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SEMILINEAR ELLIPTIC EQUATIONS WITH GENERALIZED CUBIC NONLINEARITIES

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Abstract. A semilinear elliptic equation with generalized cubic nonlinearity is studied. Global bifurcation diagrams and the existence of multiple solutions are obtained and in certain cases, exact multiplicity is proved.

1. **Introduction.** We consider a semilinear elliptic equation

$$\begin{cases} \Delta u + \lambda[m(x)u + k(x)u^q - b(x)u^p] = 0, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $1 < q < p$, $m, k, b \in L^\infty(\Omega)$ and $b(x) \geq b_0 > 0$ for $x \in \overline{\Omega}$. Semilinear equations similar to (1) have been studied extensively in the past 15 years, see for example [1, 4, 6, 7, 11, 12, 16, 17, 18], and more references can be found in [5] for spike layer solutions when $b(x) < 0$. For the sublinear case (1) we consider here, Alama and Tarantello [2] use variational methods to obtain related results. In this paper, we apply a bifurcation approach in [24] to (1).

In [24], we study a more general class of semilinear equation:

$$\begin{cases} \Delta u + \lambda u f(x,u) = 0, & x \in \Omega, \\ u \geq 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

The solutions of (2) are the steady state solution of a reaction-diffusion population model, in which $u(t,x)$ is the population density function, and $f(x,u)$ is the spatially heterogeneous growth rate per capita. We assume the growth rate per capita $f(x,u)$ satisfies

(f1) For any $u \geq 0$, $f(\cdot,u) \in L^\infty(\Omega)$, and for any $x \in \overline{\Omega}$, $f(x,\cdot) \in C^1(\mathbf{R}^+)$;

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(f2) For any $x \in \overline{\Omega}$, there exists $u_2(x) \geq 0$ such that $f(x,u) \leq 0$ for $u > u_2(x)$, and there exists $M > 0$ such that $u_2(x) \leq M$ for all $x \in \overline{\Omega}$;

(f3) For any $x \in \overline{\Omega}$, there exists $u_1(x) \geq 0$ such that $f(x,\cdot)$ is increasing in $[0, u_1(x)]$, $f(x,\cdot)$ is decreasing in $[u_1(x), \infty)$, and there exists $N > 0$ such that $N \geq f(x, u_1(x))$ for all $x \in \overline{\Omega}$.

In addition, the heterogeneous growth pattern $f(x,u)$ can take one of the following five forms:

(f4a) **Logistic**. $f(x,0) > 0$, $u_1(x) = 0$, and $f(x,\cdot)$ is decreasing in $[0, u_2(x)]$;

(f4b) **Degenerate logistic**. $u_1(x) = u_2(x) = 0$, $f(x,u) \leq 0$ for all $u \geq 0$, and $f(x,\cdot)$ is decreasing in $[0,\infty)$;

(f4c) **Weak Allee effect**. $f(x,0) \geq 0$, $u_1(x) > 0$, $f(x,\cdot)$ is increasing in $[0, u_1(x)]$, $f(x,\cdot)$ is decreasing in $[u_1(x), u_2(x)]$;

(f4d) **Strong Allee effect**. $f(x,0) < 0$, $u_1(x) > 0$, $f(x,\cdot)$ is increasing in $[0, u_1(x)]$, $f(x,\cdot)$ is decreasing in $[u_1(x), u_2(x)]$;

(f4e) **Degenerate Allee effect**. $f(x,u) \leq 0$ for all $u \geq 0$, $u_2(x) = 0$, $u_1(x) > 0$, $f(x,\cdot)$ is increasing in $[0, u_1(x)]$, $f(x,\cdot)$ is decreasing in $[u_1(x), \infty)$.

For more background on the spatial population models with Allee effect, see [3, 9, 13, 24]. In this paper, we show that the general approach in [24] can be applied to (1), and the existence of multiple solutions of (1) can be proved in a much easier way compared to previous works. In Section 2, we recall the main results of [24], and in Section 3, we discuss the applications to (1) and related results.

2. **Global bifurcation of a general equation.** In this section, we recall the basic setup of bifurcation analysis and the main results of Section 2 in [24]. The proofs of these results can be found in [24]. Let $X = C^{2,\alpha}(\overline{\Omega})$, and let $Y = C^{\alpha}(\overline{\Omega})$. Then $F : \mathbf{R} \times X \to Y$ defined by $F(\lambda, u) = \Delta u + \lambda u f(x,u)$ is a continuously differentiable mapping. We denote the set of non-negative solutions of the equation by $S = \{(\lambda, u) \in \mathbf{R}^+ \times X : u \geq 0, F(\lambda, u) = 0\}$. From the maximum principle of elliptic equations, either $u \equiv 0$ or $u > 0$ on Ω. We define $S = S_0 \cup S_+$, where $S_0 = \{(\lambda, 0) : \lambda > 0\}$, and $S_+ = \{(\lambda, u) \in S : u > 0\}$.

The stability of a solution (λ, u) to (2) can be determined by the Morse index of the solution. Consider

$$\begin{cases} \Delta\psi + \lambda[f(x,u) + u f_u(x,u)]\psi = -\mu\psi, & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega. \end{cases} \tag{3}$$

The eigenvalue problem (3) has a sequence of eigenvalues $\mu_1(u) < \mu_2(u) \leq \mu_3(u) \leq \cdots \to \infty$. The number of negative eigenvalues $\mu_i(u)$ is called *Morse index* of the solution. The solution u is *stable* if $\mu_1(u) > 0$, otherwise it is *unstable.* If $\{x \in \overline{\Omega} : f(x,0) > 0\}$ is a set of positive measure, then there exists $\lambda = \lambda_1(f,\Omega) > 0$ defined as

$$\frac{1}{\lambda_1(f,\Omega)} = \sup_{u \in H_0^1(\Omega)} \left\{ \int_\Omega f(x,0) u^2(x) dx : \int_\Omega |\nabla u(x)|^2 dx = 1 \right\}. \tag{4}$$

such that when $0 < \lambda < \lambda_1(f,\Omega)$, $u = 0$ is stable, and when $\lambda \geq \lambda_1(f,\Omega)$, $u = 0$ is unstable. A local bifurcation occurs at $\lambda = \lambda_1(f,\Omega)$. More precisely, we have the following global bifurcation result ([24] Propositions 2.3 and 2.4):

Proposition 1. *Suppose that f satisfies (f1)-(f3), and we assume*

$$\{x \in \overline{\Omega} : f(x,0) > 0\} \text{ is a set of positive measure.} \tag{5}$$

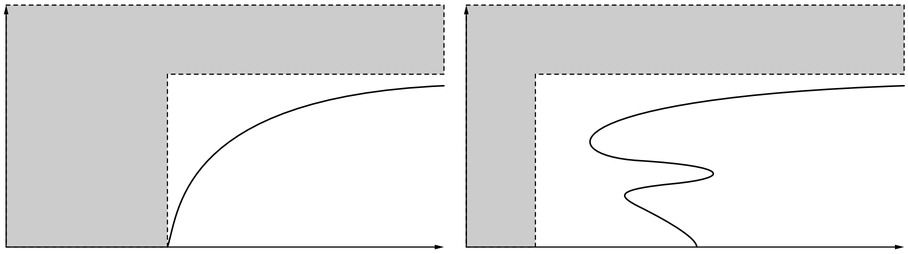

FIGURE 1. Bifurcation diagrams **a:** (left) logistic type growth; **b:** (right) weak Allee effect

Then $\lambda_1(f,\Omega) > 0$ and there is a connected component S_+^1 of S_+ satisfying

1. *The closure of S_+^1 in $\mathbf{R} \times X$ contains $(\lambda_1(f,\Omega), 0)$;*
2. *The projection of S_+^1 onto $\mathbf{R}$ via $(\lambda, u) \mapsto \lambda$ contains the interval $(\lambda_1(f,\Omega), \infty)$.*
3. *There exist $\alpha, \beta > 0$ such that for $B = \{|\lambda - \lambda_1(f,\Omega)| < \alpha, ||u||_X < \beta, u > 0\}$,*
$$S_+ \bigcap B = S_+^1 \bigcap B = \{(\lambda(s), u(s)) : 0 < s < \delta\}, \tag{6}$$
where $\delta > 0$ is a constant, $\lambda(s) = \lambda_1(f,\Omega) + \eta(s)$, $u(s) = s\varphi_1 + sv(s)$, $0 < s < \delta$, $\eta(0) = 0$ and $v(0) = 0$, and $\eta(s)$ and $v(s)$ are continuous.
4. *When $f(x,u)$ is of logistic or degenerate logistic type for almost all $x \in \overline{\Omega}$, then the bifurcation at $(\lambda_1(f,\Omega), 0)$ is supercritical;*
5. *When $f(x,u)$ is of weak, strong or degenerate Allee effect type for almost all $x \in \overline{\Omega}$, then the bifurcation at $(\lambda_1(f,\Omega), 0)$ is subcritical.*

Here a bifurcation is said to be *supercritical* bifurcation if $\lambda(s) > \lambda_1(f,\Omega)$ for $s \in (0,\delta)$, and it is subcritical if $\lambda(s) < \lambda_1(f,\Omega)$ for $s \in (0,\delta)$. Proposition 1 is an application of local and global bifurcation results in [10, 21]. Indeed when f is of logistic or degenerate logistic type for almost all $x \in \overline{\Omega}$, a precise global bifurcation diagram can be drawn (see Figure 1-a):

Theorem 1. *Suppose that the growth per capita $f(x,u)$ satisfies (f1)-(f3), and $f(x,u)$ is of logistic or degenerate logistic type for almost all $x \in \overline{\Omega}$ (but not degenerate for almost all x, i.e.* (5) *is satisfied). Then*

1. *For each $\lambda > \lambda_1(f,\Omega)$, there exists a unique solution $u(\lambda, x)$ of* (2)*;*
2. *S_+ can be parameterized as $S_+^1 = \{(\lambda, u(\lambda,x)) : \lambda > \lambda_1(f,\Omega)\}$, $\lim_{\lambda \to \lambda_1(f,\Omega)^+} u(\lambda) = 0$, and $\lambda \mapsto u(\lambda, \cdot)$ is differentiable;*
3. *For any $\lambda > \lambda_1(f,\Omega)$, $u(\lambda, x)$ is stable;*
4. *If in addition, we assume that $f(x,u) \geq 0$ for almost all (x,u). Then for $\lambda_a > \lambda_b > \lambda_1(f,\Omega)$, $u(\lambda_a, x) > u(\lambda_b, x)$ for all $x \in \Omega$.*

Theorem 1 is well-known, see for example [8] and [23]. In the case of (weak, strong or degenerate) Allee effect and (5) is satisfied (thus weak Allee effect must present in a set of positive measure), the bifurcation diagram is more complicated. In general, we can only obtain the following result (see Figure 1-b):

Theorem 2. *Suppose that the growth per capita $f(x,u)$ satisfies (f1)-(f3), and $f(x,u)$ is of weak, strong or degenerate Allee effect type for almost all $x \in \overline{\Omega}$ (but not degenerate or strong Allee effect for almost all x, i.e.* (5) *is satisfied). Then in addition to the results in Proposition 1,*

1. *There exists* $\lambda_*(f,\Omega)$ *such that*
$$\lambda_1(f,\Omega) > \lambda_*(f,\Omega) > 0, \tag{7}$$
(2) *has no solution when* $\lambda < \lambda_*(f,\Omega)$*, and when* $\lambda \geq \lambda_*(f,\Omega)$*,* (2) *has a maximal solution* $u_m(\lambda,x)$ *such that for any solution* $v(\lambda,x)$ *of* (2)*,* $u_m(\lambda,x) \geq v(\lambda,x)$ *for* $x \in \Omega$*;*
2. *If in addition,* $f(x,u) \geq 0$ *for almost all* (x,u)*, then* $u_m(\lambda,x)$ *is increasing with respect to* λ*, the map* $\lambda \mapsto u_m(\lambda,\cdot)$ *is right continuous for* $\lambda \in [\lambda_*(f,\Omega),\infty)$*, i.e.* $\lim_{p\to\lambda^+} ||u_m(p,\cdot) - u_m(\lambda,\cdot)||_X = 0$*, and* (2) *has at least two solutions when* $\lambda \in (\lambda_*(f,\Omega),\lambda_1(f,\Omega))$*.*

When the domain Ω is a ball in $\mathbf{R}^n$, and the growth rate per capita is independent of the spatial variable x, a precise global bifurcation diagram can be obtained. Consider
$$\begin{cases} \Delta u + \lambda u f(u) = 0, & x \in B^n, \\ u > 0, \ x \in B^n, & u = 0, \ x \in \partial B^n, \end{cases} \tag{8}$$
where $B^n = \{x \in \mathbf{R}^n : |x| < 1\}$, and $f(u)$ is the growth rate per capita. From a remarkable result of Gidas, Ni and Nirenberg [14], all solutions of (8) are radially symmetric if $f(u)$ is Lipschitz continuous, and satisfy an ordinary differential equation. In this case, we can simplify the conditions (f1)-(f4) to

(ff) There exist u_1 and u_2 such that $u_2 > u_1 > 0$, $f(u) > 0$ for $u \in [0,u_2)$, $f(u_2) = 0$, f is increasing on $(0,u_1)$ and f is decreasing on (u_1,u_2).

When the spatial dimension $n = 1$, the following result is proved in [24] Theorem 3.1 (see Figure 2):

Theorem 3. *Suppose that* $f(u)$ *satisfies (ff) and*

(f5) $f \in C^2(\mathbf{R}^+)$*, there exists* $u_3 \in (0,u_2)$ *such that* $(uf(u))''$ *is non-negative on* $(0,u_3)$*, and* $(uf(u))''$ *is non-positive on* (u_3,u_2)*.*

Then the equation
$$\begin{cases} u'' + \lambda u f(u) = 0, & r \in (-1,1), \\ u > 0, \ r \in (-1,1), & u(-1) = u(1) = 0, \end{cases} \tag{9}$$
has no solution when $\lambda < \lambda_*(f)$*, has exactly one solution when* $\lambda = \lambda_*$ *and* $\lambda \geq \lambda_1(f)$*, and has exactly two solutions when* $\lambda_*(f) < \lambda < \lambda_1(f)$*, where* $\lambda_*(f) = \lambda_*(f,I)$ *and* $\lambda_1(f) = \lambda_1(f,I)$*,* $I = (-1,1)$*, are same as the constants defined in Theorem 2. Moreover*

1. *All solutions lie on a single smooth curve which bifurcates from* $(\lambda,u) = (\lambda_1(f),0)$*;*
2. *The solution curve can be parameterized by* $d = u(0) = \max_{x\in I} u(x)$ *and it can be represented as* $(\lambda(d),d)$*, where* $d \in (0,u_2)$*;*
3. *Let* $\lambda(d_1) = \lambda_*(f)$*, then for* $d \in (0,d_1)$*,* $\lambda'(d) < 0$ *and the corresponding solution* $(\lambda(d),u)$ *is unstable, and for* $d \in (d_1,u_2)$*,* $\lambda'(d) > 0$ *and and the corresponding solution* $(\lambda(d),u)$ *is stable.*

When $\Omega = B^n$ with $n \geq 2$, we have (Theorem 3.2 in [24])

Theorem 4. *Suppose that* $n \geq 2$*,* $f(u)$ *satisfies (ff), (f5) and*

(f6) $2[(uf(u))']^2 - nuf(u)(uf(u))'' \geq 0$ *for* $u \in (0,u_2)$*.*

Then all the conclusions in Theorem 3 hold for (8)*.*

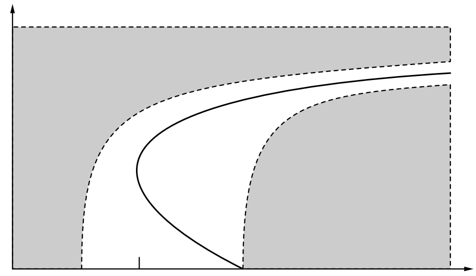

FIGURE 2. Bifurcation diagrams for weak Allee effect with exact $\subset$-shape.

The proof of Theorem 2 is based on a bifurcation approach and a result in critical point theory in Struwe [25], and Theorems 3 and 4 are based on the results in [19, 20, 15].

3. **Equation with generalized cubic nonlinearity.** In most of literatures, logistic growth is referred to the growth rate per capita $f(x, u) = m(x) - b(x)u$, where $b(x) > 0$, although later a more general logistic growth is defined qualitatively as $f(x, u)$ decreasing on u. In Section 2, we take a general approach on the growth with Allee effect, but to illustrate our general results with an example simplest possible, we consider in this section a growth rate per capita function $f(x, u) = m(x) + k(x)u^{q-1} - b(x)u^{p-1}$, where $1 < q < p$ and $b(x) > 0$ on $\overline{\Omega}$. Indeed in the earlier work on Allee effect, algebraically the simplest function which has Allee effect is $f(u) = a + bu + cu^2$, where $c < 0$ and $b > 0$. It is of weak Allee effect type if $a > 0$, and it is of strong Allee effect type if $a < 0$.

We now fit (1) into the framework of Section 2:

Proposition 2. *Let* $f(x, u) = m(x) + k(x)u^{q-1} - b(x)u^{p-1}$, *where* $1 < q < p$, $m, k, b \in L^\infty(\Omega)$ *and* $b(x) \geq b_0 > 0$ *for* $x \in \overline{\Omega}$.

1. *If* $m(x) > 0$ *and* $k(x) \leq 0$, *then* f *is of logistic type at* x;
2. *If* $m(x) \leq 0$ *and* $k(x) \leq 0$, *then* f *is of degenerate logistic type at* x;
3. *If* $m(x) > 0$ *and* $k(x) > 0$, *then* f *is of weak Allee effect type at* x;
4. *If* $m(x) < 0$, $k(x) > 0$ *and* $f(x, u_*(x)) > 0$ *where* $u_*(x) = \left(\frac{(q-1)k(x)}{(p-1)b(x)}\right)$, *then* f *is of strong Allee effect type at* x;
5. *If* $m(x) < 0$, $k(x) > 0$ *and* $f(x, u_*(x)) \leq 0$, *then* f *is of degenerate Allee effect type at* x.

Proof. By elementary calculations. ☐

Thus we can easily apply Theorem 1 to the logistic case case:

$$\{x \in \Omega : m(x) > 0\} \text{ is a set of positive measure, and } k(x) \leq 0, \text{ for all } x \in \Omega; \quad (10)$$

and apply Theorem 2 to the Allee effect case:

$$\{x \in \Omega : m(x) > 0\} \text{ is a set of positive measure, and } k(x) > 0, \text{ for all } x \in \Omega. \quad (11)$$

Indeed, because of the special form of equation (1), a result can also be obtained for sign-changing $k(x)$:

Theorem 5. *Suppose that* $m, k, b \in L^\infty(\Omega)$, $b(x) \geq b_0 > 0$ *for* $x \in \overline{\Omega}$, *and*

$$\{x \in \Omega : m(x) > 0\} \textit{ is a set of positive measure.} \quad (12)$$

Let $\lambda_1(m,\Omega)$ and φ_1 be the positive principal eigenvalue and the corresponding positive eigenfunction of $\Delta\varphi_1 + \lambda m(x)\varphi_1 = 0$, $x \in \Omega$, and $\varphi_1 = 0$, $x \in \partial\Omega$.

1. *If*
$$\int_\Omega k(x)\varphi_1^{q+1}(x)dx > 0, \tag{13}$$
then there exists $\lambda_* \in (0, \lambda_1(m,\Omega))$, such that (1) has at least one solution when $\lambda \geq \lambda_*$, and (1) has no solution when $\lambda < \lambda_*$; if in addition, $m(x) \geq 0$ for almost all $x \in \Omega$, then (1) has at least two solutions when $\lambda_* < \lambda < \lambda_1(m,\Omega)$;
2. *If*
$$\int_\Omega k(x)\varphi_1^{q+1}(x)dx \leq 0, \tag{14}$$
then (1) has at least one solution when $\lambda > \lambda_1(m,\Omega)$.

Proof. From Proposition 1, $(\lambda, u) = (\lambda_1(m,\Omega), 0)$ is a bifurcation point. Let φ_1 be the positive eigenfunction of
$$\Delta\varphi_1 + \lambda_1(f,\Omega)f(x,0)\varphi_1 = 0, \;\; x \in \Omega, \;\; \varphi_1 = 0, \; x \in \partial\Omega. \tag{15}$$
From (2) and (15), we obtain
$$[\lambda(s) - \lambda_1(f,\Omega)]\int_\Omega u(s)\varphi_1 f(x,0)dx + \lambda(s)\int_\Omega u(s)\varphi_1[f(x,u(s)) - f(x,0)]dx = 0. \tag{16}$$
Then the direction of the bifurcation curve can be determined by (16), as the integral
$$\int_\Omega u(s)\varphi_1[f(x,u(s)) - f(x,0)]dx \to s^q \int_\Omega k(x)\varphi_1^{q+1}(x)dx,$$
when $s \to 0^+$. Thus the bifurcation is subcritical if (13) holds, and the proof of Theorem 2.9 in [24] can be carried out. The bifurcation is supercritical if (14) holds, and the existence of a solution is implied by Proposition 1. □

We also apply Theorems 3 and 4 to the corresponding equation on the ball and $f(u) = m + ku^{q-1} - bu^{p-1}$:

Theorem 6. *Let $f(u) = m + ku^{q-1} - bu^{p-1}$ where constants $m, k, b > 0$, and $1 < q < p$.*

1. *If $n = 1$, then the conclusions of Theorem 3 holds for this $f(u)$ for all $m, k, b > 0$, and $1 < q < p$;*
2. *If $2 \leq n \leq 4$, then the conclusions of Theorem 3 holds for this $f(u)$ for all $m, k, b > 0$, and $q \leq 2 < p$.*

Proof. Let $g(u) \equiv uf(u) = mu + ku^q - bu^p$. Then
$$g'(u) = m + kqu^{q-1} - bpu^{p-1}, \;\; g''(u) = kq(q-1)u^{q-2} - bp(p-1)u^{p-2}, \tag{17}$$
and (f5) is satisfied for all $2 \leq q < p$. Then Theorem 3 can be applied for the case of $2 \leq q < p$ and $n = 1$. When $1 < q < 2$, all conditions in Theorem 3 are satisfied except that g is not C^2 at $u = 0$. But an alternate method in [20] Appendix B can be used in this case, and we can still get the results in Theorem 3. This completes the proof for part 1. For part 2, we apply Theorem 4. Again when $q < 2$, g is not C^2 at $u = 0$ and we shall use the approach in [20] Appendix B. We only need to check the condition (f6). Let $M(u) = 2[g'(u)]^2 - ng(u)g''(u)$, let u_* be the point where $g''(u)$ changes sign, and recall that there exists $u_2 > u_*$ such that $g(u) > 0$

on $(0, u_2)$ and $g(u_2) = 0$. Then $g''(u) \leq 0$ and thus $M(u) > 0$ for $u \in (u_*, u_2)$. Since

$$g'''(u) = \begin{cases} kq(q-1)(q-2)u^{q-3} - bp(p-1)(p-2)u^{p-3} < 0, & q < 2; \\ -bp(p-1)(p-2)u^{p-3} < 0, & q = 2, \end{cases} \quad u > 0, \tag{18}$$

then $M'(u) = (4-n)g'(u)g''(u) - ng(u)g'''(u) > 0$ for $u \in (0, u_*)$, and $M(0) = 2m^2 > 0$, hence $M(u) > 0$ for $u \in (0, u_*)$. Therefore the result claimed can be proved by applying Theorem 4. □

Corollary 1. *Let $f(u) = m + ku - bu^2$, where constants $m, k, b > 0$. Then the conclusions of Theorem 3 holds for this $f(u)$ for all $m, k, b > 0$ and $1 \leq n \leq 4$.*

We use some numerically calculated bifurcation diagrams to conclude this section. In the following numerical study, we use $f(u) = 1 + ku - u^2$ and $n = 1$. Recall that $m = f(0)$ is the growth rate when the population is at zero, and b measures the crowding effect. Here we fix both m and b, but use k, which measures the strength of the Alee effect, as the bifurcation parameter. The bifurcation diagrams in Figure 3 are generated by a simple `Maple` program.

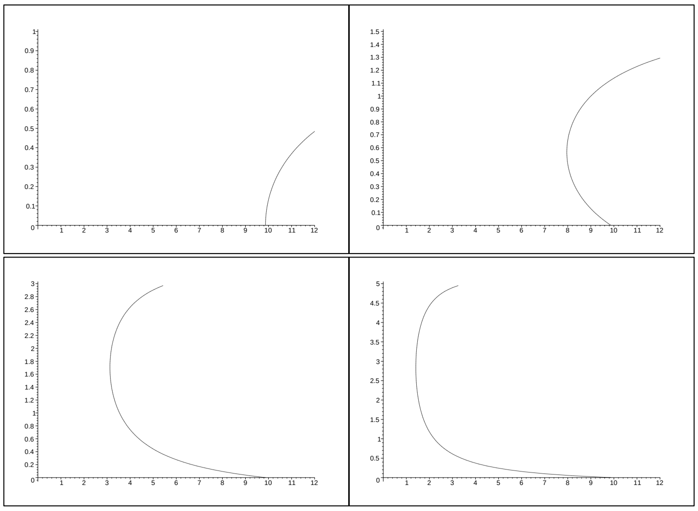

FIGURE 3. Numerical bifurcation diagrams for (8) with $f(u) = 1 + ku - u^2$ and $n = 1$, and the parameter k in the equation is chosen as $k = 0, 1, 3$ and 5 from left to right, then the next row. (Note that $k = 0$ corresponds to a logistic growth.)

In Figure 3, the bifurcation point is always at $\lambda_1 = \pi^2$ since $b = 1$; when $k > 0$, there is always a unique turning point $(\lambda_*(k), u_*(k))$, and $\lambda_*(k)$ is decreasing with respect to k (in fact, we can show that $\lambda_*(k) \to 0$ as $k \to \infty$). An amusing fact from the numerical result is that $u_*(k) \approx 0.57k$ for any k which we have experimented, so we conjecture that $u_*(k) = ck$ for a constant c which is around 0.57. This is rather surprising considering that u_* arises from a nonlinear problem.

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