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Uniqueness of the positive solution for a class of semilinear elliptic systems

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Abstract

We prove the uniqueness of the positive radially symmetric solution to the following problem:

$$\begin{cases} \Delta u + \lambda v^{p_1} = 0, & x \in B_1, \\ \Delta v + \lambda w^{p_2} = 0, & x \in B_1, \\ \Delta w + \lambda u^{p_3} = 0, & x \in B_1, \\ u = v = w = 0, & x \in \partial B_1, \end{cases}$$

where $p_i > 0$ $(i = 1, 2, 3)$ and B_1 is the unit ball in $\mathcal{R}^n$.

MSC: 35J55; 35B32

Keywords: Semilinear elliptic systems; Positive solution; Uniqueness

1. Introduction

In this paper we study the positive radially symmetric solutions of the semilinear elliptic system

$$\begin{cases} \Delta u + \lambda v^{p_1} = 0, & x \in B_1, \\ \Delta v + \lambda w^{p_2} = 0, & x \in B_1, \\ \Delta w + \lambda u^{p_3} = 0, & x \in B_1, \\ u = v = w = 0, & x \in \partial B_1, \end{cases} \tag{1.1}$$

where B_1 is the unit ball in $\mathcal{R}^n$, $n \geq 1$, and $p_i > 0$ $(i = 1, 2, 3)$. By a transformation $U(y) = u(\lambda^{-\frac{1}{2}} y)$, $V(y) = v(\lambda^{-\frac{1}{2}} y)$, $W(y) = w(\lambda^{-\frac{1}{2}} y)$, we can convert (1.1) to

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$$\begin{cases} \Delta U + V^{p_1} = 0, & y \in B_R, \\ \Delta V + W^{p_2} = 0, & y \in B_R, \\ \Delta W + U^{p_3} = 0, & y \in B_R, \\ U = V = W = 0, & y \in \partial B_R. \end{cases} \tag{1.2}$$

We shall study (1.1) instead of (1.2) since the structure of the solution set of (1.2) is same as that of (1.1). Our approach to the uniqueness is based on two ingredients: (a) the parameterization of the set of all solutions; and (b) the scaling of the homogeneous equation (1.1). To illustrate the ideas, we consider the positive radial solutions of a scalar equation:

$$\begin{cases} \Delta u + \lambda u^p = 0, & x \in B_1, \\ u = 0, & x \in \partial B_1. \end{cases} \tag{1.3}$$

Like for (1.2), the solutions of (1.3) are equivalent to those of

$$\begin{cases} \Delta U + U^p = 0, & y \in B_R, \\ U = 0, & y \in \partial B_R, \end{cases} \tag{1.4}$$

via the same change of variables as above. Then from the uniqueness of the initial value problem of the ordinary differential equation, the radius R in (1.4) is uniquely determined by $U(0) = \max_{y \in \overline{B_R}} U(y)$, and so is $\lambda = R^2$. Thus the solution set of (1.4) is parameterized by a single parameter $U(0)$. On the other hand, if $u_1(x)$ is a solution of (1.3) with $\lambda = 1$, then $u_\lambda(x) = \lambda^{1/(1-p)} u_1(x)$ is a solution of (1.3) for general $\lambda > 0$, and the range of $\{u_\lambda(0)\}$ is $\mathcal{R}^+$. The curve $\Sigma = \{(\lambda, u_\lambda) : \lambda > 0\}$ is monotone; hence we obtain the uniqueness of the solution for each $\lambda > 0$.

We follow a similar approach for the uniqueness of solution to the system (1.1). We generalize an idea of Dalmasso [2] and Korman and Shi [3] to prove that the solution set $\{(\lambda, u, v, w)\}$ of (1.1) (or equivalently $\{(R, U, V, W)\}$ of (1.2)) can be parameterized by a single variable u_0 (or U_0 respectively) under certain conditions. In particular, we prove the uniqueness of the solution of (1.1) for any fixed λ when $p_i > 0$ $(i = 1, 2, 3)$, which generalizes results of [1,2,4]. In [5], the uniqueness of positive solution is obtained when the exponents are sublinear.

We state our main result.

Theorem 1.1. *We assume $p_i > 0$ $(i = 1, 2, 3)$ and there exists $\lambda_0 > 0$ such that* (1.1) *has a positive radially symmetric solution $(u_{\lambda_0}, v_{\lambda_0}, w_{\lambda_0})$. Then:*

1. *If $1 - p_1 p_2 p_3 \neq 0$, then for each $\lambda > 0$, there exists exactly one positive radially symmetric solution $(u_\lambda, v_\lambda, w_\lambda)$.*
2. *If $1 - p_1 p_2 p_3 = 0$, then* (1.1) *has no positive radially symmetric solution for any $\lambda > 0$ and $\lambda \neq \lambda_0$, and* (1.1) *has infinitely many positive radially symmetric solutions at $\lambda = \lambda_0$, which can be represented as*

$$\left\{ \left(k u_{\lambda_0}, k^{\frac{1+p_2+p_2 p_3}{1+p_1+p_1 p_2}} v_{\lambda_0}, k^{\frac{1+p_3+p_1 p_3}{1+p_1+p_1 p_2}} w_{\lambda_0} \right) : k > 0 \right\}.$$

For (1.1), we call the system sublinear when $1 - p_1 p_2 p_3 > 0$, and we call it superlinear when $1 - p_1 p_2 p_3 < 0$. We prove the uniqueness of the solution in both sublinear and superlinear cases.

Our uniqueness result is proved under the assumption that a positive solution exists. The existence results for the system in (1.1) on general domains for the exponents satisfying $1 - p_1 p_2 p_3 > 0$ have been proved in [5] (see Theorem 1.2 below), but the existence for the superlinear case is still not known.

Theorem 1.2. *Consider the problem*

$$\begin{cases} \Delta u + v^{p_1} = 0, & x \in D, \\ \Delta v + w^{p_2} = 0, & x \in D, \\ \Delta w + u^{p_3} = 0, & x \in D, \\ u = v = w = 0, & x \in \partial D, \end{cases} \tag{1.5}$$

where D is a ball in $\mathcal{R}^n$, $p_i > 0$ $(i = 1, 2, 3)$ and $1 - p_1 p_2 p_3 > 0$. Then (1.5) *admits a positive classical solution (u, v, w).*

2. Proof of main theorems

Before giving the proofs of our main theorems, we prepare two lemmas.

Lemma 2.1. *Assume that (u_1, v_1, w_1) and (u_2, v_2, w_2) are two radially symmetric solutions of (1.1) with the same parameter $\lambda > 0$. If $u_1(0) = u_2(0)$, then*

$$v_1(0) = v_2(0), \qquad w_1(0) = w_2(0).$$

Proof. Suppose this is not true. We will show a contradiction for each possible case. In the following, δ is a small positive constant chosen appropriately as needed. First we assume that $(v_1 - v_2)(0) > 0$ and $(w_1 - w_2)(0) > 0$. Then $(v_1 - v_2)(r) > 0$ and $(w_1 - w_2)(r) > 0$ for $r \in (0, \delta)$. On the other hand $\Delta(u_1 - u_2) = -\lambda(v_1^{p_1} - v_2^{p_1}) < 0$, $\nabla(u_1 - u_2)(0) = 0$, and $(u_1 - u_2)(0) = 0$; then $(u_1 - u_2)(r) < 0$ for $r \in (0, \delta)$. Define

$$r_1 = \sup\{r \le 1 : (u_1 - u_2)(s) < 0, (v_1 - v_2)(s) > 0, (w_1 - w_2)(s) > 0, \text{ for } s \in (0, r)\}.$$

We claim that $(u_1 - u_2)(r_1) < 0$ and $(w_1 - w_2)(r_1) > 0$. If $(u_1 - u_2)(r_1) = 0$, $\Delta(u_1 - u_2) = -\lambda(v_1^{p_1} - v_2^{p_1}) < 0$ for $x \in B_{r_1}$, and $u_1 - u_2 = 0$ on ∂B_{r_1}; hence $u_1 - u_2 > 0$ for $x \in B_{r_1}$ from the maximum principle, which contradicts with $(u_1 - u_2)(r) < 0$ for $r \in (0, \delta)$. Similarly we can show $(w_1 - w_2)(r_1) > 0$. Thus $(u_1 - u_2)(r_1) < 0$ and $(w_1 - w_2)(r_1) > 0$. In particular this implies $r_1 < 1$ and $(v_1 - v_2)(r_1) = 0$.

Since $V = v_1 - v_2$ satisfies

$$V'' + \frac{n-1}{r} V' + \lambda(w_1^{p_2} - w_2^{p_2}) = 0,$$

then $(v_1 - v_2)(r) < 0$ for $r \in (r_1, r_1 + \delta)$. Obviously we also have $(u_1 - u_2)(r) < 0$ and $(w_1 - w_2)(r) > 0$ for $r \in (r_1, r_1 + \delta)$. Define

$$r_2 = \sup\{r_1 < r \le 1 : (u_1 - u_2)(s) < 0, (v_1 - v_2)(s) < 0, (w_1 - w_2)(s) > 0, \text{ for } s \in (r_1, r)\}.$$

Like in the proof above, we claim that $(v_1 - v_2)(r_2) < 0$ and $(w_1 - w_2)(r_2) > 0$. If $(v_1 - v_2)(r_2) = 0$, $\Delta(v_1 - v_2) = -\lambda(w_1^{p_2} - w_2^{p_2}) < 0$ for $x \in B_{r_2} \setminus \overline{B_{r_1}}$ and $v_1 - v_2 = 0$ for $x \in \partial(B_{r_2} \setminus \overline{B_{r_1}})$, then $v_1 - v_2 > 0$ for $r \in (r_1, r_2)$ from the maximum principle, which is a contradiction. If $(w_1 - w_2)(r_2) = 0$, $\Delta(w_1 - w_2) = -\lambda(u_1^{p_3} - u_2^{p_3}) > 0$ for $x \in B_{r_2}$, $w_1 - w_2 = 0$ for $x \in \partial B_{r_2}$, then $w_1 - w_2 < 0$ for $r \in (r_1, r_2)$ from the maximum principle, which is a contradiction again. Thus $(v_1 - v_2)(r_2) < 0$ and $(w_1 - w_2)(r_2) > 0$. In particular this implies $r_2 < 1$ and $(u_1 - u_2)(r_2) = 0$.

We can continue this argument to show that there exists an infinite sequence r_i such that

$$\begin{aligned} &0 < r_1 < r_2 < r_3 < \cdots < r_i < r_{i+1} < \cdots < 1, \\ &v_1(r_{3k+1}) - v_2(r_{3k+1}) = 0, \qquad u_1(r_{3k+2}) - u_2(r_{3k+2}) = 0, \\ &w_1(r_{3k+3}) - w_2(r_{3k+3}) = 0, \end{aligned} \tag{2.1}$$

for $k \ge 0$. Let $\lim_{i\to\infty} r_i = r_* \le 1$. Then from the differentiability of u_i, v_i, w_i, we must have $u_1(r_*) - u_2(r_*) = v_1(r_*) - v_2(r_*) = w_1(r_*) - w_2(r_*) = 0$ and $u_1'(r_*) - u_2'(r_*) = v_1'(r_*) - v_2'(r_*) = w_1'(r_*) - w_2'(r_*) = 0$. From the uniqueness of the initial value problem of the ordinary differential equation, we conclude that $u_1 - u_2 = v_1 - v_2 = w_1 - w_2 \equiv 0$ for $r \in [0, 1]$, which contradicts $(v_1 - v_2)(0) > 0$ and $(w_1 - w_2)(0) > 0$.

If we assume $(v_1 - v_2)(0) > 0$ and $(w_1 - w_2)(0) < 0$; or $(v_1 - v_2)(0) < 0$ and $(w_1 - w_2)(0) < 0$; or $(v_1 - v_2)(0) < 0$ and $(w_1 - w_2)(0) > 0$, the above proof clearly still holds. If $(v_1 - v_2)(0) = 0$ and $(w_1 - w_2)(0) > 0$, then we can still prove that for small $\delta > 0$, $(w_1 - w_2)(r) > 0$, $(v_1 - v_2)(r) < 0$ and $(u_1 - u_2)(r) > 0$ for $r \in (0, \delta)$; hence the arguments above can still be carried over to reach a contradiction. This comment also applies to other cases when two values among $(u_1 - u_2)(0)$, $(v_1 - v_2)(0)$ and $(w_1 - w_2)(0)$ are equal to 0, while the other is not. Therefore we must have $(v_1 - v_2)(0) = 0$ and $(w_1 - w_2)(0) = 0$. □

Lemma 2.2. *The set of positive radial solutions of (1.1) can be parameterized by $d_1 = u(0)$, i.e. for each $d_1 > 0$, there exists at most one solution (λ, u, v, w) such that $u(0) = d_1$.*

Proof. Let $d_1 > 0$. From Lemma 2.1, there is at most a pair (d_2, d_3) such that (1.1) has a solution with $(u(0), v(0), w(0)) = (d_1, d_2, d_3)$.

If such a solution exists, λ can be uniquely determined. With a change of variables, we have

$$\begin{cases} \Delta U + V^{p_1} = 0, & x \in B_R, \\ \Delta V + W^{p_2} = 0, & x \in B_R, \\ \Delta W + U^{p_3} = 0, & x \in B_R, \\ U = V = W = 0, & x \in \partial B_R, \end{cases}$$

where $R = \sqrt{\lambda}$. R can be uniquely determined from the initial value problem:

$$\begin{cases} U'' + \dfrac{n-1}{t} U' + V^{p_1} = 0, & t > 0, \\ V'' + \dfrac{n-1}{t} V' + W^{p_1} = 0, & t > 0, \\ W'' + \dfrac{n-1}{t} W' + U^{p_1} = 0, & t > 0, \\ U'(0) = V'(0) = W'(0) = 0, \\ U(0) = d_1, V(0) = d_2, W(0) = d_3. \quad \square \end{cases}$$

Proof of Theorem 1.1. (I) First we assume that $1 - p_1 p_2 p_3 \neq 0$. Suppose that there exists a $\lambda_0 > 0$ such that (1.1) has a solution $(u_{\lambda_0}, v_{\lambda_0}, w_{\lambda_0})$; then it is easy to verify that

$$\left(u_1 = \lambda_0^{-\frac{1+p_1+p_1p_2}{1-p_1p_2p_3}} u_{\lambda_0}, v_1 = \lambda_0^{-\frac{1+p_2+p_2p_3}{1-p_1p_2p_3}} v_{\lambda_0}, w_1 = \lambda_0^{-\frac{1+p_3+p_1p_3}{1-p_1p_2p_3}} w_{\lambda_0} \right)$$

is a solution of (1.1) with $\lambda = 1$.

For each $\lambda > 0$, we define $\begin{cases} u_\lambda = \lambda^{\frac{1+p_1+p_1p_2}{1-p_1p_2p_3}} u_1 \\ v_\lambda = \lambda^{\frac{1+p_2+p_2p_3}{1-p_1p_2p_3}} v_1 \\ w_\lambda = \lambda^{\frac{1+p_3+p_1p_3}{1+p_1p_2p_3}} w_1 \end{cases}$; then it is a solution of (1.1) with the given λ. We define the map $P : \lambda \mapsto u_\lambda(0)$, which is smooth and monotone. Clearly $P(\lambda) = \lambda^{\frac{1+p_1+p_1p_2}{1-p_1p_2p_3}} P(1)$; thus it is strictly increasing when $1 - p_1p_2p_3 > 0$, and it is strictly decreasing when $1 - p_1p_2p_3 < 0$. The range of the map P is $(0, \infty)$. From Lemma 2.2, for each $d_1 > 0$, there is at most one solution of (1.1) with $u(0) = d_1$, which proves the uniqueness of the solution for each $\lambda > 0$.

(II) If $1 - p_1p_2p_3 = 0$, one verifies that $(ku_{\lambda_0}, k^{\frac{1+p_2+p_2p_3}{1+p_1+p_1p_2}} v_{\lambda_0}, k^{\frac{1+p_3+p_1p_3}{1+p_1+p_1p_2}} w_{\lambda_0})$ is a solution of (1.1), where $k > 0$. We define the map $Q : k \mapsto ku_{\lambda_0}(0)$; for $k > 0$, the range of Q is $(0, \infty)$. From Lemma 2.2, for each $d > 0$, there is at most one solution of (1.1), with $u(0) = d$. Therefore (1.1) has no other solutions besides the ones on $\{(ku_{\lambda_0}, k^{\frac{1+p_2+p_2p_3}{1+p_1+p_1p_2}} v_{\lambda_0}, k^{\frac{1+p_3+p_1p_3}{1+p_1+p_1p_2}} w_{\lambda_0}) : k > 0\}$. $\square$

Proof of Theorem 1.2. We know that $\begin{pmatrix} 1 & -p_1 & 0 \\ 0 & 1 & -p_2 \\ -p_3 & 0 & 1 \end{pmatrix}$ is a non-singular irreducible M-matrix when $1 - p_1p_2p_3 > 0$. See [5] Theorem 1.1; we complete the proof of existence. $\square$

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