Saddle Solutions of the Balanced Bistable Diffusion Equation

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Abstract

We prove that $\Delta u + f(u) = 0$ has a unique entire solution $u(x, y)$ on $\mathbb{R}^2$ that has the same sign as the function xy, where f is a balanced bistable function like $f(u) = u - u^3$. But we neither assume f is odd nor assume the monotonicity properties of $f(u)/u$. Our result generalizes a previous result by Dang, Fife, and Peletier [12]. Our approach combines bifurcation methods and recent results on the qualitative properties for elliptic equations in unbounded domains by Berestycki, Caffarelli, and Nirenberg [5, 6]. © 2002 Wiley Periodicals, Inc.

1 Introduction

Consider a semilinear elliptic equation in the whole plane

$$\Delta u + f(u) = 0\,, \quad \mathbf{x} = (x, y) \in \mathbb{R}^2\,. \tag{1.1}$$

A *saddle solution* u of (1.1) is a solution that has the same sign as the function $g(x, y) = xy$, thus the origin is a saddle point of the function u. Saddle solutions were first studied by Dang, Fife, and Peletier [12]. They proved the following:

THEOREM 1.1 [12, theorem 3] *Suppose that* $f \in C^2([-1, 1])$, $f(-1) = f(0) = f(1) = 0$, $f'(0) > 0$, $f'(\pm 1) < 0$, $f(-u) = -f(u)$, *and that* $f(u)/u$ *is a strictly monotone decreasing function for* $u \in (0, 1)$; *then* (1.1) *has a unique solution* $u(\mathbf{x})$ *satisfying*

$$u(x, -y) = -u(x, y)\,, \quad u(-x, y) = -u(x, y)\,, \quad \textit{for } (x, y) \in \mathbb{R}^2\,, \tag{1.2}$$

and $0 < u(x, y) < 1$ *if* $x > 0$ *and* $y > 0$.

In this paper, we extend the result in [12] to a wider class of functions. We assume that f satisfies

(F1) $f \in C^2([m, M])$ for some $\infty < m < M < \infty$, there exists $\alpha \in (m, M)$ such that $f(m) = f(M) = f(\alpha) = 0$, $f'(\alpha) > 0$, $f'(m) < 0$, $f'(M) < 0$, and $f(u)(u-\alpha) > 0$ for $u \in (m, M)\backslash\{\alpha\}$;

(F2) $$\int_m^M f(u)du = 0\,.$$

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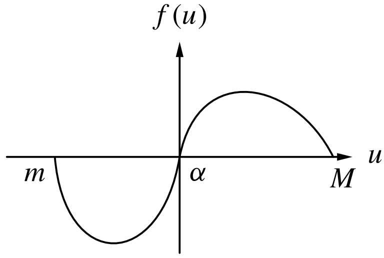

FIGURE 1.1. Balanced bistable $f(u)$.

We call $f(u)$ a *balanced bistable function* if it satisfies (F1) and (F2). Clearly the function in Theorem 1.1 satisfies (F1) and (F2). But we do not assume f to be an odd function, and we do not assume that $f(u)/u$ is monotone on (α, M) or (m, α). Our main result is the following:

THEOREM 1.2 *Suppose that f satisfies* (F1) *and* (F2); *then* (1.1) *has a unique solution $u(\mathbf{x}) \in C^2(\mathbb{R}^2)$ satisfying*

$$u(x, y) = \alpha \qquad \text{if } xy = 0\,, \tag{1.3}$$

$$M > u(x, y) > \alpha \qquad \text{if } xy > 0\,, \tag{1.4}$$

$$\alpha > u(x, y) > m \qquad \text{if } xy < 0\,, \tag{1.5}$$

$$u(y, x) = u(x, y)\,, \quad u(-y, -x) = u(x, y)\,. \tag{1.6}$$

In particular, when $\alpha = 0$, then u has the same sign as xy; thus u is a saddle solution. Note that (1.6) is implicitly contained in Theorem 1.1 because of the uniqueness of the solution. But (1.2) is not true in general if f is not an odd function.

Our approach to the existence of the saddle solutions is different from that in [12]. In [12] a Dirichlet boundary value problem

$$\Delta u + f(u) = 0\,,\ \mathbf{x} \in Q_1 \equiv \mathbb{R}^+ \times \mathbb{R}^+, \quad u = \alpha\,,\ \mathbf{x} \in \partial Q_1\,, \tag{1.7}$$

is considered, and the radially symmetric solutions on finite balls are used as subsolutions in a comparison method. That method relies on the monotonicity of $f(u)/u$ in constructing subsolutions as well as on the oddness of f to extend the solution to the whole plane. In our approach, we also use the solutions on bounded domains as approximations. But instead of the Dirichlet problem, we consider a Neumann boundary value problem

$$\Delta u + \lambda f(u) = 0\,,\ \mathbf{x} \in S \equiv (0, a) \times (0, a)\,, \quad \frac{\partial u}{\partial n} = 0\,,\ \mathbf{x} \in \partial S\,. \tag{1.8}$$

Using the ideas in [23], we prove that for some $\overline{\lambda}$, when $\lambda > \overline{\lambda}$, (1.8) has a solution $u_\lambda(\mathbf{x})$ whose nodal set (the set $\mathcal{N}(u) \equiv \{u = \alpha\}$) is nearly the diagonal line $x + y = a$. If we extend u_λ evenly across the boundary to $(0, 2a) \times (-a, a)$, then the blowup sequence $v_\lambda(\mathbf{z}) = u_\lambda(\lambda^{-1/2}\mathbf{z} - \mathbf{x}_0)$, where $\mathbf{x}_0 = (a, 0)$, has a limit in $C^2(\mathbb{R}^2)$ as $\lambda \to \infty$, which is the desired saddle solution after a rotation of 45°.

The key of the construction is the existence of u_λ. In [23], the author studies the bifurcation problem and asymptotic behavior of solutions to (1.8) when $\lambda \to \infty$ with the domain $R = (0, a) \times (0, b)$. Among other things, we prove the following result, which plays an essential role in the construction of u_λ:

THEOREM 1.3 [23, theorem 6.6] *Suppose that f satisfies* (F1) *and* (F2). *Then there exists $\bar{\lambda} > 0$ such that for $\lambda > \bar{\lambda}$, if (λ, u) is a solution of*

$$\Delta u + \lambda f(u) = 0\,, \ \mathbf{x} \in R \equiv (0, a) \times (0, b)\,, \quad \frac{\partial u}{\partial n} = 0\,, \ \mathbf{x} \in \partial R\,, \tag{1.9}$$

such that $\partial u/\partial x \le 0$ and $\partial u/\partial y \le 0$ for $\mathbf{x} \in \overline{R}$, then u must be one of the following forms:

(i) *a constant solution $u = m, \alpha, M$*;

(ii) *a semitrivial solution $u(x, y) = v(x)$, where v is the unique monotone decreasing solution of*

$$\begin{cases} u'' + \lambda f(u) = 0\,, & x \in (0, a)\,, \\ u'(0) = u'(a) = 0\,; \end{cases} \tag{1.10}$$

(iii) *a semitrivial solution $u(x, y) = v(ay/b)$, where v is the unique monotone decreasing solution of* (1.10); *or*

(iv) *$a = b$, u is a solution such that $\mathcal{N}(u) \cap \partial\mathbb{R} = \{(a, 0), (0, a)\}$ and $\mathcal{N}(u)$ intersects both vertices at* 45°, *where $\mathcal{N}(u) \equiv \{u = \alpha\}$.*

The studies of bounded solutions of (1.1) have recently attracted a lot of attention, and the focus is on the proof of De Giorgi's conjecture. In 1978 De Giorgi [13] made the following conjecture:

CONJECTURE (De Giorgi's Conjecture) Let u be a solution of

$$\Delta u + u - u^3 = 0\,, \quad \mathbf{x} \in \mathbb{R}^n\,, \tag{1.11}$$

such that

$$|u| \le 1\,, \quad \frac{\partial u}{\partial x_n} > 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^n\,. \tag{1.12}$$

Is it true that all level sets $\{u = k\}$ of u are hyperplanes, at least if $n \le 8$?

When $n = 2$, the conjecture was recently proved by Ghoussoub and Gui [15]:

THEOREM 1.4 [15] *Let u be a bounded solution of* (1.1), *where f is any C^1-function. If*

$$\frac{\partial u}{\partial x} > 0\,, \quad \mathbf{x} \in \mathbb{R}^2\,, \tag{1.13}$$

then u is a function of one variable; *that is, there exist $a, b \in \mathbb{R}$ such that $u(x, y) = u(ax + by)$.*

The proof of Theorem 1.4 uses some powerful tools developed by Berestycki, Caffarelli, and Nirenberg in their work on the qualitative properties of elliptic equations in unbounded domains [5]. Very recently the conjecture has also been solved in dimension 3 by Ambrosio and Cabré [2] for (1.11) and by Alberti, Ambrosio, and Cabré [1] for general C^1 nonlinearities. The conjecture remains open in dimension $n \geq 4$. If one adds an additional assumption to De Giorgi's conjecture—namely, $u(x', x_n)$ converges to ± 1 uniformly for $x' \in \mathbb{R}^{n-1}$ when $x_n \to \pm\infty$—then the weaker conjecture (which is sometimes called *Gibbons' conjecture*) for all dimensions n has been proven independently by Barlow, Bass, and Gui [3], Berestycki, Hamel, and Monneau [7], and Farina [14]. Here we only want to point out that the proof of Theorem 1.3 [23, theorem 6.6] uses the result of Theorem 1.4. In some sense, Theorem 1.3 is a version of De Giorgi's conjecture in a two-dimensional bounded domain. We also mention that in [23], we prove that (1.9) may have monotone solutions with nonflat interfaces if λ is not so large.

We also notice that Theorem 1.4 proves the uniqueness of the monotone solution of (1.1). If f satisfies (F1), then the necessary and sufficient condition of the existence of such a solution is exactly (F2). (For a complete necessary and sufficient condition without assuming (F1), see [2].) Here we prove that this is also true for saddle solutions:

PROPOSITION 1.5 *Suppose that f satisfies* (F1). *Then* (1.1) *has a solution $u(\mathbf{x})$ satisfying* (1.3)–(1.5) *if and only if* (F2) *is satisfied.*

In fact, when (F2) is replaced by

$$\text{(F2}'\text{)} \qquad \int_m^M f(u)du > 0\,,$$

which we call a *unbalanced bistable* nonlinearity, it is well-known that (1.1) has a radially symmetric solution $V(\mathbf{x}) = V(|\mathbf{x}|)$ satisfying

$$V'(0) = 0\,, \quad V'(r) < 0\,, \quad r \in (0, \infty)\,, \quad \lim_{r\to\infty} V(r) = m\,, \tag{1.14}$$

and

$$M > V(0) > \theta \quad \text{where } \theta \text{ satisfies } \int_m^\theta f(u)du = 0\,. \tag{1.15}$$

Such a solution $V(\mathbf{x})$ is called a *ground state solution*; see [4, 8, 20] for more information. As a comparison to Proposition 1.5, we also prove the following:

PROPOSITION 1.6 *Suppose that f satisfies* (F1). *Then* (1.1) *has a ground state solution $V(\mathbf{x}) = V(|\mathbf{x}|)$ satisfying* (1.14) *and* (1.15) *if and only if* (F2′) *is satisfied.*

Certainly (F2′) can also be replaced by $\int_m^M f(u)du < 0$; then we have a ground state that is monotone increasing in $(0, \infty)$.

The saddle solution that we obtain in Theorem 1.2 has the following properties:

PROPOSITION 1.7 *Suppose that* $u(\mathbf{x})$ *is the saddle solution in Theorem* 1.2. *Then we have the following*:

(i) (Monotonicity) *In* $Q_1 \equiv \mathbb{R}^+ \times \mathbb{R}^+$, $\partial u/\partial x(\mathbf{x}) > 0$, $\partial u/\partial y(\mathbf{x}) > 0$, *and along each ray* $y = kx + b$, $k > 0$, $x \geq \max\{0, -b/k\}$, $u(\mathbf{x})$ *is strictly monotone increasing*; *i.e.,* $v'(x) > 0$, *where* $v(x) = u(x, kx + b)$, *and* $\lim_{x\to\infty} v(x) = M$; *similar results hold on* $Q_2 \equiv -\mathbb{R}^+ \times \mathbb{R}^+$: $\partial u/\partial x(\mathbf{x}) > 0$, $\partial u/\partial y(\mathbf{x}) < 0$, *and* "$k > 0$" *and* "*increasing*" *are replaced by* "$k < 0$" *and* "*decreasing,*" *respectively.*

(ii) (Decaying) *There exist constants* $C, K > 0$ *such that*

$$|M - u(\mathbf{x})| \leq Ce^{-K\|\mathbf{x}\|} \textit{if } xy > 0\,, \tag{1.16}$$

$$|m - u(\mathbf{x})| \leq Ce^{-K\|\mathbf{x}\|} \textit{if } xy < 0\,, \tag{1.17}$$

where $\|\mathbf{x}\| = \min\{|x|, |y|\}$.

(iii) (Instability) *The linear operator* $L_0 = -\Delta - f'(u)$, *where u is the saddle solution, has negative spectrum. More precisely, there exists* $\zeta \in C_0^\infty(\mathbb{R}^2)$ *such that* $\int_{\mathbb{R}^2}(|\nabla\zeta|^2 - f'(u)\zeta^2)d\mathbf{x} < 0$.

The monotonicity and decaying properties were also proven in [12] for the special case, and Schatzman [22] proved the instability for the special case in [12]. In [22], it was also numerically shown that the negative eigenvalue is unique if $f(u) = 2u - 2u^3$. The proofs that we give here are much simpler, and essentially they are the corollaries of recent work by Berestycki, Caffarelli, and Nirenberg [6] (for the monotonicity and the decaying properties) and [5] (for the instability).

In Section 2 we study the bifurcations of the solutions of (1.8), and we prove Theorem 1.2 in Section 3. The properties of the saddle solutions and the proofs of Propositions 1.5, 1.6, and 1.7 are given in Section 4. Throughout the paper, we use the notation $\mathbf{x} = (x, y) \in \mathbb{R}^2$, and in the following we use u_x and u_y to denote the partial derivatives of function u.

2 Bifurcations on a Square

In this section we consider

$$\begin{cases} \Delta u + \lambda f(u) = 0\,, & (x, y) \in S \equiv (0, a) \times (0, a)\,, \\ \frac{\partial u}{\partial n} = 0\,, & (x, y) \in \partial S\,, \end{cases} \tag{2.1}$$

where $a > 0$ and $\lambda > 0$. We assume that f satisfies (F1) and (F2). Since $f'(\alpha) > 0$, there are bifurcation points along the line of the trivial solutions $\Sigma_{0,0} = \{(\lambda, \alpha) : \lambda > 0\}$. The eigenvalue problem

$$\begin{cases} \Delta\Psi + \eta f'(\alpha)\Psi = 0\,, & (x, y) \in S \equiv (0, a) \times (0, a)\,, \\ \frac{\partial \Psi}{\partial n} = 0\,, & (x, y) \in \partial S\,, \end{cases} \tag{2.2}$$

has eigenpairs with the form ($k, l \in \{0\} \cup \mathbb{N}$)

$$\eta_{k,l} = \frac{(k^2 + l^2)\pi^2}{f'(\alpha)a^2}\,, \quad \Psi_{k,l}(x, y) = \cos\left(\frac{k\pi x}{a}\right)\cos\left(\frac{l\pi y}{a}\right).$$

When $kl \neq 0$, the eigenfunctions satisfy the following periodicity condition:

$$\Psi_{k,l}\left(x + \frac{2a}{k}, y\right) = \Psi_{k,l}(x, y)\,, \tag{2.3}$$

$$\Psi_{k,l}\left(x, y + \frac{2a}{l}\right) = \Psi_{k,l}(x, y)\,, \tag{2.4}$$

$$\Psi_{k,l}\left(\frac{a}{k} - x, \frac{a}{l} - y\right) = \Psi_{k,l}(x, y)\,, \tag{2.5}$$

and when $k = 0$ or $l = 0$, the periodicity becomes the homogeneity:

$$\Psi_{k,0}(x, y_1) = \Psi_{k,0}(x, y_2)\,, \tag{2.6}$$

$$\Psi_{0,k}(x_1, y) = \Psi_{0,k}(x_2, y)\,, \tag{2.7}$$

for any $x, x_i \in (0, a)$ and $y, y_i \in (0, a)$. A solution of (2.1) satisfies

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial S\,. \tag{2.8}$$

Let $X = C^{2,\alpha}(\overline{S})$ and $Y = C^{\alpha}(\overline{S})$. We define for $k \geq 1, l \geq 1$,

$$X_{k,l} = \{u \in X : u \text{ satisfies (2.3), (2.4), (2.5), and (2.8)}\}\,,$$
$$X_{k,0} = \{u \in X : u \text{ satisfies (2.3), (2.6), and (2.8)}\}\,,$$
$$X_{0,l} = \{u \in X : u \text{ satisfies (2.4), (2.7), and (2.8)}\}\,.$$

We also define $Y_{k,l}$ by replacing X by Y in all definitions above. Since the periodicity and the homogeneity defined above are all linear properties, and since $X_{k,l}$ is closed in X, then $X_{k,l}$ is a well-defined Banach space itself. In [23], we prove the following result regarding the bifurcations with symmetry from $\Sigma_{0,0}$:

THEOREM 2.1 *Suppose that f satisfies* (F1). *For any $k, l \in \{0\} \cup \mathbb{N}$, $k + l > 0$, $(\eta_{k,l}, \alpha)$ is a bifurcation point for* (2.1)*, and there is a continuum of the nontrivial solutions $\Sigma_{k,l} \subset \mathbb{R}^+ \times X_{k,l}$ bifurcating from $(\eta_{k,l}, \alpha)$. Near $(\eta_{k,l}, \alpha)$, $\Sigma_{k,l} = \{(\lambda_{k,l}(t), \alpha + t\Psi_{k,l} + o(|t|)) : t \in (-\delta, \delta)\}$ with $\lambda_{k,l}(0) = \eta_{k,l}$. Moreover, either $\Sigma_{k,l}$ is unbounded in $\mathbb{R}^+ \times X_{k,l}$, or $\Sigma_{k,l}$ meets $\mathbb{R}^+ \times \{\alpha\}$ at $\eta \neq \eta_{k,l}$.*

We notice that none of $\eta_{k,l}$ is a simple eigenvalue when we consider the bifurcations in X since $\eta_{k,l} = \eta_{l,k}$, but $\eta_{k,l}$ is a simple eigenvalue of $\Delta\Psi + \eta f'(\alpha)\Psi = 0$ when Ψ is restricted to $X_{k,l}$. The symmetry group of the square is

$$D_4 = \{I, \theta, \theta^2, \theta^3, \psi, \psi\theta, \psi\theta^2, \psi\theta^3\}\,, \tag{2.9}$$

where θ is the rotation of $\pi/2$ with center $(a/2, a/2)$, and ψ is the reflection with respect to the axis $x = y$. If $u(\mathbf{x})$ is a solution of (2.1), so is $u(g(\mathbf{x}))$ for any $g \in D_4$. For any $k, l \in \{0\} \cup \mathbb{N}$, $k + l > 0$, we define

$$\Sigma^+_{k,l} = \{(\lambda, v) \in \Sigma_{k,l} : v(0, 0) > \alpha\}\,, \tag{2.10}$$

$$\Sigma^-_{k,l} = \{(\lambda, v) \in \Sigma_{k,l} : v(0, 0) < \alpha\}\,. \tag{2.11}$$

When $kl \neq 0$, the orbit of $\Sigma^+_{k,l}$ under the operation of the group D_4 is exactly

$$\left\{\Sigma^+_{k,l}, \Sigma^-_{k,l}, \Sigma^+_{l,k}, \Sigma^-_{l,k}\right\}, \tag{2.12}$$

and u is invariant under θ^2 because of (2.5). When $k = 0$ or $l = 0$, the orbit of $\Sigma^+_{k,0}$ is the same as (2.12) (with $l = 0$), but u is invariant under either $\psi\theta$ or $\psi\theta^3$ (the reflection with respect to the axis $x = a/2$ or $y = a/2$).

Next we study the bifurcation solutions that are invariant under ψ, the reflection with respect to the diagonal line $y = x$. For our purpose of studying the saddle solutions, it is sufficient to consider the solutions bifurcating from $\eta_{1,0}$. Near $\eta_{1,0}$, there are two solution curves $\Sigma_{1,0}$ and $\Sigma_{0,1}$ by Theorem 2.1, and there are exactly four solutions on these two curves for λ belonging to a left or right neighborhood of $\eta_{1,0}$ depending on whether the bifurcation is supercritical or subcritical. All these solutions depend on only one variable since they satisfy (2.6) or (2.7), so they are called *semitrivial solutions*. On the other hand, the eigenfunction $\Psi_{1,+}(x, y) = \cos(\pi x/a) + \cos(\pi y/a)$ is invariant under ψ, and the invariance can be characterized by

$$\Psi_{1,+}(y, x) = \Psi_{1,+}(x, y). \tag{2.13}$$

Similarly, the eigenfunction $\Psi_{1,-}(x, y) = \cos(\pi x/a) - \cos(\pi y/a)$ is invariant under $\psi\theta^2$, and the invariance can be characterized by

$$\Psi_{1,-}(a - y, a - x) = \Psi_{1,-}(x, y). \tag{2.14}$$

Define

$$X_{1,+} = \{u \in X : u \text{ satisfies (2.13) and (2.8)}\},$$
$$X_{1,-} = \{u \in X : u \text{ satisfies (2.14) and (2.8)}\}.$$

We also define $Y_{1,\pm}$ by replacing X by Y in the definitions.

THEOREM 2.2 *Suppose that f satisfies* (F1).

(i) *There exists a global continuum $\Sigma_{1,+} \subset \mathbb{R}^+ \times X_{1,+}$ of nontrivial solutions to* (2.1) *bifurcating from* $(\lambda, u) = (\eta_{1,0}, \alpha)$; *near* $(\eta_{1,0}, \alpha)$, $\Sigma_{1,+} = \{(\lambda_{1,+}(t), \alpha + t\Psi_{1,+} + o(|t|)) : t \in (-\delta, \delta)\}$ *with* $\lambda_{1,+}(0) = \eta_{1,0}$.

(ii) $\Sigma_{1,+}$ *is unbounded in* $\mathbb{R}^+ \times X_{1,+}$ *and can be extended to all* $\lambda \in (\eta_{1,0}, \infty)$.

(iii) *For each* $(\lambda, u) \in \Sigma_{1,+}$, $u_x \neq 0$ *and* $u_y \neq 0$ *for* $(x, y) \in (0, a) \times (0, a)$.

(iv) *Similarly, there exists another global continuum* $\Sigma_{1,-} \subset \mathbb{R}^+ \times X_{1,-}$ *with similar properties.*

PROOF: Define $F(\lambda, u) = \Delta u + \lambda f(u)$, where $\lambda > 0$ and $u \in X_{1,+}$. Note that $F(\lambda, \cdot)$ maps $X_{1,+}$ into $Y_{1,+}$. At $(\eta_{1,0}, \alpha)$, $\mathcal{N}(D_u F(\eta_{1,0}, \alpha)) = \operatorname{span}\{\Psi_{1,+}\}$ and $\mathcal{R}(D_u F(\eta_{1,0}, \alpha)) = \{v \in Y_{1,+} : \int_S v\Psi_{1,+} d\mathbf{x} = 0\}$. Here $\mathcal{N}(L)$ is the null space of the linear operator L, and $\mathcal{R}(L)$ is the range space of L. Finally, $F_{\lambda u}(\eta_{1,0}, \alpha)\Psi_{1,+} \notin \mathcal{R}(D_u F(\eta_{1,0}, \alpha))$ since $\int_S f'(\alpha)\Psi^2_{1,+} d\mathbf{x} \neq 0$. From theorem 1.7 of [11], we obtain the local bifurcation result in (i). Since $\Delta^{-1} : Y_{1,+} \to X_{1,+}$ is compact, by the global bifurcation result of [21], $\Sigma_{1,+}$ is a global continuum, which either is unbounded or meets $\Sigma_{0,0}$ at another $(\eta_{k,l}, \alpha)$.

We show that each (λ, u) on $\Sigma_{1,+}$ is monotone in both the x- and y-directions. Let $\Sigma_{1,+}^{+}$ be the subcontinuum of $\Sigma_{1,+}$ containing $\{(\lambda(t), u(t)) : t \in (0, \delta)\}$, and let $\Sigma_{1,+}^{-}$ be the subcontinuum containing $\{(\lambda(t), u(t)) : t \in (-\delta, 0)\}$. We prove that for any solution $(\lambda, u) \in \Sigma_{1,+}^{+}$, $u_x < 0$ and $u_y < 0$ for $(x, y) \in S$. We extend u to the infinite strip $S_a = \{0 < x < a,\ -\infty < y < \infty\}$. Then u_x satisfies

$$\Delta u_x + \lambda f'(u) u_x = 0\,,\ \mathbf{x} \in S_a, \quad u_x = 0,\ \mathbf{x} \in \partial S_a\,. \tag{2.15}$$

We claim that $u_x < 0$ for $\mathbf{x} \in S_a$. Near the bifurcation point $(\eta_{1,0}, \alpha)$, this is true since $u(t) = \alpha + t\Psi_{1,+} + o(|t|)$. We also notice that for any direction s entering S_a transversally along ∂S_a, we have

$$\frac{\partial u_x}{\partial s}(\mathbf{x}) < 0\,, \quad \mathbf{x} \in \partial S_a\,, \tag{2.16}$$

again when (λ, u) is near the bifurcation point.

We prove $u_x < 0$ for $\mathbf{x} \in S_a$ when $(\lambda, u) \in \Sigma_{1,+}^{+}$. Suppose that the claim is not true for some $(\lambda, u) \in \Sigma_{1,+}^{+}$; then from the connectedness of $\Sigma_{1,+}^{+}$, there exists $(\lambda^*, u^*) \in \Sigma_{1,+}^{+}$ such that either

(a) $\exists \mathbf{x}_0 \in S_a$ such that $u_x^*(\mathbf{x}_0) = 0$, or
(b) $\exists \mathbf{x}_0 \in \partial S_a$ such that $(\partial u_x^*/\partial s)(\mathbf{x}_0) = 0$ for some s transversal to ∂S_a.

However, from the maximum principle and the Hopf boundary lemma, neither can happen. Thus the claim is true for any $(\lambda, u) \in \Sigma_{1,+}^{+}$. In particular, $u_x < 0$ for any $(\lambda, u) \in \Sigma_{1,+}^{+}$. Similarly, we can prove $u_y < 0$.

Since each (λ, u) on $\Sigma_{1,+}$ is monotone in both the x- and y-directions, then $\Sigma_{1,+}$ cannot meet $\Sigma_{0,0}$ at another point $(\eta_{k,l}, \alpha)$ since near that point the solutions are of form $\alpha + t\Psi + o(|t|)$, where Ψ is an eigenfunction at $\eta_{k,l}$. But only at $\eta_{1,0}$ can the eigenfunction be a monotone function in both the x- and y-directions. Therefore $\Sigma_{1,+}$ must be an unbounded continuum.

Finally, since any solution (λ, u) of (2.1) satisfies $m \le u(\mathbf{x}) \le M$, then $\|u\|_{L^p(S)} \le C_1$ for $p > 1$ and some $C_1 > 0$ independent of $\lambda > 0$. Thus for any bounded interval $[0, K]$, if (λ, u) is a solution on $\Sigma_{1,+}$ and $\lambda \in [0, K]$, then $\|u\|_{C^{2,\alpha}(\overline{S})} \le C_2$ for $C_2 > 0$ independent of λ. Thus $\Sigma_{1,+}$ can be extended to all $\lambda \in (\eta_{1,0}, \infty)$. The proof for $\Sigma_{1,+}^{-}$ is also similar. □

The fact that the nodal structure of the solutions on a global branch is preserved (when the domain is a subset of $\mathbb{R}^2$) was first observed by Healey and Kielhöfer [17].

From the proof above, we can define the subcontinua of $\Sigma_{1,\pm}$ as follows:

$$\Sigma_{1,+}^{+} = \{(\lambda, v) \in \Sigma_{1,+} : v(0,0) > \alpha\}\,, \tag{2.17}$$

$$\Sigma_{1,+}^{-} = \{(\lambda, v) \in \Sigma_{1,+} : v(0,0) < \alpha\}\,, \tag{2.18}$$

$$\Sigma_{1,-}^{+} = \{(\lambda, v) \in \Sigma_{1,-} : v(a,0) > \alpha\}\,, \tag{2.19}$$

$$\Sigma_{1,-}^{-} = \{(\lambda, v) \in \Sigma_{1,-} : v(a,0) < \alpha\}\,. \tag{2.20}$$

Then the orbit of $\Sigma_{1,+}^{+}$ under the operation of the group D_4 is exactly

$$\left\{\Sigma_{1,+}^{+},\, \Sigma_{1,+}^{-},\, \Sigma_{1,-}^{+},\, \Sigma_{1,-}^{-}\right\}, \tag{2.21}$$

and the solutions on $\Sigma_{1,+}$ are invariant under ψ while the solutions on $\Sigma_{1,-}$ are invariant under $\psi\theta^2$. From Theorems 2.1 and 2.2, we conclude that near bifurcation points, (2.1) has eight possible distinctive monotone solutions with a connected interface (nodal line). Listed below are the nodal diagrams of these 8 solutions:

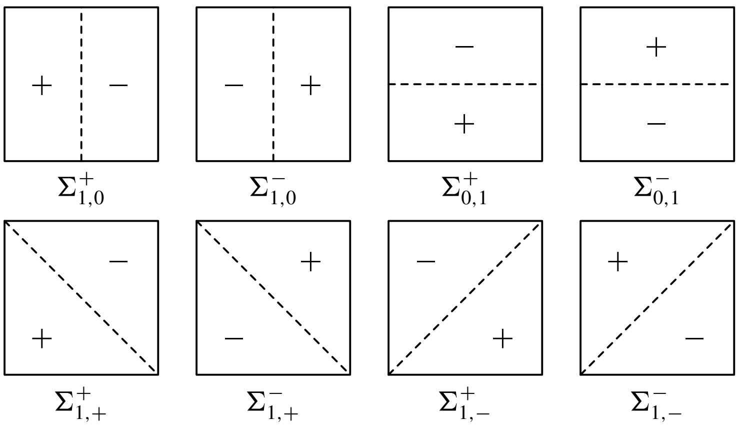

FIGURE 2.1. Nodal lines of solutions bifurcating from $\eta_{0,1} = \eta_{1,0}$.

The dotted lines in the diagrams above are approximately the nodal lines $\mathcal{N}(u) \equiv \{\mathbf{x} \in \overline{\Omega} : u(\mathbf{x}) = \alpha\}$. In fact, the nodal lines of the semitrivial solutions are indeed straight lines, while the nodal lines of solutions on $\Sigma_{1,\pm}$ are approximately the diagonal lines. The nodal lines of solutions on $\Sigma_{1,\pm}$ can be significantly different from the diagonal lines when λ is far away from the bifurcation point. However, from Theorem 1.3, we have the following:

PROPOSITION 2.3 *There exists $\overline{\lambda} > 0$ such that for any $\lambda > \overline{\lambda}$ if (λ, u) is a solution on $\Sigma_{1,+}^{+}$, then $\mathcal{N}(u) = \{(x, q_\lambda(x)) : x \in [0,a]\}$ where*

$$\begin{gathered} q_\lambda'(x) < 0\,, \quad x \in (0,a)\,, \\ q_\lambda'(0) = q_\lambda'(a) = -1\,, \quad q_\lambda(0) = a,\ q_\lambda(a) = 0\,, \end{gathered} \tag{2.22}$$

and

$$\lim_{\lambda \to \infty} q_\lambda(x) = a - x\,. \tag{2.23}$$

PROOF: From Theorem 2.2, a solution (λ, u) on $\Sigma_{1,+}^{+}$ satisfies $u_x < 0$ and $u_y < 0$ for $(x, y) \in S$. Then from Theorem 1.3, u must be a solution with an approximately diagonal nodal line. $q_\lambda'(x) < 0$ because $u_x < 0$ and $u_y < 0$, $q_\lambda'(0) = q_\lambda'(a) = -1$ is from a result by [10] (see the details in [23]), and (2.23) can be obtained by using the proof of theorem 6.6 in [23]. □

3 Existence of Saddle Solutions on $\mathbb{R}^2$

In this section, we construct a saddle solution of (1.1) for balanced bistable nonlinearity f, by using the bifurcation results in Section 2. Suppose that $\{\lambda_j\}_{j=1}^{\infty}$ is a sequence such that $\lambda_j \to \infty$ as $j \to \infty$, and $(\lambda_j, u_j(\mathbf{x})) \subset \Sigma_{1,+}^{+}$ is a solution of (2.1) with $\lambda = \lambda_j$. We extend u_j to $\Omega_1 \equiv [0, 2a] \times [-a, a]$ by extending u_j evenly:

$$u_j(x, -y) = u_j(x, y), \quad u_j(2a - x, y) = u_j(x, y), \tag{3.1}$$

for $(x, y) \in [0, a] \times [0, a]$. It is standard to show that (λ_j, u_j) is a classical solution of

$$\begin{cases} \Delta u + \lambda f(u) = 0, & (x, y) \in \Omega_1, \\ \frac{\partial u}{\partial n} = 0, & (x, y) \in \partial\Omega_1, \end{cases} \tag{3.2}$$

and $u_j \in C^{2,\alpha}(\overline{\Omega_1})$. On the other hand, we can assume that for all j, $\lambda_j > \overline{\lambda}$, which is defined in Proposition 2.3. Thus the nodal set $\mathcal{N}(u_j)$ is a pair of crossing curves that are perpendicular to each other at $(a, 0)$. More precisely, the nodal set in $[0, a] \times [0, a]$ can be written as $\{(x, q_j(x)) : x \in [0, a]\}$ and q_j satisfies (2.22) and (2.23). We also define

$$\Omega_1^+(u) = \{\mathbf{x} \in \Omega_1 : u(\mathbf{x}) > \alpha\} \quad \text{and} \quad \Omega_1^-(u) = \{\mathbf{x} \in \Omega_1 : u(\mathbf{x}) < \alpha\}.$$

We recall the following lemma from [23], which is based on the sweeping principle:

LEMMA 3.1 *Suppose that f satisfies* (F1). *For any $\tau \in (\alpha, M)$, there exists constants $K_\tau > 0$ and $\lambda_\tau > 0$ such that for any solution (λ, u) of* (3.2) *with $\lambda > \lambda_\tau$, if $\mathbf{x} \in \Omega_1^+(u)$ and*

$$\operatorname{dist}(\mathbf{x}, \mathcal{N}(u)) \geq K_\tau \lambda^{-1/2},$$

then $u(\mathbf{x}) \geq \tau$. A similar result holds for $\tau \in (m, \alpha)$ with $u(\mathbf{x}) \leq \tau$.

THEOREM 3.2 *Suppose that f satisfies* (F1) *and* (F2); *then* (1.1) *has a solution $U(\mathbf{x})$ such that*

$$U(-x, y) = U(x, y), \quad U(x, -y) = U(x, y), \tag{3.3}$$

$$\alpha < U(x, y) < M \quad \text{if } |x| > |y|, \tag{3.4}$$

$$m < U(x, y) < \alpha \quad \text{if } |x| < |y|, \tag{3.5}$$

$$U(x, y) = \alpha \quad \text{if } |x| = |y|. \tag{3.6}$$

PROOF: Let $\mathbf{x}_0 = (a, 0)$, $\varepsilon_j = \lambda_j^{-1/2}$, and

$$S(r) = \{(z_1, z_2) : -r < z_1 < r,\ -r < z_2 < r\}\,. \tag{3.7}$$

We define $v_j(\mathbf{z}) = u_j(\mathbf{x}_0 + \varepsilon_j \mathbf{z})$ for $\mathbf{z} = (z_1, z_2) \in S(\varepsilon_j^{-1} a)$. Then v_j satisfies the equation

$$\Delta v_j(\mathbf{z}) + f(v_j(\mathbf{z})) = 0\,, \quad \mathbf{z} \in S\big(\varepsilon_j^{-1} a\big)\,. \tag{3.8}$$

We claim that $\{v_j\}$ is bounded in $C^{2,\alpha}(\overline{K})$ for any $\alpha \in (0,1)$ and any compact subset K of $\mathbb{R}^2$. Let $K_1 \supset K$ be another compact subset of $\mathbb{R}^2$. From the maximum principle, we know that $m < u_j(\mathbf{x}) < M$ for $\mathbf{x} \in \Omega_1$. Thus $\|v_j\|_{C^0(\overline{K_1})} \le C_1$ and function $|f(u)| \le C_2$ for all $u \in [m, M]$; then by the interior Schauder estimates of the elliptic equations, we have

$$\|v_j\|_{C^{2,\alpha}(\overline{K})} \le C_3\big(\|v_j\|_{C^0(\overline{K_1})} + C_2\big) \le C_4\,. \tag{3.9}$$

Therefore $\{v_j\}$ is a relatively compact set in $C^2(\overline{K})$, and by a diagonal process, we can obtain a subsequence (still denoted by $\{v_j\}$) such that

$$v_j \to U \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^2)\,.$$

The limit function $U \in C^2(\mathbb{R}^2)$ satisfies

$$\Delta U(\mathbf{z}) + f(U(\mathbf{z})) = 0\,, \quad \mathbf{z} \in \mathbb{R}^2\,. \tag{3.10}$$

Since u_j satisfies (3.1), U satisfies (3.3). Since the nodal line $(x, q_j(x))$ of u_j satisfies (2.23), we obtain (3.6). Because $(u_j)_x < 0$ and $(u_j)_y < 0$ for $x < 0$ and $y > 0$, we have $U_x(\mathbf{z}) \le 0$ and $U_y(\mathbf{z}) \le 0$ for $z_1 \le 0$ and $z_2 \ge 0$. Thus $M \ge U(x,y) \ge \alpha$ if $|x| > |y|$ and $\alpha \ge U(x,y) \ge m$ if $|x| < |y|$.

Let $Q_j(z_1) = q_j(\varepsilon_j z_1)$ be the nodal line function of v_j. We choose any $\tau \in (\alpha, M)$; then by Lemma 3.1, there exists $K_\tau > 0$ such that $v_j(\mathbf{x}) \ge \tau$ whenever $\operatorname{dist}(\mathbf{x}, \mathcal{N}(v_j)) \ge K_\tau$ and $|x| > |y|$. Since $\lim_{j\to\infty} Q_j(z_1) = z_1$, there exists $j_1 \ge 1$ such that

$$|Q_j(z_1) - z_1| \le \frac{K_\tau}{2}\,, \quad 0 \le z_1 \le 2K_\tau,\ j \ge j_1\,.$$

In particular, for $\mathbf{x}_\tau = (-2K_\tau, 0)$, $\operatorname{dist}(\mathbf{x}_\tau, \mathcal{N}(v_j)) \ge \frac{3\sqrt{2}}{4} K_\tau > K_\tau$; thus $v_j(\mathbf{x}_\tau) \ge \tau$ when $j \ge j_1$, and we get $U(\mathbf{x}_\tau) \ge \tau$ when passing to the limit. So $U \not\equiv \alpha$. Similarly, we can also show that for each $\tilde{\tau} \in (m, \alpha)$, there exists (x, y) such that $U(x,y) < \tilde{\tau}$.

We prove that (3.4) is true. First, by theorem 1 in [18], if there exists $\mathbf{x}$ such that $U(\mathbf{x}) = M$, then $U \equiv M$. So $U < M$ for $|x| > |y|$ since $U \not\equiv M$. On the other hand, suppose that there exists (x_*, y_*) such that $|x_*| > |y_*|$ and $U(x_*, y_*) = \alpha$. Without loss of generality, we assume $x_* < 0$ and $y_* > 0$. Since $U_x(\mathbf{z}) \le 0$ and $U_y(\mathbf{z}) \le 0$ for $z_1 \le 0$ and $z_2 \ge 0$, we have $U(\mathbf{z}) \equiv \alpha$ if $\mathbf{z}$ is in the triangle formed by (x_*, y_*), $(x_*, -x_*)$, and $(-y_*, y_*)$, which implies $U(\mathbf{z}) \equiv \alpha$ for $\mathbf{z} \in \mathbb{R}^2$. We reach a contradiction. So (3.4) is true and (3.5) can be shown similarly. □

PROOF OF THEOREM 1.2: The saddle solution described in Theorem 1.2 is easily obtained if we rotate $U(\mathbf{x})$ in Theorem 3.2 by $\pi/4$. So we only need to show the uniqueness. Let u_1 be the restriction of u on $Q_1 \equiv \mathbb{R}^+ \times \mathbb{R}^+$. Then u_1 is a solution of (1.7). Since Q_1 is an unbounded Lipschitz domain, by theorem 1.2(d) in [6], u_1 is the unique solution of (1.7). (Note that the conditions of the theorem in [6] are satisfied if (F1) and (F2) are satisfied.) Similarly, $u_2 = u|_{Q_2}$, where $Q_2 = -\mathbb{R}^+ \times \mathbb{R}^+$ is also unique. Therefore u must also be unique. $\square$

Note that while the uniqueness of u_1 and u_2 imply the uniqueness of u, the existence of both u_1 and u_2 does not imply the existence of u, since u_1 and u_2 cannot be glued together if (F2) does not hold. In fact, that is exactly the content of Proposition 1.5.

4 Monotonicity, Decay, Instability, and Other Characteristics

We recall the following lemma proven by Modica [18]; see also [9].

LEMMA 4.1 *Suppose that f satisfies* (F1) *and* (F2), *and that u is a solution of* (1.1). *Then*

$$\frac{1}{2}|\nabla u(\mathbf{x})|^2 \leq F(M) - F(u(\mathbf{x}))\,, \tag{4.1}$$

where $F(u) = \int_\alpha^u f(t)dt$.

Note that the equation in [18] and [9] is $\Delta u - f(u) = 0$, while ours is $\Delta u + f(u) = 0$, and in [18] and [9] it is assumed that $F(u) \geq 0$, so Lemma 4.1 here is the version of the result in [18] after an obvious transformation.

We first prove Proposition 1.7.

PROOF OF PROPOSITION 1.7: (i) The monotonicity of the solution along each ray $y = kx + b$, $k > 0$, $b \in \mathbb{R}$, $x \geq \max\{0, -b/k\}$, is a direct consequence of theorems 1.1 and 1.2(e) in [6]. Therefore, for any $x > 0$, $y > 0$, $h > 0$, and $k > 0$, we have $u(x+h, y+k) > u(x, y)$. Then $u(x+h, y) \geq u(x, y)$ by taking the limit of a sequence $(x + h, y + k_n) \to (x + h, y)$ and $u_x(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in Q_1$. By the maximum principle, $u_x(\mathbf{x}) > 0$ for all $\mathbf{x} \in Q_1$; otherwise $u(\mathbf{x}) \equiv \alpha$, which is not the case. Similarly, $u_y(\mathbf{x}) > 0$ for all $\mathbf{x} \in Q_1$.

(ii) We choose $\tau > 0$ such that $f'(u) < -\delta$ for some $\delta > 0$ and $u \in (\tau, M)$. Then by a nonparameterized version of Lemma 3.1, there exists $K_\tau > 0$ such that if $\operatorname{dist}(\mathbf{x}, \mathcal{N}(u)) \geq K_\tau$ and $xy > 0$, then $u(\mathbf{x}) \geq \tau$. Note here that $\operatorname{dist}(\mathbf{x}, \mathcal{N}(u)) = \|\mathbf{x}\|$ because $\mathcal{N}(u) = \{xy = 0\}$. In the region $\{\|x\| \geq K_\tau\}$, we have $f'(u(\mathbf{x})) < -\delta$, and $M - u(\mathbf{x})$ satisfies the equation

$$\Delta(M-u) + \frac{f(M) - f(u)}{M-u}(M-u) = 0\,. \tag{4.2}$$

From the mean value theorem, we have

$$\frac{f(M) - f(u(\mathbf{x}))}{M - u(\mathbf{x})} = f'(\theta) \leq -\delta < 0 \quad \text{for } \|\mathbf{x}\| \geq K_\tau\,; \tag{4.3}$$

then by the well-known decay estimates of the elliptic equation (see, for example, [19, lemma 4.3, p. 840]),

$$\begin{aligned} |M - u(\mathbf{x})| &\leq 2\Big(\max_{\|\mathbf{z}\| \geq K_\tau} |M - u(\mathbf{z})|\Big) e^{-Kd_1(\mathbf{x})} \\ &\leq 2(M - \tau) e^{-Kd_1(\mathbf{x})}\,, \quad \|\mathbf{x}\| \geq K_\tau\,, \end{aligned} \tag{4.4}$$

where $d_1(\mathbf{x}) = \|x\| - K_\tau$ is the distance from $\mathbf{x}$ to the boundary of $\{\|x\| \geq K_\tau\}$ and $K > 0$ is independent of u and $\mathbf{x}$. From (4.4), we obtain (1.16). The estimate for $xy < 0$ is similar.

(iii) Since u_x is a solution of $\Delta\psi + f'(u)\psi = 0$, u_x is bounded by Lemma 4.1, and u_x changes sign in $\mathbb{R}^2$, the instability holds by theorem 1.7 in [5]. □

PROOF OF PROPOSITION 1.5: Suppose that f satisfies (F1) and $u(\mathbf{x})$ is a saddle solution satisfying (1.3)–(1.5). We first prove that

$$\lim_{y\to\infty} u_x(0, y) = \sqrt{2F(M)}\,. \tag{4.5}$$

For a sequence $\{y_j\}_{j=1}^{\infty}$ such that $y_j \to \infty$ as $j \to \infty$, we define $u_j(x) = u(x, y_j)$ for $x \in [0, \infty)$. We claim

$$\lim_{j\to\infty} u_j(x) = v(x) \quad \text{in } C^2_{\text{loc}}(\overline{\mathbb{R}^+})\,,$$

where v is the unique solution of

$$\begin{cases} v'' + f(v) = 0\,, & x \in \mathbb{R}^+\,,\ v(0) = \alpha\,, \\ v'(x) > 0\,, & \lim_{x\to\infty} v(x) = M\,. \end{cases} \tag{4.6}$$

To prove the claim, we cite lemma 4.2 of [6], in which it is shown that u is uniformly Hölder-continuous in $\overline{Q_1}$, and we can easily extend it to get uniform Hölder continuity of u in $\mathbb{R}^2$. We choose a sequence of balls $\{B_i\}$ such that $B_{i+1} \supset B_i$, $R_{i+1} = 2R_i$, where R_i is the radius of B_i, and $\bigcup_i B_i \supset \overline{Q_1}$. Then by the interior Schauder's estimates [16, theorem 4.6] and the fact that $f'(u)$ is uniformly bounded, we have

$$\|u\|'_{C^{2,\alpha}(\overline{B_i})} \leq C\Big(\|u\|_{C^0(\overline{B_{i+1}})} + R_{i+1}^2 \|u\|'_{C^\alpha(\overline{B_{i+1}})}\Big)\,, \tag{4.7}$$

where

$$\|u\|'_{C^\alpha(\overline{\Omega})} = d^\alpha \sup_{\mathbf{x},\mathbf{z}\in\Omega,\mathbf{x}\neq\mathbf{z}} \frac{|u(\mathbf{x}) - u(\mathbf{z})|}{|\mathbf{x} - \mathbf{z}|^\alpha}\,,$$

$$\|u\|'_{C^{2,\alpha}(\overline{\Omega})} = \sum_{k=0}^{2} d^k \sup_{\mathbf{x}\in\Omega} |D^k u| + d^2 \|D^2 u\|'_{C^\alpha(\overline{\Omega})}\,,$$

and d is the diameter of Ω and C is a constant depending only on α. Since u is uniformly Hölder-continuous in $\mathbb{R}^2$, then (4.7) implies that for any $[0, R]$, $R > 0$, u_j, u_j', and u_j'' are uniformly bounded, and u'' is also uniformly Hölder-continuous. Therefore by Arzela's theorem, there is a subsequence of $\{u_j\}$ (still denoted by itself) converging to a limit v in $C^2([0, R])$. By a diagonal process, we have $u_j \to v$ in $C^2_{\text{loc}}([0, \infty))$ by taking a further subsequence. On the other hand, for fixed x, $u_{yy}(x, y) \to 0$ as $y \to \infty$, and the convergence is uniform in $[0, R]$ by (4.7). Therefore the limit v must satisfy (4.6) since $u_x > 0$. It is well-known that the solution of (4.6) is unique; thus, in fact, we have

$$\lim_{y\to\infty} u(x, y) = v(x) \quad \text{in } C^2_{\text{loc}}(\overline{\mathbb{R}^+})\,. \tag{4.8}$$

Integrating v, we obtain

$$\frac{1}{2}[v'(R)]^2 - \frac{1}{2}[v'(0)]^2 + F(v(R)) = 0 \tag{4.9}$$

for $R > 0$. So $v'(0) = \sqrt{2F(M)}$ by taking $R \to \infty$ in (4.9), and we obtain (4.5) from the convergence in (4.8).

However, if we apply the same arguments in Q_2, we obtain

$$\lim_{y\to\infty} u_x(0, y) = \sqrt{2F(m)}\,. \tag{4.10}$$

Therefore a necessary condition for the existence of u is $F(m) = F(M)$, which is equivalent to (F2). On the other hand, we prove the existence of a saddle solution when (F2) is satisfied in Theorem 1.2. So (F2) is a necessary and sufficient condition. □

PROOF OF PROPOSITION 1.6: Suppose f satisfies (F1) and (F2) and $V(\mathbf{x}) = V(|\mathbf{x}|)$ is a radially symmetric solution of (1.1) satisfying (1.14) and (1.15). Then V satisfies

$$V'' + \frac{1}{r}V' + f(V) = 0\,, \quad V'(0) = 0\,,\ V'(r) < 0\,,\ r \in (0, \infty)\,. \tag{4.11}$$

We multiply the equation in (4.11) by V' and integrate on $(0, R)$ to obtain

$$\frac{1}{2}[V'(R)]^2 + \int_0^R \frac{1}{r}[V'(r)]^2\,dr + F(V(R)) - F(V(0)) = 0\,. \tag{4.12}$$

From the maximum principle, $\alpha < V(0) < M$. Since $V' < 0$, then $V'(R) \to 0$ and $V(R) \to m$ as $R \to \infty$. Therefore $F(V(0)) = \int_0^\infty r^{-1}[V'(r)]^2 dr$. But $F(u) < 0$ for any $\alpha < u < M$, and that is a contradiction. So u does not exist if (F2) is satisfied. On the other hand, the existence of u when (F2$'$) is satisfied is well-known; see for example, [8, 20]. □

Remark 4.2. (i) From theorem 1.2(c) in [6], we also know that there exist positive constants C, ρ, and h such that $u(\mathbf{x}) - \alpha \geq C\|\mathbf{x}\|^\rho$ if $\|\mathbf{x}\| \leq h$ and $xy > 0$.

(ii) In the proof of Proposition 1.5, we actually prove

$$\lim_{y\to\infty} u(x,y) = v(x) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}),$$

where v is the unique solution of

$$\begin{cases} v'' + f(v) = 0,\ x \in \mathbb{R},\ v(0) = \alpha, \\ \lim_{x\to-\infty} v(x) = m,\ \lim_{x\to\infty} v(x) = M, \end{cases} \tag{4.13}$$

the homoclinic solution of $v'' + f(v) = 0$ connecting m and M.

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