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Existence and instability of spike layer solutions to singular perturbation problems

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Abstract

An abstract framework is given to establish the existence and compute the Morse index of spike layer solutions of singularly perturbed semilinear elliptic equations. A nonlinear Lyapunov–Schmidt scheme is used to reduce the problem to one on a normally hyperbolic manifold, and the related linearized problem is also analyzed using this reduction. As an application, we show the existence of a multi-peak spike layer solution with peaks on the boundary of the domain, and we also obtain precise estimates of the small eigenvalues of the operator obtained by linearizing at a spike layer solution.

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1. Introduction

In recent years, singularly perturbed elliptic equations have been the topic of many interesting papers (see [BDS, BFu, DY1, DY2, G, GW1, GW2, GWW, Ko, Li, LNT, NT1, NT2, NT3, N, Waz, We1, We2, WW1, WW2],

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for instance). The model equation is

$$\begin{cases} \varepsilon^2 \Delta v - av + f(v) = 0, & x \in \Omega, \\ \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $a > 0$, $f(v)$ is a smooth positive function such that $f(0) = f'(0) = 0$, $\varepsilon > 0$ is a small positive parameter and Ω is a smooth bounded domain in $\mathbf{R}^n$ for $n \geqslant 2$. The stationary Cahn–Hilliard equation and nonlinear Schrödinger equation are also considered.

A significant discovery is that (1.1) possesses spike layer solutions. A spike layer solution can be characterized as being almost constant in most of the domain Ω, and having several "spikes" at points in the interior or on the boundary of Ω. Each spike is approximately equal to $w(\varepsilon^{-1}(x - P))$, where $P \in \bar{\Omega}$ is the location of the spike, and $w(\cdot)$ is a ground state solution of

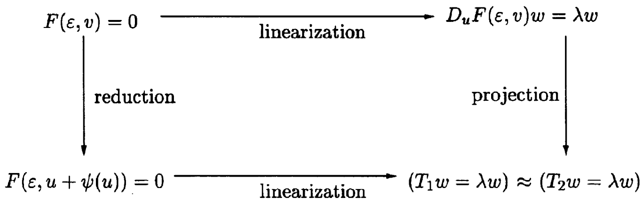
$$\begin{cases} \Delta w - aw + f(w) = 0, & x \in \mathbf{R}^n, \\ w(0) = \max w(x), & \\ w(x) \to 0, & |x| \to \infty. \end{cases} \tag{1.2}$$

In this paper, we set up an abstract framework to study the existence and stability of spike layer solutions of singular perturbation problems that is applicable to many of the previous results. In particular, we give sufficient conditions under which a Lyapunov–Schmidt reduction can be applied. The differential equations which we consider can be formulated as abstract equations

$$F(\varepsilon, u) = 0, \tag{1.3}$$

where $\varepsilon \in (0, \varepsilon_0)$, and $F(\varepsilon, \cdot) : X_\varepsilon \to Y_\varepsilon$ is a differentiable mapping from a Banach space X_ε to another Banach space Y_ε. Also there exists a finite-dimensional differentiable manifold M_ε in X_ε such that for any $u \in M_\varepsilon$, $F(\varepsilon, u)$ is approximately equal to 0. We show that if F is normally hyperbolic (see definition in Section 2) with respect to the manifold M_ε, then a Lyapunov–Schmidt reduction can be performed at each point $u \in M_\varepsilon$, and the equation is reduced to $F(\varepsilon, u + \psi(u)) = 0$, where ψ is a mapping to a subspace "normal" to M_ε. Then we solve the finite-dimensional problem $G(u) \equiv F(\varepsilon, u + \psi(u)) = 0$ for $u \in M_\varepsilon$. The Lyapunov–Schmidt reduction which we introduce here is a generalization of the classical technique, in which the equation is reduced to a problem in a fixed closed linear subspace.

Suppose that we obtain a solution $v = u + \psi(u)$ of the equation, then the next question is: what is the (in)stability of this solution? Usually, the normal hyperbolicity of M_ε implies that $v = u + \psi(u)$ is a nondegenerate saddle point in the normal direction. So the precise instability is determined by the tangential stability of v along M_ε. We obtain a result regarding the precise instability of such a solution. In fact, the idea of Lyapunov–Schmidt reduction is applied here again, and we show that the linearized equation can be projected to a finite-dimensional problem on $T_u M_\varepsilon$, the tangent space of M_ε at u. Moreover, we show that this linear equation is closely related to $DG(u)$, the linearized operator of the nonlinear finite-dimensional problem. The relation can be represented in the following diagram:



In the diagram, $T_1 = DG(u)$ and T_2 (the projected linear operator) are both linear operators on a finite-dimensional space. In general, the diagram does not commute, but we show that the two resulting linear problems are almost the same under reasonable assumptions (see Theorem 2.3).

The spike layer solutions for semilinear elliptic equations have been studied extensively in recent years. There are two principal ways to construct spike layer solutions. One is a direct variational method used by Lin et al. [LNT], also by Wang [Waz], and Gui [G] for (1.1), and by Rabinowitz [R2], Wang [Wax], and del Pino and Felmer [DF] for the nonlinear Schödinger equation:

$$\varepsilon^2 \Delta u - V(x)u + f(u) = 0, \quad x \in \mathbf{R}^n. \tag{1.4}$$

The basic idea of the direct variational method is to apply the Mountain Pass Theorem and its variants to obtain a nontrivial solution to the equation (usually a least energy solution), then use a delicate analysis to show that such a solution is a spike layer solution (see [NT2,NT3,Wax]). The advantage of this approach is that we do not need to know a priori the profile of the solution, and the spike layer solution arises in a natural way. The drawback is that one can only obtain spike layer solutions which satisfy a certain minimizing property.

Another approach is based on a Lyapunov–Schmidt reduction, and the abstract results in this paper provide the general version of this reduction. This method was first used by Floer and Weinstein [FW] for (1.4) with $n = 1$ and $f(u) = |u|^2 u$, and was extended by Oh [O1,O2] to the higher dimensional version of (1.4). Later this method was widely utilized for (1.1) and the stationary Cahn–Hilliard equation by Wei and Winter [WW1,WW2], Wei [We1], Bates et al. [BDS], Li [Li], Dancer and Yan [DY1,DY2], Gui and Wei [GW1,GW2], Gui et al. [GWW] and many others. A slightly different reduction method was used in Bates and Fusco [BFu], Kowalczyk [Ko], (see also [AK]). More related work can be found in the references of these papers.

The purpose of our abstract results for the Lyapunov–Schmidt reduction is to show that the reductions from infinite to finite dimensions for different problems share some common characters, and we also formulate some sufficient conditions for reduction which are easy to check (see Section 2).

Then the nonlinear equation on the manifold (of approximate solutions) determines the nature of the spike layer solutions.

The stability of the spike layer solutions to (1.1) is considered in [BDS] (see also [BFi] for the case of $n = 1$). The spectrum of the linearized operator at the spike layer solution is closely related to the spectrum of the ground state solution. For example, let u_ε be a single-peak boundary spike layer solution with peak at $P_\varepsilon \in \partial\Omega$, and suppose $P_\varepsilon \to P_0$ as $\varepsilon \to 0$. Suppose that $\lambda_{k,\varepsilon}$ and $\lambda_{k,0}$ are the eigenvalues of the following two eigenvalue problems:

$$\begin{cases} \varepsilon^2 \Delta\phi - a\phi + f'(u_\varepsilon)\phi = \lambda_{k,\varepsilon}\phi, & x \in \Omega, \\ \dfrac{\partial\phi}{\partial n} = 0, & x \in \partial\Omega \end{cases} \tag{1.5}$$

and

$$\begin{cases} \Delta\varphi - a\varphi + f'(w)\varphi = \lambda_{k,0}\varphi, & x \in \mathbf{R}^n_+, \\ \dfrac{\partial\varphi}{\partial x_n} = 0, & x \in \partial\mathbf{R}^n_+, \end{cases} \tag{1.6}$$

where $\mathbf{R}^n_+ = \{(x', x_n) \in \mathbf{R}^n\colon x_n > 0\}$ is the half-space in $\mathbf{R}^n$, and w is the ground state solution (restricted to the half-space). For the case $f(u) = |u|^{p-1}u$, $1 < p < (n+2)/(n-2)$ or the case $-au + f(u) = -du(u-b)(u-c)$ with $c > 2b > 0$ and $d = a/(bc)$ (for more general $f(u)$, see Section 5.6), the spectrum of (1.6) is well known: the principal eigenvalue $\lambda_{1,0} > 0$ is simple, with a unique (up to a constant scale) positive radially symmetric eigenfunction φ_1; 0 is the second eigenvalue, and the eigen-space is

$$\left\{ \frac{\partial w}{\partial x_j},\ j = 1, 2, \ldots, n-1 \right\};$$

other eigenvalues may exist in $(-a, 0)$, and $(-\infty, -a]$ is the essential spectrum (see more details in Section 5.6). We prove in [BDS] that $\lambda_{1,\varepsilon}$ is always positive, and in fact, a k-peak solution has at least k positive eigenvalues. So the spike layer solutions for (1.1) are unstable. In this paper, we obtain more precise information: as $\varepsilon \to 0$ (for a single peak solution)

(a) $\lambda_{1,\varepsilon} \to \lambda_{1,0}$, $\lambda_{k,\varepsilon} \to 0$, $(2 \leqslant k \leqslant n)$, $\lambda_{n+1,\varepsilon} < -C$;
(b) $\lambda_{k+1,\varepsilon} = C\varepsilon^2\mu_k + o(\varepsilon^2)$, $(1 \leqslant k \leqslant n-1)$, where μ_k is the kth eigenvalue of the Hessian of the mean curvature function of the boundary manifold $D^2H(P)\colon T_{P_0}(\partial\Omega) \times T_{P_0}(\partial\Omega) \to \mathbf{R}$, and C is a positive constant.

Recall that the mean curvature function $H\colon \partial\Omega \to \mathbf{R}$ is a smooth function on the $(n-1)$-dimensional differentiable manifold $\partial\Omega$, and the Hessian of $H(\cdot)$ is a symmetric bilinear form on the tangent space $T_P(\partial\Omega)$ which does not depend on the coordinate system. We also show the convergence of corresponding eigenspaces. The rough estimates in (a) can be directly obtained from the fact that the spike layer solution is approximately the ground state (see Lemma 5.1). The delicate estimates of small eigenvalues are based on (i) the abstract results on the reduction of the eigenvalue problem; (ii) the estimates of the finite-dimensional eigenvalue problem. Similar results hold for multi-peak boundary solutions. A similar estimate of the small eigenvalues is also obtained by Wei [We2] by a different method, but the approach for our abstract results seems to be more general.

We mention that the abstract setting in Section 2 can also be used to consider the related evolution equation:

$$\frac{du}{dt} = F(\varepsilon, u). \tag{1.7}$$

The manifold M_ε defined above is an approximately invariant manifold for system (1.7), and although the manifold $\{u + \psi(u)\colon u \in M_\varepsilon\}$ is also not invariant, it is more nearly so and it contains the solutions of (1.3), which are the equilibrium solutions of (1.7). An interesting question is whether there is a real invariant manifold near M_ε. Very recently the question is answered positively in Bates et al. [BLZ] in a more general setting. We remark that in the application to spike layer solutions, the calculation of small eigenvalues in this paper can be carried over to the local dynamics of the reduced system on the invariant manifold near equilibrium points.

We introduce the abstract setting and prove the general results in Section 2. Applications to boundary spike layer solutions are given in Section 3. In Section 4, we discuss the instability problem. All detailed technical estimates are given in Section 5. In the paper, C or c stands for a generic positive constant. $A \circ B$ is the composition of operators A and B, and $D_u F(\varepsilon, u)$ is the partial derivative of F with respect to u.

2. Reduction and instability of nonlinear operator equations

We define a singular perturbation problem. For $\varepsilon \in (0, \varepsilon_0)$, suppose that X_ε and Y_ε are Banach spaces, X_ε is compactly embedded in Y_ε, X_ε is dense in Y_ε,

$$\|u\|_{Y_\varepsilon} \leqslant \|u\|_{X_\varepsilon} \quad \text{for } u \in X_\varepsilon. \tag{2.1}$$

$F(\varepsilon, \cdot) : X_\varepsilon \to Y_\varepsilon$ is a continuously differentiable map. We assume that there is a differential manifold (possibly with boundary) M_ε in X_ε, $dim(M_\varepsilon) = n < \infty$, such that $u \in M_\varepsilon$ is an approximate solution of $F(\varepsilon, u) = 0$. Precisely, we assume that

(A1) For any $\delta > 0$, there exists $\varepsilon_1(\delta) \in (0, \varepsilon_0)$ such that for $\varepsilon \in (0, \varepsilon_1(\delta))$, and $u_\varepsilon \in M_\varepsilon$, we have

$$\|F(\varepsilon, u_\varepsilon)\|_{Y_\varepsilon} \leqslant \delta. \tag{2.2}$$

We seek the solution of $F(\varepsilon,\cdot)=0$ near the manifold M_ε. We assume that M_ε is *normally hyperbolic* in the following sense:

(A2) For any $u=u_\varepsilon\in M_\varepsilon$, there exist splittings of X_ε and Y_ε:

$$X_\varepsilon = X_\varepsilon^{su}(u)\oplus X_\varepsilon^c(u),\quad Y_\varepsilon = Y_\varepsilon^{su}(u)\oplus X_\varepsilon^c(u), \tag{2.3}$$

where $X_\varepsilon^{su}(u)$ and $X_\varepsilon^c(u)$ are closed subspaces of X_ε, $Y_\varepsilon^{su}(u)$ is a closed subspace of Y_ε, and $X_\varepsilon^c(u)=T_uM_\varepsilon$, the tangent space of M_ε at u.

(A3) Associated with the decomposition in (A2), the projections

$$\begin{aligned}&P_\varepsilon^{su}(u): X_\varepsilon\to X_\varepsilon^{su}(u),\quad P_\varepsilon^c(u): X_\varepsilon\to X_\varepsilon^c(u),\\ &Q_\varepsilon^{su}(u): Y_\varepsilon\to Y_\varepsilon^{su}(u),\quad Q_\varepsilon^c(u): Y_\varepsilon\to X_\varepsilon^c(u)\end{aligned} \tag{2.4}$$

are well defined. We assume that the mappings: $u\mapsto P_\varepsilon^\alpha(u)$ and $u\mapsto Q_\varepsilon^\alpha(u)$, $\alpha=su,c$, are of class C^1 from M_ε to $\mathscr{L}(X_\varepsilon)$ and $\mathscr{L}(Y_\varepsilon)$, respectively, where $\mathscr{L}(Z)$ is the space of continuous linear operators on Z.

(A4) For any $u\in M_\varepsilon$, the linear mapping

$$T_1(u)=Q_\varepsilon^{su}(u)\circ D_uF(\varepsilon,u)|_{X_\varepsilon^{su}(u)}: X_\varepsilon^{su}(u)\to Y_\varepsilon^{su}(u) \tag{2.5}$$

is an isomorphism, where $D_uF(\varepsilon,u)$ is the partial derivative with respect to the second argument. Moreover,

$$\|T_1(u)\psi\|_{Y_\varepsilon}\geqslant h\|\psi\|_{X_\varepsilon}\quad \text{for } \psi\in X_\varepsilon^{su}(u), \tag{2.6}$$

where $h>0$ does not depend on ε or u.

Also associated with the decompositions of X_ε and Y_ε in (A2) are vector bundles

$$X_\varepsilon^\alpha=\{(u,X_\varepsilon^\alpha(u)):\ u\in M_\varepsilon\},\quad \alpha=c,su$$

and

$$Y_\varepsilon^{su}=\{(u,Y_\varepsilon^{su}(u)):\ u\in M_\varepsilon\}. \tag{2.7}$$

Theorem 2.1 (Lyapunov–Schmidt Reduction). *For $\varepsilon\in(0,\varepsilon_0)$, suppose that X_ε, Y_ε, $F(\varepsilon,\cdot)$ and M_ε satisfy* (A1)–(A4) *and*

(A5) *$F(\varepsilon,u)$ is uniformly differentiable with respect to u for $\varepsilon\in(0,\varepsilon_0)$ and u belonging to a neighborhood of M_ε in X_ε. More precisely, for some $R>0$, define $M_\varepsilon^R=\{u+v:\ u\in M_\varepsilon, \|v\|_{X_\varepsilon}\leqslant R\}$; for any $\eta>0$, there exists $\delta_1=\delta_1(\eta)$ (independent of ε and u) such that if $u,u+\psi\in M_\varepsilon^R$, and $\|\psi\|_{X_\varepsilon}\leqslant\delta_1$, then*

$$\|F(\varepsilon,u+\psi)-F(\varepsilon,u)-D_uF(\varepsilon,u)\psi\|_{Y_\varepsilon}\leqslant\eta\|\psi\|_{X_\varepsilon}. \tag{2.8}$$

Then there exist $\varepsilon_2,\delta_2>0$ such that for any $\varepsilon\in(0,\varepsilon_2)$, and for any $u\in M_\varepsilon$, there exists a unique $\psi(u)\in\{\phi\in X_\varepsilon^{su}(u):\ \|\phi\|_{X_\varepsilon}\leqslant\delta_2\}$ such that

$$Q_\varepsilon^{su}(u)\circ F(\varepsilon,u+\psi(u))=0, \tag{2.9}$$

$\psi: M_\varepsilon\to X_\varepsilon^{su}$ is C^1, and

$$\|\psi(u)\|_{X_\varepsilon}\leqslant C\|F(\varepsilon,u)\|_{Y_\varepsilon}. \tag{2.10}$$

Proof. Fix $u \in M_\varepsilon$ and for simplicity, write $Q = Q_\varepsilon^{su}(u)$. We seek a solution of

$$0 = Q \circ F(\varepsilon, u + \psi) = Q \circ F(\varepsilon, u) + Q \circ D_u F(\varepsilon, u)\psi + Q \circ N(\varepsilon, u, \psi),$$

where $N(\varepsilon, u, \psi) = F(\varepsilon, u + \psi) - F(\varepsilon, u) - D_u F(\varepsilon, u)\psi$. That is equivalent to

$$\psi = -[T_1(u)]^{-1}[Q \circ F(\varepsilon, u) + Q \circ N(\varepsilon, u, \psi)] \equiv K(\psi). \tag{2.11}$$

We define $K(\psi)$ for ψ belonging to

$$O(\varepsilon, u, \delta) = \{\psi \in X_\varepsilon^{su}(u) \colon \|\psi\|_{X_\varepsilon} \leqslant \delta\}. \tag{2.12}$$

Let $\eta_1 = h/2$, where h is defined in (2.6). From (A5), there exists $\delta_2 = \delta_1(\eta_1) > 0$ independent of ε, u such that if $\|\psi\|_{X_\varepsilon} \leq \delta_2$, then $\|N(\varepsilon, u, \psi)\|_{Y_\varepsilon} \leqslant \eta_1 \delta_2$. Consequently,

$$\|[T_1(u)]^{-1} \circ Q \circ N(\varepsilon, u, \psi)\|_{X_\varepsilon} \leqslant h^{-1}\eta_1\delta_2 \leqslant \frac{\delta_2}{2}$$

for $\psi \in O(\varepsilon, u, \delta_2)$. Then from (A1), there exists $\varepsilon_2 > 0$ such that for any $\varepsilon \in (0, \varepsilon_2)$, $\|F(\varepsilon, u)\|_{Y_\varepsilon} \leqslant h\delta_2/2$. Thus for $\|\psi\|_{X_\varepsilon} \leq \delta_2$, and $\varepsilon \in (0, \varepsilon_2)$, $\|K(\psi)\|_{X_\varepsilon} \leqslant \delta_2$, and $K(O(\varepsilon, u, \delta_2)) \subset O(\varepsilon, u, \delta_2)$. Similarly, we have

$$\begin{aligned} \|K(\psi_1) - K(\psi_2)\|_{X_\varepsilon} &\leqslant h^{-1}\|N(\varepsilon, u, \psi_1) - N(\varepsilon, u, \psi_2)\|_{Y_\varepsilon} \\ &= h^{-1}\|F(\varepsilon, u + \psi_1) - F(\varepsilon, u + \psi_2) \\ &\quad - D_u F(\varepsilon, u)(\psi_1 - \psi_2)\|_{Y_\varepsilon} \\ &\leqslant \eta_1 h^{-1}\|\psi_1 - \psi_2\|_{X_\varepsilon} \\ &= \tfrac{1}{2}\|\psi_1 - \psi_2\|_{X_\varepsilon}, \end{aligned} \tag{2.13}$$

so K is a contraction on $O(\varepsilon, u, \delta_2)$. By the Contraction Mapping Principle, K has a unique fixed point $\psi(u)$ in $O(\varepsilon, u, \delta_2)$. The differentiability of ψ can be obtained by combining (2.11), (2.13) and the differentiability of F. From (2.11), we have

$$\|\psi(u)\|_{X_\varepsilon} \leqslant h^{-1}\|F(\varepsilon, u)\|_{Y_\varepsilon} + \tfrac{1}{2}\|\psi(u)\|_{X_\varepsilon},$$

which implies (2.10). □

From Theorem 2.1, the solvability of $F(\varepsilon, u) = 0$ near M_ε is reduced to solving

$$T_2(\varepsilon, u) \equiv Q_\varepsilon^c(u) \circ F(\varepsilon, u + \psi(u)) = 0, \tag{2.14}$$

where $u \in M_\varepsilon$. The solvability of the equation $T_2(\varepsilon, u) = 0$ depends on the nature of the original singular perturbation problem. Under appropriate conditions, degree theory or variational methods can be quite efficient in solving such problems. Note that in the proof of Theorem 2.1, we do not need the assumption that $X_\varepsilon^c(u) = T_u M_\varepsilon$.

Next, we consider the stability of a solution obtained by the reduction method. We assume that for any $\varepsilon \in (0, \varepsilon_2)$, $T_2(\varepsilon, u) = 0$ has a solution $u(\varepsilon) \in M_\varepsilon$. Our main concern here is the relationship between the stability of

$u(\varepsilon)+\psi(u(\varepsilon))$ with respect to the equation $F(\varepsilon,u)=0$ and the linear operator $D_uT_2(\varepsilon,u(\varepsilon))$.

Here we restrict our attention to a more special case. We assume that X_ε and Y_ε are Hilbert spaces, and that $D_uF(\varepsilon,u):X_\varepsilon\to Y_\varepsilon$ is a linear unbounded self-adjoint operator in Y_ε with domain X_ε. We denote the inner-products in X_ε and Y_ε by $\langle\cdot,\cdot\rangle_{X_\varepsilon}$ and $\langle\cdot,\cdot\rangle_{Y_\varepsilon}$. We assume that $D_uF(\varepsilon,u)$ is bounded from above (uniformly for ε) in the sense that for any $\psi\in X_\varepsilon$, we have $\langle D_uF(\varepsilon,u)\psi,\psi\rangle_{Y_\varepsilon}\leqslant C_0\langle\psi,\psi\rangle_{Y_\varepsilon}$ for some $C_0>0$ independent of ε and $u\in M_\varepsilon^R$. We also replace (A2) and (A3) by a stronger assumption:

(A6) For any $u=u_\varepsilon\in M_\varepsilon$, there exist orthogonal (under the norm of Y_ε) splittings of X_ε and Y_ε:

$$X_\varepsilon=X_\varepsilon^u(u)\oplus X_\varepsilon^c(u)\oplus X_\varepsilon^s(u)$$

and

$$Y_\varepsilon=X_\varepsilon^u(u)\oplus X_\varepsilon^c(u)\oplus Y_\varepsilon^s(u), \tag{2.15}$$

where $X_\varepsilon^\alpha(u)$, $\alpha=u,c,s$, are closed subspaces of X_ε, $Y_\varepsilon^s(u)$ is a closed subspace of Y_ε, and $X_\varepsilon^c(u)=T_uM_\varepsilon$; there exist projections P_ε^α and Q_ε^α for $\alpha=u,c,s$ similar to those in (A3), and all projections are C^1. Moreover, there exist constants $C_1,C_2>0$ independent of ε and u, and a constant $C_3(\varepsilon)>0$ independent of u such that

$$\langle D_uF(\varepsilon,u)\psi,\psi\rangle_{Y_\varepsilon}\geqslant C_1\|\psi\|_{Y_\varepsilon}^2\quad\text{for }\psi\in X_\varepsilon^u(u), \tag{2.16}$$

$$\langle D_uF(\varepsilon,u)\psi,\psi\rangle_{Y_\varepsilon}\leqslant -C_2\|\psi\|_{Y_\varepsilon}^2\quad\text{for }\psi\in X_\varepsilon^s(u) \tag{2.17}$$

and

$$\|D_uF(\varepsilon,u)\psi\|_{Y_\varepsilon}\leqslant C_3(\varepsilon)\|\psi\|_{Y_\varepsilon}\quad\text{for }\psi\in X_\varepsilon^c(u). \tag{2.18}$$

The following lemma reveals the precise relationship between (A6) and (A4).

Lemma 2.2. *For $\varepsilon\in(0,\varepsilon_0)$, suppose that X_ε, Y_ε, $F(\varepsilon,\cdot)$ and M_ε satisfy* (A1) *and* (A6). *Then $T_1(u)$ defined in* (A4) *is injective, and*

$$\|T_1(u)\psi\|_{Y_\varepsilon}\geqslant\min(C_1,C_2)\|\psi\|_{Y_\varepsilon}\quad\textit{for }\psi\in X_\varepsilon^{su}(u). \tag{2.19}$$

Moreover,

1. *if $D_uF(\varepsilon,u)$ is a Fredholm operator with index* 0, *then $T_1(u)$ is also surjective*;
2. *if $D_uF(\varepsilon,u)=L(\varepsilon,u)+B(\varepsilon,u)$, where $B(\varepsilon,u):X_\varepsilon\to Y_\varepsilon$ is a linear operator satisfying*

$$\|B(\varepsilon,u)\psi\|_{Y_\varepsilon}\leqslant C_4\|\psi\|_{Y_\varepsilon}\quad\textit{for }\psi\in X_\varepsilon \tag{2.20}$$

for a constant $C_4>0$ independent of ε and u, and $L(\varepsilon,u):X_\varepsilon\to Y_\varepsilon$ is an unbounded linear operator which satisfies a uniform inequality

$$\|\psi\|_{X_\varepsilon}\leqslant C_5(\|L(\varepsilon,u)\psi\|_{Y_\varepsilon}+\|\psi\|_{Y_\varepsilon})\quad\textit{for }\psi\in X_\varepsilon \tag{2.21}$$

for a constant $C_5 > 0$ *independent of* ε *and* u, *and* $C_3(\varepsilon) \leqslant C_6$ *for all* $\varepsilon \in (0, \varepsilon_0)$ *and some* $C_6 > 0$, *then* (2.6) *holds.*

Proof. Let $A = D_uF(\varepsilon, u)$. Let $u = u^s + u^u \in X_\varepsilon^s(u) \oplus X_\varepsilon^u(u) = X_\varepsilon^{su}(u)$, and $Q_\varepsilon^{su} \circ A(u^s + u^u) = a(v^s + v^u)$, where $a \in \mathbf{R}$, $u^s, v^s \in X_\varepsilon^s(u)$, $u^u, v^u \in X_\varepsilon^u(u)$, and $\|u^s + u^u\|_{Y_\varepsilon}^2 = \|u^s\|_{Y_\varepsilon}^2 + \|u^u\|_{Y_\varepsilon}^2 = 1$, $\|v^s + v^u\|_{Y_\varepsilon}^2 = \|v^s\|_{Y_\varepsilon}^2 + \|v^u\|_{Y_\varepsilon}^2 = 1$. Then

$$Q_\varepsilon^s(u) \circ A(u^s + u^u) = av^s \quad \text{and} \quad Q_\varepsilon^u(u) \circ A(u^s + u^u) = av^u. \tag{2.22}$$

Thus,

$$\begin{aligned} \langle Q_\varepsilon^u(u) \circ Au^s, u^u \rangle_{Y_\varepsilon} &= -\langle Q_\varepsilon^u(u) \circ Au^u, u^u \rangle_{Y_\varepsilon} + a \langle v^u, u^u \rangle_{Y_\varepsilon} \\ &\leqslant -C_1 \|u^u\|_{Y_\varepsilon}^2 + |a| \cdot \|v^u\|_{Y_\varepsilon} \|u^u\|_{Y_\varepsilon}. \end{aligned} \tag{2.23}$$

Similarly, we have

$$\langle Q_\varepsilon^s(u) \circ Au^u, u^s \rangle_{Y_\varepsilon} \geqslant C_2 \|u^s\|_{Y_\varepsilon}^2 - |a| \cdot \|v^s\|_{Y_\varepsilon} \|u^s\|_{Y_\varepsilon}. \tag{2.24}$$

On the other hand, since A is self-adjoint, then

$$\begin{aligned} \langle Q_\varepsilon^u(u) \circ Au^s, u^u \rangle_{Y_\varepsilon} &= \langle Au^s, u^u \rangle_{Y_\varepsilon} \\ &= \langle u^s, Au^u \rangle_{Y_\varepsilon} = \langle Q^s(u) \circ Au^u, u^s \rangle_{Y_\varepsilon}. \end{aligned} \tag{2.25}$$

Therefore, we obtain

$$-C_1 \|u^u\|_{Y_\varepsilon}^2 + |a| \cdot \|v^u\|_{Y_\varepsilon} \|u^u\|_{Y_\varepsilon} \geqslant C_2 \|u^s\|_{Y_\varepsilon}^2 - |a| \cdot \|v^s\|_{Y_\varepsilon} \|u^s\|_{Y_\varepsilon}, \tag{2.26}$$

and by Schwarz inequality,

$$\begin{aligned} |a| &\geqslant |a|(\|v^u\|_{Y_\varepsilon} \|u^u\|_{Y_\varepsilon} + \|v^s\|_{Y_\varepsilon} \|u^s\|_{Y_\varepsilon}) \\ &\geqslant C_1 \|u^u\|_{Y_\varepsilon}^2 + C_2 \|u^s\|_{Y_\varepsilon}^2 \\ &\geqslant \min(C_1, C_2). \end{aligned} \tag{2.27}$$

Thus for any $\psi \in X_\varepsilon^{su}(u)$, $\|Q_\varepsilon^{su}(u) \circ D_uF(\varepsilon, u)\psi\|_{Y_\varepsilon} \geqslant \min(C_1, C_2) \|\psi\|_{Y_\varepsilon}$. In particular, $T_1(u)$ is injective.

Suppose $A = D_uF(\varepsilon, u)$ is a Fredholm operator with index 0. Let $R(T_1(u))$ be the range of $T_1(u)$. From the assumptions, $T_1(u) : X_\varepsilon^{su}(u) \to Y_\varepsilon^{su}(u)$ is a densely defined closed operator on $Y_\varepsilon^{su}(u)$ with $D(T_1(u)) = X_\varepsilon^{su}(u)$, thus $R(T_1(u))$ is closed in $Y_\varepsilon^{su}(u)$. Moreover,

$$T_1(u) = D_uF(\varepsilon, u) - Q_\varepsilon^c(u) \circ D_uF(\varepsilon, u).$$

$Q_\varepsilon^c(u) \circ D_uF(\varepsilon, u)$ is $D_uF(\varepsilon, u)$-compact in the sense of Kato [Ka, p. 194], since $X_\varepsilon^c(u)$ is finite dimensional. Therefore by Theorem 5.26 in [Ka, p. 238], $T_1(u)$ is also a Fredholm operator with the same index as $D_uF(\varepsilon, u)$. So $R(T_1(u)) = Y_\varepsilon^{su}(u)$.

Finally, we assume that $A = L + B$, where $L = L(\varepsilon, u)$ and $B = B(\varepsilon, u)$. We first prove that for any $\psi \in X_\varepsilon^{su}(u)$, (recall $n = dim(M_\varepsilon)$)

$$\|Q_\varepsilon^c(u) \circ D_uF(\varepsilon, u)\psi\|_{Y_\varepsilon} \leqslant nC_3(\varepsilon) \|\psi\|_{Y_\varepsilon}. \tag{2.28}$$

Let $\{\phi_j\colon 1\leqslant j\leqslant n\}$ be an orthonormal basis of $X^c_\varepsilon(u)$. Then by (2.18) and the self-adjointness of A, we have

$$\begin{aligned}\|Q^c_\varepsilon(u)\circ A\psi\|_{Y_\varepsilon} &= \left\|\sum_{j=1}^{n}\langle Q^c_\varepsilon(u)\circ A\psi,\phi_j\rangle_{Y_\varepsilon}\phi_j\right\|_{Y_\varepsilon}\\ &\leqslant \sum_{j=1}^{n}|\langle\psi,A\phi_j\rangle_{Y_\varepsilon}|\leqslant nC_3(\varepsilon)\|\psi\|_{Y_\varepsilon}. \qquad (2.29)\end{aligned}$$

Therefore, by (2.21), for $\psi\in X^{su}_\varepsilon(u)$, we get

$$\begin{aligned}\|\psi\|_{X_\varepsilon} &\leqslant C_5(\|L(\varepsilon,u)\psi\|_{Y_\varepsilon}+\|\psi\|_{Y_\varepsilon})\\ &\leqslant C_5[\|D_uF(\varepsilon,u)\psi\|_{Y_\varepsilon}+(1+C_4)\|\psi\|_{Y_\varepsilon}]\\ &\leqslant C_5(\|Q^{su}_\varepsilon(u)\circ A\psi\|_{Y_\varepsilon}+\|Q^c_\varepsilon(u)\circ A\psi\|_{Y_\varepsilon})+C\|\psi\|_{Y_\varepsilon}\\ &\leqslant C_5\|Q^{su}_\varepsilon(u)\circ A\psi\|_{Y_\varepsilon}+C\|\psi\|_{Y_\varepsilon}\quad \text{(by (2.28))}\\ &\leqslant C\|Q^{su}_\varepsilon(u)\circ A\psi\|_{Y_\varepsilon}\quad \text{(by (2.19))}. \qquad (2.30)\end{aligned}$$

The constant C above changes from line to line, but it is independent of ε and u if $C_3(\varepsilon)$ has a uniform upper bound for all $\varepsilon\in(0,\varepsilon_0)$. □

We consider two eigenvalue problems:

$$K_1\theta\equiv D_uF(\varepsilon,u+\psi(u))\theta=\lambda\theta \qquad (2.31)$$

and

$$K_2\xi\equiv Q^c_\varepsilon(u)\circ D_uF(\varepsilon,u+\psi(u))[\xi+D_u\psi(u)\xi]=\lambda\xi. \qquad (2.32)$$

Note that the operator in (2.32) is the linearization of (2.14) at $u+\psi(u)$, a solution to $F(\varepsilon,v)=0$. In fact, if we define $G: M_\varepsilon\to X^c_\varepsilon$ by

$$G(u)=Q^c_\varepsilon(u)\circ F(\varepsilon,u+\psi(u)), \qquad (2.33)$$

then

$$\begin{aligned}D_uG(u)h = {}&[D_uQ^c_\varepsilon(u)h]\circ F(\varepsilon,u+\psi(u))\\ &+ Q^c_\varepsilon(u)\circ D_uF(\varepsilon,u+\psi(u))[\xi+D_u\psi(u)\xi]. \qquad (2.34)\end{aligned}$$

But by our assumption $F(\varepsilon,u+\psi(u))=0$, and $D_uQ^c_\varepsilon(u)h\in\mathscr{L}(Y_\varepsilon)$ is linear, thus $[D_uQ^c_\varepsilon(u)h]\circ F(\varepsilon,u+\psi(u))=0$, and $D_uG(u)h=K_2h$. We also note that $K_2\xi$ is indeed the same as $D_uF(\varepsilon,u+\psi(u))[\xi+D_u\psi(u)\xi]$, since

$$Q^{su}_\varepsilon(u)\circ D_uF(\varepsilon,u+\psi(u))[\xi+D_u\psi(u)\xi]\equiv 0 \qquad (2.35)$$

by differentiating (2.9). Finally, we point out that if $D_uF(\varepsilon, u+\psi(u))$ is a self-adjoint operator, so is K_2. In fact, for any θ_1, $\theta_2 \in X_\varepsilon^c(u)$, we have

$$\begin{aligned}\langle K_2\theta_1, \theta_2 \rangle_{Y_\varepsilon} &= \langle D_uF(\varepsilon, u+\psi(u))[\theta_1 + D_u\psi(u)\theta_1], \theta_2 \rangle_{Y_\varepsilon} \\ &= \langle D_uF(\varepsilon, u+\psi(u))[\theta_1 + D_u\psi(u)\theta_1], \theta_2 + D_u\psi(u)\theta_2 \rangle_{Y_\varepsilon} \\ &\quad \text{(by (2.35))} \\ &= \langle \theta_1 + D_u\psi(u)\theta_1, D_uF(\varepsilon, u+\psi(u))[\theta_2 + D_u\psi(u)\theta_2] \rangle_{Y_\varepsilon} \\ &= \langle \theta_1, K_2\theta_2 \rangle_{Y_\varepsilon}.\end{aligned}$$

For a linear self-adjoint operator T on a Hilbert space, we decompose the spectrum of $T : X(\subset Y) \to Y$ as

$$\sigma(T) = \sigma^u(T) \cup \sigma^c(T) \cup \sigma^s(T), \tag{2.36}$$

where $\sigma^u(T) = \{\lambda \in \sigma(T) : \lambda > 0\}$, $\sigma^c(T) = \{\lambda \in \sigma(T) : \lambda = 0\}$ and $\sigma^s(T) = \{\lambda \in \sigma(T) : \lambda < 0\}$. And $X^s(T), X^c(T), X^u(T)$ are corresponding subspaces of X associated with the spectral decomposition, which exist if these spectral subsets are closed. We define

$$i^u(T) = dim(X^u(T)), \quad i^c(T) = dim(X^c(T)), \tag{2.37}$$

and the index is infinity if the space is infinite dimensional. In the following theorem, $\lambda_{k,\varepsilon}(K_m)$ are eigenvalues of K_m, $1 \leqslant k \leqslant n$, $m = 1, 2$, $\phi_{k,\varepsilon}(K_m)$ is the unit eigenfunction corresponding to $\lambda_{k,\varepsilon}(K_m)$, and $P_{i,i+j}(K_m) : X_\varepsilon^c(u) \to X_\varepsilon^c(u)$ is the projection onto the subspace generated by the eigenfunctions $\phi_{k,\varepsilon}(K_m)$, $i \leqslant k \leqslant i+j$, $m = 1, 2$.

Theorem 2.3. *Suppose that X_ε, Y_ε, $F(\varepsilon, \cdot)$ and M_ε satisfy* (A1), (A4)–(A6), *and $v_\varepsilon = u + \psi(u)$ is a solution of the equation $F(\varepsilon, v) = 0$. Then $i^c(K_1) = i^c(K_2)$. If in addition to* (A6), *we assume that*

(I)

$$C_3(\varepsilon) \to 0 \quad as\ \varepsilon \to 0, \tag{2.38}$$

where $C_3(\varepsilon)$ is defined in (A6);

(II) *there exists a continuous increasing function $q(\varepsilon) : [0, \varepsilon_1) \to \mathbf{R}^+$ with $q(0) = 0$ such that all the eigenvalues of K_2 satisfy*

$$|\lambda_{i,\varepsilon}(K_2)| \leqslant q(\varepsilon) \quad (1 \leqslant i \leqslant n) \tag{2.39}$$

and K_1 has exactly n eigenvalues $\lambda_{i,\varepsilon}(K_1)$ satisfying

$$|\lambda_{i,\varepsilon}(K_1)| \leqslant q(\varepsilon) \quad (1 \leqslant i \leqslant n). \tag{2.40}$$

Then $\lambda_{i,\varepsilon}(K_1)$ can be rearranged so that

$$\lambda_{i,\varepsilon}(K_1) = \lambda_{i,\varepsilon}(K_2) + o(q(\varepsilon)), \tag{2.41}$$

as $\varepsilon \to 0$. Moreover, if $\lambda_{i,\varepsilon}(K_2) = \lambda_{i+1,\varepsilon}(K_2) = \cdots \lambda_{i+j,\varepsilon}(K_2)$, then

$$\|P_{i,i+j}(K_2) - P_{i,i+j}(K_1)\| = o(1), \tag{2.42}$$

as $\varepsilon \to 0$.

Remark 2.4. (1) In the last part of the statement in the theorem, if $j = 0$, i.e. $\lambda_i(K_2)$ is a simple eigenvalue, then the corresponding normalized eigenfunctions of K_1 and K_2 can be chosen so that

$$\|\phi_i(K_1) - \phi_i(K_2)\|_{X_\varepsilon} = o(1). \tag{2.43}$$

(2) The function $q(\varepsilon)$ controls the order of the small eigenvalues. For example, in the case of spike layer solutions (see Section 4), we shall show that $q(\varepsilon) = K\varepsilon^2$ for some $K > 0$. So our result here shows that when the small eigenvalues of the infinite-dimensional problem and finite-dimensional problem converge to zero with ε at a certain order, then they are identical up to that order.

(3) If $X_\varepsilon^u(u)$ is also finite dimensional, then it is easy to show that there are exactly n eigenvalues of K_1 which converge to 0 as $\varepsilon \to 0$ (which we assume in Theorem 2.3) by the Raleigh quotient representation of eigenvalues. However, there are examples of Schrödinger operators which have spectra of the form $(-\infty, -a] \cup \{\lambda_1, \dots, \lambda_n\} \cup [b, c]$, where $a, b, c > 0$. (see [Da, p. 96]).

To prove the theorem, we need the following Gerŝgorin disc theorem of linear algebra (see [HJ, Theorem 6.1.1, p. 344]):

Lemma 2.5. *Let $A = [a_{ij}]$ be an $n \times n$ complex matrix, and let*

$$R_i(A) = \sum_{j=1, j \neq i}^{n} |a_{ij}|, \quad 1 \leqslant i \leqslant n$$

denote the deleted absolute row sums of A. Then all eigenvalues of A are located in the union of n discs

$$\bigcup_{i=1}^{n} \{z \in \mathbf{C} : |z - a_{ii}| \leqslant R_i(A)\}.$$

Furthermore, if a union of k of these n discs forms a connected region that is disjoint from all the remaining $n - k$ discs, then there are precisely k eigenvalues of A in this region.

Proof of Theorem 2.3. First we show that $\lambda = 0$ has the same multiplicity for the two problems. We claim that (2.31) with $\lambda = 0$ can be reduced to an equation

$$Q_\varepsilon^c(u) \circ D_u F(\varepsilon, u + \psi(u))(\theta^c + P\theta^c) = 0, \tag{2.44}$$

where $\theta^c \in X_\varepsilon^c(u) = T_u M_\varepsilon$. In fact, from (A4), $T_1(u) = Q_\varepsilon^{su}(u) \circ D_u F(\varepsilon, u)$ is an isomorphism on $X_\varepsilon^{su}(u)$, so when ε is small enough, $T_3(u) = Q_\varepsilon^{su}(u) \circ D_u F(\varepsilon, u + \psi(u))|_{X_\varepsilon^{su}(u)}$ is also an isomorphism on $X_\varepsilon^{su}(u)$ since $D_u F$ is continuous and $\psi(u) \to 0$ uniformly for $u \in M_\varepsilon$ as $\varepsilon \to 0$. Define

$$P\theta^c = -[T_3(u)]^{-1} \circ Q_\varepsilon^{su}(u) \circ D_u F(\varepsilon, u + \psi(u))\theta^c. \tag{2.45}$$

Then we have for any $\theta^c \in X_\varepsilon^c$,

$$Q_\varepsilon^{su}(u) \circ D_u F(\varepsilon, u + \psi(u))(\theta^c + P\theta^c) = 0. \tag{2.46}$$

So (2.31) reduces to (2.44), and the solutions of (2.32) with $\lambda = 0$ and (2.44) are in one-to-one correspondence, with $\theta^c = \xi$, $P\theta^c = D_u\psi(u)\xi$. Thus $i^c(K_1) = i^c(K_2)$. (This proof is similar to an argument in p. 16 of Dancer [D1].)

Next we show that for a nonvanishing eigenvalue, a similar procedure can be applied, only giving a weaker result. Suppose that (λ, θ) is an eigen-pair of K_1 with $|\lambda| \leqslant q(\varepsilon)$. Let $\theta = \theta^c + \theta^{su}$ be the decomposition in $X_\varepsilon^c \oplus X_\varepsilon^{su}$. Then we have

$$Q_\varepsilon^{c}(u) \circ D_u F(\varepsilon, u + \psi(u))(\theta^c + \theta^{su}) = \lambda\theta^c, \tag{2.47}$$

$$Q_\varepsilon^{su}(u) \circ D_u F(\varepsilon, u + \psi(u))(\theta^c + \theta^{su}) = \lambda\theta^{su}. \tag{2.48}$$

Eq. (2.48) can be rewritten as

$$Q_\varepsilon^{su}(u) \circ D_u F(\varepsilon, u + \psi(u))\theta^{su} - \lambda\theta^{su} = -Q_\varepsilon^{su}(u) \circ D_u F(\varepsilon, u + \psi(u))\theta^c.$$

Combining with (2.35), we get

$$\begin{aligned} &Q_\varepsilon^{su}(u) \circ D_u F(\varepsilon, u + \psi(u))\theta^{su} - \lambda\theta^{su} \\ &\quad = Q_\varepsilon^{su}(u) \circ D_u F(\varepsilon, u + \psi(u)) D_u\psi(u)\theta^c \end{aligned}$$

or

$$\theta^{su} - \lambda[T_3(u)]^{-1}\theta^{su} = D_u\psi(u)\theta^c.$$

Thus $\theta^{su} = (I - \lambda[T_3(u)]^{-1})^{-1} D_u\psi(u)\theta^c \equiv \vartheta(\lambda) D_u\psi(u)\theta^c$, and the eigenvalue problem is reduced to (2.47):

$$K_3(\lambda)\theta^c \equiv Q_\varepsilon^{c}(u) \circ D_u F(\varepsilon, u + \psi(u))[\theta^c + \vartheta(\lambda) D_u\psi(u)\theta^c] = \lambda\theta^c. \tag{2.49}$$

We show the following estimates:

$$\sup_{\theta \in X_\varepsilon^c(u)\setminus\{0\}} \frac{\|D_u\psi(u)\theta\|_{Y_\varepsilon}}{\|\theta\|_{Y_\varepsilon}} \to 0 \quad \text{as } \varepsilon \to 0, \tag{2.50}$$

$$\|(I - \vartheta(\lambda))\theta\|_{Y_\varepsilon} \leqslant C|\lambda| \cdot \|\theta\|_{Y_\varepsilon}, \quad \theta \in X_\varepsilon^{su}(u) \quad \text{for } \lambda \text{ small}. \tag{2.51}$$

We first prove (2.50). Recall from Lemma 2.2, we have

$$\|\psi\|_{Y_\varepsilon} \geqslant \min(C_1, C_2) \|[T_1(u)]^{-1}\psi\|_{Y_\varepsilon}$$

for any $\psi \in X_\varepsilon^{su}(u)$. Since $T_3(u)$ is a perturbation of $T_1(u)$, then the same estimate holds for $T_3(u)$ except the constant may be smaller. From (2.35), we have $D_u\psi(u)\theta = -[T_3(u)]^{-1} \circ Q_\varepsilon^{su}(u) \circ D_u F(\varepsilon, u + \psi(u))\theta$. Thus for any $\theta \in X_\varepsilon^c(u)\setminus\{0\}$,

$$\begin{aligned} \|D_u\psi(u)\theta\|_{Y_\varepsilon} &= \|[T_3(u)]^{-1} \circ Q_\varepsilon^{su}(u) \circ D_u F(\varepsilon, u + \psi(u))\theta\|_{Y_\varepsilon} \\ &\leqslant C\|Q_\varepsilon^{su}(u) \circ D_u F(\varepsilon, u + \psi(u))\theta\|_{Y_\varepsilon} \leqslant C\|D_u F(\varepsilon, u + \psi(u))\theta\|_{Y_\varepsilon} \\ &\leqslant C \cdot C_3(\varepsilon)\|\theta\|_{Y_\varepsilon} = o(\|\theta\|_{Y_\varepsilon}) \quad \text{(by (2.38))}. \end{aligned}$$

For (2.51), we have

$$\|I - \vartheta(\lambda)\| = \|I - (I - \lambda[T_3(u)]^{-1})^{-1}\| = \left\| I - \sum_{k=0}^{\infty} \lambda^k [T_3(u)]^{-k} \right\|$$
$$= \left\| \sum_{k=1}^{\infty} \lambda^k [T_3(u)]^{-k} \right\| \leqslant |\lambda| \sum_{k=1}^{\infty} |\lambda|^{k-1} \|[T_3(u)]^{-1}\|^k \leqslant C|\lambda|$$

if $|\lambda| < 1/\|[T_3(u)]^{-1}\|$.

Now we prove (2). Since K_2 is self-adjoint, there exists a Y_ε-orthonormal basis $\{\theta_1, \ldots, \theta_n\}$ of $X_\varepsilon^c(u)$ such that the matrix representation of K_2 with respect to this basis is diagonal:

$$K_2 = [\langle K_2\theta_i, \theta_j \rangle_{Y_\varepsilon}] = diag(\lambda_{1,\varepsilon}(K_2), \ldots, \lambda_{n,\varepsilon}(K_2)) = [a_{ij}].$$

On the other hand, $\lambda_{i,\varepsilon}(K_1)$ $(1 \leqslant i \leqslant n)$ are the eigenvalues of $K_3(\lambda)\theta = \lambda\theta$. Let $[k_{ij}(\lambda)]$ be the matrix representation of $K_3(\lambda)$ under the basis $\{\theta_i\}$. Then

$$k_{ij}(\lambda) = a_{ij} - \langle Q_\varepsilon^c(u) \circ D_u F(\varepsilon, u + \psi(u)) \circ (I - \vartheta(\lambda)) \circ D_u \psi(u)\theta_j, \theta_i \rangle_{Y_\varepsilon}.$$

The remainder term is estimated as follows

$$\begin{aligned} &|\langle Q_\varepsilon^c(u) \circ D_u F(\varepsilon, u + \psi(u)) \circ (I - \vartheta(\lambda)) \circ D_u \psi(u)\theta_j, \theta_i \rangle_{Y_\varepsilon}| \\ &\quad = \left| \langle D_u F(\varepsilon, u + \psi(u)) \circ (I - \vartheta(\lambda)) \circ D_u \psi(u)\theta_j, \theta_i \rangle_{Y_\varepsilon} \right| \\ &\quad = \left| \langle (I - \vartheta(\lambda)) \circ D_u \psi(u)\theta_j, D_u F(\varepsilon, u + \psi(u))\theta_i \rangle_{Y_\varepsilon} \right| \\ &\quad \leqslant \|(I - \vartheta(\lambda)) \circ D_u \psi(u)\theta_j\|_{Y_\varepsilon} \|D_u F(\varepsilon, u + \psi(u))\theta_i\|_{Y_\varepsilon} \\ &\quad \leqslant |\lambda| \cdot o(\|\theta_j\|_{Y_\varepsilon}) o(\|\theta_i\|_{Y_\varepsilon}) \quad \text{(by (2.50) and (2.51))} \\ &\quad = |\lambda| o(1) \quad (\text{as } \varepsilon \to 0). \end{aligned} \tag{2.52}$$

Therefore,

$$|k_{ij}(\lambda) - a_{ij}| = o(|\lambda|) \tag{2.53}$$

for any $\lambda \in \mathbf{R}$ as $\varepsilon \to 0$.

Let $\lambda \in (-q(\varepsilon), q(\varepsilon))$. We consider the eigenvalue problem $K_3(\lambda)\xi = \mu(\lambda)\xi$. From (2.53), we have

$$|k_{ij}(\lambda) - a_{ij}| = o(|q(\varepsilon)|) \quad \text{as } \varepsilon \to 0. \tag{2.54}$$

For any $\delta > 0$, there exists $\varepsilon_3 > 0$ such that for $\varepsilon \in (0, \varepsilon_3)$, we have

$$|k_{ij}(\lambda) - a_{ij}| \leqslant \delta q(\varepsilon). \tag{2.55}$$

By (2.55) and Lemma 2.5, the eigenvalues $\mu_{i,\varepsilon}(\lambda)$ of $K_3(\lambda)$ lie in the union of disks

$$D_\varepsilon \equiv \bigcup_{i=1}^{n} D_{i,\varepsilon} \equiv \bigcup_{i=1}^{n} \{z \in \mathbf{C} : |z - \lambda_{i,\varepsilon}(K_2)| \leqslant n\delta q(\varepsilon)\}. \tag{2.56}$$

In particular, all $\lambda_{i,\varepsilon}(K_1)$ $(1 \leqslant i \leqslant n)$ also lie in $\bigcup_{i=1}^{n} D_{i,\varepsilon}$ since $\lambda_{i,\varepsilon}(K_1) \in (-q(\varepsilon), q(\varepsilon))$ and $\lambda_{i,\varepsilon}(K_1) = \lambda_{i,\varepsilon}(K_3(\lambda)) = \mu_{j,\varepsilon}(\lambda_{i,\varepsilon}(K_1))$ for some j.

We show that if the union of k discs $D_{\varepsilon,i}$ forms a connected region, then there are precisely k eigenvalues $\lambda_{i,\varepsilon}(K_1)$ in this region. First, we define an extension $\widetilde{K_2}$ of K_2 to the whole space X_ε: $\widetilde{K_2}|_{X_\varepsilon^c(u)} = K_2$, and $\widetilde{K_2}|_{X_\varepsilon^{su}(u)} = Q_\varepsilon^{su}(u)\circ K_1$. We consider a family of operators $H(\varepsilon,t) = \widetilde{K_2} + t(K_1 - \widetilde{K_2})$ for $t\in[0,1]$. Note that $\lambda_{i,\varepsilon}(K_2)$ is also an eigenvalue of $\widetilde{K_2}$. We can see that $H(\varepsilon,t)$ and K_1 share all spectral estimates, and with a careful checking of the previous proof, we obtain that $H(\varepsilon,t)$ has exactly n eigenvalues which converge to 0 as $\varepsilon\to 0$, and these n eigenvalues all lie in D_ε. Let D'_ε be a connected component of D_ε which is the union of k $D_{\varepsilon,i}$'s containing $\lambda_{i,\varepsilon}(K_2),\dots,\lambda_{i+k-1,\varepsilon}(K_2)$. If we rearrange the zero-approaching eigenvalues of $H(\varepsilon,t)$ such that $\lambda_1(\varepsilon,t)\geqslant\cdots\geqslant\lambda_n(\varepsilon,t)$, then for $i\leqslant j\leqslant i+k-1$, $t\mapsto\lambda_j(\varepsilon,t)$ is a continuous curve which lies entirely in D'_ε since $\lambda_j(\varepsilon,0) = \lambda_{j,\varepsilon}(K_2)\in D'_\varepsilon$. In particular, this implies K_1 has exactly k eigenvalues in region D'_ε. Therefore, we obtain

$$|\lambda_{i,\varepsilon}(K_1) - \lambda_{i,\varepsilon}(K_2)|\leqslant(2n-1)n\delta q(\varepsilon), \tag{2.57}$$

where $(2n-1)n$ is a constant which can be achieved if all $D_{i,\varepsilon}$'s are connected. Since δ can be chosen arbitrarily, then we obtain

$$|\lambda_{i,\varepsilon}(K_1) - \lambda_{i,\varepsilon}(K_2)| = o(q(\varepsilon)), \quad 1\leqslant i\leqslant n. \tag{2.58}$$

The closeness of eigenspace of K_2 and $K_3(\lambda)$ can be established in a standard way once the closeness of eigenvalues are proved (see [Ka, Chapter II] for details). Here we note that $K_3(\lambda)$ is an analytic perturbation of K_2. If θ^c is an eigenvector of $K_3(\lambda)$, then the eigenvector for K_1 is $\theta^c + \vartheta(\lambda)D_u\psi(u)\theta^c = \theta^c + o(\|\theta^c\|_{Y_\varepsilon})$, thus (2.42) can be proved. □

3. Application to spike layer solutions

As an application of the abstract results, we consider

$$\begin{cases} \varepsilon^2\Delta v - av + f(v) = 0, & x\in\Omega, \\ \dfrac{\partial v}{\partial n} = 0, & x\in\partial\Omega, \end{cases} \tag{3.1}$$

where $a>0$, $\varepsilon>0$ is a positive parameter and Ω is a bounded domain in $\mathbf{R}^n$ for $n\geqslant 2$ with C^4 boundary. Let $\Omega_\varepsilon = \{y\in\mathbf{R}^n : \varepsilon y\in\Omega\}$. If $v(x)$ is a solution of (3.1), then $u(z) = v(\varepsilon z)$ is a solution of

$$\begin{cases} \Delta u - au + f(u) = 0, & z\in\Omega_\varepsilon, \\ \dfrac{\partial u}{\partial n} = 0, & z\in\partial\Omega_\varepsilon. \end{cases} \tag{3.2}$$

Now we fit (3.2) into the abstract framework developed in Section 2. Consider an equation in the whole space:

$$\begin{cases} \Delta w - aw + f(w) = 0, & y \in \mathbf{R}^n, \\ w(0) = \max w(y), & w > 0, \\ w(y) \to 0, & |y| \to \infty. \end{cases} \tag{3.3}$$

We assume that f and (3.3) satisfy

(B1) $f \in C^1(\overline{\mathbf{R}^+})$, $f(0) = f'(0) = 0$.

(B2) Eq. (3.3) has a radially symmetric solution $w \in H^2(\mathbf{R}^n) \cap C^2(\mathbf{R}^n)$, and there exists $C, K > 0$ such that

$$|D^\alpha w(y)| \leqslant Ce^{-K|y|} \quad \text{for } y \in \mathbf{R}^n, \quad |\alpha| \leqslant 2. \tag{3.4}$$

(B3) Let

$$L_0 = \Delta - aI + f'(w)I : H^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n).$$

Then $\sigma(L_0) \cap (-b, \infty) = \{\lambda_1, 0\}$, where $\sigma(L_0)$ is the spectrum of L_0, $b \in (0, a)$ is a constant and $\lambda_1 > 0$ is the principal eigenvalue. Moreover, the eigenspace associated with the eigenvalue $\lambda = 0$ is spanned by

$$\left\{ \frac{\partial w}{\partial y_j}, \ j = 1, 2, \ldots, n \right\}. \tag{3.5}$$

Remark 3.1. All conditions (B1)–(B3) are satisfied by the prototype nonlinearities $f(u) = |u|^{p-1}u$, $1 < p < (n+2)/(n-2)$ and $g(u) = -au + f(u) = -du(u-b)(u-c)$ with $d > 0$ and $c > 2b > 0$. In fact, (B1)–(B3) can be verified for more general nonlinearities, see Section 5.6 for the details. Be advised that (B1), as it stands, is sufficient for the reduction but is strengthened somewhat in (B1a–d) below (see Section 4) in order to obtain the existence result and to calculate the Morse index.

Let

$$X_\varepsilon = \left\{ u \in H^2(\Omega_\varepsilon) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega_\varepsilon \right\}, \qquad Y_\varepsilon = L^2(\Omega_\varepsilon), \tag{3.6}$$

and consider a Fréchet differentiable map $F(\varepsilon, \cdot) : X_\varepsilon \to Y_\varepsilon$,

$$F(\varepsilon, u) = \Delta u - au + f(u). \tag{3.7}$$

Then a solution u of (3.2) is a solution of $F(\varepsilon, u) = 0$. For $P \in \mathbf{R}^n$ we define $W_{\varepsilon,P}$ to be the solution of

$$\begin{cases} \Delta v - av + f(w(z-P)) = 0, & z \in \Omega_\varepsilon, \\ \dfrac{\partial v}{\partial n} = 0, & z \in \partial \Omega_\varepsilon. \end{cases} \tag{3.8}$$

The function $W_{\varepsilon,P}$ was first introduced by Ni and Wei [NW] when studying a similar Dirichlet boundary value problem.

We use $W_{\varepsilon,P}$ to construct a differentiable manifold in X_ε. Let U_i $(1 \leqslant i \leqslant k)$ be k disjoint open subsets of $\partial\Omega$, and let $U_i^\varepsilon = \varepsilon^{-1}U_i = \{\varepsilon^{-1}x: x \in U_i\} \subset \partial\Omega_\varepsilon$. Then U_i^ε is a $(n-1)$-dimensional differentiable manifold embedded in $\mathbf{R}^n$ and

$$\widetilde{M}_{\varepsilon,k} \equiv U_1^\varepsilon \times U_2^\varepsilon \times \cdots \times U_k^\varepsilon$$

is a $k(n-1)$-dimensional differentiable manifold. The basis for the tangent space at a point on $\widetilde{M}_{\varepsilon,k}$ can be viewed as the union of the bases for the tangent spaces at the corresponding points on the submanifolds U_i^ε. For $\mathbf{P} = (P_1, P_2, \dots, P_k) \in \widetilde{M}_{\varepsilon,k}$, we define

$$W_{\varepsilon,\mathbf{P}} = \sum_{j=1}^{k} W_{\varepsilon,P_j} \quad \text{and} \quad F_1(\mathbf{P}) = W_{\varepsilon,\mathbf{P}} \tag{3.9}$$

and

$$M_{\varepsilon,k} = F_1(\widetilde{M}_{\varepsilon,k}) = \{W_{\varepsilon,\mathbf{P}}: \ \mathbf{P} \in \widetilde{M}_{\varepsilon,k}\}. \tag{3.10}$$

Lemma 3.2. *There exists $\varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$, $M_{\varepsilon,k}$ is a differentiable submanifold of X_ε diffeomorphic to $\widetilde{M}_{\varepsilon,k}$, and dim$(M_{\varepsilon,k}) = k(n-1)$.*

Proof. The mapping $F_1: \mathbf{P} \mapsto W_{\varepsilon,\mathbf{P}}$ is differentiable, and we prove that DF_1 is an isomorphism. For $P_i \in U_i^\varepsilon$, $(1 \leqslant i \leqslant k)$, we choose an orthonormal basis $\{\partial_j^i: 1 \leqslant j \leqslant n-1\}$ of $T_{P_i}U_i^\varepsilon$, then $DF_1(\partial_j^i) = \partial_j^i W_{\varepsilon,\mathbf{P}} = \partial_j^i W_{\varepsilon,P_i}$ is a solution of

$$\begin{cases} \Delta\partial_j^i W_{\varepsilon,P_i} - a\partial_j^i W_{\varepsilon,P_i} + f'(w(y-P_i))\partial_j^i w(y-P_i) = 0, & y \in \Omega_\varepsilon, \\ \dfrac{\partial(\partial_j^i W_{\varepsilon,P_i})}{\partial n} = 0, & y \in \partial\Omega_\varepsilon. \end{cases} \tag{3.11}$$

Note that (3.11) is obtained by applying ∂_j^i to (3.8), and $\Delta\partial_j^i = \partial_j^i\Delta$ since Δ is spatial derivative and ∂_j^i is parametric derivative. Also $\partial_n\partial_j^i = \partial_j^i\partial_n$ on the boundary.

The matrix representation of DF_1 under the basis $\{\partial_j^i\}$ of $T_{\mathbf{P}}\widetilde{M}_{\varepsilon,k}$ is

$$(\langle \partial_j^i W_{\varepsilon,\mathbf{P}}, \partial_m^l W_{\varepsilon,\mathbf{P}} \rangle_{L^2(\Omega_\varepsilon)})_{k(n-1)\times k(n-1)}.$$

The invertiability of DF_1 can be deduced from the following estimates (the proofs are given at the end of Section 5.1):

$$\langle \partial_j^i W_{\varepsilon,\mathbf{P}}, \partial_j^i W_{\varepsilon,\mathbf{P}} \rangle_{L^2(\Omega_\varepsilon)} = \int_{\mathbf{R}_+^n} \left[\frac{\partial w}{\partial y_j}(y)\right]^2 dy + o(1), \quad (\varepsilon \to 0), \tag{3.12}$$

$$\langle \partial_j^i W_{\varepsilon,\mathbf{P}}, \partial_l^i W_{\varepsilon,\mathbf{P}} \rangle_{L^2(\Omega_\varepsilon)} = o(1), \quad (j \neq l) \quad (\varepsilon \to 0), \tag{3.13}$$

$$\langle \partial_j^i W_{\varepsilon,\mathbf{P}}, \partial_l^m W_{\varepsilon,\mathbf{P}} \rangle_{L^2(\Omega_\varepsilon)} = O(e^{-c/\varepsilon}), \quad (i \neq m) \quad (\varepsilon \to 0). \tag{3.14}$$

The matrix is a multiple of $I + o(1)$ and so there exists $\varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$, DF_1 is invertible, and F_1 is a local diffeomorphism at any $\mathbf{P}$. It remains to prove that F_1 is injective. In fact, for any $\mathbf{P} \in \widetilde{M}_{\varepsilon,k}$, F_1 is a diffeomorphism on a neighborhood O of $\mathbf{P} = (P_1, P_2, \ldots, P_k)$ in $\widetilde{M}_{\varepsilon,k}$, where

$$O = \{\mathbf{Q} = (Q_1, Q_2, \ldots, Q_k) \in \widetilde{M}_{\varepsilon,k} : |Q_i - P_i| < \eta_3\}, \tag{3.15}$$

$|Q_i - P_i|$ is the standard Euclidian distance on $\mathbf{R}^n$ between P_i and Q_i, and $\eta_3 > 0$ is a constant independent of ε such that $Q_i \in U_i^\varepsilon$. Thus if $F_1(\mathbf{P}^1) = F_1(\mathbf{P}^2)$, there exists $m \in \{1, 2, \ldots, k\}$ such that $|P_m^1 - P_m^2| \geqslant \eta_3$. Then

$$\begin{aligned} 0 &= |(F_1(\mathbf{P}^1) - F_1(\mathbf{P}^2))(P_m^1)| \\ &\geqslant |W_{\varepsilon, P_m^1}(P_m^1) - W_{\varepsilon, P_m^2}(P_m^1)| - O(e^{-c/\varepsilon}) \\ &\geqslant |w(0) - w(P_m^1 - P_m^2)| - o(1) > 0, \end{aligned} \tag{3.16}$$

if we choose ε small enough, which is a contradiction. In estimate (3.16), we use the fact that

$$\|W_{\varepsilon,P}(z) - w(z - P)\|_{L^\infty(\Omega_\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0, \tag{3.17}$$

which we prove at the end of Section 5.1. Therefore F_1 is a diffeomorphism onto $M_{\varepsilon,k}$. $\square$

Next we define the splittings of X_ε and Y_ε at $u \in M_{\varepsilon,k}$. We shall show that (A6) is satisfied in our situation here. We consider the properties of the linearized operator:

$$L_\varepsilon(u) = D_u F(\varepsilon, u) = \Delta - aI + f'(u)I : X_\varepsilon \to Y_\varepsilon. \tag{3.18}$$

It is easy to verify that L_ε is a Fredholm operator with index zero, and the spectrum of L_ε consists of an unbounded real eigenvalues $\lambda_{i,\varepsilon} \to -\infty$ as $i \to \infty$. The spectrum of L_ε is closely related to that of L_0, as shown in the following lemma:

Lemma 3.3. *Recall that* $b > 0$ *is the constant defined in* (B3). *Suppose that* $W_{\varepsilon,\mathbf{P}_\varepsilon} = F_1(\mathbf{P}_\varepsilon) \in M_{\varepsilon,k}$, $\mathbf{P}_\varepsilon = (P_{\varepsilon,1}, P_{\varepsilon,2}, \ldots, P_{\varepsilon,k})$, *and* $(\lambda_{i,\varepsilon}, \phi_{i,\varepsilon})$ *is the i-th eigenpair of* $L_\varepsilon(W_{\varepsilon,\mathbf{P}_\varepsilon})$ *such that* $\|\phi_{i,\varepsilon}\|_{L^2(\Omega_\varepsilon)} = 1$, *then*

$$|\lambda_{i,\varepsilon} - \lambda_1| \to 0, \quad \left\| \phi_{i,\varepsilon} - \sum_{j=1}^k a_j^i \phi_1(\cdot - P_{\varepsilon,j}) \right\|_{H^1(\Omega_\varepsilon)} \to 0 \quad (1 \leqslant i \leqslant k), \tag{3.19}$$

$$|\lambda_{i,\varepsilon}| \to 0, \quad \left\| \phi_{i,\varepsilon} - \sum_{j=1}^k b_j^i \bar{D}_j^i w(\cdot - P_{\varepsilon,j}) \right\|_{H^1(\Omega_\varepsilon)} \to 0 \quad (k+1 \leqslant i \leqslant kn), \tag{3.20}$$

$$\lambda_{i,\varepsilon} \leqslant -b, \quad (i \geqslant kn+1), \quad \textit{as } \varepsilon \to 0, \tag{3.21}$$

where $a^i_j, b^i_j \in \mathbf{R}$,

$$\bar{D}^i_j w = \sum_{m=1}^{n} \tau^i_m(P_{\varepsilon,j}) \frac{\partial w}{\partial z_m}, \tag{3.22}$$

where $\tau^i(P_{\varepsilon,j}) = (\tau^i_1(P_{\varepsilon,j}), \ldots, \tau^i_n(P_{\varepsilon,j}))$ *is a unit vector tangent to* $\partial\Omega_\varepsilon$ *at* $P_{\varepsilon,j}$.

A natural thought is that we use the eigenspaces in Lemma 3.3 as the subspaces in the splittings of X_ε and Y_ε. But the profiles of eigenfunctions are not as clear as the functions $\partial^i_j W_{\varepsilon,P_i}$ defined in the proof of Lemma 3.2. So instead, we use *span*$\{\partial^i_j W_{\varepsilon,P_i}\}$ as $X^c_\varepsilon(W_{\varepsilon,\mathbf{P}_\varepsilon})$ and modify the other two subspaces using the Gram–Schmidt orthonormal procedure.

Lemma 3.4. *Suppose that* $W_{\varepsilon,\mathbf{P}_\varepsilon} = F_1(\mathbf{P}_\varepsilon) \in M_{\varepsilon,k}$, $\mathbf{P}_\varepsilon = (P_{\varepsilon,1}, \ldots, P_{\varepsilon,k})$. *Let*

$$X^c_\varepsilon(W_{\varepsilon,\mathbf{P}_\varepsilon}) = T_{W_{\varepsilon,\mathbf{P}_\varepsilon}} M_{\varepsilon,k} = \{\partial^i_j W_{\varepsilon,\mathbf{P}_\varepsilon} : 1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant n-1\}. \tag{3.23}$$

Then there exist $X^s_\varepsilon(W_{\varepsilon,\mathbf{P}_\varepsilon})$, $X^u_\varepsilon(W_{\varepsilon,\mathbf{P}_\varepsilon})$ *and* $Y^s_\varepsilon(W_{\varepsilon,\mathbf{P}_\varepsilon})$ *such that* (A6) *holds.*

The proof of Lemmas 3.3 and 3.4 are given in Section 5.2. Next, we verify that (A1) and (2.21) are satisfied.

Lemma 3.5. *Suppose that* $F(\varepsilon, u)$ *is defined as in* (3.7), *and* $W_{\varepsilon,\mathbf{P}_\varepsilon} = F_1(\mathbf{P}_\varepsilon) \in M_{\varepsilon,k}$, $\mathbf{P}_\varepsilon = (P_{\varepsilon,1}, \ldots, P_{\varepsilon,k})$. *Then*

(1)
$$\|F(\varepsilon, W_{\varepsilon,\mathbf{P}_\varepsilon})\|_{L^2(\Omega_\varepsilon)} = O(\varepsilon). \tag{3.24}$$

(2) *For any* $\psi \in X_\varepsilon$, *we have*

$$\|\psi\|_{H^2(\Omega_\varepsilon)} \leqslant C_4(\|\Delta\psi\|_{L^2(\Omega_\varepsilon)} + \|\psi\|_{L^2(\Omega_\varepsilon)}), \tag{3.25}$$

for a constant $C_4 > 0$ *independent of* ε.

Proof. Estimate (3.24) is proved in Lemma 3.4 of [BDS], so we omit it here. Estimate (3.25) is proved in [WW1, Appendix B] (see also [BDS, p. 26]).

Finally, we show (A5) holds if some extra conditions on f are satisfied.

Lemma 3.6. *Suppose that in addition to* (B1)–(B3), $f(u)$ *also satisfies*

(B4) *There exists constants* C_6, $C_7 > 0$ *such that*

$$|f'(u)| \leqslant C_6 + C_7|u|^s, \tag{3.26}$$

where $0 \leqslant s < 4/(n-4)$ *if* $n \geqslant 5$, *and* $0 \leqslant s < \infty$ *if* $1 \leqslant n \leqslant 4$.

Then for any neighborhood $M^R_\varepsilon = \{u + \psi\colon\ u \in M_{\varepsilon,k}, \|\psi\|_{X_\varepsilon} \leqslant R\}$, (2.8) *holds.*

The proof of Lemma 3.6 is given in Section 5.3. Summarizing Lemmas 3.2–3.6, and applying Theorem 2.1 and Lemma 2.2, we obtain

Theorem 3.7. *Suppose that* (B1)–(B4) *hold*, $F(\varepsilon, u)$ *is defined as in* (3.7). *Then there exists* $\varepsilon_1 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_1)$ *and any* $W_{\varepsilon,\mathbf{P}} \in M_{\varepsilon,k}$, *there exists a unique* $\psi(W_{\varepsilon,\mathbf{P}}) \in X_\varepsilon^{su}(W_{\varepsilon,\mathbf{P}})$ *such that* $Q_\varepsilon^{su}(W_{\varepsilon,\mathbf{P}}) \circ F(\varepsilon, W_{\varepsilon,\mathbf{P}} + \psi(W_{\varepsilon,\mathbf{P}})) = 0$.

Remark 3.8. (1) Theorem 3.7 was proved in [BDS] by an indirect argument (see [BDS, Proposition 3.5]). Though our proof here is in the same spirit, the proof is slightly different and more direct. Moreover, the assumptions (B1)–(B4) are weaker than the ones in [BDS]. (In [BDS], we assume that $f \in C^3(\mathbf{R})$).

(2) The condition (B4) implies that

$$|f(u)| \leqslant C_8 + C_9 |u|^r, \tag{3.27}$$

where $0 \leqslant r < n/(n-4)$ if $n \geqslant 5$, and $0 \leqslant r < \infty$ if $1 \leqslant n \leqslant 4$. Note that

$$\frac{n}{n-4} > \frac{n+2}{n-2}$$

when $n \geqslant 5$, but to get Theorem 3.7 in the case of $f(u) = |u|^{p-1}u$, we still need $p < (n+2)/(n-2)$ since this is a necessary condition for (3.3) to have a solution in $H^2(\mathbf{R}^n)$.

(3) (B4) can also be relaxed if we have an a prior estimate of the solutions of (3.1). In the case of $-au + f(u) = -du(u-b)(u-c)$ with $0 < 2b < c$, all solutions satisfy $0 < u < c$, and so we can modify $f(u)$ outside of $[0, c]$ so that it satisfies (B4). Thus, in that case (B4) is not necessary.

4. Estimates of small eigenvalues

In the last section, finding the solutions of $F(\varepsilon, u) = 0$ is reduced to a finite-dimensional problem on the manifold $\widetilde{M}_{\varepsilon,k}$:

$$G_1(\mathbf{P}) \equiv Q_\varepsilon^c(W_{\varepsilon,\mathbf{P}}) \circ F(\varepsilon, W_{\varepsilon,\mathbf{P}} + \psi(W_{\varepsilon,\mathbf{P}})) = 0. \tag{4.1}$$

The range space of the map G_1 is $X_\varepsilon^c(W_{\varepsilon,\mathbf{P}})$, which depends on $\mathbf{P}$, but $X_\varepsilon^c(W_{\varepsilon,\mathbf{P}})$ is generated by functions $\{\partial_j^i W_{\varepsilon,P_i}: 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant n-1\}$, thus $G_1(\mathbf{P}) = 0$ is equivalent to

$$G_{ij}(\mathbf{P}) \equiv \langle F(\varepsilon, W_{\varepsilon,\mathbf{P}} + \psi(W_{\varepsilon,\mathbf{P}})), \partial_j^i W_{\varepsilon,P_i} \rangle_{L^2(\Omega_\varepsilon)} = 0. \tag{4.2}$$

So instead of considering the map G_1, we consider $G : \widetilde{M}_{\varepsilon,k} \to \mathbf{R}^{(n-1)k}$ given by $G(\mathbf{P}) = (G_{ij}(\mathbf{P}))$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n-1$. In Proposition 4.1 of [BDS], we prove that

$$\langle F(\varepsilon, W_{\varepsilon,\mathbf{P}} + \psi(W_{\varepsilon,\mathbf{P}})), \partial_j^i W_{\varepsilon,P_i} \rangle_{L^2(\Omega_\varepsilon)} = \varepsilon^2 \gamma \partial_j^i H(\varepsilon P_i) + o(\varepsilon^2), \tag{4.3}$$

where $H : \partial\Omega \to \mathbf{R}$ is the mean curvature function on $\partial\Omega$, and

$$\gamma = \frac{1}{3}\int_{\partial\mathbf{R}^n_+} \left[\frac{w'(|y|)}{|y|}\right]^2 y_j^4\, dy > 0. \tag{4.4}$$

Note that here we use the fact that w is radially symmetric, $w(y) = w(|y|)$ and $w'(\cdot)$ is the derivative in the radial direction. Also in the coordinate system of $\partial\Omega$, $\partial^i_j H(\varepsilon P_i) = D^i_j H(P_i)$ if we define a basis of the tangent space of $\partial\Omega$ at P_i by $D^i_j = \varepsilon\partial^i_j$. In [BDS], in addition to (B1)–(B3), we also assume that $f \in C^3(\mathbf{R})$, since we consider the Cahn–Hilliard equation there. But in fact, we can still obtain (4.3) if f satisfies (B2), (B3) and one of the following:

(B1a) $f \in C^{2,\alpha}(\overline{\mathbf{R}^+})$, $0 < \alpha \leqslant 1$, and $f(0) = f'(0) = 0$; or

(B1b) $f \in C^{1,\alpha}(\overline{\mathbf{R}^+}) \cap C^2(\mathbf{R}^+)$, $0 < \alpha \leqslant 1$, $f(0) = f'(0) = 0$, and

$$\lim_{u \to 0^+} u^{1-\alpha} f''(u) = C > 0. \tag{4.5}$$

One can see that for $f(u) = u^p$, (B1a) is satisfied when $p \geqslant 2$, and (B1b) is satisfied when $1 < p < 2$. For f satisfying (B1a), (B2) and (B3), the proof of (4.3) is basically the same as the proof given in [BDS]; for f satisfying (B1b), (B2) and (B3), certain integrals in the proof of [BDS] involving $f''(u)$ will be improper, but by using (4.5) and the exponential decaying properties of w and other related functions, we can still show all improper integrals are convergent, and the proof goes through. In the latter case, a similar estimate to (4.3) is also proved in Wei [We1, Lemma 4.1] and Li [Li, Theorem 3.1] for the special case of $f(u) = u^p$.

So if we choose each U_i in the definition of $\widetilde{M}_{\varepsilon,k}$ to be a small neighborhood of a particular nondegenerate critical point of $H(P)$ on $\partial\Omega$, and use degree theory, we obtain a solution $\mathbf{P}_\varepsilon$ of $G(\mathbf{P}) = 0$ and therefore the following result (see [BDS] for more details, and also similar results in [WW2,Li]):

Theorem 4.1. *If f satisfies* (B2), (B3), *and* (B1a) *or* (B1b), *then* (3.1) *has a solution of form $u_\varepsilon = W_{\varepsilon,\mathbf{P}} + \psi(W_{\varepsilon,\mathbf{P}})$, which has exactly k local maximum points, each of them is on $\partial\Omega$ and is near a distinct nondegenerate critical point of $H(P)$.*

In this section, we consider the eigenvalue problem:

$$\begin{cases} \Delta\phi - a\phi + f'(u_\varepsilon)\phi = \lambda_{i,\varepsilon}\phi, & z \in \Omega_\varepsilon, \\ \dfrac{\partial\phi}{\partial n} = 0, & z \in \partial\Omega_\varepsilon. \end{cases} \tag{4.6}$$

First we notice that $u_\varepsilon = W_{\varepsilon,\mathbf{P}} + \psi(W_{\varepsilon,\mathbf{P}})$ is an order $O(\varepsilon)$ perturbation of $W_{\varepsilon,\mathbf{P}}$ (see Section 5.1, (5.10)), thus the spectral estimates in Lemma 3.3 are also true for $L_\varepsilon(u_\varepsilon)$. With a careful check of the proof of Lemma 3.3, we have the following result:

Proposition 4.2. *The results in Lemma* 3.3 *hold if we replace* $W_{\varepsilon,\mathbf{P}}$ *by* $u_\varepsilon = W_{\varepsilon,\mathbf{P}} + \psi(W_{\varepsilon,\mathbf{P}})$.

We call the eigenvalues $\lambda_{i,\varepsilon}$ $(k+1 \leqslant i \leqslant kn)$ *small eigenvalues* since they approach zero as $\varepsilon \to 0$. The main goal of this section is to obtain a precise estimate of these small eigenvalues. First we have to impose stronger smoothness assumption on f: instead of (B1a) or (B1b), we assume that f satisfies

(B1c) $f \in C^{3,\alpha}(\overline{\mathbf{R}^+})$, $0<\alpha\leqslant 1$, and $f(0) = f'(0) = 0$; or

(B1d) $f \in C^{1,\alpha}(\overline{\mathbf{R}^+}) \cap C^3(\mathbf{R}^+)$, $0<\alpha<1$, $f(0) = f'(0) = 0$, and

$$\lim_{u\to 0^+} u^{1-\alpha} f''(u) = \lim_{u\to 0^+} (\alpha-1)^{-1} u^{2-\alpha} f'''(u) = C > 0. \tag{4.7}$$

We remark that even though (B1c) and (B1d) are stronger than (B1a) and (B1b), $f(u) = -du(u-b)(u-c)$ with $c > 2b > 0$ is still included as an example of (B1c), and $f(u) = u^p$ with $p > 1$ satisfies (B1c) or (B1d) depending on whether $p \geqslant 3$ or $1 < p < 3$. In the remaining part of this section, we assume that f satisfies (B2), (B3) and (B1c) or (B1d).

Our proof consists of three steps:

(a) Use direct estimates to show that these eigenvalues are of order $O(\varepsilon^2)$.
(b) Study the reduced finite-dimensional eigenvalue problem.
(c) Apply Theorem 2.3 to obtain the precise form of the small eigenvalues of the infinite-dimensional problem.

Step (a) is needed to justify the assumption (2.40) in Theorem 2.3. Our estimate is

Proposition 4.3. *Let* u_ε *be the solution obtained in Theorem* 4.1, *and let* $\lambda_{i,\varepsilon}$ *be the eigenvalues of* (4.6). *In addition we assume that* (B1c) *or* (B1d) *is satisfied. Then for* $k+1 \leqslant i \leqslant kn$, $|\lambda_{i,\varepsilon}| = O(\varepsilon^2)$.

The proof of Proposition 4.3 will be given in Section 5.4. Next we study the finite-dimensional eigenvalue problem. From the discussions in Section 2, we consider a finite-dimensional linear operator on $X_\varepsilon^c(W_{\varepsilon,\mathbf{P}})$:

$$K\xi \equiv Q_\varepsilon^c(u) \circ D_u F(\varepsilon, u + \psi(u))(\xi + D_u \psi(u)\xi). \tag{4.8}$$

The matrix representation of K is

$$K_{i,j,l,m} = \langle D_u F(\varepsilon, u + \psi(u))[\partial_j^i W_{\varepsilon,P_i} + D_u \psi(u) \partial_j^i W_{\varepsilon,P_i}], \partial_m^l W_{\varepsilon,P_l} \rangle_{L^2(\Omega_\varepsilon)},$$

where $K_{i,j,l,m}$ is the entry of the matrix at the position $(j + i(n-1), m + l(n-1))$, $1 \leqslant i, l \leqslant k$, $1 \leqslant j, m \leqslant n-1$.

For the calculation of the eigenvalues of K, we prove the following two lemmas:

Lemma 4.4. *Let* K_i $(1 \leqslant i \leqslant k)$ *be the* $(n-1)\times(n-1)$ *block in matrix* K *consisting of* $K_{i,j,i,m}$, $1 \leqslant j, m \leqslant n-1$. *Then*

$$K_i = \varepsilon^2 \gamma D^2 H(P_i) + o(\varepsilon^2), \tag{4.9}$$

where γ *is defined in* (4.4), $D^2 H(P_i) = (\partial^i_m \partial^i_j H(\varepsilon P_i))$ *is the matrix representation of the Hessian of* $H(P_i)$ *under the basis* $\{\partial^i_j : 1 \leqslant j \leqslant n-1\}$ *of* $T_{P_i}\partial\Omega_\varepsilon$, *and* $o(\varepsilon^2)$ *is taken in matrix-operator norm* $\|E\| = \max\{\|Ex\| : \|x\| = 1\}$.

Lemma 4.5. *Let* K_i *be as defined in Lemma* 4.4. *Then*

$$K = \bigoplus_{i=1}^{k} K_i + O(e^{-c/\varepsilon}), \tag{4.10}$$

where $K_i \oplus K_l$ *is the direct product of matrices*:

$$\begin{pmatrix} K_i & 0 \\ 0 & K_l \end{pmatrix}.$$

In particular, the eigenvalues of K *have the form* $\mu_{i,j} = \varepsilon^2 \gamma \eta_{i,j} + o(\varepsilon^2)$, *where* $\eta_{i,j}$, $(1 \leqslant j \leqslant n-1)$, *are the eigenvalues of* $D^2 H(P_i)$, $1 \leqslant i \leqslant k$.

Lemmas 4.4 and 4.5 will be proved in Section 5.5 once we establish the technical framework and various estimates in Section 5.1. We notice here that

$$K_{i,j,l,m} = \partial^i_j G_{lm}(\mathbf{P}), \tag{4.11}$$

where $G_{lm}(\mathbf{P})$ is defined in (4.2). But estimate (4.9) cannot be directly obtained from (4.3), since the higher order term $o(\varepsilon^2)$ is taken in the L^∞ norm, not the C^1 norm. So essentially (4.9) is an improvement of (4.3) from the L^∞ norm to the C^1 norm.

Combining the abstract results in Section 2, we obtain the following results on the small eigenvalues of a k-peak solution:

Theorem 4.6. *Suppose that* f *satisfies* (B2), (B3), *and* (B1c) *or* (B1d). *Let* u_ε *be the solution obtained in Theorem* 4.1, *and let* $\lambda_{i,\varepsilon}$ $(i \geqslant 1)$ *be the eigenvalues of* (4.6). *Then*

$$\lambda_{i,\varepsilon} = \varepsilon^2 \gamma \eta_i + o(\varepsilon^2), \quad k+1 \leqslant i \leqslant nk, \tag{4.12}$$

where $\{\eta_i : k+1 \leqslant i \leqslant nk\}$ *is the decreasing rearrangement of* $\{\eta_{m,j} : 1 \leqslant j \leqslant n-1, 1 \leqslant m \leqslant k\}$, *the set of eigenvalues of* $D^2 H(P_m)$, $1 \leqslant m \leqslant k$.

Proof. From Proposition 4.2, (4.6) has exactly $k(n-1)$ eigenvalues $\lambda_{i,\varepsilon}$ $(k+1 \leqslant i \leqslant nk)$ such that $|\lambda_{i,\varepsilon}| \to 0$ as $\varepsilon \to 0$. Moreover, by Proposition 4.3, these eigenvalues satisfy $|\lambda_{i,\varepsilon}| \leqslant C\varepsilon^2$ for some $C > 0$. On the other hand, from Lemma 4.5, K has $k(n-1)$ eigenvalues satisfying $|\mu_{i,j}| \leqslant B\varepsilon^2$. Therefore Theorem 2.3, Lemmas 4.4 and 4.5 imply (4.12). □

We remark that Theorem 4.6 implies local uniqueness of k-peak spike layer solution with one peak at each nondegenerate critical point of $H(P)$. More precisely, we have

Proposition 4.7. *Suppose that f satisfies* (B2), (B3), *and* (B1c) *or* (B1d). *Let $\{P_m \in \partial\Omega : 1 \leqslant m \leqslant k\}$ be a set of critical points of $H(P)$ on $\partial\Omega$ such that $D^2H(P_m)$ is nondegenerate at each P_m. Then for sufficiently small $\varepsilon > 0$,* (3.1) *has a unique solution u_ε such that u_ε has exactly k local maximum points, which can be arranged so that there is exactly one local maximum in $N_m = \{P \in \bar{\Omega} : |P - P_m| \leqslant c\varepsilon\}$ for some $c > 0$ independent of ε.*

Proof. We have shown the existence of a solution u_ε in Theorem 1.1 of [BDS], and in the proof there, we also showed that u_ε has exactly k local maximum points. Moreover, for each local maximum point P, $P \in N_m \cap \partial\Omega$, and $g(u_\varepsilon(P)) > 0$ where $g(u) = -au + f(u)$. So we only need to show the uniqueness of the solution. Suppose that v_ε is a solution of (3.2) satisfying the description in the statement. Then by a blow-up argument, one can show that

$$\|v_\varepsilon - W_{\varepsilon,\mathbf{P}_\varepsilon}\|_{H^2(\Omega_\varepsilon)} = O(\varepsilon), \quad \text{as } \varepsilon \to 0, \tag{4.13}$$

where $\mathbf{P}_\varepsilon = (\widetilde{P_{1,\varepsilon}}, \ldots, \widetilde{P_{k,\varepsilon}})$, $\widetilde{P_{m,\varepsilon}}$ $(1 \leqslant m \leqslant k)$ are the local maximum points of v_ε. Therefore, v_ε must be in an $O(\varepsilon)$ neighborhood of $M_{\varepsilon,k}$, and by Theorem 3.7 and estimate (5.10), v_ε must be of the form of $v_\varepsilon = W_{\varepsilon,\mathbf{P}} + \psi(W_{\varepsilon,\mathbf{P}})$ as in Theorem 3.7. By using a degree argument as in the proof of Theorem 1.1 of [BDS], and the nondegeneracy of v_ε from Theorem 4.6, we now can conclude that v_ε is locally unique in a neighborhood of $W_{\varepsilon,\mathbf{P}}$ in X, where $\mathbf{P} = (P_1, \ldots, P_k)$. □

5. Technical estimates

5.1. Geometry of the boundary and the asymptotic expansion

In this subsection, we recall the geometric setup of the boundary manifold and the expansion of related functions in terms of ε from [BDS], where proofs can be found. Let $\mathbf{R}^n_+ = \{(x', x_n) \in \mathbf{R}^n : x_n > 0\}$, $B(P, r) = \{x \in \mathbf{R}^n : |x - P| < r\}$, and let $B'(r) = \{x' \in \mathbf{R}^{n-1} : |x'| < r\}$. Let Ω be a bounded smooth domain with at least C^4 boundary $\partial\Omega$, and $\Omega_\varepsilon = \varepsilon^{-1}\Omega$.

We introduce a diffeomorphism which straightens the boundary in a neighborhood of $P\in\partial\Omega$. Through a rotation of the coordinate system we may assume that the inner normal to $\partial\Omega$ at P is $(0,\dots,0,1)$, and in these coordinates write $P=(P',P_n)$. We can find $\delta_1>0$ and a smooth function $\rho: B'(\delta_1)\to\mathbf{R}$ such that for some neighborhood N_P of P,

(1) $\rho(0)=0$, $D\rho(0)=0$,
(2) $D^2\rho(0)=diag(\rho_{11}(0),\rho_{22}(0),\dots,\rho_{(n-1)(n-1)}(0))$,
(3) $\Omega\cap N_P=\{(x',x_n):\ \delta_1>x_n-P_n>\rho(x'-P')\}$ and $\partial\Omega\cap N_P=\{(x',x_n): x_n-P_n=\rho(x'-P')\}$,

where the subscripts on ρ denote partial derivatives. We define a mapping $y=\Psi(x)=(\Psi_1(x),\dots,\Psi_n(x))$ for $x\in B(P,\delta_1)$, and

$$\Psi_i(x)=\begin{cases} x_i-P_i, & i=1,2,\dots,n-1,\\ x_n-P_n-\rho(x'-P'), & i=n.\end{cases} \tag{5.1}$$

We also define a mapping which is a rescaling of Ψ:

$$\Psi_\varepsilon(x)=(\Psi_{\varepsilon 1},\dots,\Psi_{\varepsilon n})=\varepsilon^{-1}\Psi(x). \tag{5.2}$$

We calculate that $D\Psi(P)=I$, the identity mapping. Thus, Ψ has an inverse mapping $x=\Phi(y)=\Psi^{-1}(y)$ for $y\in B(\delta_2)\subset\Psi(B(P,\delta_1))$, where δ_2 is a positive constant. Let $\Psi^{-1}(B(\delta_2))\cap\Omega=\Omega_P$. Therefore, we have defined a local diffeomorphism $\Psi:\Omega_P\to B(\delta_2)$ such that

(1) $\Psi(P)=0$, $D\Psi(P)=I$,
(2) $\Psi(\Omega_P)=\mathbf{R}^n_+\cap B(\delta_2)$ and $\Psi(\partial\Omega\cap\overline{\Omega_P})=\partial\mathbf{R}^n_+\cap\overline{B(\delta_2)}$.

We also define $\Phi_\varepsilon(y)=\Psi_\varepsilon^{-1}(y)=\Phi(\varepsilon y)$. If we want to specify the location of the diffeomorphism, we will use notations Ψ_P and Φ_P, or $\Psi_{\varepsilon,P}$ and $\Phi_{\varepsilon,P}$. In this section, we use the following coordinate conversion:

$$x\in\Omega,\quad z\in\Omega_\varepsilon,\quad y\in\mathbf{R}^n_+,\quad z=\frac{x}{\varepsilon},\quad y=\Psi_{\varepsilon,P}(x). \tag{5.3}$$

Let M_P and O_P also be neighborhoods of P satisfying $M_P\subset\subset O_P\subset\Psi^{-1}(B(\delta_2))$. Then we can define a smooth cut-off function $\chi_P(x):\ \mathbf{R}^n\to[0,1]$ such that

(1) $\chi_P(x)=1$ for $x\in M_P$,
(2) $\chi_P(x)=0$ for $x\in\bar{\Omega}\backslash O_P$.

We recall that the mean curvature of $\partial\Omega$ at P is $H(P)=\frac{1}{n-1}\sum_{i=1}^{n-1}\rho_{ii}(0)$, and we have the Taylor expansion of ρ

$$\begin{aligned}\rho(x'-P')=&\frac{1}{2}\sum_{i=1}^{n-1}\rho_{ii}(0)(x_i-P_i)^2\\ &+\frac{1}{6}\sum_{i,j,k=1}^{n-1}\rho_{ijk}(0)(x_i-P_i)(x_j-P_j)(x_k-P_k)\\ &+O(|x'-P'|^4).\end{aligned} \tag{5.4}$$

Next we recall from [BDS] the estimates of the difference between the ground state solution w and the function $W_{\varepsilon,P}$ (defined in (3.8)). Let $Q\in\partial\Omega$ and let $P=\varepsilon^{-1}Q\in\partial\Omega_\varepsilon$. We assume that the inner normal to $\partial\Omega_\varepsilon$ at P is $(0,0,\dots,1)$. Let $\{\partial_j:\ 1\leqslant j\leqslant n-1\}$ be an orthonormal basis of $T_P\partial\Omega_\varepsilon$ such that for the projection function $\pi_j(z_1,z_2,\dots,z_n)=z_j$, $\partial_j(\pi_i)=\delta_{ij}$ where δ_{ij} is the Kronecker symbol. Then from Proposition 2.2 of [BDS], we have

$$\begin{aligned}W_{\varepsilon,P}(z)=w(z-P)&-\varepsilon\chi_Q(\varepsilon z)v_{1,Q}(\Psi_{\varepsilon,Q}(\varepsilon z))\\&-\varepsilon^2\chi_Q(\varepsilon z)v_{2,Q}(\Psi_{\varepsilon,Q}(\varepsilon z))-\varepsilon^3e_1(z),\end{aligned}\tag{5.5}$$

$$\partial_jW_{\varepsilon,P}(z)=\partial_jw(z-P)-\varepsilon\chi_Q(\varepsilon z)u_{0,Q}(\Psi_{\varepsilon,Q}(\varepsilon z))-\varepsilon^2e_2(z),\tag{5.6}$$

where $z\in\Omega_\varepsilon$, $\|e_i\|_{H^1(\Omega_\varepsilon)}\leqslant C$ for $i=1,2$ and $C>0$. Here $v_{1,Q}$, $v_{2,Q}$, and $u_{0,Q}$ are, respectively, the unique solutions in $H^1(\mathbf{R}^n_+)$ of

$$\begin{cases}\Delta v-av=0, & y\in\mathbf{R}^n_+,\\ \dfrac{\partial v}{\partial y_n}=-\dfrac{w'(|y|)}{2|y|}\sum_{p=1}^{n-1}\rho_{pp}(0)y_p^2, & y\in\partial\mathbf{R}^n_+,\end{cases}\tag{5.7}$$

$$\begin{cases}\Delta v-av-2\sum_{i=1}^{n-1}\rho_{ii}(0)y_i\dfrac{\partial^2v_{1,Q}}{\partial y_i\partial y_n}-\Delta\rho(0)\dfrac{\partial v_{1,Q}}{\partial y_n}=0, & y\in\mathbf{R}^n_+,\\ \dfrac{\partial v}{\partial y_n}=\sum_{i=1}^{n-1}\rho_{ii}(0)y_i\dfrac{\partial v_{1,Q}}{\partial y_i} & \\ \qquad-\dfrac{w'(|y|)}{3|y|}\sum_{i,j,k=1}^{n-1}\rho_{ijk}(0)y_iy_jy_k, & y\in\partial\mathbf{R}^n_+\end{cases}\tag{5.8}$$

and

$$\begin{cases}\Delta v-av=0, & y\in\mathbf{R}^n_+,\\ \dfrac{\partial v}{\partial y_n}=-\dfrac{|y|w''(|y|)-w'(|y|)}{2|y|^3}\sum_{p=1}^{n-1}\rho_{pp}(0)y_p^2y_j & \\ \qquad-\dfrac{w'(|y|)}{|y|}\rho_{jj}(0)y_j, & y\in\partial\mathbf{R}^n_+,\end{cases}\tag{5.9}$$

where $\rho_{ii}(0),\rho_{ijk}(0)$ depend on the location $Q\in\partial\Omega$.

In Theorem 3.7, we have shown that there is a spike layer solution of the form $u_\varepsilon=W_{\varepsilon,\mathbf{P}_\varepsilon}+\psi(W_{\varepsilon,\mathbf{P}_\varepsilon})$. Suppose that $\mathbf{P}_\varepsilon=(P_{\varepsilon,1},P_{\varepsilon,2},\dots,P_{\varepsilon,k})$, and $\mathbf{Q}_\varepsilon=\varepsilon\mathbf{P}_\varepsilon=(Q_{\varepsilon,1},Q_{\varepsilon,2},\dots,Q_{\varepsilon,k})$. Then $\|\psi(W_{\varepsilon,\mathbf{P}_\varepsilon})\|_{H^2(\Omega_\varepsilon)}\leqslant C\varepsilon$ and

$$\psi(W_{\varepsilon,\mathbf{P}_\varepsilon})(z)=\varepsilon\sum_{i=1}^k\chi_{Q_{\varepsilon,i}}(\varepsilon z)\psi_{0,Q_{\varepsilon,i}}(\Psi_{\varepsilon,Q_{\varepsilon,i}}(\varepsilon z))+\varepsilon^2e_3(z),\tag{5.10}$$

where $\psi_{0,Q}$ is the unique solution of

$$\begin{cases} \Delta v - av + f'(w(y))v - f'(w(y))v_{1,Q} = 0, & y \in \mathbf{R}^n_+, \\ \dfrac{\partial v}{\partial y_n} = 0, & y \in \partial \mathbf{R}^n_+, \\ v \in K_0^\perp, \end{cases} \tag{5.11}$$

$$K_0^\perp = \left\{ v \in L^2(\mathbf{R}^n_+) : \int_{\mathbf{R}^n_+} v \frac{\partial w}{\partial y_i} dy = 0, \ i = 1, 2, \ldots, n-1 \right\},$$

and $||e_3||_{H^2(\Omega_\varepsilon)} \leqslant C$ for some C independent of ε.

Let $P \in \partial\Omega_\varepsilon$, and let $Q = \varepsilon P$. If $\{\partial_j : 1 \leqslant j \leqslant n-1\}$ is as defined before, then we have the following estimate:

$$\partial_j w(z-P) = \frac{\partial w(y)}{\partial y_j} - \chi_Q(\varepsilon z)\varepsilon \left[\rho_{jj}(0) y_p \frac{\partial w(y)}{\partial y_n} + \frac{|y|w''(|y|) - w'(|y|)}{2|y|^3} \sum_{p=1}^{n-1} \rho_{pp}(0) y_p^2 y_j y_n \right] + \varepsilon^2 e_4(z), \tag{5.12}$$

where $z \in \Omega_\varepsilon$, $y = \Psi_{\varepsilon,Q}(\varepsilon z)$, and $||e_4||_{L^2(\Omega_\varepsilon)} \leqslant C$. (This is (4.1) in [BDS].)

A function $f : \mathbf{R}_n^+ \to \mathbf{R}$ is called an *odd function* if for each $1 \leqslant i \leqslant n-1$, $f(y_1, \ldots, -y_i, \ldots, y_{n-1}, y_n) = -f(y_1, \ldots, y_i, \ldots, y_{n-1}, y_n)$. If f is an odd function, then $\int_{\mathbf{R}_n^+} f(y)\, dy = 0$. Similarly, a function is an *even function* if for each $1 \leqslant i \leqslant n-1$, $f(y_1, \ldots, -y_i, \ldots, y_{n-1}, y_n) = f(y_1, \ldots, y_i, \ldots, y_{n-1}, y_n)$.

To conclude this subsection, we prove (3.12)–(3.14) and (3.17).

Proof of (3.12)–(3.14). From (5.6) and (5.12), we have

$$\langle \partial_j^i W_{\varepsilon,\mathbf{P}}, \partial_j^i W_{\varepsilon,\mathbf{P}} \rangle_{L^2(\Omega_\varepsilon)} = \int_{\mathbf{R}^n_+} \left(\frac{\partial w}{\partial z_j}(z) \right)^2 dz + O(\varepsilon) \tag{5.13}$$

and

$$\langle \partial_j^i W_{\varepsilon,\mathbf{P}}, \partial_l^i W_{\varepsilon,\mathbf{P}} \rangle_{L^2(\Omega_\varepsilon)} = \int_{\mathbf{R}^n_+} \frac{\partial w}{\partial z_j}(z) \frac{\partial w}{\partial z_l}(z)\, dz + O(\varepsilon) = O(\varepsilon), \tag{5.14}$$

if $j \neq l$ since the integrand is an odd function. Finally, (3.14) can be obtained from the fact that $|P_i - P_j| \geqslant \eta\varepsilon^{-1}$ and $\partial_j^i W_{\varepsilon,\mathbf{P}}$ is exponentially decaying. $\square$

Proof of (3.17). Let $v_\varepsilon(z) = W_{\varepsilon,P}(z) - w(z-P)$, and $u_\varepsilon(z) = v_\varepsilon(z+P)$. Then u_ε satisfies

$$\begin{cases} \Delta u_\varepsilon - a u_\varepsilon = 0, & z \in \Omega_\varepsilon + P, \\ \dfrac{\partial u_\varepsilon}{\partial n} = \dfrac{\partial w}{\partial n}, & z \in \partial(\Omega_\varepsilon + P). \end{cases} \tag{5.15}$$

Then by a standard argument (see [NT2, pp. 830–832]),

$$u_\varepsilon \to u \quad \text{in } C^2_{\mathrm{loc}}(\mathbf{R}^n_+),$$

where u is a solution to the problem

$$\begin{cases} \Delta u - au = 0, & y \in \mathbf{R}^n_+, \\ \dfrac{\partial u}{\partial y_n} = 0, & y \in \partial \mathbf{R}^n_+, \\ v \in H^2(\mathbf{R}^n_+), \end{cases} \tag{5.16}$$

after a rotation. But $u \equiv 0$, so (3.17) is proved. $\square$

5.2. Asymptotic behavior of eigenvalues

In this subsection, we prove Lemmas 3.3 and 3.4. First we prove another lemma. Here we assume that $\{\varepsilon_j\}$ is a sequence such that $\varepsilon_j \downarrow 0$ as $j \to \infty$, $\mathbf{P}_j = (P_{j,1}, P_{j,2}, \ldots, P_{j,k}) \in \widetilde{M}_{\varepsilon,k}$ such that for $1 \leqslant i \leqslant k$,

$$\Omega_{i,\infty} = \lim_{j \to \infty} (\Omega_{\varepsilon_j} - P_{j,i})$$

exists and is a half-space orthogonal to a unit vector v_i. We can always assume the existence of these limits by taking subsequences.

Lemma 5.1. *Recall that $b > 0$ is the constant defined in* (B3). *If $(\varepsilon_j, \mathbf{P}_j)$ satisfies the above assumptions, if (λ_j, ψ_j) is an eigen-pair of $L_{\varepsilon_j}(W_{\varepsilon_j,\mathbf{P}_j})\psi = \lambda\psi$ with $\|\psi_j\|_{H^1(\Omega_{\varepsilon_j})} = 1$, and if $\lambda_j > -b$ for $j \geqslant 1$, then there exists a subsequence of $\{\varepsilon_j\}$ (still denoted by itself), and at most k eigen-pairs (λ_∞, ϕ^i) $(i = 1, 2, \ldots, k)$ which satisfy*

$$\begin{cases} \Delta \phi^i - a\phi^i + f'(w(y))\phi^i = \lambda_\infty \phi^i, & y \in \Omega_{i,\infty}, \\ \dfrac{\partial \phi^i}{\partial v_i} = 0, & y \in \partial\Omega_{i,\infty}, \end{cases} \tag{5.17}$$

such that

$$\varepsilon_j \to 0, \quad \lambda_j \to \lambda_\infty,$$

and

$$\left\| \psi_j(\cdot) - \sum_{i=1}^{k} \phi^i(\cdot - P_{j,i}) \right\|_{H^1(\Omega_{\varepsilon_j})} \to 0 \quad \text{as } j \to \infty. \tag{5.18}$$

Proof. We prove the lemma in several steps.

Step 1. There exist constants $R, C_1 > 0$, a sequence of points $\{z_j^{(1)}\}$, and a subsequence of $\{\psi_j\}$ (still denoted by $\{\psi_j\}$), such that

$$\int_{B(z_j^{(1)},R) \cap \Omega_{\varepsilon_j}} |\psi_j(z)|^2 \, dz > C_1. \tag{5.19}$$

First we present a well-known lemma: (The proof can be found in, for example, [CR, Lemma 2.18], and also [L,G].)

Lemma 5.2. *Suppose that* $\|\psi_j\|_{H^1(\Omega_{\varepsilon_j})} \leqslant C$ *and there exists an* $R>0$ *such that*

$$\lim_{j\to\infty} \sup_{z\in\mathbf{R}^n} \int_{B(z,R)\cap\Omega_{\varepsilon_j}} |\psi_j(z)|^2\, dz = 0.$$

Then $\|\psi_j\|_{L^q(\Omega_{\varepsilon_j})} \to 0$ *as* $j\to\infty$ *for all* $q\in(2, 2n/(n-2))$.

Suppose that (5.19) is not true, then by Lemma 5.2, we have $\|\psi_j\|_{L^q(\Omega_{\varepsilon_j})} \to 0$ as $j\to\infty$ for all $q\in(2,2n/(n-2))$. On the other hand, by the equation,

$$\int_{\Omega_{\varepsilon_j}} |\nabla\psi_j|^2\, dz + (a+\lambda_j)\int_{\Omega_{\varepsilon_j}} \psi_j^2\, dz - \int_{\Omega_{\varepsilon_j}} f'(W_{\varepsilon_j,\mathbf{P}_j})\psi_j^2\, dz = 0. \tag{5.20}$$

In assumption (B4), since $f'(0)=0$, then the constant $C_6>0$ can be chosen arbitrarily small as long as we choose C_7 larger. So here we assume that $0<C_6<(a-b)/2$, where $b>0$ is defined in (B3). Then

$$\left|\int_{\Omega_{\varepsilon_j}} f'(W_{\varepsilon_j,\mathbf{P}_j})\psi_j^2\, dz\right| \leqslant C_6\int_{\Omega_{\varepsilon_j}} \psi_j^2\, dz + C_7\int_{\Omega_{\varepsilon_j}} |W_{\varepsilon_j,\mathbf{P}_j}|^s|\psi_j|^2\, dz. \tag{5.21}$$

By Hölder's inequality,

$$\left|\int_{\Omega_{\varepsilon_j}} |W_{\varepsilon_j,\mathbf{P}_j}|^s\psi_j^2\, dz\right| \leqslant \left(\int_{\Omega_{\varepsilon_j}} |W_{\varepsilon_j,\mathbf{P}_j}|^{sp}\, dz\right)^{\frac{1}{p}} \left(\int_{\Omega_{\varepsilon_j}} |\psi_j|^{2q}\, dz\right)^{\frac{1}{q}} \tag{5.22}$$

for any $p,q>1$ such that $1/p+1/q=1$. Since $W_{\varepsilon,\mathbf{P}}$ is L^∞ bounded (see [BDS, Proposition 2.3]), and w is bounded in $L^2(\mathbf{R}^n)$, using (3.17) we have

$$\int_{\Omega_{\varepsilon_j}} |W_{\varepsilon_j,\mathbf{P}_j}(z)|^{sp}\, dz \leqslant C(p)$$

for any $p>0$, $C(p)$ is a constant which only depends on $p>0$. Therefore, we obtain

$$\begin{aligned}\int_{\Omega_{\varepsilon_j}} |f'(W_{\varepsilon_j,\mathbf{P}_j})\psi_j^2|\, dz &\leqslant C_6\int_{\Omega_{\varepsilon_j}} \psi_j^2\, dz + C(\|\psi_j\|_{L^{2q}(\Omega_{\varepsilon_j})})^2\\ &\leqslant (C_6+\delta)\int_{\Omega_{\varepsilon_j}} \psi_j^2\, dz \quad (j\to\infty)\end{aligned} \tag{5.23}$$

for any small $\delta>0$. Hence, by (5.20) and (5.23),

$$\int_{\Omega_{\varepsilon_j}} |\nabla\psi_j|^2\, dz + (a+\lambda_j)\int_{\Omega_{\varepsilon_j}} \psi_j^2\, dz \leqslant (C_6+\delta)\int_{\Omega_{\varepsilon_j}} \psi_j^2\, dz \quad (j\to\infty).$$

Since $a+\lambda_j \geqslant a-b$ and $0<C_6<(a-b)/2$, we get a contradiction with $\|\psi_j\|_{H^1(\Omega_{\varepsilon_j})}=1$. Thus (5.19) is true.

Step 2. We prove that, by choosing $R>0$ larger, there exists a subsequence of $\{\varepsilon_j\}$ (still denoted by $\{\varepsilon_j\}$), such that we can assume that $z_j^{(1)}=P_{j,i}$ for some $1\leqslant i\leqslant k$.

We first prove that $\{\lambda_j\}$ is bounded. In fact, by assumption, $\lambda_j\geqslant -b$, and

$$\lambda_j\leqslant -\operatorname*{Min}_{v\in H^1(\Omega_\varepsilon)\setminus\{0\}}\frac{\int_{\Omega_{\varepsilon_j}}(|\nabla v|^2+av^2-f'(W_{\varepsilon_j,\mathbf{P}_j})v^2)\,dz}{\int_{\Omega_{\varepsilon_j}}v^2\,dz}$$
$$\leqslant -a+C_2,$$

where $C_2=\max_u|f'(u)|$ is a constant independent of ε. Thus $\{\lambda_j\}$ is bounded.

Next we extend ψ_j to $\mathbf{R}^n\setminus\Omega_{\varepsilon_j}$ such that $\|\psi_j\|_{H^1(\mathbf{R}^n)}\leqslant C$ for some positive constant $C>0$ independent of ε. This can be done since Ω_ε is a scaling of Ω, then the constants in the extension theorem (see [GT, Lemma 6.37, Theorem 7.25]) can be chosen independent of ε whenever $\varepsilon<1$. Therefore $v_j(z)\equiv\psi_j(z+z_j^{(1)})$ is bounded in $H^1(\mathbf{R}^n)$, and we can choose a subsequence of $\{v_j\}$ (still denoted by $\{v_j\}$), such that $\lambda_j\to\lambda_\infty$, and $v_j(z)$ converges to v_∞ weakly in $H^1(\mathbf{R}^n)$ and strongly in $L^2_{\text{loc}}(\mathbf{R}^n)$. Note that $\|v_\infty\|_{H^1(\mathbf{R}^n)}>0$ because of (5.19) and that $v_j(z)$ satisfies

$$\begin{cases}\Delta v_j-av_j+f'(W_{\varepsilon_j,\mathbf{P}_j}(z+z_j^{(1)}))v_j=\lambda_jv_j, & z\in\Omega_{\varepsilon_j}-z_j^{(1)},\\ \dfrac{\partial v_j}{\partial n}=0, & z\in\partial(\Omega_{\varepsilon_j}-z_j^{(1)}).\end{cases}\tag{5.24}$$

From the proof of Theorem 1.1 in [BDS], we know that $W_{\varepsilon_j,\mathbf{P}_j}(\cdot+z_j^{(1)})$ converges to some function u weakly in $H^1(\mathbf{R}^n)$ and strongly in $L^2_{\text{loc}}(\mathbf{R}^n)$, and if $\min\{|z_j^{(1)}-P_{j,i}|:1\leqslant i\leqslant k\}\to\infty$ as $j\to\infty$, then $u\equiv 0$. In that case, the limit v_∞ satisfies

$$\begin{cases}\Delta v_\infty-(a+\lambda_\infty)v_\infty=0, & z\in\Omega_\infty^{(1)},\\ \dfrac{\partial v_\infty}{\partial n}=0, & z\in\partial\Omega_\infty^{(1)},\end{cases}\tag{5.25}$$

where $\Omega_\infty^{(1)}=\lim_{j\to\infty}(\Omega_{\varepsilon_j}-z_j^{(1)})$, is either $\mathbf{R}^n$ or $\mathbf{R}^n_+$ up to a rotation and a translation. (When $\Omega_\infty^{(1)}=\mathbf{R}^n$, we do not have any boundary condition in (5.25).) Since $a+\lambda_\infty>a-b>0$, and $v_\infty\in H^1(\Omega_\infty^{(1)})$, then $v_\infty\equiv 0$. Therefore,

$$R_\infty=\sup_{j\geqslant 1}\min\{|z_j^{(1)}-P_{j,i}|:\ 1\leqslant i\leqslant k\}<\infty.$$

Through a diagonal procedure, we can assume that for some i and a subsequence of $\{\varepsilon_j\}$, we have $|z_j^{(1)}-P_{j,i}|\leqslant R_\infty+1$ for all j. Then we can replace R in (5.19) by $R+R_\infty+1$ and $z_j^{(1)}$ by $P_{j,i}$, and (5.19) becomes

$$\int_{B(P_{j,i},R)\cap\Omega_{\varepsilon_j}} |\psi_j(z)|^2\,dz > C_1. \tag{5.26}$$

By relabeling $P_{j,i}$, we can assume that $i=1$.

Step 3. There exists a subsequence of $\{\varepsilon_j\}$ (still denoted by $\{\varepsilon_j\}$), such that $v_j(z)\equiv\psi_j(z+P_{j,1})$ converges to $\phi^1(z)$ weakly in $H^1(\mathbf{R}^n)$ and strongly in $L^2_{\mathrm{loc}}(\mathbf{R}^n)$, where (λ_∞,ϕ^1) satisfies (5.17) with $\lambda_\infty>-b$, $\|\phi^1\|_{H^1(\Omega_{1,\infty})}\neq 0$.

We repeat the arguments in Step 2 by replacing $z_j^{(1)}$ by $P_{j,i}$, and we take the limit in (5.24) as $j\to\infty$, $v_\infty=\lim_{j\to\infty}v_j$, then we obtain

$$\begin{cases} \Delta v_\infty - av_\infty + f'(u(\cdot))v_\infty = \lambda_\infty v_\infty, & z\in\Omega_{1,\infty},\\ \dfrac{\partial v_\infty}{\partial n}=0, & z\in\partial\Omega_{1,\infty}, \end{cases} \tag{5.27}$$

where $u=\lim_{j\to\infty}W_{\varepsilon_j,\mathbf{P}_j}(z+P_{j,1})$ and $\Omega_{1,\infty}=\lim_{j\to\infty}(\Omega_{\varepsilon_j}-P_{j,1})$. Since $P_{j,1}\in\partial\Omega_{\varepsilon_j}$, then $\Omega_{1,\infty}$ is a half-space. Since $W_{\varepsilon_j,\mathbf{P}_j}(z+P_{j,1})$ converges weakly to $w(z)$ in $H^1(\mathbf{R}^n)$ as $j\to\infty$, then $u=w$ and $v_\infty=\phi^1$ with (λ_∞,ϕ^1) satisfying (5.17). This completes Step 3.

Step 4. Repeat and modify arguments in Steps 1–3 to prove the lemma.

Let $\psi_j^{(2)}(z)=\psi_j(z)-\phi^1(z-P_{j,1})$. If $\limsup_{j\to\infty}\|\psi_j^{(2)}\|_{H^1(\Omega_{\varepsilon_j})}=0$, then we are done, since $\phi^1\neq 0$ and we take $\phi^i\equiv 0$ for $i=2,3,\dots,k$. Otherwise, there exists a subsequence of $(\varepsilon_j,\psi_j^{(2)})$, such that $\|\psi_j^{(2)}\|_{H^1(\Omega_{\varepsilon_j})}\geqslant C_1>0$. Obviously, there also exists a $C_2>0$ such that $\|\psi_j^{(2)}\|_{H^1(\Omega_{\varepsilon_j})}\leqslant C_2$.

Repeating a previous argument, there exist constants $R,C>0$, a sequence of points $\{z_j^{(2)}\}$, and a subsequence of $\{\psi_j^{(2)}\}$ (still denoted by $\{\psi_j^{(2)}\}$), such that

$$\int_{B(z_j^{(2)},R)\cap\Omega_{\varepsilon_j}} |\psi_j^{(2)}(z)|^2\,dz > C. \tag{5.28}$$

In fact, by the definition of $\psi_j^{(2)}$, it satisfies

$$\begin{aligned} \Delta\psi_j^{(2)} - a\psi_j^{(2)} + f'(W_{\varepsilon_j,\mathbf{P}_j})\psi_j^{(2)} &= \lambda_j\psi_j^{(2)} + (\lambda_j-\lambda_\infty)\phi^1(z-P_{j,1})\\ &\quad - [f'(W_{\varepsilon_j,\mathbf{P}_j}(z)) - f'(w(z-P_{j,1}))]\\ &\quad \times\phi^1(z-P_{j,1}) \end{aligned} \tag{5.29}$$

for $z\in\Omega_{\varepsilon_j}$. Multiplying (5.29) by $\psi_j^{(2)}$ and integrating over Ω_{ε_j}, we obtain

$$\int_{\Omega_{\varepsilon_j}} |\nabla\psi_j^{(2)}|^2\,dz + (a+\lambda_j)\int_{\Omega_{\varepsilon_j}} |\psi_j^{(2)}|^2\,dz - \int_{\Omega_{\varepsilon_j}} f'(W_{\varepsilon_j,\mathbf{P}_j})|\psi_j^{(2)}|^2\,dz + R(\varepsilon_j) = 0,$$

where

$$R(\varepsilon_j) = \int_{\partial\Omega_{\varepsilon_j}} \psi_j^{(2)} \frac{\partial\phi^1(z-P_{j,1})}{\partial n}\, ds + (\lambda_j - \lambda_\infty)\int_{\Omega_{\varepsilon_j}} \psi_j^{(2)}(z)\phi^1(z-P_{j,1})\, dz$$
$$- \int_{\Omega_{\varepsilon_j}} [f'(W_{\varepsilon_j,\mathbf{P}_j}(z)) - f'(w(z-P_{j,1}))]\phi^1(z-P_{j,1})\psi_j^{(2)}(z)\, dz$$
$$\equiv I_1 + I_2 + I_3.$$

We have

$$I_1 = \int_{\Omega_{\varepsilon_j}} \nabla\psi_j^{(2)}(z)\cdot\nabla\phi^1(z-P_{j,1})\, dz + \int_{\Omega_{\varepsilon_j}} \Delta\phi^1(z-P_{j,1})\psi_j^{(2)}(z)\, dz$$
$$= \int_{\Omega_{\varepsilon_j}} \nabla\psi_j^{(2)}(z)\cdot\nabla\phi^1(z-P_{j,1})\, dz$$
$$+ \int_{\Omega_{\varepsilon_j}} [a - f'(w(z-P_{j,1})) + \lambda_\infty]\phi^1(z-P_{j,1})\psi_j^{(2)}(z)\, dz$$
$$\to 0,$$

since $\psi_j^{(2)}(z+P_{j,1})$ converges to 0 weakly in $H^1(\mathbf{R}^n)$. Similarly $I_2, I_3 \to 0$ as $j\to\infty$. Hence $R(\varepsilon_j)\to 0$ as $j\to\infty$. Then we can repeat the arguments in Step 1 to prove (5.28). By the same argument as in Step 2, we can choose $z_j^{(2)}$ to be $P_{j,2}$ since (5.28) does not hold with $z_j^{(2)}$ replaced by $P_{j,1}$.

Then $v_j^{(2)}(z) \equiv \psi_j^{(2)}(z+P_{j,2})$ is bounded in $H^1(\mathbf{R}^n)$, and we can choose a subsequence of $\{v_j^{(2)}\}$ (still denoted by $\{v_j^{(2)}\}$), such that $\lambda_j\to\lambda_\infty$, and $v_j^{(2)}(z)$ converges to $\phi^2(z)$ weakly in $H^1(\mathbf{R}^n)$ and strongly in $L^2_{\text{loc}}(\mathbf{R}^n)$ with $\phi^2\not\equiv 0$ because of (5.28). We show that ϕ^2 satisfies (5.17). Notice that $v_j^{(2)}(z)$ satisfies

$$\Delta v_j^{(2)} - a v_j^{(2)} + f'(W_{\varepsilon_j,\mathbf{P}_j}(z+P_{j,2}))v_j^{(2)} - \lambda_j v_j^{(2)} = s_j(z)$$

for $z\in\Omega_{\varepsilon_j} - P_{j,2}$, where

$$s_j(z) = (\lambda_j - \lambda_\infty)\phi^1(z+P_{j,2}-P_{j,1}) - [f'(W_{\varepsilon_j,\mathbf{P}_j}(z+P_{j,2}))$$
$$- f'(w(z+P_{j,2}-P_{j,1}))]\phi^1(z+P_{j,2}-P_{j,1}).$$

Since $|P_{j,2}-P_{j,1}|\to\infty$ as $j\to\infty$, $\phi^1(z+P_{j,2}-P_{j,1})$ converges to 0 weakly in $H^1(\mathbf{R}^n)$ and strongly in $L^2_{\text{loc}}(\mathbf{R}^n)$, and so does $s_j(z)$. Therefore, ϕ^2 satisfies (5.17).

Repeating the above procedure, it must terminate after at most k times. If it terminates after $\hat{k}<k$ times, then $\phi^i\equiv 0$ for $\hat{k}+1\leqslant i\leqslant k$. This completes the proof of Lemma 5.1. □

From (B3), the limit λ_∞ in Lemma 5.1 can only be either λ_1 or 0. If $\lambda_\infty = \lambda_1$, then the solution space of (5.17) is one dimensional: *span*$\{\phi_1\}$; if

$\lambda_\infty = 0$, then the solution space of (5.17) is $n-1$ dimensional:

$$\left\{ \sum_{j=1}^{n} k_j \frac{\partial w(z)}{\partial z_j} : k \cdot v_i = 0, \ k = (k_1, \ldots, k_n) \right\}.$$

Proof of Lemma 3.3. The eigenvalue $\lambda_{i,\varepsilon}$ is represented by the Rayleigh quotient:

$$-\lambda_{i,\varepsilon} = \operatorname*{Min}_i \ \operatorname*{Max}_i \frac{\int_{\Omega_\varepsilon} (|\nabla v|^2 + av^2 - f'(W_{\varepsilon,\mathbf{P}})v^2)\, dz}{\int_{\Omega_\varepsilon} v^2\, dz}. \tag{5.30}$$

Max_i is over all $v \in T_i$, and Min_i is over all subspaces T_i of $H^1(\Omega_\varepsilon)$ of dimension i. Using the same proof as that of Proposition 5.1 of [BDS], one can prove $\lambda_{i,\varepsilon} \leqslant \lambda_1 + O(\varepsilon)$ for $1 \leqslant i \leqslant k$, and applying the same idea, one can also show that $\lambda_{i,\varepsilon} \leqslant O(\varepsilon)$ for $k+1 \leqslant i \leqslant kn$ (by using test function $\partial_j^i W_{\varepsilon,\mathbf{P}}$ in the Rayleigh quotient). Applying Lemma 5.1 yields (3.19). Next we claim that $\lambda_{k+1,\varepsilon} \to 0$ as $\varepsilon \to 0$. Suppose not, then by Lemma 5.1, there exists a subsequence $\{\varepsilon_j\}$ such that $\lambda_{k+1,\varepsilon_j} \to \lambda_1$ as $\varepsilon_j \to 0$. From Lemma 5.1 and the fact that the solution space of (5.17) when $\lambda_\infty = \lambda_1$ is one dimensional, we have

$$\left\| \phi_{i,\varepsilon_j}(\cdot) - \sum_{m=1}^{k} a_{i,m} \phi_1(\cdot - P_{j,m}) \right\|_{H^1(\Omega_{\varepsilon_j})} \to 0,$$

where ϕ_{i,ε_j} is the eigenfunction corresponding to $\lambda_{i,\varepsilon_j}$ for $1 \leqslant i \leqslant k+1$. On the other hand, since ϕ_1 is radially symmetric and exponentially decaying by Theorem 5.4, then

$$\begin{aligned} &\langle \phi_1(\cdot - P_{j,m}), \phi_1(\cdot - P_{l,m}) \rangle_{L^2(\Omega_{\varepsilon_j})} \\ &\quad = \begin{cases} \int_{\mathbf{R}_n^+} [\phi_1(z)]^2\, dz + o(1) & \text{if } m = l, \\ O(\varepsilon_j) & \text{if } m \neq l. \end{cases} \end{aligned} \tag{5.31}$$

Let $A_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,k})$ for $1 \leqslant i \leqslant k+1$. Since $\{\phi_{i,\varepsilon_j}\}$ are mutually orthogonal in $L^2(\Omega_{\varepsilon_j})$, then A_i must be also mutually orthogonal. This contradiction implies $\lambda_{k+1,\varepsilon} \to 0$ as $\varepsilon \to 0$. Similarly, one can prove (3.20) and (3.21) using the fact of the solution space of (5.17) when $\lambda_\infty = 0$ is $n-1$ dimensional. $\square$

Proof of Lemma 3.4. Let $(\lambda_{i,\varepsilon}, \phi_{i,\varepsilon})$ be the eigenpairs of $L_\varepsilon(W_{\varepsilon,\mathbf{P}})\psi = \lambda\psi$, and let $Q_\varepsilon^c(W_{\varepsilon,\mathbf{P}})$ be the L^2 projection into $X_\varepsilon^c(W_{\varepsilon,\mathbf{P}})$. We define

$$\tilde{\phi}_{i,\varepsilon} = \phi_{i,\varepsilon} - Q_\varepsilon^c(W_{\varepsilon,\mathbf{P}})\phi_{i,\varepsilon}, \quad 1 \leqslant i \leqslant k, \tag{5.32}$$

$X_\varepsilon^u(W_{\varepsilon,\mathbf{P}}) = span\{\tilde{\phi}_{i,\varepsilon} : 1 \leqslant i \leqslant k\}$, $Y_\varepsilon^s(W_{\varepsilon,\mathbf{P}})$ is the L^2 complement of $X_\varepsilon^u(W_{\varepsilon,\mathbf{P}}) \oplus X_\varepsilon^c(W_{\varepsilon,\mathbf{P}})$, and $X_\varepsilon^s(W_{\varepsilon,\mathbf{P}}) = Y_\varepsilon^s(W_{\varepsilon,\mathbf{P}}) \cap X_\varepsilon$. Obviously, these subspaces form orthogonal decompositions, and the projections are smooth with respect to $W_{\varepsilon,\mathbf{P}}$. For the spectral estimates, (2.18) is achieved with $C_3(\varepsilon) = O(\varepsilon)$:

$$\begin{aligned}&\|L_\varepsilon(W_{\varepsilon,\mathbf{P}})\partial_j^i W_{\varepsilon,\mathbf{P}}\|_{L^2(\Omega_\varepsilon)}\\&\quad= \|f'(W_{\varepsilon,\mathbf{P}})\partial_j^i W_{\varepsilon,\mathbf{P}} - f'(w(z-P_i))\partial_j w(z-P_i)\|_{L^2(\Omega_\varepsilon)} = O(\varepsilon).\end{aligned}$$

Estimate (2.16) holds for $\psi = \phi_{i,\varepsilon}$ with $C_1 = -\lambda_1 - \delta$ for some $\delta > 0$ as long as ε is small, and it also holds for $\tilde{\phi}_{i,\varepsilon}$ since $\|Q_\varepsilon^c(W_{\varepsilon,\mathbf{P}})\phi_{i,\varepsilon}\|_{L^2(\Omega_\varepsilon)} = o(\|\phi_{i,\varepsilon}\|_{L^2(\Omega_\varepsilon)})$ because ϕ_1 and $\partial w/\partial v_j$ are orthogonal to each other in the limit half-space. Finally, (2.16) is true for any $\psi \in X_\varepsilon^u(W_{\varepsilon,\mathbf{P}})$ since $\phi_{i,\varepsilon}$ and $\phi_{j,\varepsilon}$ are orthogonal to each other if $i \neq j$. Inequality (2.17) can be obtained similarly using Lemma 3.3. □

5.3. *Uniform differentiability of Nemytskii operator*

In this subsection, we prove Lemma 3.6. The notion of uniform differentiability of a differentiable functional on a bounded set was introduced by Krasnoselskii [Kr, p. 68], and here we consider the same property for operators instead of functionals. In Rabinowitz [R1], the uniform differentiability of the functional $E : H_0^1(\Omega) \to \mathbf{R}$ defined by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx - \int_\Omega F(u(x))\,dx \tag{5.33}$$

was proved under the condition:

$$|F'(u)| \leqslant a + b|u|^s, \tag{5.34}$$

where $a, b > 0$, $0 \leqslant s < (n+2)/(n-2)$ if $n \geqslant 3$ and $0 \leqslant s < \infty$ if $n = 1, 2$. Although the operator $F(\varepsilon, u)$ we consider here is the gradient operator of a functional similar to $E(u)$ (but with a different domain), the condition which we impose on $f(u)$ is slightly weaker than that of Rabinowitz. On the other hand, our proof follows the same line as that of Proposition B10 in [R1, pp. 90–94] (see also [BR]).

Proof of Lemma 3.6. First we recall the Sobolev embedding theorem for $H^2(\Omega_\varepsilon)$: there exists a constant $c_1 > 0$ such that for any $u \in H^2(\Omega_\varepsilon)$,

$$\|u\|_{L^t(\Omega_\varepsilon)} \leqslant c_1\|u\|_{H^2(\Omega_\varepsilon)}, \tag{5.35}$$

where $t \in [1, 2n/(n-4)]$, and c_1 is independent of $\varepsilon > 0$. Note that c_1 can be chosen independent of ε since $\partial\Omega_\varepsilon$ has the same geometry for all $\varepsilon > 0$.

For the operator $F(\varepsilon, u) = \Delta u - au + f(u)$, the linear part $\Delta - a$ is clearly uniformly differentiable, so we only need to show the Nemytskii operator:

$$f : H^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon),\ \ u(\cdot) \mapsto f(u(\cdot)) \tag{5.36}$$

is uniformly differentiable. To be more precise, we need to show that, given any $\eta > 0$, there exists $\delta = \delta(\eta, \|u\|_{H^2(\Omega_\varepsilon)})$ such that for $u, \psi \in H^2(\Omega_\varepsilon)$,

$$I \equiv \int_{\Omega_\varepsilon} |f(u+\psi) - f(u) - f'(u)\psi|^2 \, dx \leqslant \eta \|\psi\|^2_{H^2(\Omega_\varepsilon)} \tag{5.37}$$

provided that $\|\psi\|_{H^2(\Omega_\varepsilon)} \leqslant \delta$. Having δ depend only on $\|u\|_{H^2(\Omega_\varepsilon)}$ instead of u implies the uniform differentiability on bounded set in $H^2(\Omega_\varepsilon)$, and thus proves Lemma 3.6 since M_ε^R is bounded in $H^2(\Omega_\varepsilon)$ with the bound independent of ε. In fact for any $u + \psi \in M_\varepsilon^R$, $\|u+\psi\|_{H^2(\Omega_\varepsilon)} \leqslant \|W_{\varepsilon,\mathbf{P}}\|_{H^2(\Omega_\varepsilon)} + R \leqslant \|w\|_{H^2(\mathbf{R}^n_+)} + R + o(1)$. We shall prove (5.37) for a fixed ε, then we show the choice of δ can be made independent of ε.

Let $\Psi(x) = f(u(x) + \psi(x)) - f(u(x)) - f'(u(x))\psi(x)$, and let

$$\Omega_1 = \{x \in \overline{\Omega_\varepsilon} : |u(x)| \geqslant \beta\}, \quad \Omega_2 = \{x \in \overline{\Omega_\varepsilon} : |\psi(x)| \geqslant \gamma\}$$

and

$$\Omega_3 = \{x \in \overline{\Omega_\varepsilon} : |u(x)| \leqslant \beta \text{ and } |\psi(x)| \leqslant \gamma\},$$

where $\beta, \gamma > 0$ are to be determined. Then

$$I \leqslant \sum_{i=1}^{3} \int_{\Omega_i} |\Psi(x)|^2 \, dx \equiv \sum_{i=1}^{3} I_i. \tag{5.38}$$

In the following, we assume that $\|u\|_{H^2(\Omega_\varepsilon)} = M_0$ is fixed and $n \geqslant 5$. We first estimate I_1. From the mean value theorem, there exists $0 \leqslant \xi(x) \leqslant 1$ such that $f(u(x) + \psi(x)) - f(u(x)) = f'(u(x) + \xi(x)\psi(x))\psi(x)$. Then

$$\begin{aligned}
I_{11} &\equiv \int_{\Omega_1} |f(u(x) + \psi(x)) - f(u(x))|^2 \, dx = \int_{\Omega_1} |f'(u + \xi\psi)|^2 \psi^2 \, dx \\
&\leqslant \int_{\Omega_1} [C_6 + C_7(|u| + |\psi|)^s]^2 \psi^2 \, dx \\
&\leqslant 2C_6^2 \int_{\Omega_1} \psi^2 \, dx + 2^{2s+1} C_7^2 \int_{\Omega_1} (|u|^{2s} + |\psi|^{2s}) \psi^2 \, dx \\
&\leqslant 2C_6^2 \|\psi\|^2_{L^{2n/(n-4)}(\Omega_\varepsilon)} |\Omega_1|^{4/n} \\
&\quad + 2^{2s+1} C_7^2 |\Omega_1|^{1/\sigma} \|\psi\|^2_{L^{2n/(n-4)}(\Omega_\varepsilon)} (\|u\|^{2s}_{L^{2(s+1)}(\Omega_\varepsilon)} + \|\psi\|^{2s}_{L^{2(s+1)}(\Omega_\varepsilon)})
\end{aligned}$$

and

$$\frac{n-4}{n} + \frac{s}{s+1} + \frac{1}{\sigma} = 1, \tag{5.39}$$

and $\sigma > 0$ since $s < 4/(n-4)$. In the last line of the above estimate, we use the generalized Hölder's inequality:

$$\int_{\Omega} |u_1| \cdot |u_2| \cdot |u_3| \, dx \leqslant \|u_1\|_{L^p(\Omega)} \|u_2\|_{L^q(\Omega)} \|u_3\|_{L^r(\Omega)},$$

where

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

From (5.35), we obtain

$$I_{11} \leqslant c_2 \|\psi\|^2_{H^2(\Omega_\varepsilon)} [|\Omega_1|^{4/n} + |\Omega_1|^{1/\sigma} (\|u\|^{2s}_{H^2(\Omega_\varepsilon)} + \|\psi\|^{2s}_{H^2(\Omega_\varepsilon)})], \tag{5.40}$$

since $2(s+1) < 2n/(n-4)$. Similarly, we can show that

$$\begin{aligned} I_{12} &\equiv \int_{\Omega_1} |f'(u)|^2 \psi^2 \, dx \\ &\leqslant c_2 \|\psi\|^2_{H^2(\Omega_\varepsilon)} [|\Omega_1|^{4/n} + |\Omega_1|^{1/\sigma} (\|u\|^{2s}_{H^2(\Omega_\varepsilon)} + \|\psi\|^{2s}_{H^2(\Omega_\varepsilon)})]. \end{aligned}$$

Thus,

$$\begin{aligned} I_1 &\leqslant 2I_{11} + 2I_{12} \\ &\leqslant c_3 \|\psi\|^2_{H^2(\Omega_\varepsilon)} [|\Omega_1|^{4/n} + |\Omega_1|^{1/\sigma} (\|u\|^{2s}_{H^2(\Omega_\varepsilon)} + \|\psi\|^{2s}_{H^2(\Omega_\varepsilon)})]. \end{aligned} \tag{5.41}$$

On the other hand,

$$\|u\|_{H^2(\Omega_\varepsilon)} \geqslant \|u\|_{L^2(\Omega_1)} \geqslant \beta |\Omega_1|^{1/2},$$

so

$$|\Omega_1|^{1/\sigma} \leqslant \left(\frac{\|u\|_{H^2(\Omega_\varepsilon)}}{\beta} \right)^{2/\sigma} \equiv M_1, \quad |\Omega_1|^{4/n} \leqslant \left(\frac{\|u\|_{H^2(\Omega_\varepsilon)}}{\beta} \right)^{8/n} \equiv M_2,$$

and $M_1, M_2 \to 0$ as $\beta \to \infty$. So if we assume that $\delta \leqslant 1$, then we can choose β so large that

$$M_2 + M_1 (\|u\|^{2s}_{H^2(\Omega_\varepsilon)} + 1) \leqslant \frac{\varepsilon}{3},$$

and so

$$I_1 \leqslant \frac{\varepsilon}{3} \|\psi\|^2_{H^2(\Omega_\varepsilon)}. \tag{5.42}$$

Similarly,

$$\begin{aligned} I_2 &\leqslant 4 \int_{\Omega_2} [C_6 + C_7 (|u| + |\psi|)^s]^2 \psi^2 \, dx \\ &\leqslant 4 \left(\int_{\Omega_2} [C_6 + C_7 (|u| + |\psi|)^s]^{2(s+1)/s} \, dx \right)^{s/(s+1)} \|\psi\|^2_{L^{2(s+1)}(\Omega_2)} \\ &\leqslant c_4 (|\Omega_2|^{s/(s+1)} + \|u\|^{2s}_{L^{2(s+1)}(\Omega_\varepsilon)} + \|\psi\|^{2s}_{L^{2(s+1)}(\Omega_\varepsilon)}) \\ &\quad \cdot \left(\int_{\Omega_2} |\psi|^{2(s+1)} \left(\frac{|\psi(x)|}{\gamma} \right)^{m-2(s+1)} dx \right)^{1/(s+1)} \\ &\leqslant c_5 (\gamma^{-2s/(s+1)} \|\psi\|^{2s/(s+1)}_{H^2(\Omega_\varepsilon)} + \|u\|^{2s}_{H^2(\Omega_\varepsilon)} + \|\psi\|^{2s}_{H^2(\Omega_\varepsilon)}) \\ &\quad \cdot \gamma^{(2s+2-m)/(s+1)} \|\psi\|^{m/(s+1)}_{L^m(\Omega_2)} \\ &\leqslant c_5 \gamma^{(2s+2-m)/(s+1)} (1 + \|u\|^{2s}_{H^2(\Omega_\varepsilon)} + \|\psi\|^{2s}_{H^2(\Omega_\varepsilon)}) \|\psi\|^{m/(s+1)}_{H^2(\Omega_\varepsilon)}. \end{aligned} \tag{5.43}$$

Here $m = 2n/(n-4)$, $|\Omega_2|$ is estimated using

$$\|\psi\|_{H^2(\Omega_\varepsilon)} \geqslant \|\psi\|_{L^2(\Omega_2)} \geqslant \gamma |\Omega_2|^{1/2},$$

and we require $\delta < \gamma^{2s/(s+1)}$. Since $f \in C^1(\mathbf{R})$, then for any $\bar{\eta}, \bar{\beta} > 0$, there exists $\bar{\gamma} = \bar{\gamma}(\bar{\eta}, \bar{\beta}) > 0$ such that

$$|f(v+h) - f(v) - f'(v)h| \leqslant \bar{\eta}|h|, \tag{5.44}$$

whenever $|v| \leqslant \bar{\beta}$, $|h| \leqslant \bar{\gamma}$. In particular, if $\bar{\beta} = \beta$ and $\bar{\eta} = \sqrt{\eta/3}$, then for $\gamma = \bar{\gamma}(\bar{\eta}, \bar{\beta})$, we have

$$I_3 \leqslant \bar{\eta}^2 \int_{\Omega_3} |\psi(x)|^2 \, dx \leqslant \tfrac{1}{3} \eta \|\psi\|^2_{H^2(\Omega_\varepsilon)}. \tag{5.45}$$

With this choice of γ, if we choose δ small enough such that

$$\delta < \gamma^{2s/(s+1)}, \quad \delta \leqslant 1,$$

and

$$c_5 \gamma^{(2s+2-m)/(s+1)} \left(2 + \|u\|^{2s}_{H^2(\Omega_\varepsilon)}\right) \delta^{(m-2s-2)/(s+1)} \leqslant \frac{\eta}{3},$$

then we obtain (5.37). Note that $(m - 2s - 2)/(s+1) > 0$.

The proof for $1 \leqslant n \leqslant 4$ is similar. We can see from the proof that the choice of δ only depends on the Sobolev embedding constant c_1 in (5.35), which we have shown be to independent of ε, therefore δ can be chosen independent of $\varepsilon > 0$. $\square$

5.4. *Refined estimates of eigenvalues*

Proof of Proposition 4.3. From the proofs of Lemmas 3.3, 3.4 and 5.1, we can see that there is not much more difficulty in proving the eigenvalue results for a multi-peak solution than that of a single-peak solution. So to simplify the notation, we only give the proof for a single-peak solution, and the case for a multi-peak solution can be easily adapted from it.

We first prove the case when f satisfies (B1c). Let $u_\varepsilon = W_{\varepsilon,P_\varepsilon} + \psi(W_{\varepsilon,P_\varepsilon})$ be a single-peak solution, and let $(\lambda_\varepsilon, \phi_\varepsilon)$ be a normalized eigen-pair of (4.6) such that as $\varepsilon \to 0$,

$$\lambda_\varepsilon \to 0, \quad \left\| \phi_\varepsilon(z) - \sum_{i=1}^{n-1} a_{i,\varepsilon} \frac{\partial w(z - P_\varepsilon)}{\partial z_i} \right\|_{H^1(\Omega_\varepsilon)} \to 0. \tag{5.46}$$

Here we assume that $T_{P_\varepsilon} \partial\Omega_\varepsilon = \partial \mathbf{R}^n_+ = \{(z', z_n) : z_n = 0\}$. In the following, we use $w_\varepsilon(z)$ to denote $w(z - P_\varepsilon)$. Let

$$t_\varepsilon(z) = \frac{1}{\varepsilon} \left[\phi_\varepsilon(z) - \sum_{i=1}^{n-1} a_{i,\varepsilon} \frac{\partial w_\varepsilon(z)}{\partial z_i} \right] \quad \text{and} \quad \bar{\lambda}_\varepsilon = \varepsilon^{-1} \lambda_\varepsilon,$$

then $(\bar{\lambda}_\varepsilon, t_\varepsilon)$ satisfies

$$\begin{cases} \Delta t_\varepsilon - at_\varepsilon + f'(u_\varepsilon)t_\varepsilon + \dfrac{f'(u_\varepsilon) - f'(w_\varepsilon)}{\varepsilon} \sum_{i=1}^{n-1} a_{i,\varepsilon} \dfrac{\partial w_\varepsilon}{\partial z_i} \\ \quad = \bar{\lambda}_\varepsilon \phi_\varepsilon, & z \in \Omega_\varepsilon, \\ \dfrac{\partial t_\varepsilon}{\partial n} = -\varepsilon^{-1} \sum_{i=1}^{n-1} a_{i,\varepsilon} \dfrac{\partial}{\partial n} \dfrac{\partial w_\varepsilon}{\partial z_i}, & z \in \partial\Omega_\varepsilon. \end{cases} \tag{5.47}$$

First we claim that $\{\bar{\lambda}_\varepsilon\}$ is bounded. In fact, multiplying (5.47) by ϕ_ε, (4.6) by t_ε, and integrating over Ω_ε, we obtain

$$\begin{aligned} &\bar{\lambda}_\varepsilon \int_{\Omega_\varepsilon} \phi_\varepsilon^2 \, dz - \bar{\lambda}_\varepsilon \int_{\Omega_\varepsilon} \varepsilon^{-1} t_\varepsilon \phi_\varepsilon \, dz \\ &\quad = \int_{\partial\Omega_\varepsilon} \frac{\partial t_\varepsilon}{\partial n} \phi_\varepsilon \, ds + \varepsilon^{-1} \sum_{i=1}^{n-1} a_{i,\varepsilon} \int_{\Omega_\varepsilon} [f'(u_\varepsilon) - f'(w_\varepsilon)] \phi_\varepsilon \frac{\partial w_\varepsilon}{\partial z_i} \, dz. \end{aligned} \tag{5.48}$$

Since $\|\phi_\varepsilon\|_{L^2(\Omega_\varepsilon)} = 1$, then by (5.46),

$$\left| \int_{\Omega_\varepsilon} \varepsilon^{-1} t_\varepsilon \phi_\varepsilon \, dz \right| \leqslant \|\varepsilon^{-1} t_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\phi_\varepsilon\|_{L^2(\Omega_\varepsilon)} = o(1). \tag{5.49}$$

From the proof of Proposition 2.2 in [BDS], we have that for $z \in \partial\Omega_\varepsilon \cap B(P_\varepsilon, R)$ for some $R > 0$,

$$\begin{aligned} &\varepsilon^{-1} \frac{\partial}{\partial n} \frac{\partial w_\varepsilon}{\partial z_i} \\ &\quad = \left[\frac{|y| w''(|y|) - w'(|y|)}{2|y|^3} \sum_{p=1}^{n-1} \rho_{pp}(0) y_p^2 y_i + \frac{w'(|y|)}{|y|} \rho_{ii}(0) y_i \right] + O(\varepsilon e^{-k|y|}) \\ &\quad \equiv J(y) + O(\varepsilon e^{-k|y|}). \end{aligned} \tag{5.50}$$

Then,

$$\begin{aligned} \left| \int_{\partial\Omega_\varepsilon} \frac{\partial t_\varepsilon}{\partial n} \phi_\varepsilon \, ds \right| &\leqslant \|\phi_\varepsilon\|_{L^2(\partial\Omega_\varepsilon)} \left\| \frac{\partial t_\varepsilon}{\partial n} \right\|_{L^2(\partial\Omega_\varepsilon)} \\ &\leqslant \|\phi_\varepsilon\|_{H^1(\partial\Omega_\varepsilon)} \left(\int_{\mathbf{R}_+^n} J^2(y) \, dy \right)^{1/2} + O(\varepsilon) = O(1). \end{aligned} \tag{5.51}$$

Note that $\|\phi_\varepsilon\|_{H^1(\partial\Omega_\varepsilon)} = O(1)$ since ϕ_ε is an eigenfunction for an eigenvalue approaching 0 and $\|\phi_\varepsilon\|_{L^2(\Omega_\varepsilon)} = 1$. Finally from (5.5) and (5.10), we obtain

$$\begin{aligned}
&\left|\varepsilon^{-1}\int_{\Omega_\varepsilon}[f'(u_\varepsilon)-f'(w_\varepsilon)]\phi_\varepsilon\frac{\partial w_\varepsilon}{\partial z_i}\,dz\right| \\
&\quad\leqslant \varepsilon^{-1}\max|f''(u)|\int_{\Omega_\varepsilon}\left|(u_\varepsilon-w_\varepsilon)\frac{\partial w_\varepsilon}{\partial z_i}\phi_\varepsilon\right|dz \\
&\quad\leqslant C\|\varepsilon^{-1}(u_\varepsilon-w_\varepsilon)\|_{L^2(\Omega_\varepsilon)}\left\|\frac{\partial w_\varepsilon}{\partial z_i}\phi_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}=O(1).
\end{aligned}\tag{5.52}$$

Then from (5.49), (5.51), (5.52) and (5.48), $\{\bar\lambda_\varepsilon\}$ is bounded.

So we can assume that $\bar\lambda_\varepsilon\to\bar\lambda$ as $\varepsilon\to 0$. Then the left-hand side of (5.48) approaches $\bar\lambda$ as $\varepsilon\to 0$, and the behavior of $\bar\lambda$ is determined by the right-hand side of (5.48).

From (5.46) and (5.50), we have

$$\begin{aligned}
\int_{\partial\Omega_\varepsilon}\frac{\partial t_\varepsilon}{\partial n}\phi_\varepsilon\,ds &= -\int_{\partial\Omega_\varepsilon}\left(\varepsilon^{-1}\sum_{i=1}^{n-1}a_{i,\varepsilon}\frac{\partial}{\partial n}\frac{\partial w_\varepsilon}{\partial z_i}\right)\left(\sum_{j=1}^{n-1}a_{j,\varepsilon}\frac{\partial w_\varepsilon}{\partial z_j}+\varepsilon t_\varepsilon\right)ds \\
&= -\sum_{i,j=1}^{n-1}a_{i,\varepsilon}a_{j,\varepsilon}\int_{\partial\mathbf{R}^n_+}\left[\frac{|y|w''(|y|)-w'(|y|)}{2|y|^3}\sum_{p=1}^{n-1}\rho_{pp}(0)y_p^2y_i\right. \\
&\qquad\left.+\frac{w'(|y|)}{|y|}\rho_{ii}(0)y_i\right]\frac{\partial w(y)}{\partial y_j}\,dy+O(\varepsilon).
\end{aligned}\tag{5.53}$$

On the other hand, with $v_{1,Q}$ and $\psi_{0,Q}$ defined in (5.7) and (5.11),

$$\begin{aligned}
f'(u_\varepsilon(z))-f'(w_\varepsilon(z)) &= f''(w_\varepsilon(z))[u_\varepsilon(z)-w_\varepsilon(z)]+E_1(z) \\
&= [f''(w(y))+E_2(z)][\varepsilon\psi_{0,Q_\varepsilon}(y)-\varepsilon v_{1,Q_\varepsilon}(y)+E_3(z)] \\
&\quad + E_1(z),
\end{aligned}\tag{5.54}$$

where $Q_\varepsilon=\varepsilon P_\varepsilon\in\partial\Omega$,

$$|E_1(z)|\leqslant\tfrac{1}{2}\max|f'''(u)|\cdot|u_\varepsilon(z)-w_{\varepsilon,P_\varepsilon}(z)|^2 \tag{5.55}$$

and

$$|E_2(z)|\leqslant\max|f'''(u)|\cdot|(z-P_\varepsilon)-y|\leqslant C\varepsilon^2|y|^2. \tag{5.56}$$

Thus $\|E_1\|_{H^1(\Omega_\varepsilon)}=O(\varepsilon^2)$ by Proposition 2.2 and Lemma 3.6 of [BDS], and (5.56) follows from (A.11) of [BDS]. Also $\|E_3\|_{H^1(\Omega_\varepsilon)}=O(\varepsilon^2)$ from (5.5) and (5.10). Hence from (5.46) and (5.54), we obtain

$$\begin{aligned}
&\varepsilon^{-1}\sum_{i=1}^{n-1}a_{i,\varepsilon}\int_{\Omega_\varepsilon}[f'(u_\varepsilon)-f'(w_\varepsilon)]\phi_\varepsilon\frac{\partial w_\varepsilon}{\partial z_i}\,dz \\
&\quad=\sum_{i,j=1}^{n-1}a_{i,\varepsilon}a_{j,\varepsilon}\int_{\mathbf{R}^n_+}f''(w(y))[\psi_{0,Q_\varepsilon}(y)-v_{1,Q_\varepsilon}(y)]\frac{\partial w(y)}{\partial y_i}\frac{\partial w(y)}{\partial y_j}\,dy \\
&\qquad+O(\varepsilon).
\end{aligned}\tag{5.57}$$

Each integral term in (5.53) and (5.57) is zero if $i \neq j$ since the integrand is an odd function on $\mathbf{R}^n_+$ when $i \neq j$. Therefore, summarizing (5.53) and (5.57), we obtain that

$$\bar{\lambda} = -\left[\sum_{i=1}^{n-1} a_{i,\varepsilon}^2 \int_{\mathbf{R}^n_+} \left(\frac{\partial w(y)}{\partial y_i}\right)^2 dy\right]^{-1} \sum_{i=1}^{n-1} a_{i,\varepsilon}^2 I_i + O(\varepsilon),$$

where

$$\begin{aligned} I_i \equiv & -\int_{\partial \mathbf{R}^n_+} \left[\frac{|y|w''(|y|) - w'(|y|)}{2|y|^3} \sum_{p=1}^{n-1} \rho_{pp}(0) y_p^2 y_i + \frac{w'(|y|)}{|y|} \rho_{ii}(0) y_i\right] \\ & \times \frac{w'(|y|)}{|y|} y_i \, dy + \int_{\mathbf{R}^n_+} f''(w)(\psi_{0,Q_\varepsilon} - v_{1,Q_\varepsilon}) \left(\frac{\partial w}{\partial y_i}\right)^2 dy, \end{aligned} \tag{5.58}$$

and where $\rho_{ii}(0)$ are the coefficients of the Taylor expansion at Q_ε. We prove $\bar{\lambda} = 0$ by showing $I_i = 0$ for $1 \leqslant i \leqslant n-1$. For simplicity, we drop the subscript Q_ε in $\psi_{0,Q_\varepsilon} - v_{1,Q_\varepsilon}$. By (5.7) and (5.11), $\psi_0 - v_1$ satisfies

$$\begin{cases} \Delta(\psi_0 - v_1) - a(\psi_0 - v_1) + f'(w(y))(\psi_0 - v_1) = 0, & y \in \mathbf{R}^n_+, \\ \dfrac{\partial(\psi_0 - v_1)}{\partial y_n} = \dfrac{w'(|y|)}{2|y|} \sum_{p=1}^{n-1} \rho_{pp}(0) y_p^2, & y \in \partial \mathbf{R}^n_+, \end{cases} \tag{5.59}$$

and on the other hand, we have

$$\Delta \frac{\partial^2 w}{\partial y_i^2} - a \frac{\partial^2 w}{\partial y_i^2} + f'(w) \frac{\partial^2 w}{\partial y_i^2} + f''(w)\left(\frac{\partial w}{\partial y_i}\right)^2 = 0, \tag{5.60}$$

therefore we have $(1 \leqslant i \leqslant n-1)$

$$\begin{aligned} & \int_{\mathbf{R}^n_+} f''(w)(\psi_0 - v_1)\left(\frac{\partial w}{\partial y_i}\right)^2 dy \\ & = \int_{\mathbf{R}^n_+} \left[(-\Delta + a - f'(w)) \frac{\partial^2 w}{\partial y_i^2}\right](\psi_0 - v_1)\, dy \\ & = \int_{\mathbf{R}^n_+} \left[(-\Delta + a - f'(w))(\psi_0 - v_1) \frac{\partial^2 w}{\partial y_i^2}\right] dy \\ & \quad - \int_{\partial \mathbf{R}^n_+} \frac{\partial^2 w}{\partial y_i^2} \frac{\partial(\psi_0 - v_1)}{\partial y_n} ds + \int_{\partial \mathbf{R}^n_+} (\psi_0 - v_1) \frac{\partial^3 w}{\partial y_i^2 \partial y_n} ds \\ & = -\int_{\partial \mathbf{R}^n_+} \frac{\partial^2 w}{\partial y_i^2} \frac{\partial(\psi_0 - v_1)}{\partial y_n} ds \end{aligned}$$

$$= -\int_{\partial\mathbf{R}^n_+}\left[\frac{w''(|y|)|y| - w'(|y|)}{|y|^3}y_i^2 + \frac{w'(|y|)}{|y|}\right]\left[\frac{1}{2}\sum_{m=1}^{n-1}\rho_{mm}(0)\frac{w'(|y|)}{|y|}y_m^2\right]ds$$

$$= -\frac{1}{2}\sum_{m=1}^{n-1}\rho_{mm}(0)\int_{\partial\mathbf{R}^n_+}\frac{|y|w''(|y|) - w'(|y|)}{|y|^4}w'(|y|)y_m^2y_i^2\,ds$$

$$-\frac{1}{2}\sum_{m=1}^{n-1}\rho_{mm}(0)\int_{\partial\mathbf{R}^n_+}\left[\frac{w'(|y|)}{|y|}\right]^2 y_m^2\,ds. \tag{5.61}$$

Hence, we have

$$-I_i = \sum_{m=1}^{n-1}\rho_{mm}(0)\int_{\partial\mathbf{R}^n_+}\frac{|y|w''(|y|) - w'(|y|)}{|y|^4}w'(|y|)y_m^2y_i^2\,ds$$

$$+\ \rho_{ii}(0)\int_{\partial\mathbf{R}^n_+}\left[\frac{w'(|y|)}{|y|}\right]^2 y_i^2\,ds + \frac{1}{2}\sum_{m=1}^{n-1}\rho_{mm}(0)\int_{\partial\mathbf{R}^n_+}\left[\frac{w'(|y|)}{|y|}\right]^2 y_m^2\,ds$$

$$= \rho_{ii}(0)\int_{\partial\mathbf{R}^n_+}\frac{|y|w''(|y|) - w'(|y|)}{|y|^4}w'(|y|)y_i^4\,ds$$

$$+\ \sum_{m\neq i}\rho_{mm}(0)\int_{\partial\mathbf{R}^n_+}\frac{|y|w''(|y|) - w'(|y|)}{|y|^4}w'(|y|)y_m^2y_i^2\,ds$$

$$+\ \rho_{ii}(0)\int_{\partial\mathbf{R}^n_+}\left[\frac{w'(|y|)}{|y|}\right]^2 y_i^2\,ds + \frac{1}{2}\sum_{m=1}^{n-1}\rho_{mm}(0)\int_{\partial\mathbf{R}^n_+}\left[\frac{w'(|y|)}{|y|}\right]^2 y_m^2\,ds$$

$$= \frac{1}{2}[(n-1)H(P_i) + 2\rho_{ii}(0)]\left(\int_{\partial\mathbf{R}^n_+}\left[\frac{w'(|y|)}{|y|}\right]^2 y_i^2\,ds\right.$$

$$\left.+\,2\int_{\partial\mathbf{R}^n_+}\frac{|y|w''(|y|) - w'(|y|)}{|y|^4}w'(|y|)y_m^2y_i^2\,ds\right). \tag{5.62}$$

Note, for any radially symmetric function $g(y) = g(|y|) : \partial\mathbf{R}^n_+ \to \mathbf{R}$, we have

$$\int_{\partial\mathbf{R}^n_+} g(|y|)y_i^4\,ds = 3\int_{\partial\mathbf{R}^n_+} g(|y|)y_i^2y_j^2\,ds \tag{5.63}$$

for $1 \leqslant j, i \leqslant n-1$ and $j \neq i$. We use (5.63) in the last step of (5.62). Finally, we claim that

$$\int_{\partial\mathbf{R}^n_+}\left[\frac{w'(|y|)}{|y|}\right]^2 y_i^2\,ds + 2\int_{\partial\mathbf{R}^n_+}\frac{|y|w''(|y|) - w'(|y|)}{|y|^4}w'(|y|)y_i^2y_j^2\,ds = 0. \tag{5.64}$$

The proofs of (5.63) and (5.64) are given separately below. Therefore we obtain $\bar{\lambda} = 0$, which implies $\lambda_\varepsilon = o(\varepsilon)$. On the other hand, from the proof above and the proof of Proposition 2.2 of [BDS], we can see that t_ε

has the form

$$t_\varepsilon(z) = \chi_{Q_\varepsilon}(\varepsilon z)\sum_{i=1}^{n-1} a_{i,\varepsilon} t^i_{1,Q_\varepsilon}(y) + \varepsilon e_5(z),$$

where $t^i_{1,Q}$ is the unique solution of

$$\begin{cases} \Delta v - av + f'(w)v + f''(w)(\psi_{0,Q} - v_{1,Q})\dfrac{\partial w}{\partial y_i} = 0, & y \in \mathbf{R}^n_+, \\ \dfrac{\partial v}{\partial y_n} = -\dfrac{|y|w''(|y|) - w'(|y|)}{2|y|^3}\sum_{m=1}^{n-1} \rho_{mm}(0)y_m^2 y_i & \\ \qquad -\dfrac{w'(|y|)}{|y|}\rho_{ii}(0)y_i, & y \in \partial\mathbf{R}^n_+ \end{cases} \tag{5.65}$$

and $\|e_5\|_{H^1(\Omega_\varepsilon)} \leqslant C$. From $I_i = 0$ and the boundedness of e_5, using the similar estimates as (5.49), (5.51) and (5.52), the right-hand side of (5.48) is of order $O(\varepsilon)$, hence we obtain $\bar{\lambda}_\varepsilon = O(\varepsilon)$ and $\lambda_\varepsilon = O(\varepsilon^2)$.

Finally, we indicate how to modify the proof above for the case when f satisfies (B1d) instead of (B1c). For simplicity, we consider the special case of $f(u) = u^p$ with $1 < p < 3$. The proof for the more general case of (B1d) can easily be adapted from this special case. In the proof above, we only need to modify the proofs of (5.52) and (5.57). For (5.52), we recall from Theorem 2 in [GNN] that

$$|w(r)|, |w'(r)| = Cr^{-(n-1)/2}e^{-\sqrt{a}r}(1 + O(r^{-1})) \quad \text{as } r \to \infty. \tag{5.66}$$

Thus, the integral in (5.52) satisfies

$$\begin{aligned} &\left|\varepsilon^{-1} p \int_{\Omega_\varepsilon} [u_\varepsilon^{p-1} - w_\varepsilon^{p-1}]\phi_\varepsilon \frac{\partial w_\varepsilon}{\partial z_i}\,dz\right| \\ &\quad \leqslant \varepsilon^{-1} p(p-1)\int_{\Omega_\varepsilon}\left|(u_\varepsilon - w_\varepsilon)(u_\varepsilon^{p-2} + w_\varepsilon^{p-2})\frac{\partial w_\varepsilon}{\partial z_i}\phi_\varepsilon\right| dz \\ &\quad \leqslant C\|\varepsilon^{-1}(u_\varepsilon - w_\varepsilon)\|_{L^2(\Omega_\varepsilon)}\left\|(u_\varepsilon^{p-2} + w_\varepsilon^{p-2})\frac{\partial w_\varepsilon}{\partial z_i}\phi_\varepsilon\right\|_{L^2(\Omega_\varepsilon)} = O(1). \end{aligned} \tag{5.67}$$

The last integral is of order $O(1)$ because

$$\int_{\Omega_\varepsilon}\left[w_\varepsilon^{p-2}\frac{\partial w_\varepsilon}{\partial z_i}\phi_\varepsilon\right]^2 dz = \sum_{j=1}^{n-1} a_{j,\varepsilon}\int_{\mathbf{R}^n_+} w^{2p-4}\left(\frac{\partial w}{\partial y_i}\right)^2\left(\frac{\partial w}{\partial y_j}\right)^2 dy + o(1),$$

and the integrals on the right are convergent by (5.66). The integral $\int_{\Omega_\varepsilon}[u_\varepsilon^{p-2}(\partial w_\varepsilon/\partial z_i)\phi_\varepsilon]^2\,dz$ is also of order $O(1)$ for the same reason. The estimates of the integral in (5.52) and (5.57) are similar. By Taylor's expansion, we have

$$u_\varepsilon^{p-1} - w_\varepsilon^{p-1} = (p-1)w_\varepsilon^{p-2}(u_\varepsilon - w_\varepsilon) + O(w_\varepsilon^{p-3}(u_\varepsilon - w_\varepsilon)^2).$$

Note that $u_\varepsilon - w_\varepsilon = \varepsilon(\psi_{0,Q} - v_{1,Q}) + \varepsilon^2(e^3 - e^1)$, where e_1 and e_3 are defined in (5.5) and (5.10). From (5.7) and (5.11), $\psi_{0,Q} - v_{1,Q}$ is a solution of $L_0 v = 0$, and from Theorem 5.4, the solution space of $L_0 v = 0$ is spanned by $\{\partial w/\partial y_i\colon 1 \leqslant i \leqslant n\}$. Thus, $\psi_{0,Q} - v_{1,Q}$ satisfies the decaying property in (5.66), which implies

$$\varepsilon^{-1} \int_{\Omega_\varepsilon} \left| w_\varepsilon^{p-3}(u_\varepsilon - w_\varepsilon)^2 \phi_\varepsilon \frac{\partial w_\varepsilon}{\partial z_i} \right| dz = O(\varepsilon). \tag{5.68}$$

Similarly,

$$\int_{\Omega_\varepsilon} w_\varepsilon^{p-2}(u_\varepsilon - w_\varepsilon) \frac{\partial w_\varepsilon}{\partial z_i} \phi_\varepsilon \, dz = \sum_{j=1}^{n-1} a_{j,\varepsilon} \int_{\mathbf{R}^n_+} w^{p-2}(\psi_{0,Q} - v_{1,Q}) \frac{\partial w}{\partial y_i} \frac{\partial w}{\partial y_j} dy + O(\varepsilon).$$

Therefore (5.57) can also be obtained. $\square$

Proof of (5.63). Without loss of generality, we assume that $y_1 = y_i$ and $y_2 = y_j$. We introduce polar coordinates for $\mathbf{R}^{n-1}$:

$$\begin{cases} y_1 = r \sin\theta_{n-2} \sin\theta_{n-3} \sin\theta_2 \sin\theta_1, \\ y_2 = r \sin\theta_{n-2} \sin\theta_{n-3} \sin\theta_2 \cos\theta_1, \\ y_3 = r \sin\theta_{n-2} \sin\theta_{n-3} \cos\theta_2, \\ \cdots, \\ y_{n-1} = r\cos\theta_{n-2}. \end{cases} \tag{5.69}$$

Note that $\mathbf{R}^{n-1} = \{(r, \theta_1, \ldots, \theta_{n-2})\colon r > 0,\ 0 \leqslant \theta_1 < 2\pi,\ 0 \leqslant \theta_k \leqslant \pi \text{ for } k = 2, \ldots, n-2\}$ and that

$$dy = r^{n-2} \sin\theta_2 \sin^2\theta_3 \cdots \sin^{n-3}\theta_{n-2}\, dr\, d\theta_1 \cdots d\theta_{n-2}.$$

Therefore, for an exponentially decaying smooth function g,

$$\int_{\mathbf{R}^{n-1}} g(|y|) y_1^4 \, dy = A_1 \int_0^\infty [g'(r)]^2 r^{n+2}\, dr \int_0^{2\pi} \sin^4\theta_1 \, d\theta_1$$

and

$$\int_{\mathbf{R}^{n-1}} g(|y|) y_1^2 y_2^2 \, dy = A_1 \int_0^\infty [g'(r)]^2 r^{n+2}\, dr \int_0^{2\pi} \sin^2\theta_1 \cos^2\theta_1 \, d\theta_1,$$

where

$$A_1 = I_2 \cdots I_{n-2},$$

and $I_k = \int_0^\pi \sin^{k+3}\theta\, d\theta$ for $k = 2, \ldots, n-2$. Then (5.63) easily follows from

$$\int_0^{2\pi} \sin^4\theta\, d\theta = \tfrac{3}{4}\pi, \quad \int_0^{2\pi} \sin^2\theta \cos^2\theta \, d\theta = \tfrac{1}{4}\pi. \qquad \square$$

Proof of (5.64). Without loss of generality, we assume that $y_1 = y_i$ and $y_2 = y_j$. We use the notation in the last proof.

$$\int_{\partial \mathbf{R}^n_+} \left[\frac{w'(|y|)}{|y|}\right]^2 y_1^2\, ds = A_2 \int_0^\infty [w'(r)]^2 r^{n-2}\, dr \int_0^{2\pi} \sin^2\theta_1\, d\theta_1$$
$$= A_2 \pi \int_0^\infty [w'(r)]^2 r^{n-2}\, dr, \tag{5.70}$$

where $A_2 = J_2 \cdots J_{n-2}$, and $J_k = \int_0^\pi \sin^{k+1}\theta\, d\theta$ for $k = 2, \dots, n-2$. Also,

$$2\int_{\partial \mathbf{R}^n_+} \frac{|y|w''(|y|) - w'(|y|)}{|y|^4} w'(|y|) y_2^2 y_1^2\, ds$$
$$= \frac{2}{n+1} \int_{\partial \mathbf{R}^n_+} \frac{|y|w''(|y|) - w'(|y|)}{|y|^4} w'(|y|) y_2^2 |y|^2\, ds$$
$$= \frac{2A_2}{n+1} \int_0^\infty \{w''(r)w'(r)r^{n-1} - [w'(r)]^2 r^{n-2}\}\, dr \int_0^{2\pi} \sin^2\theta_1\, d\theta_1$$
$$= \frac{2A_2\pi}{n+1} \int_0^\infty \{w''(r)w'(r)r^{n-1} - [w'(r)]^2 r^{n-2}\}\, dr.$$

On the other hand, by integrating by parts, we have

$$\int_0^\infty w''(r)w'(r)r^{n-1}\, dr = -\frac{n-1}{2}\int_0^\infty [w'(r)]^2 r^{n-2}\, dr.$$

Thus,

$$2\int_{\partial \mathbf{R}^n_+} \frac{|y|w''(|y|) - w'(|y|)}{|y|^4} w'(|y|) y_2^2 y_1^2\, ds$$
$$= \frac{2A_2\pi}{n+1}\left[-\frac{n-1}{2} - 1\right] \int_0^\infty [w'(r)]^2 r^{n-2}\, dr$$
$$= -A_2\pi \int_0^\infty [w'(r)]^2 r^{n-2}\, dr.$$

Therefore for $1 \leqslant j, l \leqslant n-1$ and $j \neq l$,

$$\int_{\partial \mathbf{R}^n_+} \left[\frac{w'(|y|)}{|y|}\right]^2 y_l^2\, ds + 2\int_{\partial \mathbf{R}^n_+} \frac{|y|w''(|y|) - w'(|y|)}{|y|^4} w'(|y|) y_j^2 y_l^2 = 0. \qquad \square$$

5.5. *Precise estimates of eigenvalues*

Proof of Lemma 4.4. We first assume f satisfies (B1c). Recall that

$$K_{i,j,i,m} = \partial_j^i G_{im}(\mathbf{P}), \tag{5.71}$$

where G_{im} is defined as

$$G_{im}(\mathbf{P}) = \langle F(\varepsilon, W_{\varepsilon,\mathbf{P}} + \psi(W_{\varepsilon,\mathbf{P}})), \partial_m^i W_{\varepsilon,P_i} \rangle_{L^2(\Omega_\varepsilon)}$$
$$= \int_{\Omega_\varepsilon} [\Delta u_{\varepsilon,\mathbf{P}} - a u_{\varepsilon,\mathbf{P}} + f(u_{\varepsilon,\mathbf{P}})] \partial_m^i W_{\varepsilon,P_i}\, dz, \tag{5.72}$$

where $u_{\varepsilon,\mathbf{P}} = W_{\varepsilon,\mathbf{P}} + \psi(W_{\varepsilon,\mathbf{P}})$. In Proposition 4.1 of [BDS], it is proved that

$$G_{im}(\mathbf{P}) = \varepsilon^2 \gamma \partial^i_m H(\varepsilon P_i) + o(\varepsilon^2), \tag{5.73}$$

so we only need to show that the error term $o(\varepsilon^2)$ is still $o(\varepsilon^2)$ when ∂^i_j is applied to it. In fact, we only need to show that

$$\partial^i_j G_{im}(\mathbf{P}) = \varepsilon^2 g(\mathbf{P}) + o(\varepsilon^2) \tag{5.74}$$

for some continuous function $g : \widetilde{M_{\varepsilon,k}} \to \mathbf{R}$, since in light of (5.73), $g(\mathbf{P})$ must be identical to $\gamma\partial^i_j\partial^i_m H(\varepsilon P_i)$. For simplicity, we only consider the case of $k = 1$, and the general case is similar. In this case, we have $\mathbf{P} = P$.

From the properties of the functions in the integral, we have

$$\begin{aligned}
\partial^i_j G_{im}(P) &= \int_{\Omega_\varepsilon} [\Delta\partial^i_j u_{\varepsilon,P} - a\partial^i_j u_{\varepsilon,P} + f'(u_{\varepsilon,P}\partial^i_j)]\partial^i_m W_{\varepsilon,P}\, dz, \\
&= -\int_{\Omega_\varepsilon} [\Delta\partial^i_m W_{\varepsilon,P} - a\partial^i_m W_{\varepsilon,P} + f'(w_{\varepsilon,P})\partial^i_m w_{\varepsilon,P}]\partial^i_j u_{\varepsilon,P}\, dz \\
&\quad + \int_{\Omega_\varepsilon} \partial^i_j u_{\varepsilon,P}[f'(u_{\varepsilon,P}\partial^i_j)\partial^i_m W_{\varepsilon,P} - f'(w_{\varepsilon,P})\partial^i_m w_{\varepsilon,P}]\, dz \\
&= \int_{\Omega_\varepsilon} \partial^i_j u_{\varepsilon,P}[f'(u_{\varepsilon,P}) - f'(w_{\varepsilon,P})]\partial^i_m W_{\varepsilon,P}\, dz \\
&\quad + \int_{\Omega_\varepsilon} \partial^i_j u_{\varepsilon,P} f'(w_{\varepsilon,P})(\partial^i_m W_{\varepsilon,P} - \partial^i_m w_{\varepsilon,P})\, dz.
\end{aligned} \tag{5.75}$$

We shall prove that the two integrals in the last two lines of (5.75) are both of the form of $\varepsilon g_1(P) + \varepsilon^2 g_2(P) + o(\varepsilon^2)$. From now on, we use $g_1(P)$ and $g_2(P)$ to denote various continuous functions.

We notice that all integrands are exponentially decaying at infinity. We follow the proof of Proposition 4.1 in [BDS] to estimate these integrals. The following approximation is implicitly used in all integral estimates in the proof of Proposition 4.1 in [BDS]. For each $P \in \Omega_\varepsilon$, we define $\Omega_{\varepsilon,P} = \varepsilon^{-1}\Omega_{\varepsilon P}$, where $\Omega_{\varepsilon P}$ is one of the subdomains of Ω in Section 5.1. Then $\Omega_{\varepsilon,P}$ is the subdomain of Ω_ε containing P, and for any $Q \in \Omega_\varepsilon \backslash \Omega_{\varepsilon,P}$, $|Q - P| \geqslant C\varepsilon^{-1}$. In fact, under the coordinate change $z = \varepsilon^{-1}\Phi(\varepsilon y)$, the region $\{y \in \mathbf{R}^+_n \cap B(\varepsilon^{-1}\delta_2)\}$ is mapped to $\{z \in \Omega_{\varepsilon,P}\}$. Thus for an integrable function $g:\ \mathbf{R}^n \to \mathbf{R}$ which is exponentially decaying when $|y| \to \infty$, we have

$$\begin{aligned}
\int_{\Omega_\varepsilon} g(z)\, dz &= \int_{\Omega_{\varepsilon,P}} g(z)\, dz + \int_{\Omega_\varepsilon\backslash\Omega_{\varepsilon,P}} g(z)\, dz \\
&= \int_{\mathbf{R}^+_n \cap B(\varepsilon^{-1}\delta_2)} g(\varepsilon^{-1}\Phi(\varepsilon y))\, dy \\
&\quad + \int_{\Omega_\varepsilon\backslash\Omega_{\varepsilon,P}} g(z)\, dz
\end{aligned}$$

$$= \int_{\mathbf{R}_n^+} g(\varepsilon^{-1}\Phi(\varepsilon y))\, dy + \int_{\mathbf{R}_n^+\setminus B(\varepsilon^{-1}\delta_2)} g(\varepsilon^{-1}\Phi(\varepsilon y))\, dy$$
$$+ \int_{\Omega_\varepsilon\setminus\Omega_{\varepsilon,P}} g(z)\, dz$$
$$= \int_{\mathbf{R}_n^+} g(\varepsilon^{-1}\Phi(\varepsilon y))\, dy + O(e^{-c/\varepsilon}). \tag{5.76}$$

Moreover, if g is differentiable, then from [BDS, p. 61. (A.18)], we have

$$g(z) = g(y) - \frac{\varepsilon}{2|y|}\frac{\partial g(y)}{\partial y_n}\sum_{i=1}^{n-1} \rho_{ii}(0)y_i^2 y_n + o(\varepsilon), \tag{5.77}$$

and thus

$$\int_{\mathbf{R}_n^+} g(\varepsilon^{-1}\Phi(\varepsilon y))\, dy = \int_{\mathbf{R}_n^+} g(y)\, dy - \frac{\varepsilon(n-1)H(P)}{2}\int_{\mathbf{R}_n^+} \frac{y_i^2 y_n}{|y|}\frac{\partial g(y)}{\partial y_n}\, dy + o(\varepsilon). \tag{5.78}$$

In that way, the integrals in (5.75) can be estimated if each integrand has an expansion in form

$$g(z) = \varepsilon^m \chi_Q(\varepsilon z) g_m(y) + \varepsilon^{m+1}\chi_Q(\varepsilon z) g_{m+1}(y) + o(\varepsilon^{m+1}), \tag{5.79}$$

where $m \in \mathbf{Z}$, $m \geqslant 0$, χ_Q is the cut-off function at $Q = \varepsilon P$, and g_m, $g_{m+1} \in L^1(\mathbf{R}_n^+)$. To conclude our proof, we show such expansions exist for all integrands. First from (5.6) and (5.12), we have

$$\partial_m^i W_{\varepsilon,P} = \frac{\partial w(y)}{\partial y_m} - \varepsilon\chi_Q(\varepsilon z)\Bigg[\rho_{mm}(0)y_p\frac{\partial w(y)}{\partial y_n} + \frac{|y|w''(|y|) - w'(|y|)}{2|y|^3}\sum_{p=1}^{n-1} \rho_{pp}(0)y_p^2 y_m y_n + u_{0,Q}(y)\Bigg] + o(\varepsilon). \tag{5.80}$$

Since $\partial_m^i u_{\varepsilon,P} = \partial_m^i W_{\varepsilon,P} + \partial_m^i \psi_{\varepsilon,P}$, the expansion of $\partial_m^i u_{\varepsilon,P}$ can be obtained by combining (5.80) and

$$\partial_m^i \psi_{\varepsilon,P} = \varepsilon\chi_Q(\varepsilon z)\frac{\partial \psi_{0,Q}(y)}{\partial y_m} + o(\varepsilon), \tag{5.81}$$

where $\psi_{0,Q}$ is defined in (5.11). The proof of (5.81) is similar to the proof of (5.10) (see [BDS, Lemma 3.6]) so we omit it here. Similarly, we have

$$\partial_m^i W_{\varepsilon,P} - \partial_m^i w_{\varepsilon,P} = \varepsilon\chi_Q(\varepsilon z)u_{0,Q}(y) + \varepsilon^2\chi_Q(\varepsilon z)u_{1,Q}(y) + o(\varepsilon^2). \tag{5.82}$$

Here $u_{1,Q}$ can be obtained in a similar way (but more calculation) as $u_{0,Q}$ in Proposition 2.2 in [BDS] since f has more smoothness (in (B1c)); so the expansion of more terms is possible. Note that here we do not need explicit forms of these functions in the expansion, so the calculation of these terms is not necessary. From [BDS, p. 30, (4.2)], we have

$$f'(w_{\varepsilon,P}) = f'(w(y)) - \varepsilon f''(w(y))\frac{w'(|y|)}{|y|}\sum_{p=1}^{n-1}\rho_{pp}(0)y_p^2 y_n + O(\varepsilon^2). \tag{5.83}$$

Finally, the expansion of $f'(u_{\varepsilon,P}) - f'(w_{\varepsilon,P})$ is

$$\begin{aligned} f'(u_{\varepsilon,P}) - f'(w_{\varepsilon,P}) &= f''(w_{\varepsilon,P})(u_{\varepsilon,P} - w_{\varepsilon,P}) + \tfrac{1}{2}f'''(w_{\varepsilon,P}) \\ &\quad \times (u_{\varepsilon,P} - w_{\varepsilon,P})^2 + O(|u_{\varepsilon,P} - w_{\varepsilon,P}|^{2+\alpha}) \\ &= \chi_Q(\varepsilon z)\left[f''(w) - \varepsilon f'''(w)\frac{w'(|y|)}{|y|}\sum_{p=1}^{n-1}\rho_{pp}(0)y_p^2 y_n\right] \\ &\quad \times [\varepsilon\psi_{0,Q} - \varepsilon v_{1,Q} + \varepsilon^2\psi_{1,Q} - \varepsilon^2 v_{2,Q}] \\ &\quad + \frac{\varepsilon^2}{2}f'''(w)(\psi_{0,Q} - v_{1,Q})^2 + O(\varepsilon^{2+\alpha}), \end{aligned} \tag{5.84}$$

where $\psi_{0,Q}$, $v_{1,Q}$ and $v_{2,Q}$ are defined in (5.10) and (5.5), and $\psi_{1,Q}$ is the second term in the expansion of $\psi_{\varepsilon,P}$, which can be obtained in a similar way to $\psi_{1,Q}$. Again, since we do not need the explicit form of $\psi_{1,Q}$, we omit the proof for the expansion containing $\psi_{1,Q}$. Combining (5.80)–(5.84), we conclude that integrals in (5.75) both have the form $\varepsilon g_1(P) + \varepsilon^2 g_2(P) + o(\varepsilon^2)$. This completes the proof of Lemma 4.4 when (B1c) holds. The proof when (B1d) holds is similar. In fact, by (5.83), (5.84) and estimate (5.66), one can still show that integrals in (5.75) are both of the form $\varepsilon g_1(P) + \varepsilon^2 g_2(P) + o(\varepsilon^2)$. □

Proof of Lemma 4.5. For (4.10), we only need to show that $K_{i,j,l,m} = O(e^{-c/\varepsilon})$ if $i \neq l$. But that is obvious from the facts that $|P_i - P_j| \geqslant \eta\varepsilon^{-1}$ and $\partial_j^i W_{\varepsilon,\mathbf{P}}$ (also its derivatives) are exponentially decaying. To prove the result about the eigenvalues of K, we change the basis $\{\partial_j^i W_{\varepsilon,\mathbf{P}}\colon\ 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant n-1\}$ of $X_\varepsilon^c(W_{\varepsilon,\mathbf{P}})$ to the basis generated by the eigenfunctions of K_i. Under the new basis, the matrix representation $\tilde{K}_{i,j,l,m}$ still satisfies $\tilde{K}_{i,j,l,m} = O(e^{-c/\varepsilon})$ if $i \neq l$. Hence, we obtain $\mu_{i,j} = \varepsilon^2\eta_{i,j} + o(\varepsilon^2)$ from Lemma 2.5. □

5.6. Nondegeneracy of the ground state solution

In this subsection, we verify assumption (B3) in Section 3 for two classes of nonlinear functions $g(u) = -au + f(u)$. The prototypes of the two classes of nonlinearities are $g(u) = -au + |u|^{p-1}u,\ 1 < p < (n+2)/(n-2)$ and $g(u) = -du(u-b)(u-c)$ with $d > 0$ and $c > 2b > 0$. The proofs of the various parts of the results have appeared in Kato [Ka], Berezin and Shubin [BS], Davies [Da], Kwong and Zhang [KZ], Ni and Takagi [NT3], Dancer [D2] and Ouyang and Shi [OS]. But since a complete proof of (B3) is hard to extract from these references, we sketch a proof here with some details left to the references above.

For $g \in C^1([0,\infty))$, we define $K_g(u) = ug'(u)/g(u)$ for $u \geqslant 0$ as long as $g(u) \neq 0$. We assume that g satisfies

(g1) there exists $b > 0$ such that $g(0) = g(b) = 0$, $g(u) < 0$ for $u \in (0,b)$, $g'(0) < 0$ and $g'(b) > 0$;

(g2) there exists $\theta > b$ such that $g(u) > 0$ in $(b,\theta]$, and $\int_0^\theta g(u)\,du = 0$.

We say that the function g is of class (A) if g satisfies (g1), (g2) and

(g3A) $g(u) > 0$ for all $u > b$;

(g4A) $K_g(u)$ is nonincreasing in $[\theta,\infty)$ and converges to $K_\infty \in [1,(n+2)/(n-2))$ as $u \to \infty$;

(g5A) $K_g(u) \geqslant K_g(\theta)$ for $u \in (b,\theta]$, and $K_g(u) \leqslant K_\infty$ for $u \in (0,b)$.

Similarly, we say that the function g is of class (B) if g satisfies (g1), (g2) and

(g3B) $g(u) > 0$ for all $u \in (b,c)$ for some $c > \theta$, and $g(u) < 0$ for $u > c$.

(g4B) Assume that there exists $\rho \in [b,c)$ such that $(u-\rho)g'(u) \leqslant g(u)$ in (ρ,c). If $\rho > \theta$, then (i) $K_g(u) \geqslant K_g(\theta)$ for $u \in (b,\theta]$; (ii) $K_g(u)$ is nonincreasing in $[\theta,\rho]$; (iii) $K_g(u) \leqslant K_g(\rho)$ for $u \in [\rho,c)$.

It is not hard to check that $g(u) = -au + |u|^{p-1}u$, $1 < p < (n+2)/(n-2)$ is of class (A) and $g(u) = -du(u-b)(u-c)$ with $d > 0$ and $c > 2b > 0$ is of class (B). Class (A) was first introduced by Kwong and Zhang [KZ], and class (B) was first introduced by Dancer [D2] (see also [OS]).

The ground state solution is a solution of

$$\begin{cases} \Delta u + g(u) = 0 & \text{in } \mathbf{R}^n, \\ u > 0 & \text{in } \mathbf{R}^n, \quad u \to 0, \quad |x| \to \infty, \\ \max u(x) = u(0). \end{cases} \tag{5.85}$$

The following are some well-known results about the ground state solutions:

Theorem 5.3. *Assume that g satisfies* (g1), (g2), *and either* (g3B) *or* (g3A) *and*

$$\lim_{u\to\infty} \frac{g(u)}{u^l} = 0 \quad \text{with } 0 \leqslant l < \frac{n+2}{n-2}. \tag{5.86}$$

Then

(1) *Eq.* (5.85) *has a solution* $u \in H^2(\mathbf{R}^n) \cap C^2(\mathbf{R}^n)$.

(2) *There exists* $C, K > 0$ *such that*

$$|D^\alpha u(y)| \leqslant Ce^{-K|y|} \quad \text{for } y \in \mathbf{R}^n \quad |\alpha| \leqslant 2. \tag{5.87}$$

(3) *Any solution u of* (5.85) *is radially symmetric with respect to the origin.*

The proof of the existence and the exponential-decaying properties can be found in Berestycki et al. [BLP, Theorem I.1], and the third part was proved by Gidas et al. [GNN]. We remark that in Theorem 5.3, we only need g to be locally Lipschitz continuous (see [BLP,GNN] for details), but we will need g to be C^1 in the latter part of this subsection.

Let u be a ground state solution. Define $L_0 = \Delta + g'(u) : H^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$. We have the following result about the spectrum set $\sigma(L_0)$:

Theorem 5.4. (1) $\sigma(L_0) = \sigma_p(L_0) \cup \sigma_e(L_0)$, *where* $\sigma_p(L_0)$ *is the point spectrum, and* $\sigma_e(L_0)$ *is the essential spectrum*;

(2) $\sigma_e(L_0) = (-\infty, g'(0)]$, $\sigma_p(L_0) \subset (g'(0), \infty)$.

(3) *If* $\lambda \in \sigma_p(L_0)$, *then the corresponding eigenfunction* ϕ *satisfies*

$$|\phi(y)| \leqslant C_\varepsilon e^{-\sqrt{(-g'(0)+\lambda+\varepsilon)/2}|y|} \quad \text{for } y \in \mathbf{R}^n,$$

for any small $\varepsilon > 0$ *and some* $C_\varepsilon > 0$.

(4) *If* $\lambda \in \sigma_p(L_0) \cap (0, \infty)$, *then the corresponding eigenfunction* ϕ *is radially symmetric.*

(5) *The principal eigenvalue* $\lambda_1(L_0) > 0$ *is simple, and the corresponding eigenfunction* ϕ_1 *can be chosen to be positive.*

(6) $\lambda_2(L_0) = 0$, *and the eigenspace associated with the eigenvalue* $\lambda = 0$ *is spanned by*

$$\left\{ \frac{\partial u}{\partial y_i},\ i = 1, 2, \ldots, n \right\}. \tag{5.88}$$

Proof. We sketch a proof here. Since u is uniformly bounded in $\mathbf{R}^n$, the potential $V(x) = g'(u(x))$ is uniformly bounded, and $\lim_{|x| \to \infty} V(x) = g'(0)$. Thus $L_0 = \Delta + V$ is a self-adjoint operator from the domain $D(L_0) = H^1(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$ (see [BS, Theorem 3.1.1] or [Da, Theorem 8.2.2]). Consequently, the spectrum of L_0 is real and nonempty [Da, Theorem 1.2.10]. It is well known that the spectrum of $\Delta + g'(0)$ is $(-\infty, g'(0)]$ since the symbol of $\Delta + g'(0)$ is $-|x|^2 + g'(0)$, and the closure of $\{-|x|^2 + g'(0) \colon |x| \in \mathbf{R}^n\}$ is $(-\infty, g'(0)]$, which is the spectral set (see [Da, Chapter 3]). Moreover, $(-\infty, g'(0)]$ is all essential spectrum of L_0. Since $g'(u) \to g'(0)$ as $|x| \to \infty$, then the multiplication operator induced by $g'(u) - g'(0)$ is relatively compact with respect to $\Delta + g'(0)$ (see [Ka, Lemma V.5.8]). Hence, the essential spectrum of L_0 and $\Delta + g'(0)$ are the same by Theorem V.5.7 in [Ka], and $\sigma_e(L_0) = (-\infty, g'(0)]$.

The other part of the spectrum of L_0 (if nonempty) consists of eigenvalues, and $\sigma_p(L_0) \subset (g'(0), \infty)$ [Da, Theorem 8.5.1]. In fact, L_0 is bounded from above, so there exist $c \in \mathbf{R}$ such that $\sigma_p(L_0) \subset (g'(0), c)$. Clearly, $\sigma_p(L_0) \neq \emptyset$ since $\partial u / \partial y_i$, $i = 1, 2, \ldots, n$, are eigenfunctions corresponding to the eigenvalue 0. The exponential decay of the eigenfunctions is well known, see Theorem 3.3.2 in [BS]. Let $\lambda_1(L_0)$ be the largest eigenvalue. Then by

Theorem 3.3.4 in [BS], $\lambda_1(L_0)$ is simple and the eigenfunction can be chosen to be positive. Since 0 is not a simple eigenvalue, it follows that $\lambda_1(L_0) > 0$.

We prove (4) following the proof of Lemma 4.2 in [NT3]. Let S^{n-1} be the standard unit sphere in $\mathbf{R}^n$. It is well known that the eigenvalues of the Laplace–Beltrami operator $-\Delta$ on S^{n-1} are $0 = v_0 < v_1 = v_2 = \cdots = v_n = (n-1) < v_{n+1} < \cdots$. We denote the kth eigenfunction by e_k, $k = 0, 1, 2, \ldots$, then $\{e_k\colon = 0, 1, 2, \ldots\}$ is a complete orthogonal basis of $L^2(S^{n-1})$. Let ψ be a eigenfunction of L_0 with eigenvalue $\lambda \geqslant 0$. Define

$$\psi_k(r) = \int_{S^{n-1}} \psi(r,\theta) e_k(\theta)\, d\theta.$$

Then for any $k > 0$, $\psi_k'(0) = 0$, and $\psi_k(r)$ satisfies

$$|D^\alpha \psi_k(r)| \leqslant C e^{-Kr} \quad \text{for } r \in (0,\infty) \tag{5.89}$$

with $|\alpha| \leqslant 2$. By a direct calculation, ψ_k satisfies

$$\psi_k'' + \frac{n-1}{r}\psi_k' + \left[g'(u) - \frac{v_k}{r^2}\right]\psi_k = \lambda \psi_k \tag{5.90}$$

for all $r \in (0,\infty)$. On the other hand, u_r satisfies

$$u_r'' + \frac{n-1}{r}u_r' + \left[g'(u) - \frac{n-1}{r^2}\right]u_r = 0 \tag{5.91}$$

for all $r \in (0,\infty)$ and $u_r(r) < 0$. Multiplying (5.90) by $r^{n-1}u_r$ and multiplying (5.91) by $r^{n-1}\psi_k$, then subtracting and integrating over $(0,\rho)$, we obtain

$$\begin{aligned} &\rho^{n-1}[\psi_k'(\rho)u_r(\rho) - u_r'(\rho)\psi_k(\rho)] + (n-1-v_k)\int_0^\rho r^{n-3} u_r \psi_k\, dr \\ &\quad = \lambda \int_0^\rho r^{n-1} u_r \psi_k\, dr. \end{aligned} \tag{5.92}$$

We claim that for $k > n$, $\psi_k \equiv 0$. Suppose not, then we can assume that $\psi_k > 0$ near 0. If there is a $\rho > 0$ such that $\psi_k(\rho) = 0$ and $\psi_k(r) > 0$ in $(0,\rho)$, then the left-hand side of (5.92) is $\rho^{n-1}[\psi_k'(\rho)u_r(\rho) + (n-1-v_k)\int_0^\rho r^{n-3}u_r\psi_k\, dr > 0$, and the right-hand side is $\lambda \int_0^\rho r^{n-1}u_r\psi_k\, dr \leqslant 0$. Therefore $\psi_k > 0$ for all $r > 0$. Since u_r, u_{rr}, ψ_k and ψ_k' are all exponentially decaying at infinity, we have $\rho^{n-1}[\psi_k'(\rho)u_r(\rho) - u_r'(\rho)\psi_k(\rho)] \to 0$ as $\rho \to \infty$. Let ρ in (5.92) be large enough, then the left-hand side is $(n-1-v_k)\int_0^\rho r^{n-3}u_r\psi_k\, dr + O(r^m e^{-r}) > 0$, while the right-hand side is $\lambda \int_0^\rho r^{n-1} w_r \psi_k\, dr \leqslant 0$. This is again a contradiction. Hence, for $k > n$, $\psi_k \equiv 0$.

Since $\{e_k\colon k = 0, 1, 2, \ldots\}$ is a complete orthogonal basis of $L^2(S^{n-1})$, we have

$$\psi(x) = \psi(r,\theta) = \sum_{k=0}^{n} \psi_k(r) e_k(\theta). \tag{5.93}$$

Moreover, if $\lambda > 0$, then by the above argument $\psi_k > 0$, but the left-hand side of (5.92) approaches zero as $\rho \to \infty$ for $1 \leqslant k \leqslant n$. Thus $\psi_k \equiv 0$ for $1 \leqslant k \leqslant n$, hence $\psi(x) = \psi_0(r)e_0(\theta) = \psi_0(r)$, which implies ψ is radially symmetric if $\lambda > 0$. This completes the proof of (4).

For the proof of (6), we consider the solution of

$$\varphi'' + \frac{n-1}{r}\varphi' + g'(u)\varphi = 0, \quad r \in (0, \infty), \quad \varphi'(0) = 0, \quad \varphi(0) = 1. \tag{5.94}$$

In [OS, p. 141, Proposition 7.1], it is shown that if g is of class (A) or (B), then the solution φ of (5.94) changes sign exactly once in $(0, \infty)$, and $\lim_{r \to \infty} \varphi(r) = K < 0$. The proof of this fact for the case of class (B) may be found in [OS, pp. 141–143] (see also [D2]), and the proof for class (A) can be found in [KZ]. Let ψ be a eigenfunction of L_0 with eigenvalue $\lambda = 0$. Then (5.93) holds, and ψ_0 satisfies

$$\begin{aligned} &\psi_0'' + \frac{n-1}{r}\psi_0' + g'(u)\psi_0 = 0, \quad r \in (0, \infty), \quad \psi_0'(0) = 0, \\ &\lim_{r \to \infty} \psi_0(r) = 0. \end{aligned} \tag{5.95}$$

So $\psi_0 = c\varphi$. But $\lim_{r \to \infty} \psi_0(r) = 0 = cK$ implies $c = 0$. Thus $\psi_0 \equiv 0$, and from (5.93), the solution space of $L_0\psi = 0$ is at most n dimensional. Since the solution space contains at least $span\{\partial u/\partial y_i\colon\ 1 \leqslant i \leqslant n\}$, it is precisely this set. Finally we prove $\lambda_2(L_0) = 0$. If $\lambda_2(L_0) > 0$, then from (4), the eigenfunction ϕ_2 is also radially symmetric, so it satisfies

$$\begin{aligned} &\phi_2'' + \frac{n-1}{r}\phi_2' + g'(u)\phi_2 = \lambda_2\phi_2, \quad r \in (0, \infty), \quad \phi_2'(0) = 0, \\ &\lim_{r \to \infty} \phi_2(r) = 0. \end{aligned}$$

By using Sturm's comparison lemma with ϕ_1 and ϕ_2 and using the fact that $\phi_1 > 0$ is exponentially decaying, ϕ_2 has to change sign in $(0, \infty)$. Let r_0 be a point where ϕ_2 changes sign. Then again by using Sturm's comparison lemma with ϕ_2 and φ, and the fact that ϕ_2 is exponentially decaying, we conclude that φ has to change sign once in $(0, r_0)$ and also once in (r_0, ∞). This is a contradiction to the fact that φ changes sign only once in $(0, \infty)$. Thus $\lambda_2(L_0)$ must be 0, and this completes the proof of (6). □

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