v^*) is a stable equilibrium point for the ODE (2.1), but for $\alpha^2 > 3$ and the diffusion coefficient $c > 3\alpha b/(\alpha^2 - 3)$, it becomes unstable for (3.1) through Turing instability; now we decreases b so that $b < b_0$, a Hopf bifurcation occurs, but it is not destabilizing but causes additional instability (see discussions in [11]).

Our main result in this section is that if (H_4) or (H_5) holds, and the bifurcating periodic solution is stable with respect to (2.1), then it is also stable with respect to (3.1).

Theorem 4.1. *Suppose that* $\sigma, \alpha > 0$ *so that* (H_1) *is satisfied, and let* $b_0 = (3\alpha^2 - 5)/(\sigma\alpha)$. *Then the system* (3.1) *undergoes a Hopf bifurcation at* (u^*, v^*) *when* $b = b_0$.

1. *If* (H_2') *is satisfied, then the direction of the Hopf bifurcation is supercritical and the bifurcating periodic solutions are unstable.*
2. *If* (H_3) *is satisfied, then the direction of the Hopf bifurcation is subcritical and the bifurcating periodic solutions are unstable.*
3. *If* (H_4) *or* (H_5) *is satisfied, then the direction of the Hopf bifurcation is subcritical and the bifurcating periodic solutions are orbitally asymptotically stable.*

We only need to prove part 3 of the theorem. We use the normal form method and center manifold theorem in [5] to study the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions. Let L^* be the conjugate operator of L defined as (3.2) in Section 3:

$$L^* \begin{pmatrix} u \\ v \end{pmatrix} := D \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix} + J^* \begin{pmatrix} u \\ v \end{pmatrix}, \tag{4.1}$$

where

$$J^* := \begin{pmatrix} \dfrac{3\alpha^2 - 5}{1 + \alpha^2} & \dfrac{2\sigma\alpha^2 b}{1 + \alpha^2} \\ -\dfrac{4\alpha}{1 + \alpha^2} & -\dfrac{\sigma\alpha b}{1 + \alpha^2} \end{pmatrix},$$

with domain

$$\{(u, v) \in H^2[(0, \pi)] \times H^2[(0, \pi)] | u_x(0, t) = u_x(\pi, t) = 0, v_x(0, t) = v_x(\pi, t) = 0\}.$$

Let

$$q=\begin{pmatrix} 1 \\ \dfrac{3\alpha^2-5}{4\alpha}-\dfrac{\omega(b_0)(1+\alpha^2)}{4\alpha}\mathrm{i}\end{pmatrix},\quad q^*=\frac{2\alpha}{\omega(b_0)(1+\alpha^2)\pi}\begin{pmatrix}\dfrac{\omega(b_0)(1+\alpha^2)}{4\alpha}+\dfrac{3\alpha^2-5}{4\alpha}\mathrm{i} \\ -\mathrm{i}\end{pmatrix}.$$

It is easy to see that $\langle L^*a, b\rangle=\langle a, Lb\rangle$ for any $a\in D_{L^*}$, $b\in D_L$, and $L^*q^*=-\mathrm{i}\omega_0q^*$, $Lq=\mathrm{i}\omega_0 q$, $\langle q^*, q\rangle=1$, $\langle q^*, \overline{q}\rangle=0$. Here $\langle a, b\rangle=\int_{(0,\pi)}\overline{a}^{\mathrm{T}}b\,\mathrm{d}x$ denotes the inner product in $L^2[(0,\pi)]\times L^2[(0,\pi)]$.

According to [5], write

$$(u, v)^{\mathrm{T}}=zq+\overline{zq}+w;\ z=\langle q^*, (u, v)^{\mathrm{T}}\rangle.$$

Thus,

$$u=z+\overline{z}+w_1,$$

$$v=z\left(\frac{3\alpha^2-5}{4\alpha}-\frac{\omega(b_0)(1+\alpha^2)}{4\alpha}\mathrm{i}\right)+\overline{z}\left(\frac{3\alpha^2-5}{4\alpha}+\frac{\omega(b_0)(1+\alpha^2)}{4\alpha}\mathrm{i}\right)+w_2. \tag{4.2}$$

Our system in (z, w) coordinates becomes

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\mathrm{i}\omega_0 z+\langle q^*, \widetilde{f}\rangle,$$

$$\frac{\mathrm{d}w}{\mathrm{d}t}=Lw+[\widetilde{f}-\langle q^*, \widetilde{f}\rangle q-\langle\overline{q}^*, \widetilde{f}\rangle\overline{q}], \tag{4.3}$$

with $\widetilde{f}=(f, g)^{\mathrm{T}}$, where f and g are defined as (2.3). Straightforward but tedious calculations show that

$$\langle q^*, \widetilde{f}\rangle=\frac{2\alpha}{\omega(b_0)(1+\alpha^2)}\left\{\frac{\omega(b_0)(1+\alpha^2)}{4\alpha}f-\frac{3\alpha^2-5}{4\alpha}f\mathrm{i}+g\mathrm{i}\right\}=\frac{f}{2},$$

$$\langle\overline{q}^*, \widetilde{f}\rangle=\frac{2\alpha}{\omega(b_0)(1+\alpha^2)}\left\{\frac{\omega(b_0)(1+\alpha^2)}{4\alpha}f+\frac{3\alpha^2-5}{4\alpha}f\mathrm{i}-g\mathrm{i}\right\}=\frac{f}{2},$$

$$\langle q^*, \widetilde{f}\rangle q=\frac{2\alpha}{\omega(b_0)(1+\alpha^2)}\begin{pmatrix}\dfrac{\omega(b_0)(1+\alpha^2)}{4\alpha}f \\ \left(\dfrac{3\alpha^2-5}{4\alpha}-\dfrac{\omega(b_0)(1+\alpha^2)}{4\alpha}\mathrm{i}\right)\left(\dfrac{\omega(b_0)(1+\alpha^2)}{4\alpha}f\right)\end{pmatrix},$$

$$\langle\overline{q}^*, \widetilde{f}\rangle\overline{q}=\frac{2\alpha}{\omega(b_0)(1+\alpha^2)}\begin{pmatrix}\dfrac{\omega(b_0)(1+\alpha^2)}{4\alpha}f \\ \left(\dfrac{3\alpha^2-5}{4\alpha}+\dfrac{\omega(b_0)(1+\alpha^2)}{4\alpha}\mathrm{i}\right)\left(\dfrac{\omega(b_0)(1+\alpha^2)}{4\alpha}f\right)\end{pmatrix},$$

$$\langle q^*, \widetilde{f}\rangle q+\langle\overline{q}^*, \widetilde{f}\rangle\overline{q}=\frac{2\alpha}{\omega(b_0)(1+\alpha^2)}\begin{pmatrix}\dfrac{\omega(b_0)(1+\alpha^2)}{2\alpha}f \\ \dfrac{\omega(b_0)(1+\alpha^2)}{2\alpha}g\end{pmatrix}=\begin{pmatrix} f \\ g\end{pmatrix},$$

$$H(z, \overline{z}, w):=\widetilde{f}-\langle q^*, \widetilde{f}\rangle q-\langle\overline{q}^*, \widetilde{f}\rangle\overline{q}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Write $w=(w_{20}/2)z^2+w_{11}z\overline{z}+(w_{02}/2)\overline{z}^2+\mathcal{O}(|z|^3)$ for the equation of the center manifold, we can obtain:

$$(2\mathrm{i}\omega_0-L)\omega_{20}=0,\quad (-L)\omega_{11}=0\quad\text{and}\ \omega_{02}=\overline{\omega_{20}}.$$

This implies that $w_{20} = w_{02} = w_{11} = 0$. Thus, the equation on the center manifold in z, $\overline{z}$ coordinates now is

$$\frac{dz}{dt} = i\omega_0 z + \frac{1}{2}g_{20}z^2 + g_{11}z\overline{z} + \frac{1}{2}g_{02}\overline{z}^2 + \frac{1}{2}g_{21}z^2\overline{z} + \mathcal{O}(|z|^4),$$

among of which,

$$\begin{aligned} g_{20} &= \tfrac{1}{2}[B_{20} + 2B_{11}q_2], \\ g_{11} &= \tfrac{1}{2}[B_{20} + B_{11}\overline{q_2} + B_{11}q_2], \\ g_{02} &= \tfrac{1}{2}[B_{20} + 2B_{11}\overline{q_2}], \\ g_{21} &= \tfrac{1}{2}[B_{30} + B_{21}\overline{q_2} + 2B_{21}q_2], \end{aligned} \tag{4.4}$$

where

$$B_{20} := \frac{\partial^2 f}{\partial u^2}(0,0) = \frac{8\alpha(3-\alpha^2)}{(1+\alpha^2)^2},$$

$$B_{11} := \frac{\partial^2 f}{\partial u \partial v}(0,0) = \frac{4(\alpha^2-1)}{(1+\alpha^2)^2},$$

$$B_{30} := \frac{\partial^3 f}{\partial u^3}(0,0) = \frac{24(\alpha^4-6\alpha^2+1)}{(1+\alpha^2)^3},$$

$$B_{21} := \frac{\partial^3 f}{\partial u^2 \partial v}(0,0) = \frac{8\alpha(3-\alpha^2)}{(1+\alpha^2)^3},$$

$$q_2 := \tau + \rho i := \frac{3\alpha^2-5}{4\alpha} - \frac{\omega(b_0)(1+\alpha^2)}{4\alpha}i.$$

According to [5],

$$c_1(0) = \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}.$$

Then,

$$\begin{aligned} \operatorname{Re} c_1(0) &= \operatorname{Re}\left\{\frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}\right\} \\ &= \operatorname{Re}\left\{\frac{i}{2\omega_0}g_{20}g_{11} + \frac{g_{21}}{2}\right\}. \end{aligned}$$

Since $g_{20}g_{11} = \frac{1}{4}(B_{20} + 2B_{11}\tau)^2 + \frac{1}{2}B_{11}\rho(B_{20} + 2B_{11}\tau)i$, we can get that

$$\begin{aligned} \operatorname{Re} c_1(0) &= \frac{B_{11}(B_{20} + 2B_{11}\tau)(1+\alpha^2)}{16\alpha} + \frac{1}{4}(B_{30} + 3B_{21}\tau) \\ &= \frac{2\alpha^4 - 27\alpha^2 - 5}{2\alpha^2(1+\alpha^2)^2}. \end{aligned}$$

Obvious $\operatorname{Re} c_1(0) < 0$ if and only if (H_2) holds. When (H_2) holds, either (H3), (H_4) or (H_5) is satisfied, but when (H_3) is satisfied, the equilibrium is unstable with respect to (3.1), thus the bifurcating periodic solutions are also unstable; when (H_4) or (H_5) is satisfied, (u^*, v^*) is stable with respect to (3.1) thus the above analysis implies the stability of the periodic solutions.

5. Numerical illustrations

In this section, we present some numerical simulations to illustrate our theoretical analysis, and symbolic mathematical software Matlab (Version 7.0) is used to plot numerical graphs.

The ODE model (2.1) involves three parameters: a, σ, b. First we choose parameters:

$$a = 15, \quad \sigma = 8. \tag{5.1}$$

Under the set of parameters in (5.1), we have the critical point $b_0 = \frac{11}{12}$. By Theorem 2.1(1), we know that, with (5.1), the equilibrium (u^*, v^*) is asymptotically stable when $b > b_0$. From Theorem 2.1(2), a Hopf bifurcation occurs at $b = b_0$, the direction of the bifurcation is subcritical and the bifurcating periodic solutions are asymptotically stable. These are shown in Fig. 1, where the initial condition in the left is taken at (2, 11), and the initial condition on the right is taken at (3.3, 10.5).

Secondly we choose parameters

$$a = 20, \quad \sigma = 8. \tag{5.2}$$

In this case, the Hopf bifurcation value is $b_0 = \frac{43}{32} = 1.34375$. From $\alpha = a/5 = 4 > ((27 + \sqrt{769})/4)^{1/2}$, we know that the Hopf bifurcation is supercritical and the bifurcating periodic solutions are unstable. When $b < b_0 = \frac{43}{32} = 1.34375$, the positive equilibrium is unstable, and there exists a stable limit cycle by Theorem 2.2. These are shown in Fig. 2, where the initial condition on the left is taken at (2, 5), and the initial conditions on the right are taken at (2, 5) and (3.8, 16.8), respectively.

Our PDE model (3.1) contain four parameters: a, σ, b, c. In the following, we choose two sets of parameters:

$$a = 15, \quad \sigma = 8, \quad b = 1.2, \quad c = 2, \tag{5.3}$$

$$a = 15, \quad \sigma = 5, \quad c = 2. \tag{5.4}$$

By Theorem 3.1 (H3), we know that, under the set of parameters in (5.3), the homogenous equilibrium solution (u^*, v^*) of system (3.1) is unstable. This is shown as in Fig. 3, where the initial condition is taken at $(\sin x + 3.5, \cos x + 10.5)$.

Under the set of parameters in (5.4), we have the critical point $b_0 = \frac{22}{15}$. By Theorem 4.1, Hopf bifurcation occurs at $b = b_0$, the direction of the bifurcation is subcritical, and the bifurcating periodic solutions are locally asymptotically stable. This is shown in Fig. 4, where the initial condition is taken at $(\sin x + 3, \cos x + 10)$.

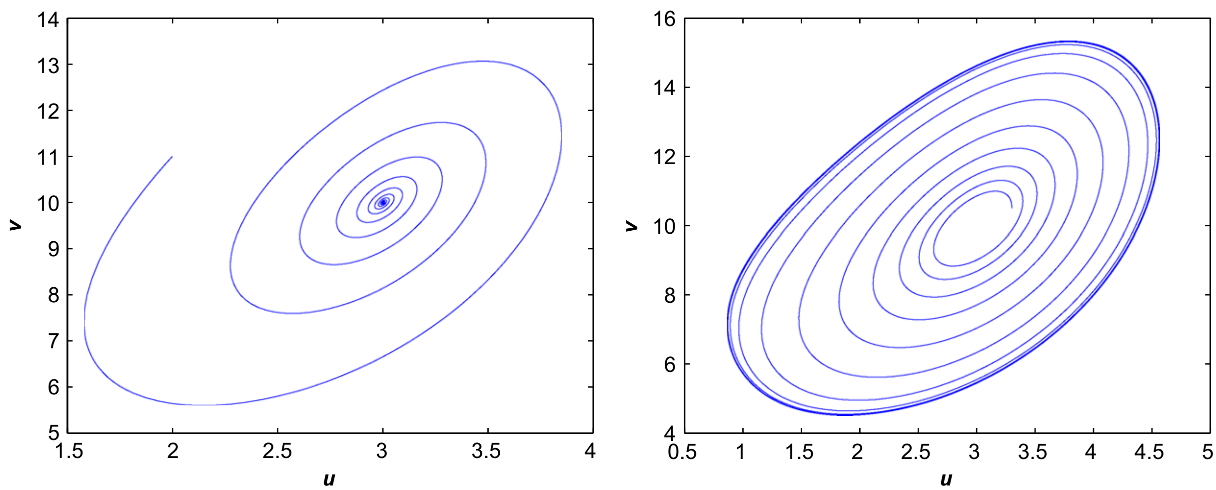

Fig. 1. Phase portraits of (2.1) with parameters in (5.1). Left: the positive equilibrium is asymptotically stable, where $b = 1.2 > b_0$; right: the bifurcating periodic solution is stable, where $b = 0.8 < b_0$.

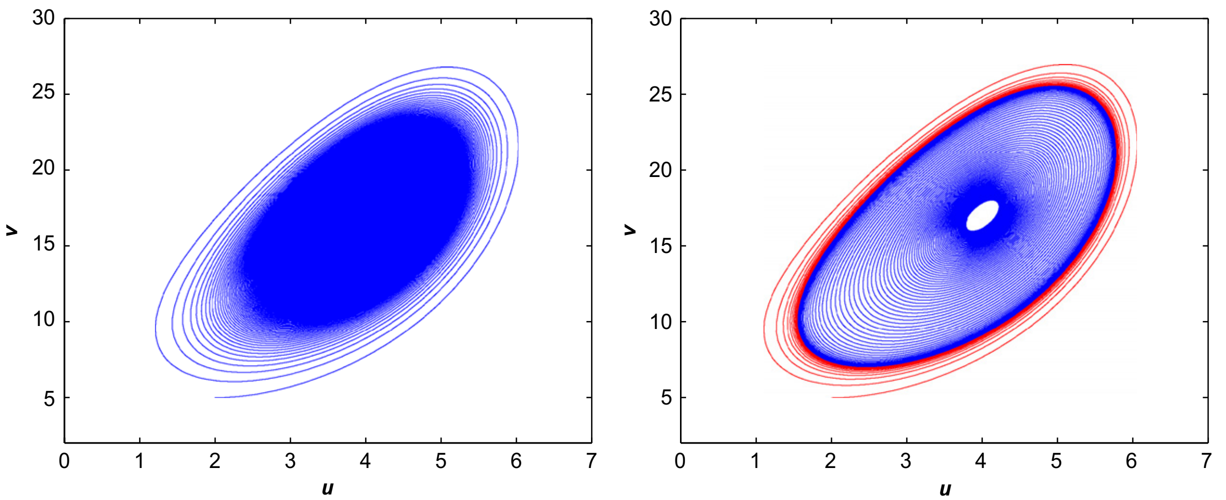

Fig. 2. Phase portraits of (2.1) with parameters in (5.2). Left: the positive equilibrium is asymptotically stable, where $b = 1.3475 > b_0$; right: the positive equilibrium is unstable, and there exists a stable limit cycle, where $b = 1.3375 < b_0$.

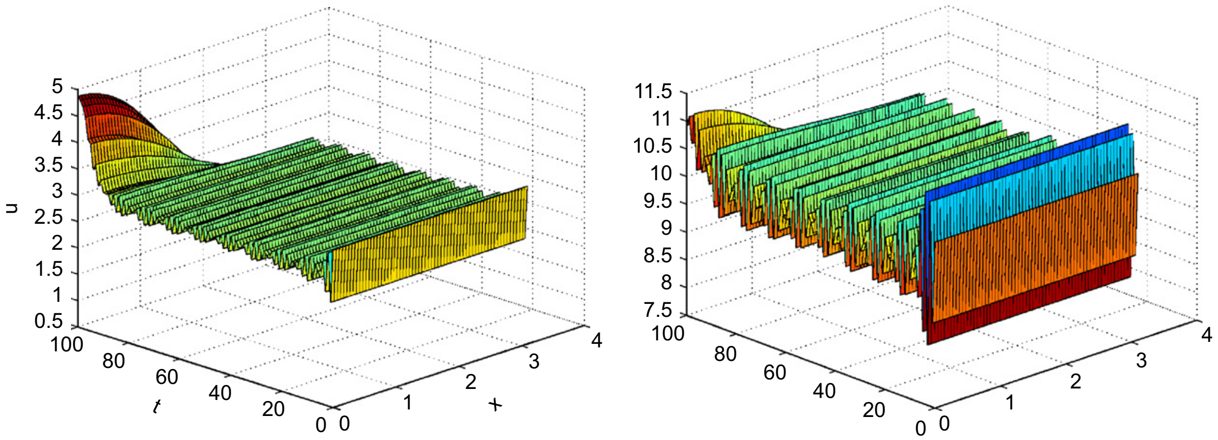

Fig. 3. Numerical simulations of an unstable homogeneous equilibrium solution of system (3.1) under (5.3). Left: component u (unstable); right: component v (unstable).

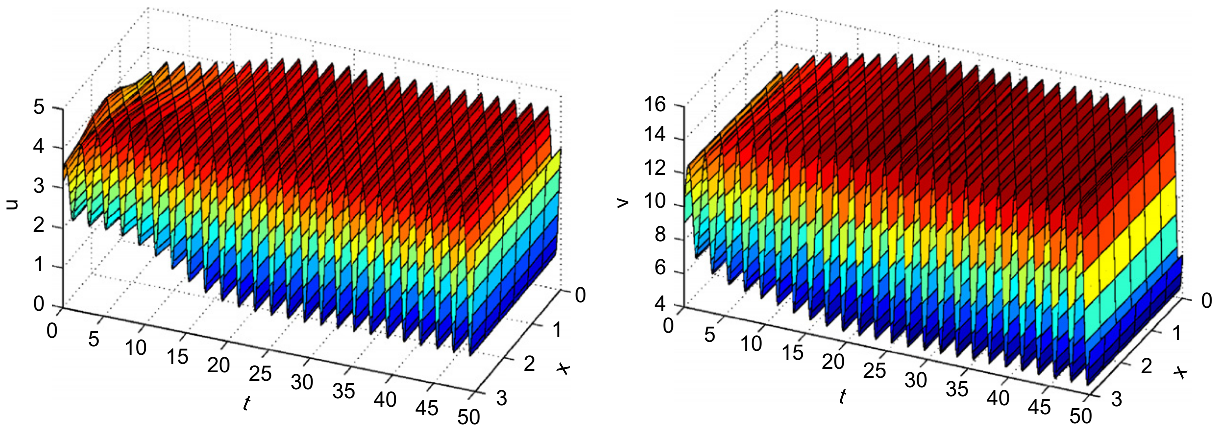

Fig. 4. Numerical simulations of inhomogeneous stable periodic solution of system (3.1) under (5.4) and $b = 0.8 < b_0 = \frac{22}{15}$. Left: orbitally stable periodic solution (component u); right: orbitally stable periodic solution (component v).

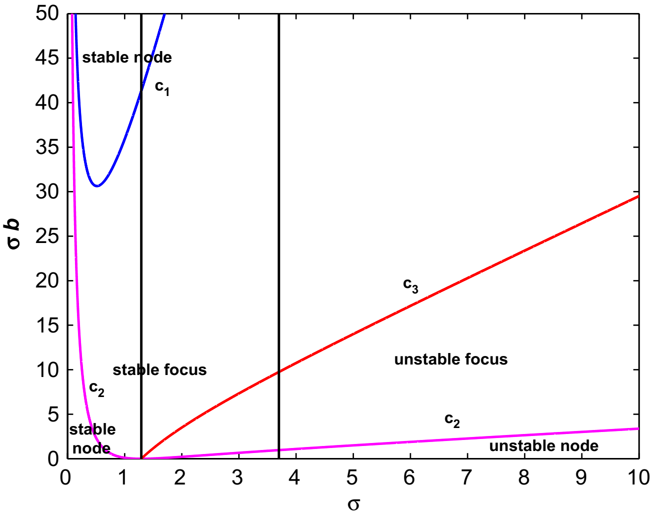

Fig. 5. Bifurcation diagram in $(\alpha, \sigma b)$ parameter space. The upper curve C_1 is: $\sigma b = 13\alpha + 5/\alpha + 4\sqrt{10\alpha^2+10}$; The lower curve C_2 is: $\sigma b = 13\alpha + 5/\alpha - 4\sqrt{10\alpha^2+10}$; The middle upward curve C_3 is: $\sigma b = (3\alpha^2 - 5)/\alpha$, which is a critical curve where Hopf bifurcation occurs; The left vertical line L_1 is: $\alpha = \sqrt{\frac{5}{3}}$; The right vertical line L_2 is: $\alpha = \sqrt{(27+\sqrt{769})/4}$, through which the direction of the bifurcation and the stability of the periodic solution change.

6. Conclusions

A rigorous investigation of the global dynamics of Lengyel–Epstein reaction–diffusion system is attempted, and the main purpose of this article is to identify the parameter ranges of stability/instability of spatial homogeneous equilibrium solution and periodic solutions. We summary our investigation in the following bifurcation diagram (see Fig. 5).

For the local system (2.1), (u^*, v^*) is a stable node when $\sigma b > 13\alpha + 5/\alpha + 4\sqrt{10\alpha^2+10}$ or $0 < \sigma b < 13\alpha + 5/\alpha - 4\sqrt{10\alpha^2+10}$ and $\alpha^2 < \frac{5}{3}$; and (u^*, v^*) is an unstable node when $0 < \sigma b < 13\alpha + 5/\alpha - 4\sqrt{10\alpha^2+10}$ and $\alpha^2 > \frac{5}{3}$. When $13\alpha + 5/\alpha - 4\sqrt{10\alpha^2+10} < \sigma b < 13\alpha + 5/\alpha + 4\sqrt{10\alpha^2+10}$, (u^*, v^*) is a stable focus if $\sigma b > (3\alpha^2 - 5)/\alpha$; and (u^*, v^*) is an unstable focus if $\sigma b < (3\alpha^2 - 5)/\alpha$. Thus, $\sigma b = (3\alpha^2 - 5)/\alpha$ is a Hopf bifurcation curve. Moreover, we obtain that the direction of the Hopf bifurcation is subcritical and the bifurcating periodic solutions are orbitally asymptotically stable if $\frac{5}{3} < \alpha^2 < (27+\sqrt{769})/4$, and the direction of the Hopf bifurcation is supercritical and the bifurcating periodic solutions are unstable if $\alpha^2 > (27+\sqrt{769})/4$. In the latter case, the system possesses at least one periodic solution for $\sigma b > (3\alpha^2 - 5)/\alpha$ and close to the bifurcation point. We shall mention that only the stability of the equilibrium and periodic solution are known, and the global stability of either solution is open.

The equilibrium and periodic solution of the ODE system (2.1) are spatial homogeneous solutions of the reaction–diffusion system (3.1). The stability of the solution can change because of the diffusion. When $\frac{5}{3} < \alpha^2 \leqslant 3$ or $\alpha^2 > 3$ and $0 < c < 3\alpha b/(\alpha^2 - 3)$ holds, the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions of (3.1) are the same as that of the local system (2.1). On the other hand, diffusion-driven instability of the equilibrium solution and bifurcating periodic solution occur when $\alpha^2 > 3$ and $c > 3\alpha b/(\alpha^2 - 3)$. The global branch of periodic solutions bifurcating from the Hopf bifurcation point needs further investigation.

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