A Mathematical Model of Economic Growth of Two Geographical Regions

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Abstract

A mathematical model of coupled differential equations is proposed to model economic growth of two geographical regions (cities, regions, continents) with flow of capital and labor between each other. It is based on two established mathematical models: the neoclassical economic growth model by Robert Solow, and the logistic population growth model. The capital flow, labor exchange and spatial heterogeneity are also incorporated in the system. The model is analyzed via equilibrium and stability analysis, and numerical simulations. It is shown that a strong attraction to the high capital region can lead to unbalanced economic growth even when the two geographical regions are similar. The model can help policy makers to decide whether the region should have an open economy or a more closed one. The results of the model can predict the trend of the trade between regions and provide a new insight into some hotly debated contemporary controversial topics.

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Chapter 1

Introduction

Mathematical models have been used by economists to study the economic development of a geographic region such as province, country or continent. The standard neoclassical model - Solow Economic Growth Model was proposed by Robert Solow [5] in the 1950s. In this model, the production of a region is relied on capital, labor and multifactor productivity of the region. The model assumes that more capital means more output, but meanwhile there is also diminishing effects of output because of depreciation associated with capital stock. The production of a region is the gap between these two opposing effect. Moreover, the Solow model assumes that the population and labor force have a linear growth and the growth in the number of workers reduces the accumulation of capital per worker. It provides some insight why some countries are poorer: the capital stock must be spread more thinly that each worker has less capital. We also incorporate technology progress into Solow model since it increases the efficiency of labor and expands the society's production capacity. The heterogeneous economic development is modeled in Figure 1.1.

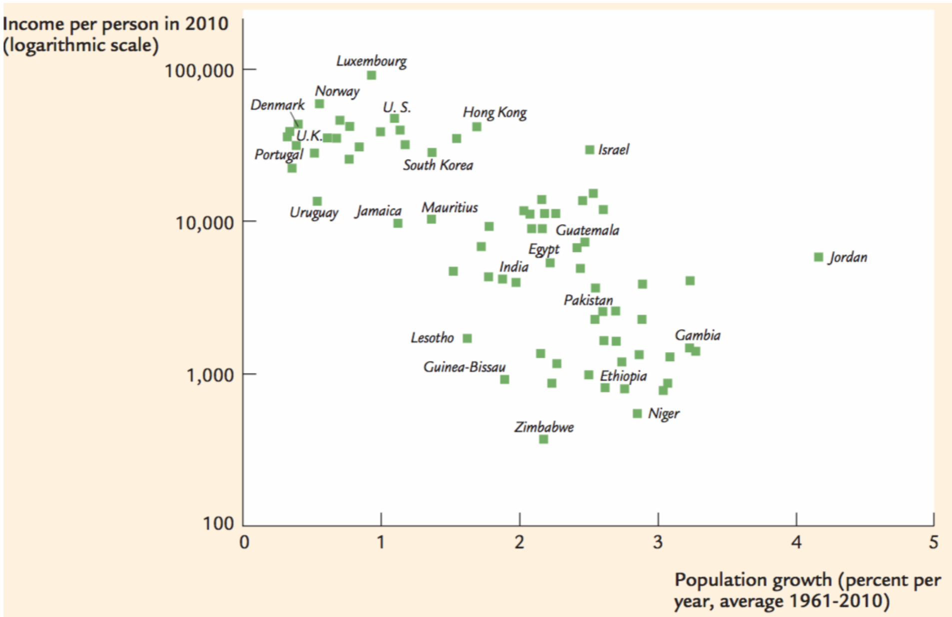

Figure 1.1: International Evidence on Population Growth and Income per Person. It shows that countries with high rates of population growth tend to have low levels of income per person as Solow model predicts. Figure from [3, Page 223]

Another model called Malthusian Population Model [2] was proposed by Thomas Robert Malthus in the 1790s to describe the population growth. In Malthusian Population Model, the population was modeled exponentially without an upper bound. Malthus argued that the power of an ever-increasing population would be greater than the power the earth could provide subsistence for man. He predicted mankind would forever live in poverty. The problem with Malthusian model is that it fails to consider technology advances that enable the production of more food. It also fails to consider birth control and shrinking populations in advanced countries such as nations in Western Europe. The population is in fact bounded by earth's carrying capacity. A modified model (Logistic model) was proposed by Pierre Francois Verhulst [7] to incorporate the growth limit and carrying capacity.

Later, production and population were studied together to establish a two dimensional system of ordinary differential equations. In this system, the standard Solow Growth Model and logistic Population Model were put together to model the growth rate of capital and labor. Both population and capital have their own limit and eventually the whole system will reach its steady state. However, it is set in a homogeneous space and does not take into account flow of capital and labor through space.

In the era of globalization, almost no countries or regions have a closed economics. International trade and capital movement connect the economics of different countries or regions. Labor force can also migrate between countries or regions in different ways. In 2014, Claeyssen and Neto [4] proposed a system of partial differential equations of capital and labor, which takes into account capital and labor flow in a continuous space-time framework based on the Cobb-Douglas production function and a logistic growth for the labor force. However, due to the property of continuity, it is not easy to study the capital and labor of discrete regions by using this model of partial differential equations. Moreover, partial differential equations are much more complex than ordinary differential equations and more difficult to solve or simulate the result.

To fix the above problems, we make some assumptions to improve the result. For capital growth, we assume that capital flow is proportional to the capital difference between the two regions. For labor growth, we assume the that labor movement in one region is proportional to the labor amount in that region and the difference of capital amount between two regions. Therefore, we shall establish a system of four coupled ordinary differential equations with respect to capital and labor growth of each of the two regions.

Chapter 2

Review on Economic and Population Models

2.1 Solow Model of capital

In 1956, Solow Economic Growth Model was proposed by Robert Solow, a renowned American economist who won the Nobel Memorial Prize in Economic Sciences in 1987. Under the assumption that production function has the constant returns to scale, the production function in Solow model is based on Cobb-Douglas function $f(K, L) = AK^{\phi}L^{1-\phi}$ $(0 < \phi < 1)$, where A stands for technology change, K stands for capital and L stands for labor. This function states that capital accumulation in a pure production economy depends on these three production factors. Since the sum of exponents of capital and labor equals to 1, the function indicates constant returns to scale. Regions will grow fast in the beginning and tend to converge to a steady state in the long term [5]. The growth of capital can be thus described by a first order nonlinear ordinary differential equation:

$$\frac{dK}{dt} = f(K, L) - \delta K, \quad K(0) = K_0, \tag{2.1}$$

where $f(K, L)$ is the Cobb-Douglas function introduced above and δK $(\delta > 0)$ is capital depreciation rate. To find the steady state, we let $f(K, L) = \delta K$ and there is a unique positive steady state $K^* = L(\frac{A}{\delta})^{\frac{1}{1-\delta}}$ in addition to trivial state $K = 0$. We know that Cobb-Douglas function satisfies the following conditions: for any fixed $L > 0$,

$$\begin{aligned} &f(0, L) = 0, \quad f_K(K, L) > 0, \quad f_{KK}(K, L) < 0, \\ &\lim_{K \to 0} f_K(K, L) = \infty, \quad \lim_{K \to \infty} f_K(K, L) = 0. \end{aligned} \tag{2.2}$$

Let $g(K) = f(K, L) - \delta K$ where $f(K, L) = AK^{\phi}L^{1-\phi}$. It is easy to see that $g(0) = g(K^*) = 0$. So $K = 0$ and $K = K^*$ are the two equilibrium points of (2.1). Furthermore, $g(K) > 0$ for $K \in (0, K^*)$ and $g(K) < 0$ for $K \in (K^*, \infty)$, where $K^* = L(\frac{A}{\delta})^{\frac{1}{1-\delta}}$. From the theory of differential equations, for any initial condition $K(0) = K_0$, the solution $K(t)$ of (2.1) converges to K^* when $t \to \infty$. We can formulate our first theorem:

Theorem 2.1. *Suppose that $A > 0$, $\delta > 0$, and $f(K, L) = AK^{\phi}L^{1-\phi}$ with $0 < \phi < 1$, then there exists a unique $K^* = L(\frac{A}{\delta})^{\frac{1}{1-\delta}} > 0$ such that for any $K_0 > 0$, the solution $K(t)$ of* (2.1) *converges to K^* when $t \to \infty$.*

2.2 Malthusian Model of Population

Another model which the research project utilizes is the Malthusian Population Model, named after Thomas Robert Malthus, who wrote "An Essay on the Principle of Population" [2], one of the most influential books on population growth. The Malthusian Model, also known as a simple exponential growth model has the form $L(t) = L_0e^{rt}$, where L_0 is the initial population size, r is the population growth rate and t is the time. The Malthusian Model predicts the population is increasing at an exponential rate without an upper bound. However in reality, since all living forms, such as human, compete for resources all the time, the population that the earth is capable of supporting is limited. Thus, the actual growth of population should be bounded and will eventually reach its carrying capacity. In 1838, Belgian mathematician Pierre Francois Verhulst [7] developed a model of population growth bounded by resource limitations and it was named as logistic function later. The logistic model takes the form

$$\frac{dL}{dt} = aL - bL^2, \quad L(0) = L_0, \tag{2.3}$$

where L represents population at time t, $a > 0$ represents the growth rate and $b > 0$ is the crowding effect or intraspecific competition. To find the equilibrium, let $H(L) = aL - bL^2$, then $H'(0) > 0$, $H'(\frac{a}{b}) < 0$. Thus $L = \frac{a}{b}$. The logistic model predicts that the population will grow exponentially initially and eventually reach its carrying capacity at $L = \frac{a}{b}$. We can present our second theorem:

Theorem 2.2. *Suppose $a > 0$, $b > 0$, and $L(t)$ satisfies the equation (2.3), then there exists a unique $L^* = \frac{a}{b}$ such that for any $L_0 > 0$, the solution $L(t)$ of (2.3) converges to $L = \frac{a}{b}$ when $t \to \infty$.*

2.3 A system of ODE model

Now, we combine the Solow Growth model (2.1) and Logistic Population Model (2.3) and get a system of ODE model in a closed economy:

$$\begin{aligned} \frac{dK}{dt} &= AK^{\phi}L^{1-\phi} - \delta K, \\ \frac{dL}{dt} &= aL - bL^2. \end{aligned} \tag{2.4}$$

We can formulate our third theorem:

Theorem 2.3. *Suppose $A > 0$, $\delta > 0$, $a > 0$, $b > 0$, $0 < \phi < 1$, and $K(t)$, $L(t)$ satisfy the equations (2.4). Then there exists a unique positive steady state*

$$(K^*, L^*) = \left((\frac{a}{b}) \cdot (\frac{A}{\delta})^{\frac{1}{1-\phi}}, \frac{a}{b} \right) \tag{2.5}$$

such that for any $K(0) = K_0 > 0, L(0) = L_0 > 0$, $\lim_{t\to\infty}(K(t), L(t)) = (K^*, L^*)$.

The proof is relies on the results from the theorems above. Since $\lim_{t\to\infty} L(t) = L^*$ from Theorem 2.2, and we know $\lim_{t\to\infty} K(t) = K^*$ from Theorem 2.1, we can substitute L in (2.1) with $L^* = \frac{a}{b}$ and get the equation for its steady state $AK^\phi(\frac{a}{b})^{1-\phi} = \delta K$. By simple algebra calculation, $K^* = (\frac{a}{b})(\frac{A}{\delta})^{\frac{1}{1-\phi}}$. The result indicates that in a closed economy without international trade, both labor and capital will eventually reach its steady state and the whole system becomes stable.

Until now, all of the models are set in a homogeneous space without movement of capital or labor. Our main interest is to study economic consequences if two regions exchange goods and services between each other. We will compare the results with that of an isolated economy to see whether the trade benefits or harms each region's economic system. We modify the two-dimensional system of ODEs (2.4) by expanding it into a four-dimensional systems of ODEs with respect to two region's capital and labor. By solving the steady state of the four dimensional system, we hope to get new equilibrium points which can be compared with old ones and achieve the conclusion that for each region, whether it is better to trade capital and labor with another region.

Chapter 3

Mathematical Model

3.1 Four-dimensional ODE system

In this chapter, we study the growth and movement of capital and labor of the two regions, by establishing a system of ODEs with four variables. We introduce two more factors that may have an influence on capital and labor: the difference of capital and the difference of labor between two regions. We assume that the rate of exchange of capitals depends on the capital difference between two regions. The more economic robust a region is compared to the other region, the more capitals it will invest in the other one. Additionally, we assume that the rate of exchange of labor depends on both the labor difference and the capital difference between two regions. This is because if the labor market in one region is saturated, it is more difficult for people to find jobs and they will look for job opportunity in another region where labor demand is higher. In addition, the region with more capitals attracts more labor since there is more opportunity in a more developed region and people will get paid higher salaries. The model we build only simulate the economic exchange between two regions and we do not consider economic influence any other region brings to the model. We propose the following model:

$$\begin{aligned}
\frac{dK_1}{dt} &= d_k(K_2 - K_1) + A_1 K_1^{\phi} L_1^{1-\phi} - \delta_1 K_1, \\
\frac{dK_2}{dt} &= d_k(K_1 - K_2) + A_2 K_2^{\phi} L_2^{1-\phi} - \delta_2 K_2, \\
\frac{dL_1}{dt} &= d_l(L_2 - L_1) + a_1 L_1 - b_1 L_1^2 - cH(L_1, L_2, K_1, K_2), \\
\frac{dL_2}{dt} &= d_l(L_1 - L_2) + a_2 L_2 - b_2 L_2^2 + cH(L_1, L_2, K_1, K_2).
\end{aligned} \tag{3.1}$$

Here $K_1(t)$ and $K_2(t)$ are the capital amount in the region 1 and region 2 respectively, and $L_1(t)$ and $L_2(t)$ are the numbers of labors in region 1 and region 2 respectively. When the region i $(i = 1, 2)$ is isolated, then the capital and labor satisfies (2.4). The movement of capital due to the difference of capital is modeled by $d_k(K_2 - K_1)$, where d_k is the capital diffusion coefficient; while the movement of labor due to the difference of labor is modeled by $d_l(L_2 - L_1)$, where d_l is the labor diffusion coefficient. Moreover the labor movement induced by capital difference is described by a function:

$$H(K_1, K_2, L_1, L_2) = \begin{cases} L_1(K_2 - K_1), & \text{if } K_2 - K_1 \geq 0, \\ L_2(K_2 - K_1), & \text{if } K_2 - K_1 < 0. \end{cases} \tag{3.2}$$

The parameter c measures the strength of capital induced labor movement. Therefore when $K_1 > K_2$, the system (3.1) becomes

$$\begin{aligned}
\frac{dK_1}{dt} &= d_k(K_2 - K_1) + A_1 K_1^{\phi} L_1^{1-\phi} - \delta_1 K_1, \\
\frac{dK_2}{dt} &= d_k(K_1 - K_2) + A_2 K_2^{\phi} L_2^{1-\phi} - \delta_2 K_2, \\
\frac{dL_1}{dt} &= d_l(L_2 - L_1) + a_1 L_1 - b_1 L_1^2 + cL_2(K_1 - K_2), \\
\frac{dL_2}{dt} &= d_l(L_1 - L_2) + a_2 L_2 - b_2 L_2^2 - cL_2(K_1 - K_2),
\end{aligned}$$

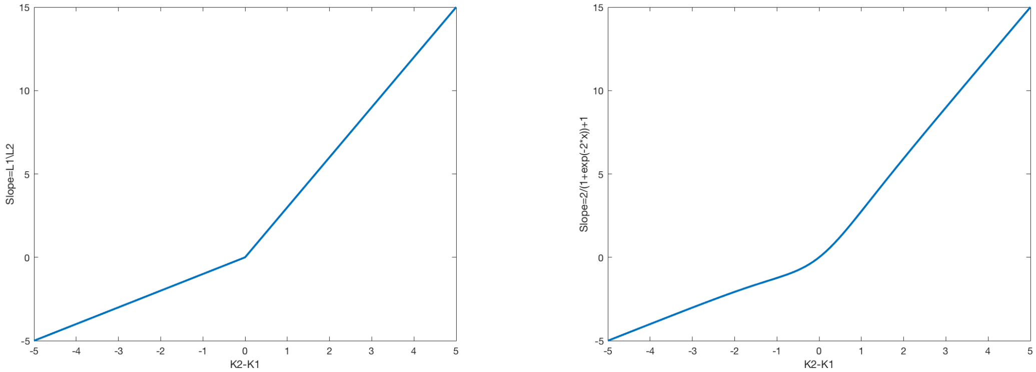

Figure 3.1: Graph of $H(K_1, K_2, L_1, L_2)/(K_2 - K_1)$ defined in (3.2) (left) and in (3.3) (right).

and when $K_1 \le K_2$, the system (3.1) becomes

$$
\begin{aligned}
\frac{dK_1}{dt} &= d_k(K_2 - K_1) + A_1 K_1^{\phi} L_1^{1-\phi} - \delta_1 K_1, \\
\frac{dK_2}{dt} &= d_k(K_1 - K_2) + A_2 K_2^{\phi} L_2^{1-\phi} - \delta_2 K_2, \\
\frac{dL_1}{dt} &= d_l(L_2 - L_1) + a_1 L_1 - b_1 L_1^2 - cL_1(K_2 - K_1), \\
\frac{dL_2}{dt} &= d_l(L_1 - L_2) + a_2 L_2 - b_2 L_2^2 + cL_1(K_2 - K_1).
\end{aligned}
$$

Since the function in (3.2) is a piecewisely defined function, it is not differentiable at $K_1 = K_2$. This could be a problem for studying the properties of equilibrium. We therefore smooth the model by approximating the non-differentiable function with a sigmoid function for future analysis. So we propose an alternative form of the function H by

$$
H(K_1, K_2, L_1, L_2) = \left(\frac{L_1 - L_2}{1 + e^{-h(K_2 - K_1)}} + L_2 \right) (K_2 - K_1), \tag{3.3}
$$

where h is a positive constant. When $K_2 - K_1 >> 0$, $H(K_1, K_2, L_1, L_2) \approx L_1(K_2 - K_1)$; and when $K_2 - K_1 << 0$, $H(K_1, K_2, L_1, L_2) \approx L_2(K_2 - K_1)$. This is the consistent with the function defined in (3.2) (see Figure 3.1). The only exception is the function in (3.2) is not smooth when $K_1 = K_2$, while the one in (3.3) is smooth. In later discussion, we only use the function defined in (3.3), but qualitatively the function in (3.2) can produce similar results.

Thus, we have a 4-dimensional ODE system that includes the capital and labor differences which affect capital and labor exchange rate of both two regions. The equation $K_1(t)$ for the capital of region 1 consists of the difference term $(K_2 - K_1)$ and $d_k > 0$ describing region 1's capital change increases proportionally to its difference from region 2. Similarly, The equation $K_2(t)$ for the capital of region 2 consists of the difference term $(K_1 - K_2)$ and $d_k > 0$ describing region 2's capital change increases proportionally to its difference from region 1.

The equation $L_1(t)$ for the labor of region 1 consists of the term $(L_2 - L_1)$ and $d_l > 0$, describing region 1's labor change increases proportionally to its difference from region 2; Similarly, the equation $L_2(t)$ for the labor of region 2 consists of the term $(L_1 - L_2)$ and $d_l > 0$, describing region 2's labor change increases proportionally to its difference from region 1; Considering different labor movement for different capital attraction, we divide the situation into two cases:

Case 1: When $K_1 \geq K_2$, the term $cL_2(K_1 - K_2)$ in $L_1(t)$ describes that labor in region 2 will flow into region 1 and the increase of labor in region 1 is proportional to its capital difference from region 2 with the effect multiplying by the current labor in region 2; the term $-cL_2(K_1 - K_2)$ in $L_2(t)$ describes that the loss of labor because of region 1's capital attraction is proportional to its capital difference from region 1 with the effect multiplying by the current labor in region 2.

Case 2: When $K_1 \leq K_2$, the term $-cL_1(K_2 - K_1)$ in $L_1(t)$ describes that labor in region 1 will flow into region 2 and the loss of labor in region 1 is proportional to its capital difference from region 2 with the effect multiplying by the current labor in region 1; the term $cL_1(K_2 - K_1)$ in $L_2(t)$ describes that the increase of labor because of region 2's capital attraction is proportional to its capital difference from region 1 with the effect multiplying by the current labor in region 1. Note that we have also used a form of equation such as

$$\begin{aligned}
\frac{dK_1}{dt} &= d_k(K_2 - K_1) + A_1 K_1^{\phi} L_1^{1-\phi} - \delta_1 K_1, \\
\frac{dK_2}{dt} &= d_k(K_1 - K_2) + A_2 K_2^{\phi} L_2^{1-\phi} - \delta_2 K_2, \\
\frac{dL_1}{dt} &= d_l(L_2 - L_1) + a_1 L_1 - b_1 L_1^2 - cL_1(K_2 - K_1) + cL_2(K_1 - K_2), \\
\frac{dL_2}{dt} &= d_l(L_1 - L_2) + a_2 L_2 - b_2 L_2^2 + cL_1(K_2 - K_1) - cL_2(K_1 - K_2),
\end{aligned} \tag{3.4}$$

Equation (4.21) is not well-posed as solution could become negative or tend to infinity in finite time. On the other hand, $-cL_1(K_2 - K_1)$ and $cL_2(K_1 - K_2)$ may not coexist. The same reason applies for the last part of equation $cL_1(K_2 - K_1)$ and $-cL_2(K_1 - K_2)$.

3.2 Existence and boundedness of solutions

We prove that system (3.1) is well-posed so that a solution exists for all time, and it remains positive and bounded.

Theorem 3.1. *Suppose that $H(K_1, K_2, L_1, L_2)$ is defined as in* (3.2) *or* (3.3)*. For any initial conditions $K_1(0) = K_{10} \geq 0$, $K_2(0) = K_{20} \geq 0$, $L_1(0) = L_{10} \geq 0$, $L_2(0) = L_{20} \geq 0$, there exists a unique solution $(K_1(t), K_2(t), L_1(t), L_2(t)) \geq 0$ of* (3.1) *for $t \in (0, \infty)$, and the solutions of* (3.1) *are uniformly bounded.*

Proof. The local existence and uniqueness of solution to (3.1) follows from standard results of ODEs [1, Page 144]. We first show that the solution remains nonnegative as long as it exists. Suppose that $(K_1(t), K_2(t), L_1(t), L_2(t))$ is the solution of (3.1), then it is nonnegative for $t \in [0, t_0]$. At $t = t_0$, one of K_1, K_2, L_1, L_2 is zero. If $K_1(t_0) = 0$, then from the first equation in (3.1), $(K_1)'(t_0) = d_k K_2(t_0) \geq 0$. If $K_2(t_0) = 0$, then from the second equation in (3.1), $(K_2)'(t_0) = d_k K_1(t_0) \geq 0$. If $L_1(t_0) = 0$, then from the third equation in (3.1), $(L_1)'(t_0) = d_l L_2 - cH(L_1, L_2, K_1, K_2)$. There are two subcases for this condition. Case1: If $K_2 - K_1 \geq 0$, $(L_1)'(t_0) = d_l L_2$; Case2: If $K_2 - K_1 < 0$, then

$(L_1)'(t_0) = d_l L_2 - L_2(K_2 - K_1) \geq 0$. If $L_2(t_0) = 0$, then from the fourth equation of (3.1), we can see there are also two subcases for this condition. Case 1: If $K_2 - K_1 \geq 0$, $(L_2)'(t_0) = d_l L_1 + L_1(K_2 - K_1) > 0$. Case 2: If $K_2 - K_1 < 0$, $(L_2)'(t_0) = d_l L_1 \geq 0$.

To prove that the solutions are bounded, we can prove $L_1 + L_2$ and $K_1 + K_2$ are bounded. We add the third equation and the fourth equation of (3.1), then we have

$$L_1' + L_2' = a_1 L_1 + a_2 L_2 - b_1 L_1^2 - b_2 L_2^2. \tag{3.5}$$

Therefore,

$$(L_1 + L_2)' \leq \max\{a_1, a_2\}(L_1 + L_2) - \min\{b_1, b_2\}(L_1^2 + L_2^2). \tag{3.6}$$

From the inequality $L_1^2 + L_2^2 \geq \frac{(L_1+L_2)^2}{2}$, we thus have

$$(L_1 + L_2)' \leq \max\{a_1 + a_2\}(L_1 + L_2) - \frac{\min\{b_1, b_2\}}{2}(L_1 + L_2)^2. \tag{3.7}$$

We can then derive from the results of (3.7) that

$$\limsup_{t\to\infty}(L_1 + L_2)(t) \leq \frac{2\max\{a_1 + a_2\}}{\min\{b_1, b_2\}} = M_1. \tag{3.8}$$

We conclude from (3.8) that $(L_1 + L_2)(t)$ is bounded. Since $L_1, L_2 > 0$, both L_1 and L_2 are bounded. We can prove $(K_1 + K_2)(t)$ is bounded in a similar way. Suppose $L_1(t)$ is bounded by N_1, $L_2(t)$ is bounded by N_2. If we add the first and second equation in (3.1), we have

$$K_1' + K_2' \leq \max\{A_1, A_2\}\max\{L_1, L_2\}^{1-\phi}(K_1^\phi + K_2^\phi) - \max\{\delta_1, \delta_2\}(K_1 + K_2). \tag{3.9}$$

We know that $K_1^\phi + K_2^\phi \leq (K_1 + K_2)^\phi$, and by substituting $(K_1^\phi + K_2^\phi)$ in the equation, we get

$$K_1' + K_2' \leq \max\{A_1, A_2\}\max\{L_1, L_2\}^{1-\phi}(K_1 + K_2)^\phi - \max\{\delta_1, \delta_2\}(K_1 + K_2). \tag{3.10}$$

Let $\max\{A_1, A_2\}\max\{L_1, L_2\}^{1-\phi} = A$ and $\max\{\delta_1, \delta_2\} = B$, then we have

$$K_1' + K_2' \leq (K_1 + K_2)^\phi \left[A - B(K_1 + K_2)^{1-\phi}\right]. \tag{3.11}$$

We can derive

$$\limsup_{t\to\infty}(K_1 + K_2)(t) \le (\frac{A}{B})^{\frac{1}{1-\phi}} = M_2. \tag{3.12}$$

Since $K_1, K_2 > 0$, both K_1 and $K_2 > 0$, we can conclude that $(K_1 + K_2)(t)$ is bounded. Since the solution is bounded, then it can be extended to all $t \in (0, \infty)$. □

Chapter 4

Model with no capital induced labor movement

4.1 Equilibrium Analysis

We first study the system (3.1) by letting $c = 0$, which tells that labor exchange between two regions is not affected by their capital differences. Then we have

$$\frac{dK_1}{dt} = d_k(K_2 - K_1) + A_1 K_1^{\phi} L_1^{1-\phi} - \delta_1 K_1, \tag{4.1}$$
$$\frac{dK_2}{dt} = d_k(K_1 - K_2) + A_2 K_2^{\phi} L_2^{1-\phi} - \delta_2 K_2, \tag{4.2}$$
$$\frac{dL_1}{dt} = d_l(L_2 - L_1) + a_1 L_1 - b_1 L_1^2, \tag{4.3}$$
$$\frac{dL_2}{dt} = d_l(L_1 - L_2) + a_2 L_2 - b_2 L_2^2. \tag{4.4}$$

We develop the theorem

Theorem 4.1. *For any* $A_1, A_2, \delta_1, \delta_2, a_1, a_2, b_1, b_2 > 0$, $d_k, d_l > 0$, *and* $\phi \in (0,1)$, *the system* (4.1)-(4.4) *has a unique positive equilibrium* $(K_1^*, K_2^*, L_1^*, L_2^*)$.

Proof. To prove that the whole system has a unique positive equilibrium, we will first prove that (4.3) and (4.4) have a unique positive equilibrium. Setting $\frac{dK_1}{dt}$ and $\frac{dK_2}{dt}$ in

(4.3) and (4.4) equal to 0, we find

$$L_2 = f(L_1) = L_1(1 - \frac{a_1}{d_l}) + \frac{b_1}{d_l}L_1^2, \tag{4.5}$$

$$L_1 = g(L_2) = L_2(1 - \frac{a_2}{d_l}) + \frac{b_2}{d_l}L_2^2. \tag{4.6}$$

Solving L_2 from (4.6), we get

$$L_2 = \frac{-(1-\frac{a_2}{d_l}) \pm \sqrt{(1-\frac{a_2}{d_l})^2 + \frac{4b_2}{d_l}L_1}}{\frac{2b_2}{d_l}}. \tag{4.7}$$

Since $L_2 > 0$, we must have

$$L_2 = \frac{-(1-\frac{a_2}{d_l}) + \sqrt{(1-\frac{a_2}{d_l})^2 + \frac{4b_2}{d_l}L_1}}{\frac{2b_2}{d_l}} \equiv g^{-1}(L_1). \tag{4.8}$$

Hence a positive equilibrium (L_1, L_2) satisfies (4.5) and (4.8) for some $L_1 > 0$. We can see from (4.5) and (4.8) that the $f(0) = 0$ and $g^{-1}(0) = 0$, and first derivatives of the two equations are that

$$\begin{aligned} f'(0) &= \frac{d_l - a_1}{d_l}, \\ (g^{-1})'(0) &= \frac{1}{g'(0)} = \frac{1}{1-\frac{a_2}{d_l}} = \frac{d_l}{d_l - a_2}. \end{aligned} \tag{4.9}$$

We can break the situation into four cases:

Case 1: $f'(0) > 0$ and $(g^{-1})'(0) > 0$. In other words, $d_l > a_1$ and $d_l > a_2$. Since $\frac{d_l}{d_l - a_2} > 1 > \frac{d_l - a_1}{d_l}$, then $f'(0) < (g^{-1})'(0)$ Together with $f(0) = g^{-1}(0) = 0$, we have $f(h) < g^{-1}(h)$ for $h > 0$ but close to $h = 0$. On the other hand, when L_1 is large,

$$\begin{aligned} \lim_{L_1 \to \infty} \frac{f(L_1)}{L_1^2} &= \frac{b_1}{d_l}, \\ \lim_{L_1 \to \infty} \frac{g^{-1}(L_1)}{\sqrt{L_1}} &= \sqrt{\frac{4b_2}{d_l}}. \end{aligned} \tag{4.10}$$

Therefore, $f(L_1) > g^{-1}(L_1)$ when L_1 is large. By the Intermediate-Value Theorem, since $f(L_1)$ and $g^{-1}(L_1)$ are both continuous functions, there exists a positive L_1^* such that

$f(L_1^*) = g^{-1}(L_1^*)$. Let $L_2^* = f(L_1^*)$, then (L_1^*, L_2^*) satisfies (4.5) and (4.6) and it is a positive equilibrium of (4.1) - (4.4). (See Figure 4.1)

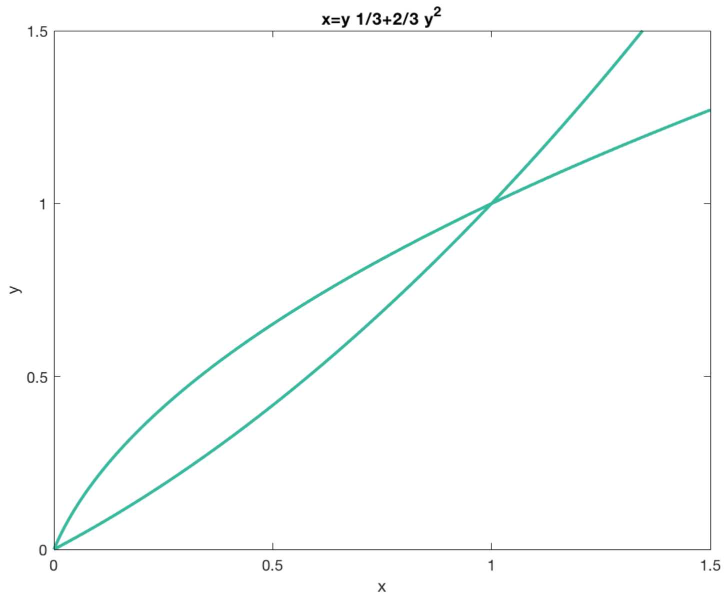

Figure 4.1: Graph of $f(L_1)$ and $g^{-1}(L_1)$ for Case 1: $a_1 = 1$, $a_2 = 2$, $d_l = 3$, $b_1 = 1$ and $b_2 = 2$.

Case 2: In this case, $f'(0) > 0$ and $(g^{-1})'(0) < 0$. In other words, $d_l > a_1$ and $d_l < a_2$. Since $g^{-1}(0) > 0 = f(0)$, $g^{-1}(L_1) > f(L_1)$ when L_1 is small. Then arguing in a similar way as in Case 1, we obtain a positive equilibrium (L_1^*, L_2^*). (See Figure 4.2)

Case 3: In this case $f'(0) < 0$ and $(g^{-1})'(0) > 0$. In other words, $d_l < a_1$ and $d_l > a_2$. Similar to Case 2, we also have $g^{-1}(L_1) > f(L_1)$ when L_1 is small. Then arguing in a similar way as in Case 2, we obtain a positive equilibrium (L_1^*, L_2^*). (See Figure 4.3)

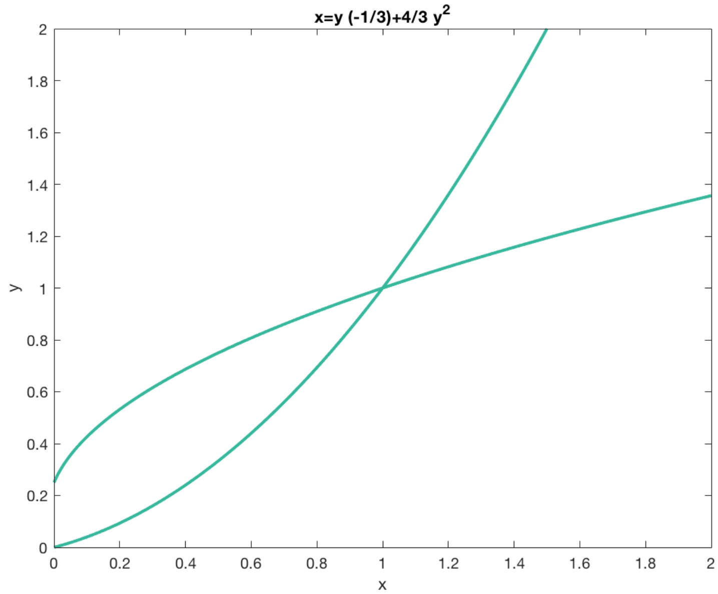

Figure 4.2: Graph of $f(L_1)$ and $g^{-1}(L_1)$ for Case 2: $a_1 = 1$, $a_2 = 2$, $d_l = 1.5$, $b_1 = 1$ and $b_2 = 2$.

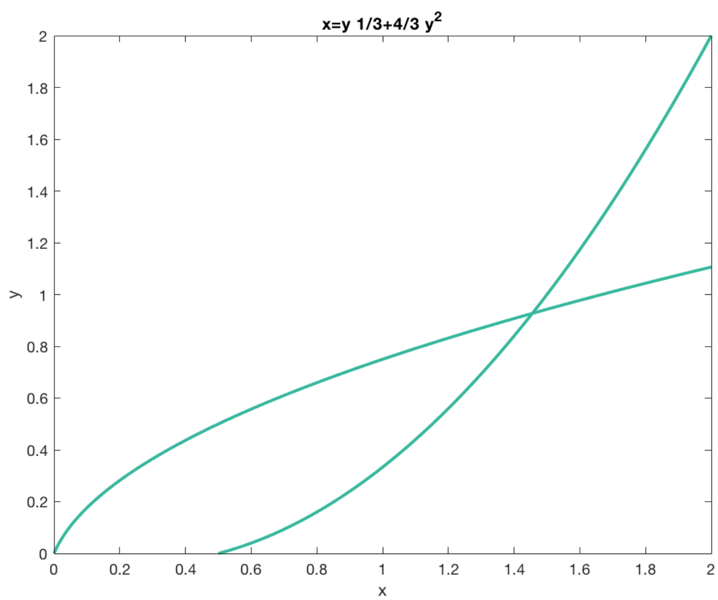

Figure 4.3: Graph of $f(L_1)$ and $g^{-1}(L_1)$ for Case 3: $a_1 = 2$, $a_2 = 1$, $d_l = 1.5$, $b_1 = 1$ and $b_2 = 2$.

Case 4: In this case $f'(0) < 0$ and $(g^{-1})'(0) < 0$. In other words, $d_l < a_1$ and $d_l < a_2$. Similar to Case 2, we also have $g^{-1}(L_1) > f(L_1)$ when L_1 is small. Then arguing in a similar way as in Case 2, we obtain a positive equilibrium (L_1^*, L_2^*). (See Figure 4.4)

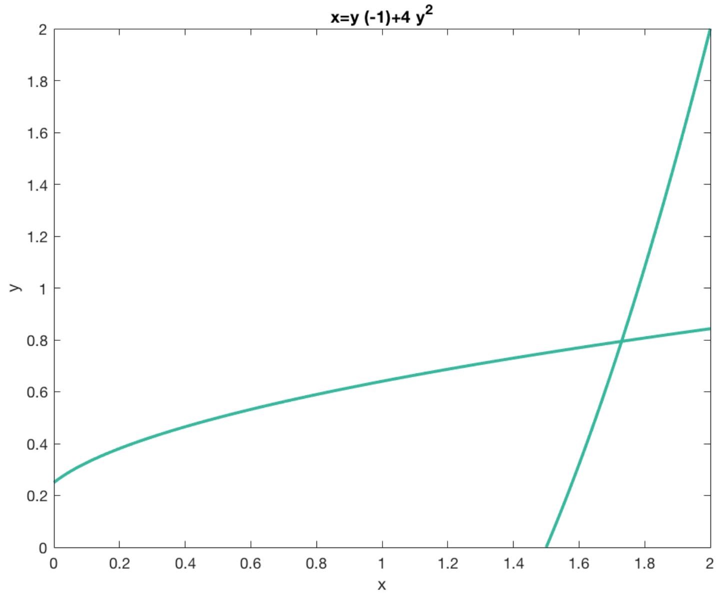

Figure 4.4: Graph of $f(L_1)$ and $g^{-1}(L_1)$ for Case 4: $a_1 = 2$, $a_2 = 1$, $d_l = 0.5$, $b_1 = 1$ and $b_2 = 2$.

Now summarizing Cases 1-4, we always have a positive equilibrium (L_1^*, L_2^*) for (4.3) and (4.4). Substituting L_1^* and L_2^* into the equations of (4.1) and (4.2), we can find unique value of K_1^* and K_2^* in a similar way. Therefore, we can see that there exists a positive equilibrium for the system.

Next, we prove that the positive equilibrium is unique. We know that $f(L_1)'' = \frac{2b_1}{d_l} > 0$. Since $f(f^{-1}(L_1)) = L_1$, we can infer that

$$f'(f^{-1}(L_1)) \cdot (f^{-1})'(L_1) = 1, \quad f''(f^{-1}(L_1))[(f^{-1})'(L_1)]^2 + (f^{-1})''(L_1) \cdot f'(f^{-1}(L_1)) = 0,$$

Since $[(f^{-1})'(L_1)]^2$ and $f'(f^{-1}(L_1))$ are positive, we can infer that $f(L_1)$ is convex and $f^{-1}(L_1)$ is concave. If we let $h(L_1) = f(L_1) - g^{-1}(L_1)$. The fact that $h'' = f''(L_1) -$

$(g^{-1})''(L_1) > 0$ excludes the possibility that $h(L_1)$ has a local minimum point. Therefore, there could only be one L_1 such that $h(L_1) = 0$. The equilibrium point for (4.3) and (4.4) is thus unique. Substituting L_1^* and L_2^*, we can again use convexity of functions to prove that equilibrium points (K_1^*, K_2^*) in (4.1) and (4.2) are also unique. $\square$

Figure 4.5 shows a typical solution of (4.1) – (4.4) converges to the unique positive equilibrium.

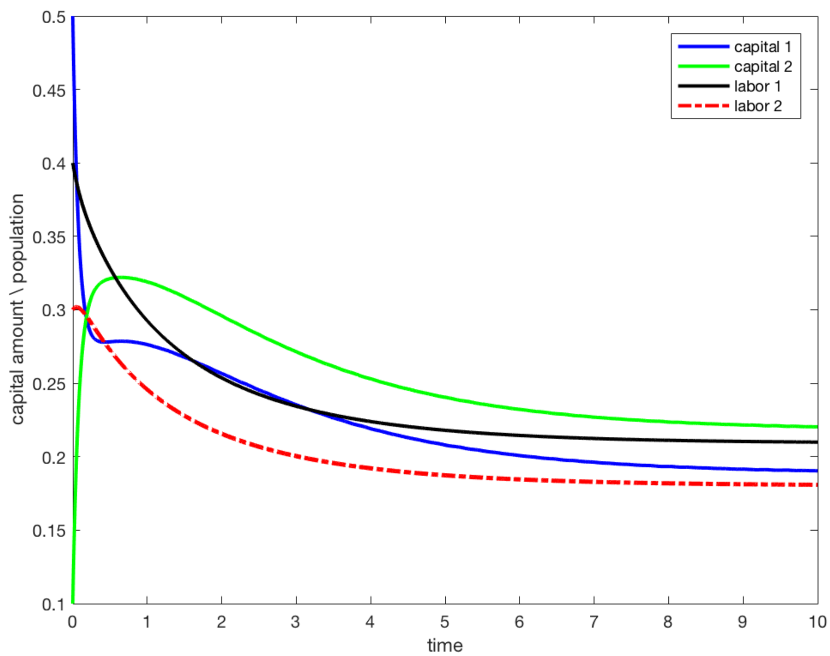

Figure 4.5: Convergence to the unique equilibrium point for (4.1)-(4.4). The solution approaches $(K_1^*, K_2^*, L_1^*, L_2^*) = (0.1904, 0.2203, 0.2099, 0.1809)$. Parameters used: $d_k = 6$, $d_l = 5$,$A_1 = 1$, $A_2 = 2$, $\phi = 0.5$, $\delta_1 = 2$, $\delta_2 = 1$, $a_1 = 0.9$, $a_2 = 0.1$, $b_1 = 1$, $b_2 = 5$. Initial value: $(K_1(0), K_2(0), L_1(0), L_2(0)) = (0.5, 0.1, 0.4, 0.3)$.

This is an asymmetric case with $A_1 \neq A_2$, $\delta_1 \neq \delta_2$, $a_1 \neq a_2$ and $b_1 \neq b_2$. We can see from the graph that if two regions with different initials, they will converge to a unique steady state. That means even if two regions with different initial capital and labor, they will eventually reach their equilibrium capital and labor amount. The interesting fact is that the region with less capital will increase its capital in the beginning, but two regions will eventually decrease to a quite close steady state. On the other hand, both regions decrease in labor from the beginning to the end with steady states close to each other.

4.2 Stability Analysis

In this section, we prove that the unique positive equilibrium $(K_1^*, K_2^*, L_1^*, L_2^*)$ is globally asymptotically stable.

Theorem 4.2. *Let* $(K_1^*, K_2^*, L_1^*, L_2^*)$ *be the unique positive equilibrium of* (4.1) – (4.4)*. Then for any initial condition* $(K_1(0), K_2(0), L_1(0), L_2(0)) = (K_{10}, K_{20}, L_{10}, L_{20})$ *satisfying* $K_{10} > 0$*,* $K_{20} > 0$*,* $L_{10} > 0$*,* $L_{20} > 0$*,* $\lim_{t\to\infty}(K_1(t), K_2(t), L_1(t), L_2(t)) = (K_1^*, K_2^*, L_1^*, L_2^*)$*.*

To prove this result, we recall a previous theorem from [6, Theorem 3.1].

Theorem 4.3. *Consider a differential system*

$$\begin{cases} x' = f(x) + d_{21}y - d_{12}x, \\ y' = g(y) - d_{21}y + d_{12}x, \\ x(0) = x_0 \geq 0, \ y(0) = y_0 \geq 0. \end{cases} \tag{4.11}$$

Here we assume that $f, g : [0, \infty) \to \mathbf{R}$ *are smooth functions, and* $d_{12}, d_{21} > 0$*. Define a function* $V : [0, \infty) \times [0, \infty) \to \mathbf{R}$ *by*

$$V(x, y) = -d_{12}F(x) - d_{21}G(y) + \frac{1}{2}(d_{12}x - d_{21}y)^2, \tag{4.12}$$

where

$$F(x) = \int_0^x f(s)ds, \quad G(y) = \int_0^y g(s)ds. \tag{4.13}$$

We suppose that $f, g : [0, \infty) \to \mathbf{R}$ *are continuously differentiable functions, and* $d_{12}, d_{21} \geq 0$*. In addition we assume that* *(i)* $f(0) \geq 0$ *and* $g(0) \geq 0$*;*

(ii) For $V(x, y)$ *defined as in* (4.12)*, there exists* $C \in \mathbf{R}$ *such that* $V(x, y) \geq C$ *for all* $x, y \geq 0$*, and the set* $S_a = \{(x, y) : x, y \geq 0, V(x, y) \leq a\}$ *is bounded for any* $a \geq C$*;*

(iii) Each equilibrium point of (4.12) *is isolated.*

Then there exists an equilibrium point $(x^*, y^*) \in [0, \infty) \times [0, \infty)$ *such that*

$$\lim_{t\to\infty} (x(t), y(t)) = (x^*, y^*).$$

Now we prove Theorem 4.2.

Proof of Theorem 4.2. Firstly, we want to prove $\lim_{t\to\infty} (L_1(t), L_2(t)) = (L_1^*, L_2^*)$. Since

$$\frac{dL_1}{dt} = d_l(L_2 - L_1) + a_1 L_1 - b_1 L_1^2, \tag{4.14}$$

$$\frac{dL_2}{dt} = d_l(L_1 - L_2) + a_2 L_2 - b_2 L_2^2. \tag{4.15}$$

We can let

$$\begin{aligned} f(x) &= a_1 L_1 - b_1 L_1^2, \\ g(y) &= a2 L_2 - b_2 L_2^2, \\ d_{21} &= d_{12} = d_l. \\ V(x, y) &= \frac{1}{2} d_l^2 (L_1 - L_2)^2 - \frac{1}{2} d_l a_1 L_1^2 + \frac{1}{3} d_l b_1 L_1^3 - \frac{1}{2} d_l a_2 L_2^2 + \frac{1}{3} d_l b_1 L_1^3, \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} F(x) &= \frac{1}{2} a_1 L_1^2 - \frac{1}{3} b_1 L_1^3, \\ G(y) &= \frac{1}{2} a_2 L_2^2 - \frac{1}{3} b_1 L_2^3, \end{aligned} \tag{4.17}$$

Since $f, g : [0, \infty) \to \mathbf{R}$ are continuously differentiable functions, and $d_l > 0$, we know that

(i) Since capital and labor are nonnegative, $f(0) \geq 0$ and $g(0) \geq 0$;

(ii) Let $min(-F(x)) = -M_1$, $min(-G(x)) = -M_2$. Then

$$V(L_1, L_2) \geq 0 - dlM_1 - dlM_2 = -dl(M_1 + M_2) \tag{4.18}$$

(iii) We have proved that (L_1^*, L_2^*) is the unique positive equilibrium. By calculating the Jacobain matrix, we get

$$J(0,0) = \begin{pmatrix} -d_l + a_1 & d_l \\ d_l & -d_l + a_2 \end{pmatrix}. \tag{4.19}$$

If $(0,0)$ is stable, then

$$\begin{aligned} TJ(0,0) &= a_1 + a_2 - 2d_l > 0 \\ detJ(0,0) &= a_1a_2 - (a_1 + a_2)d_l > 0 \end{aligned} \tag{4.20}$$

From (4.20), we can get $\frac{a_1+a_2}{2} < d_l < \frac{a_1a_2}{a_1+a_2}$, which is impossible since $(a_1 + a_2)^2 < 2a_1a_2$. Therefore, $(0,0)$ is not stable. (L_1^*, L_2^*) is the unique equilibrium point and it is globally stable from Theorem 4.3. The same theorem applies to the capital. We know that

$$\begin{aligned} \frac{dK_1}{dt} &= d_k(K_2 - K_1) + A_1K_1^{\phi}(L_1^*)^{1-\phi} - \delta_1K_1, \\ \frac{dK_2}{dt} &= d_k(K_1 - K_2) + A_2K_2^{\phi}(L_2^*)^{1-\phi} - \delta_2K_2. \end{aligned} \tag{4.21}$$

We can let

$$f(x) = d_k(K_2 - K_1) + A_1K_1^{\phi}L_1^{1-\phi} - \delta_1K_1,$$

$$g(y) = d_k(K_1 - K_2) + A_2K_2^{\phi}L_2^{1-\phi} - \delta_2K_2$$

$$d_{21} = d_{12} = d_k.$$

$$V(x,y) = \frac{1}{2}d_k^2(K_1 - K_2)^2 - d_k\frac{A_1(L_1^*)^{1-\phi}}{\phi+1}K_1^{\phi+1} + \frac{1}{2}d_k\delta_1K_1^2 - d_k\frac{A_2(L_2^*)^{1-\phi}}{\phi+1}K_2^{\phi+1} + \frac{1}{2}d_k\delta_2K_1^2, \tag{4.22}$$

where

$$\begin{aligned} F(x) &= \frac{A_1(L_1^*)^{1-\alpha}}{\alpha+1}K_1^{\alpha+1} + \frac{1}{2}\delta_1K_1^2, \\ G(y) &= \frac{A_2(L_2^*)^{1-\alpha}}{\alpha+1}K_2^{\alpha+1} + \frac{1}{2}\delta_2K_2^2, \end{aligned} \tag{4.23}$$

Since $f, g : [0, \infty) \to \mathbf{R}$ are continuously differentiable functions, and $d_l > 0$, we know that

(i) Because capital and labor are nonnegative, $f(0) \geq 0$ and $g(0) \geq 0$;

(ii) Let $min(-F_1(x)) = -M_3$, $min(-G(x)) = -M_4$,

$$V(_1, L_2) \geq 0 - d_k M_3 - d_k M_4 = -d_k(M_3 + M_4) \tag{4.24}$$

(iii) We have proved that (K_1^*, K_2^*) is the unique positive equilibrium. In addition, by calculating the Jacobian matrix

$$J(0,0) = \begin{pmatrix} -d_k - \delta_1 & d_k \\ d_k & -d_k - \delta_2 \end{pmatrix}. \tag{4.25}$$

$$\begin{aligned} Trace &= -\delta_1 - \delta_2 - 2d_k < 0, \\ Determinant &= (d_k + \delta_1)(d_k + \delta_2) - d_k^2 > 0. \end{aligned} \tag{4.26}$$

From (4.26), we can see that $(0,0)$ is a spiral source and it is not stable. Therefore, (K_1^*, K_2^*) is globally stable and $(K_1^*.K_2^*, L_1^*, L_2^*)$ is global stable steady state. □

4.3 Sensitivity Analysis

In this section we study the effects of different parameters on the dynamics of (4.1) - (4.4). In other words, holding other parameters fixed, we want to study how equilibrium points of the system change by changing the parameters we choose. The parameters we choose to study are d_l, d_k and a.

We change d_l from 0 to 5 by increasing 0.25 each time and plot the graph d_l vs *equilibrium points for the system*. The result is shown in Figure 4.6 and 4.7.

We can see that as d_l increases, labor of region 1 is always decreasing and labor of region 2 increases in the beginning and decreases until reaching the steady state. Since d_l represents the flow rate of labor exchange, we can infer from the graph that two

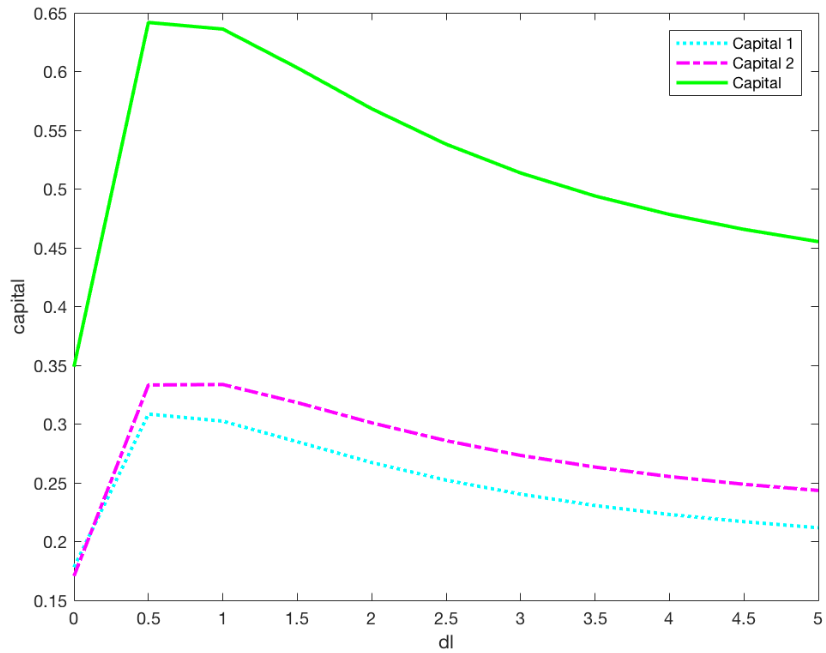

Figure 4.6: Unique Equilibrium points for (4.1)-(4.2) with different d_l values. Parameters used: $d_k = 6$, $A_1 = 1$, $A_2 = 2$, $\phi = 0.5$, $\delta_1 = 2$, $\delta_2 = 1$, $a_1 = 0.9$, $a_2 = 0.1$, $b_1 = 1$, $b_2 = 5$. Initial value: $(K_1(0), K_2(0), L_1(0), L_2(0)) = (1, 0.1, 2, 0.3)$.

regions with large difference in amount of labor in the beginning will eventually have the same amount of labor as the flow rate increases. That makes sense since with large flow rate, people in more crowded region will flood into less crowded region and quickly make two regions same amount of people. Moreover, the total labor of two regions is always decreasing no matter whether labor in region 2 increases or not. On the other hand, capital of both regions will first increase and then decrease, the same for the total capital.

We change d_k from 0 to 5 by increasing 0.25 each time and plot the graph d_k vs *equilibrium points for the system*. We plot the graph for $c = 0$ with different d_k values in Figure 4.8 and 4.9.

We can see that as d_k increases, capital of region 1 is always decreasing and capital

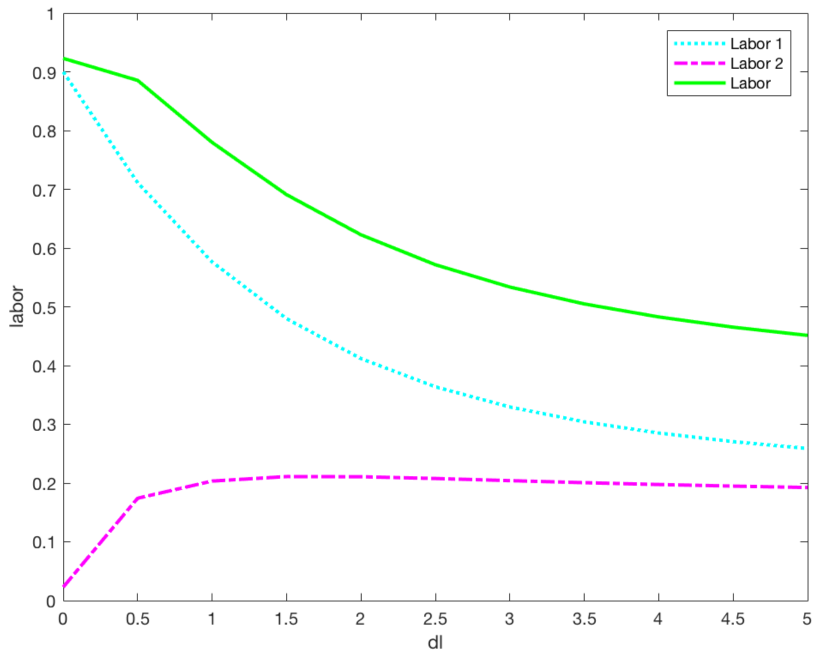

Figure 4.7: Unique Equilibrium points for (4.3)-(4.4) with different d_l values. Parameters used: $d_k = 6$, $A_1 = 1$, $A_2 = 2$, $\phi = 0.5$, $\delta_1 = 2$, $\delta_2 = 1$, $a_1 = 0.9$, $a_2 = 0.1$, $b_1 = 1$, $b_2 = 5$. Initial value: $(K_1(0), K_2(0), L_1(0), L_2(0)) = (1, 0.1, 2, 0.3)$.

of region 2 is always increasing. Since d_k represents the flow rate of capital exchange, we can infer from the graph that two regions with large difference in amount of capital in the beginning will eventually have the same amount of capital as the flow rate increases. This is because d_k promotes synchronization and capital will flow from a more developed region to a less developed region quickly and make two regions about the same amount capital. However, the total capital of two regions is always decreasing. On the other hand, since d_k does not influence labor force, the labor amount of both regions does not change.

After simulating the effect of d_l and d_k on the system, we now study the effect of a_1. For a_1, we change a_1 from 0 to 5 by increasing 0.25 each time and plot the graph a_1 vs

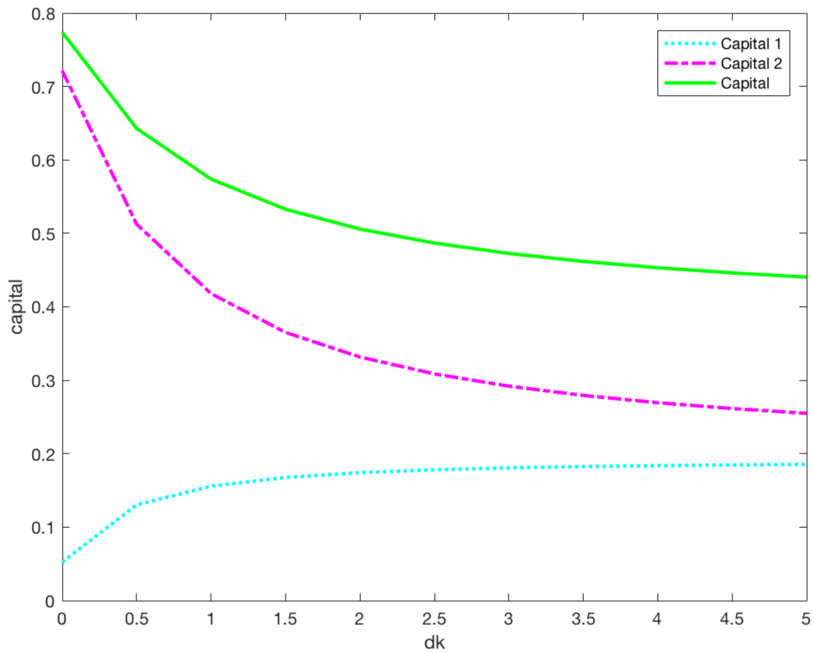

Figure 4.8: Unique Equilibrium points for (4.1)-(4.2) with different d_k values. Parameters used: $d_l = 5$, $A_1 = 1$, $A_2 = 2$, $\phi = 0.5$, $\delta_1 = 2$, $\delta_2 = 1$, $a_1 = 0.9$, $a_2 = 0.1$, $b_1 = 1$, $b_2 = 5$. Initial value: $(K_1(0), K_2(0), L_1(0), L_2(0)) = (1, 0.1, 2, 0.3)$.

equilibrium points for the system, and the results are shown in Figure 4.10 and 4.11.

We can see that as a_1 increases, labors of both region 1 and 2 are always increasing and region 1 increases much faster than region 2. The total amount of labor is thus always increasing. This is because a_1 is the growth rate and controls the carrying capacity of region 1. We know that carrying capacity of region 1 is $\frac{a_1}{b_1}$. As a_1 increases, the carrying capacity of region 1 increases. When d_l is relatively small compared to a_1, which means the labor exchange rate is less than the population growing rate, the total labor in region 1 is also expected to increase. As a result, the capitals of both regions also increase because of increasing labor force.

We now study the effect of A_1. We change A_1 from 0 to 5 by increasing 0.5 each time

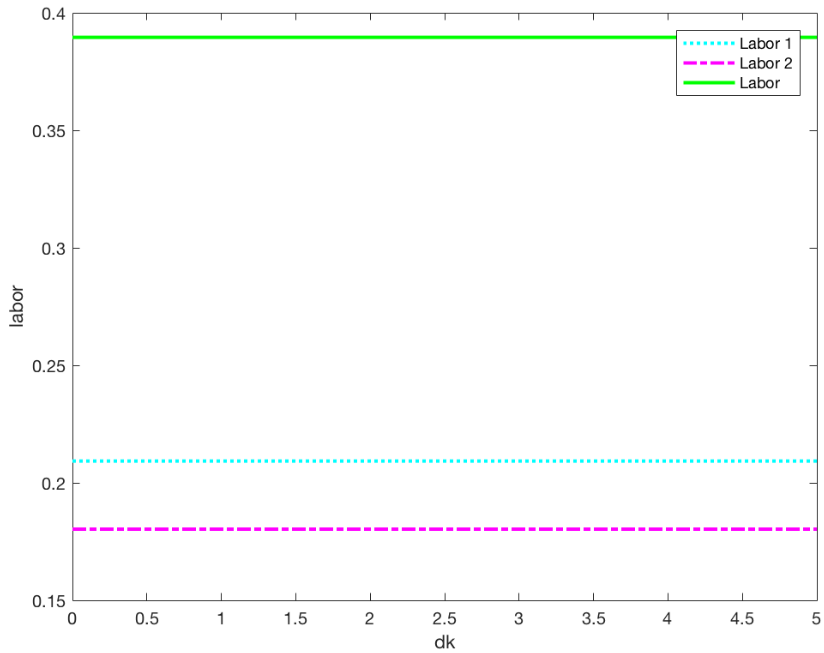

Figure 4.9: Unique Equilibrium points for (4.3)-(4.4) with different d_k values. Parameters used: $d_l = 5$, $A_1 = 1$, $A_2 = 2$, $\phi = 0.5$, $\delta_1 = 2$, $\delta_2 = 1$, $a_1 = 0.9$, $a_2 = 0.1$, $b_1 = 1$, $b_2 = 5$. Initial value: $(K_1(0), K_2(0), L_1(0), L_2(0)) = (1, 0.1, 2, 0.3)$.

and plot the graph A_1 vs *equilibrium points for the system*, and the results are shown in Figure 4.12 and 4.13.

We can see that as A_1 increases, capitals of both region 1 and 2 are always increasing and they are increasing at the relative same rate. The total amount of labor is thus always increasing. This is because A_1 represents technology advances. When A_1 increases, the labor efficiency will increase. As a result, the capital is also expected to increase. However, technology advances do not affect labor amount. From numerical perspective, the equilibrium point for capital is $L^*(\frac{A}{\delta})^{\frac{1}{1-\delta}}$. When A_1 increases, the steady state of capital for region 1 is also expected to increase. However, because d_k is relatively large, which means the exchange of capital is quick, the capital in region 2 is also increasing

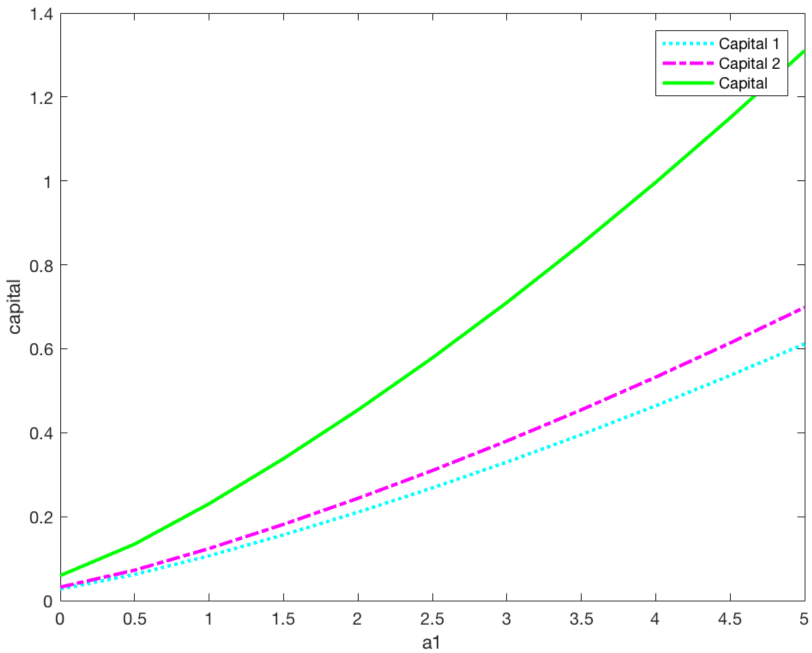

Figure 4.10: Unique Equilibrium points for (4.1)-(4.2) with different a_1 values. Parameters used: $d_k = 6,\ d_l = 5,\ A_1 = 1,\ A_2 = 2,\ \phi = 0.5,\ \delta_1 = 2,\ \delta_2 = 1,\ a_2 = 0.1,\ b_1 = 1,\ b_2 = 5$. Initial value: $(K_1(0), K_2(0), L_1(0), L_2(0)) = (1, 0.1, 2, 0.3)$.

and the capital difference between region 1 and region 2 is relatively small.

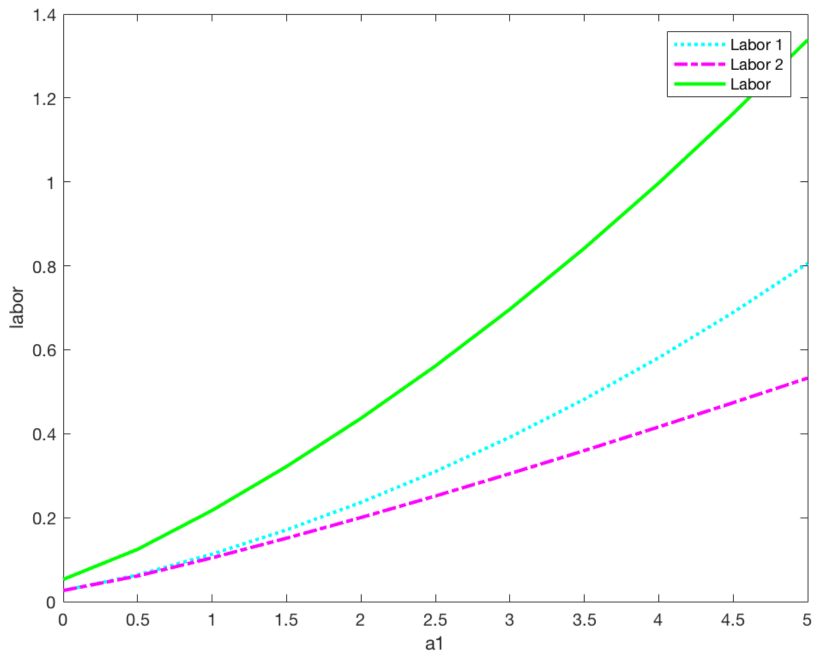

Figure 4.11: Unique Equilibrium points for (4.3)-(4.4) with different a_1 values. Parameters used: $d_k = 6$, $d_l = 5$, $A_1 = 1$, $A_2 = 2$, $\phi = 0.5$, $\delta_1 = 2$, $\delta_2 = 1$, $a_2 = 0.1$, $b_1 = 1$, $b_2 = 5$. Initial value: $(K_1(0), K_2(0), L_1(0), L_2(0)) = (1, 0.1, 2, 0.3)$.

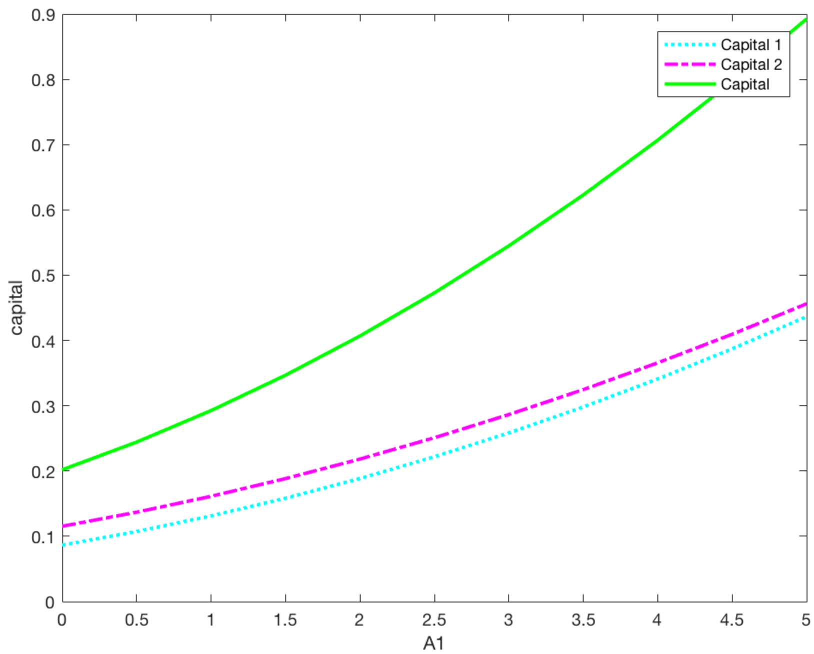

Figure 4.12: Unique Equilibrium points for (4.1)-(4.2) with different A_1 values. Parameters used: $d_k = 6$, $d_l = 5$, $A_2 = 2$, $\phi = 0.5$, $\delta_1 = 2$, $\delta_2 = 1$, $a_1 = 0.9$, $a_2 = 0.1$, $b_1 = 1$, $b_2 = 5$. Initial value: $(K_1(0), K_2(0), L_1(0), L_2(0)) = (1, 0.1, 2, 0.3)$.

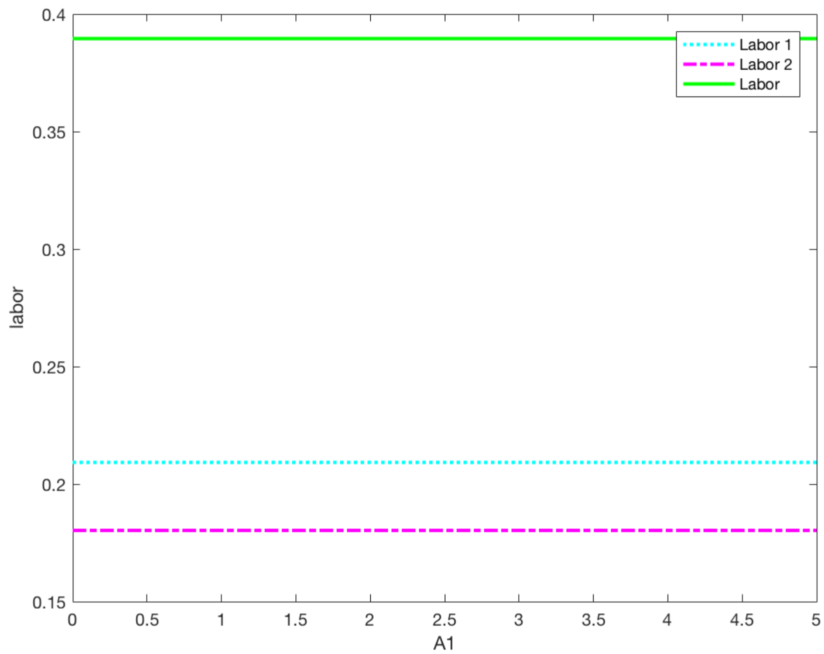

Figure 4.13: Unique Equilibrium points for (4.3)-(4.4) with different A_1 values. Parameters used: $d_k = 6$, $d_l = 5$, $A_2 = 2$, $\phi = 0.5$, $\delta_1 = 2$, $\delta_2 = 1$, $a_1 = 0.9$, $a_2 = 0.1$, $b_1 = 1$, $b_2 = 5$. Initial value: $(K_1(0), K_2(0), L_1(0), L_2(0)) = (1, 0.1, 2, 0.3)$.

Chapter 5

Model with capital induced labor movement

5.1 Equilibrium and Stability Analysis

In this chapter we consider the full system (3.1) with $c > 0$. We only consider a special case of (3.1) that $A_1 = A_2 = A, a_1 = a_2 = a, b_1 = b_2 = b, \delta_1 = \delta_2 = \delta, d_k > 0, d_l > 0$ and $c > 0$. The system is symmetric in the sense that all parameters in the two regions are identical, so the economics in two regions will reach an identical equilibrium in isolation. If $c = 0$, Theorem 4.1 shows that all solutions converge to (K^*, K^*, L^*, L^*) when $t \to \infty$. Here we prove that when $c > 0$ is large, the symmetric equilibrium is no longer stable. The system (3.1) now takes the form

$$\begin{aligned}
\frac{dK_1}{dt} &= d_k(K_2 - K_1) + A_1 K_1^{\phi} L_1^{1-\phi} - \delta_1 K_1, \\
\frac{dK_2}{dt} &= d_k(K_1 - K_2) + A_2 K_2^{\phi} L_2^{1-\phi} - \delta_2 K_2, \\
\frac{dL_1}{dt} &= d_l(L_2 - L_1) + a_1 L_1 - b_1 L_1^2 - cH(L_1, L_2, K_1, K_2), \\
\frac{dL_2}{dt} &= d_l(L_1 - L_2) + a_2 L_2 - b_2 L_2^2 + cH(L_1, L_2, K_1, K_2),
\end{aligned} \tag{5.1}$$

where $H(L_1, L_2, K_1, K_2)$ is defined in (3.3).

Theorem 5.1. *Suppose* $A_1 = A_2 = A, a_1 = a_2 = a, b_1 = b_2 = b, \delta_1 = \delta_2 = \delta, d_k > 0, d_l > 0$, *then when* $0 \leq c \leq m$, *the symmetric equilibrium point* (K^*, K^*, L^*, L^*) *is locally asymptotically stable, and when* $c > m$, *the equilibrium point is unstable where* $m = \frac{(2d_k - a_{11})(2d_l - a_{22})}{2a_{12}L^*}$ *and*

$$\begin{aligned} a_{11} &= A\phi(K^*)^{\phi-1}(L^*)^{1-\phi} - \delta, \\ a_{12} &= A(K^*)^{\phi}(1-\phi)(L^*)^{-\phi}, \quad a_{22} = a - 2bL^*. \end{aligned} \tag{5.2}$$

Proof. For stability analysis, we employ Jacobian Matrix to study whether the equilibrium is stable or not. The equilibrium is stable if all of the eigenvalues have negative parts. Otherwise, the equilibrium is unstable. The value of the parameter at which the stability changes from stable to unstable or vice versa, is called bifurcation point.

We linearize the system (3.1) and find the Jacobian for (K^*, K^*, L^*, L^*) to be:

$$A(K_1, L_1, K_2, L_2) = \begin{pmatrix} a_{11} - d_k & a_{12} & d_k & 0 \\ -cH_{K_1} & a_{22} - d_l - cH_{L_1} & -cH_{K_2} & d_l - cH_{L_2} \\ d_k & 0 & a_{11} - d_k & a_{12} \\ cH_{K_1} & d_l + cH_{L_1} & cH_{K_2} & a_{22} - d_l + cH_{L_2} \end{pmatrix}. \tag{5.3}$$

where

$$\begin{aligned} a_{11} &= A\phi(K^*)^{\phi-1}(L^*)^{1-\phi} - \delta \\ a_{12} &= A(K^*)^{\phi}(1-\phi)(L^*)^{-\phi} \\ a_{22} &= a - 2bL^* \end{aligned} \tag{5.4}$$

To calculate the determinant of the matrix, we need to calculate

$$det(A) = \begin{pmatrix} a_{11} - d_k - \lambda & a_{12} & d_k & 0 \\ -cH_{K_1} & a_{22} - d_l - cH_{L_1} - \lambda & -cH_{K_2} & d_l - cH_{L_2} \\ d_k & 0 & a_{11} - d_k - \lambda & a_{12} \\ cH_{K_1} & d_l + cH_{L_1} & cH_{K_2} & a_{22} - d_l + cH_{L_2} - \lambda \end{pmatrix}. \tag{5.5}$$

The determinant will not change if we add row two to row four and add row one to row three. (5.5) becomes

$$det(A) = \begin{pmatrix} a_{11} - d_k - \lambda & a_{12} & d_k & 0 \\ -cH_{K_1} & a_{22} - d_l - cH_{L_1} - \lambda & -cH_{K_2} & d_l - cH_{L_2} \\ a_{11} - \lambda & a_{12} & a_{11} - \lambda & a_{12} \\ 0 & a_{22} - \lambda & 0 & a_{22} - \lambda \end{pmatrix}. \tag{5.6}$$

Then we multiply column four by -1 and add to column two, and multiply column three by -1 to column one, (5.6) becomes

$$det(A) = \begin{pmatrix} a_{11} - 2d_k - \lambda & a_{12} & d_k & 0 \\ -cH_{K_1} + cH_{K_2} & a_{22} - 2d_l - cH_{L_1} + cH_{L_2} - \lambda & -cH_{K_2} & d_l - cH_{L_2} \\ 0 & 0 & a_{11} - \lambda & a_{12} \\ 0 & 0 & 0 & a_{22} - \lambda \end{pmatrix}. \tag{5.7}$$

$$det(A) = (a_{11}-\lambda)(a_{22}-\lambda)\left[(\lambda + 2d_k - a_{11})(\lambda + 2d_l - a_{22} + cH_{L_1} - cH_{L_2}) - a_{12}(-cH_{K_1} + cH_{K_2})\right] \tag{5.8}$$

We can conclude that $\lambda_1 = a_{11}$, $\lambda_2 = a_{22}$, and λ_3, λ_4 satisfying $(\lambda + 2d_k - a_{11})(\lambda + 2d_l - a_{22} + cH_{L_1} - cH_{L_2}) - a_{12}(-cH_{K_1} + cH_{K_2}) = 0$. We get

$$\begin{aligned} a_{11} &= A\phi(K^*)^{\phi-1}(L^*)^{1-\phi} - \delta, \\ &= \phi(A(K^*)^{\phi-1}(L^*)^{1-\phi} - \delta) - (1-\phi)\delta \\ &= -(1-\phi)\delta < 0, 0 < \phi < 1 \end{aligned} \tag{5.9}$$

Because $aL - bL^2 = L(a - bL) = 0$, for $a, b, L > 0$,

$$a_{22} = a - 2bL^* < 0 \tag{5.10}$$

There are two cases for stability analysis:

Case 1: When $c = 0$, $\lambda_3 = a_{11} - 2d_k < 0$, $\lambda_4 = a_{22} - 2d_l < 0$. Therefore,it is stable.

Case 2: When $c \neq 0$, the system is unstable if q in $\lambda^2 + p\lambda + q = 0$ is negative.

$$\begin{aligned} q &= (2d_k - a_{11})(2d_l - a_{22} + cH_{L_1} - cH_{L_2}) - a_{12}(-cH_{K_1} + cH_{K_2}), \\ &= (2d_k - a_{11})(2d_l - a_{22}) + c(2d_k - a_{11})(H_{L_1} - H_{L_2}) + ca_{12}(H_{K_1} - H_{K_2}) \end{aligned} \tag{5.11}$$

Since from (3.3),

$$H(K_1, K_2, L_1, L_2) = \left(\frac{L_1 - L_2}{1 + e^{-h(K_2 - K_1)}} + L_2 \right) (K_2 - K_1), \tag{5.12}$$

we know that $K_2 - K_1 = 0$, therefore $(H_{L_1} - H_{L_2}) = 0$.

$$\begin{aligned} \frac{dH}{dK_1} &= -L_2, \\ \frac{dH}{dK_2} &= L_2, \end{aligned} \tag{5.13}$$

Thus $H_{K_1} - H_{K_2} = -2L_2$. $q = (2d_k - a_{11})(2d_l - a_{22}) - 2ca_{12}L^*$ where L^* is the equilibrium point for L. When $c > \frac{(2d_k - a_{11})(2d_l - a_{22})}{2a_{12}L^*}$, the equilibrium is a saddle point and $\lambda_3 < 0 < \lambda_4$. Since not all eigenvalues are negative, the system becomes unstable. When $c \leq m = \frac{(2d_k - a_{11})(2d_l - a_{22})}{2a_{12}L^*}$, the equilibrium is a sink and the system is stable, but when $c > m$, the equilibrium becomes unstable. □

5.2 Convergence of asymmetric equilibrium

Next we use numerical simulation to show that when capital induced labor movement is present, solutions may not always converge to the symmetric equilibrium (K^*, K^*, L^*, L^*). We first observe that when we change t from 0 to 20 and plot the graph t vs *equilibrium points for the system*, the result 5.1

This is an symmetric case with $A_1 = A_2$, $\delta_1 = \delta_2$, $a_1 = a_2$ and $b_1 = b_2$. We can see from the graph that when c is small, two regions with different capital and labor amount in the beginning will eventually converge to the same labor steady state and the same capital steady state. Since region 2 has more capital at the start, the capital will flow into

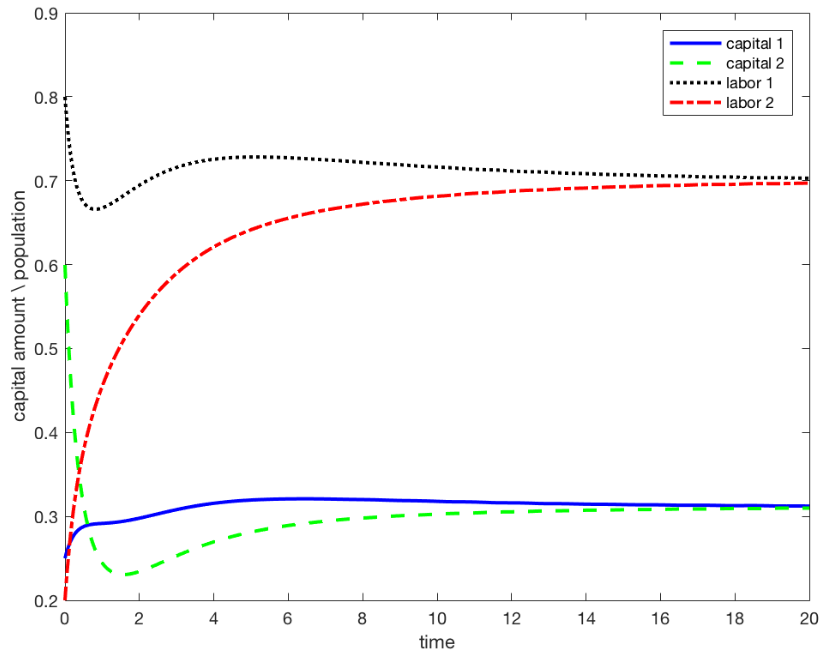

Figure 5.1: Equilibrium points for $c = 1.5$. The solution approaches $(K_1, K_2, L_1, L_2) = (0.3111, 0.3111, 0.7000, 0.7000)$. Parameters used: $d_k = 0.1$, $d_l = 0.2$, $A = 2$, $\phi = 0.5$, $\delta = 3$, $a = 0.7$, $b = 1$. Initial value: $(K_{10}, K_{20}, L_{10}, L_{20}) = (0.25, 0.6, 0.8, 0.2)$.

region 1 and make its capital increase. Therefore, capital in region 1 increases and capital in region 2 decreases when t is small. They converge to the same amount of capital when t is large. Since region 1 has more capital at the start, the capital will flow into region 2 and make its capital increase. Therefore, capital in region 1 decreases and capital in region 2 increases when t is small. They converge to the same amount of capital when t is large.

The steady states of capital in two regions are the same when c is small. However, when c is large, the steady states of capital in two regions are no longer the same. Capital in region 2 will decrease and then increase. Labor in region 1 will increase and then decrease to a steady state that is much lower than of region 1. That makes sense because

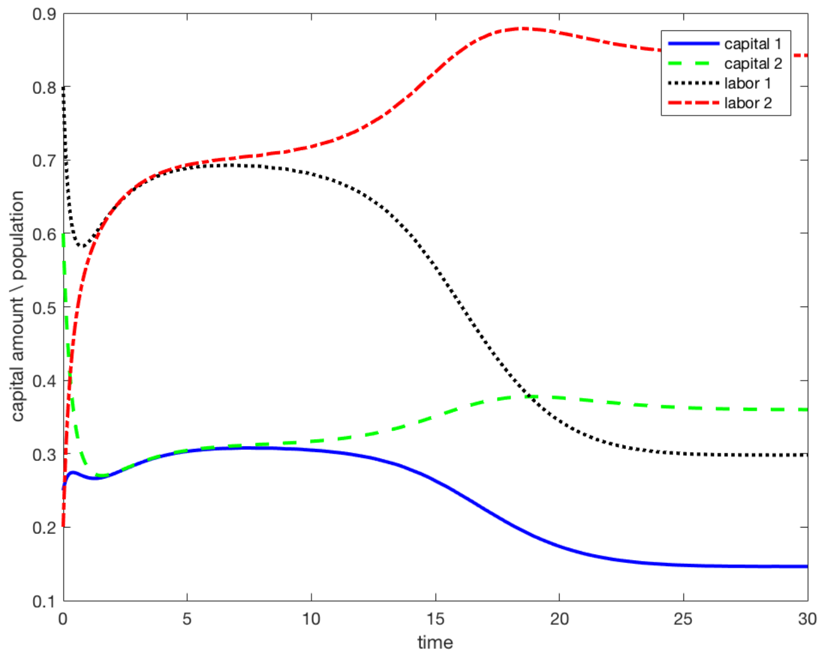

Figure 5.2: Equilibrium points for $c = 3.5$. The solution approaches $(K_1, K_2, L_1, L_2) = (0.3111, 0.3111, 0.7000, 0.7000)$. Parameters used: $d_k = 0.1$, $d_l = 0.2$, $A = 2$, $\phi = 0.5$, $\delta = 3$, $a = 0.7$, $b = 1$. Initial value: $(K_{10}, K_{20}, L_{10}, L_{20}) = (0.25, 0.6, 0.8, 0.2)$.

region 2 has more capital than region 1, and d_k will allow capital in region 2 to flow into region 1. However, when c is large, it breaks this capital synchronization and large amount of people flow into region 2 because of capital-induced labor movement. Since $K^* = L^*(\frac{A}{\delta})^{\frac{1}{1-\delta}}$, the flow of people into region 2 will increase region 2's labor force and thus increase capital in region 2.

The parameter c represents that capital-induced labor flow rate on the system. In other words, the labor exchange rate of a region is proportional to the capital difference between two regions and the population of that region. We want to know the effects of parameter c on the whole system. By holding other parameters fixed, we want to study how equilibrium points of the system change by changing c.

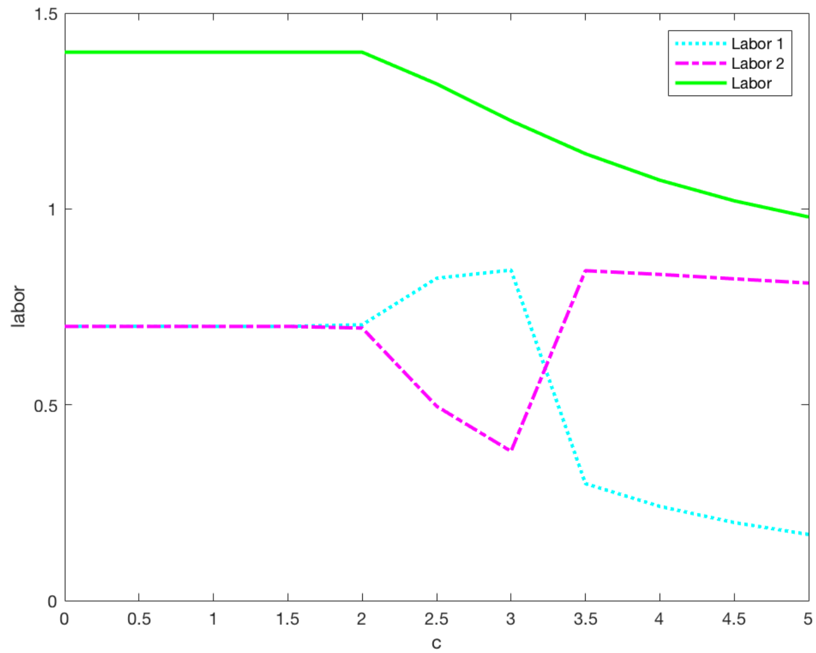

Figure 5.3: Equilibrium points of labor in two regions for different c values. Parameters used: $d_k = 0.1$, $d_l = 0.2$, $A = 2$, $\phi = 0.5$, $\delta = 3$, $a = 0.7$, $b = 1$. Initial value: $(K_{10}, K_{20}, L_{10}, L_{20}) = (0.25, 0.6, 0.8, 0.2)$.

We change c from 0 to 5 by increasing 0.5 each time and plot the graph c vs *equilibrium points of labor in two regions* and c vs *equilibrium points of capital in two regions* The result is shown in the following graphs for different initial values.

We can see that steady states of labor and capital in two regions are the same when c is less than 2. As c is beyond 2, steady state of labor in two regions is no longer the same. This is consistent with Theorem 5.1 when we substitute values and get cutting off point is 2.00. We observe that when c is large and $L_{10} > L_{20}$ and $K_{10} < K_{20}$, the equilibrium points of capital and labor of regions would be sensitive to the choice of initial values. The capital and labor of region 1 can be either larger or smaller than region 2 at equilibrium state, and there could exist switch-over of capital and labor amount when we

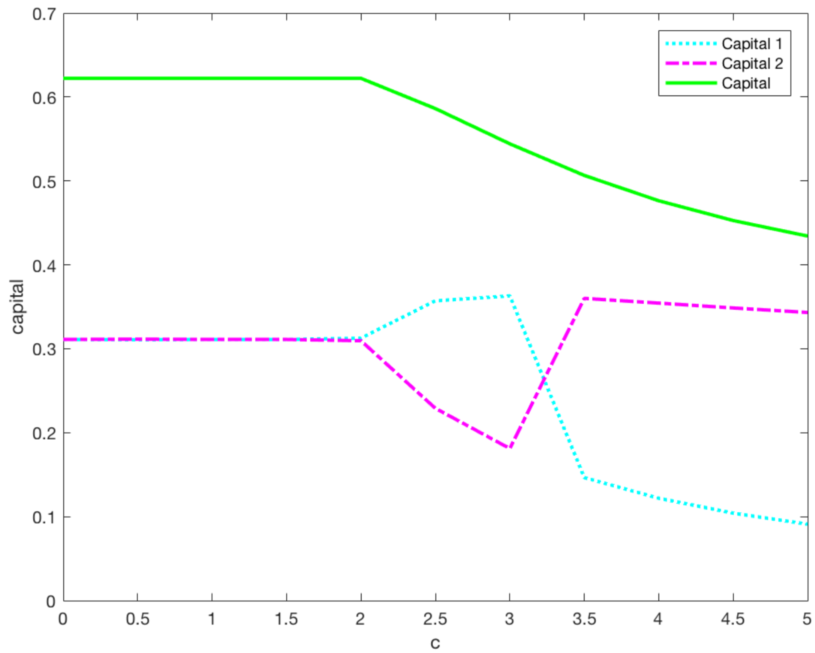

Figure 5.4: Equilibrium points of capital in two regions for different c values. Parameters used: $d_k = 0.1$, $d_l = 0.2$, $A = 2$, $\phi = 0.5$, $\delta = 3$, $a = 0.7$, $b = 1$. Initial value: $(K_{10}, K_{20}, L_{10}, L_{20}) = (0.25, 0.6, 0.8, 0.2)$.

increase c (see Figure 5.3 and 5.4). Nevertheless, the total capital and labor amount would always decrease, which implies that capital induced labor movement no longer benefits the economy.

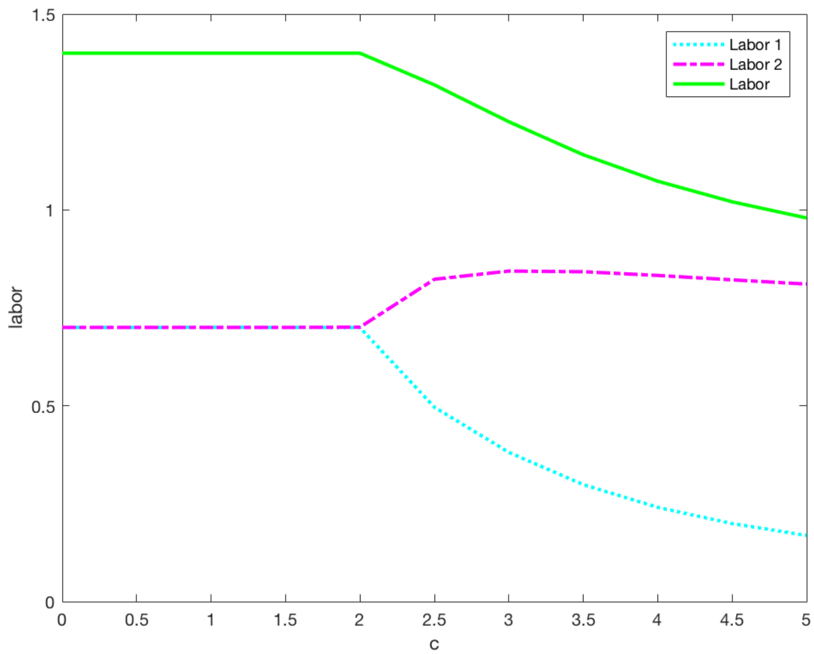

Figure 5.5: Equilibrium points of labor in two regions for different c values. Parameters used: $d_k = 0.1$, $d_l = 0.2$, $A = 2$, $\phi = 0.5$, $\delta = 3$, $a = 0.7$, $b = 1$. Initial value: $(K_{10}, K_{20}, L_{10}, L_{20}) = (0.25, 0.6, 0.7, 0.3)$.

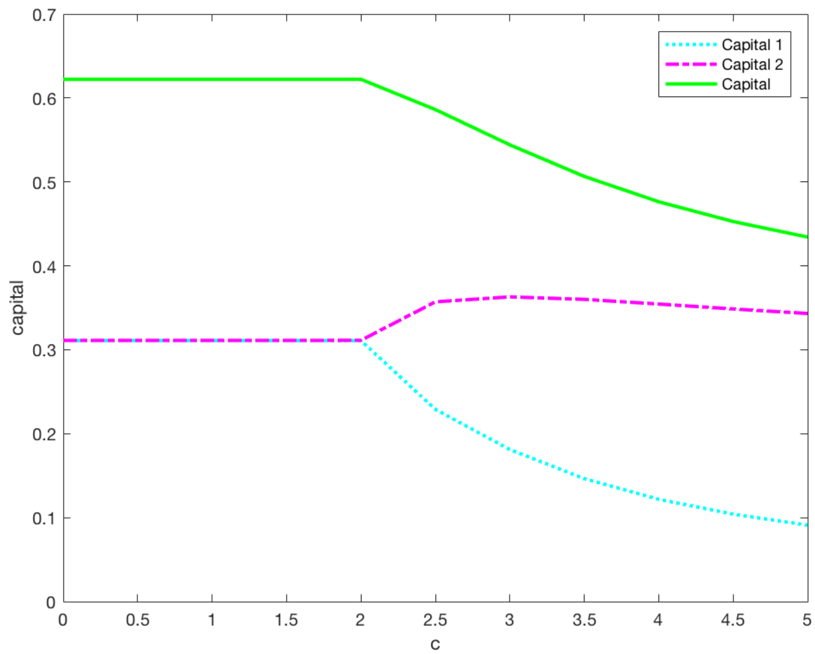

Figure 5.6: Equilibrium points of capital in two regions for different c values. Parameters used: $d_k = 0.1$, $d_l = 0.2$, $A = 2$, $\phi = 0.5$, $\delta = 3$, $a = 0.7$, $b = 1$. Initial value: $(K_{10}, K_{20}, L_{10}, L_{20}) = (0.25, 0.6, 0.7, 0.3)$.

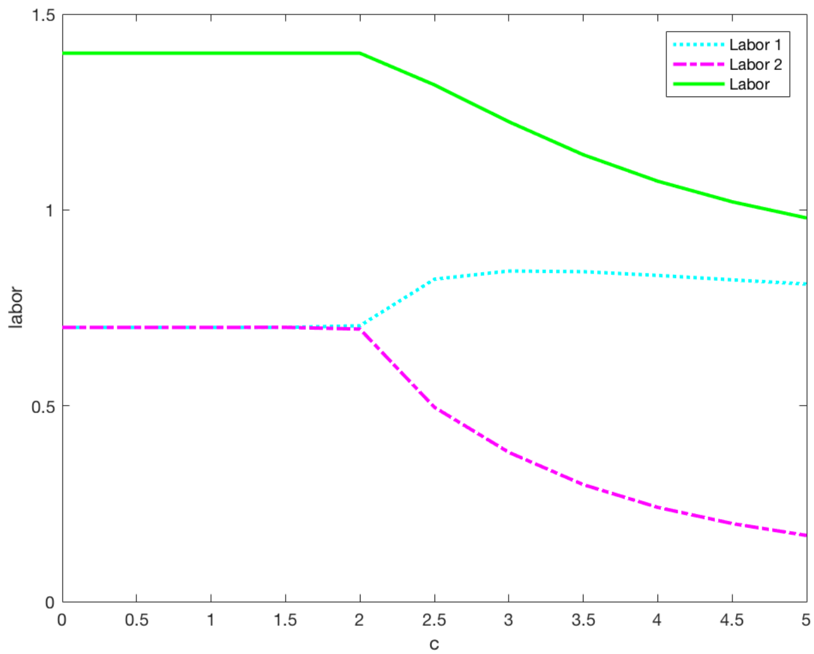

Figure 5.7: Equilibrium points of labor in two regions for different c values. Parameters used: $d_k = 0.1$, $d_l = 0.2$, $A = 2$, $\phi = 0.5$, $\delta = 3$, $a = 0.7$, $b = 1$. Initial value: $(K_{10}, K_{20}, L_{10}, L_{20}) = (0.25, 0.6, 0.9, 0.1)$.

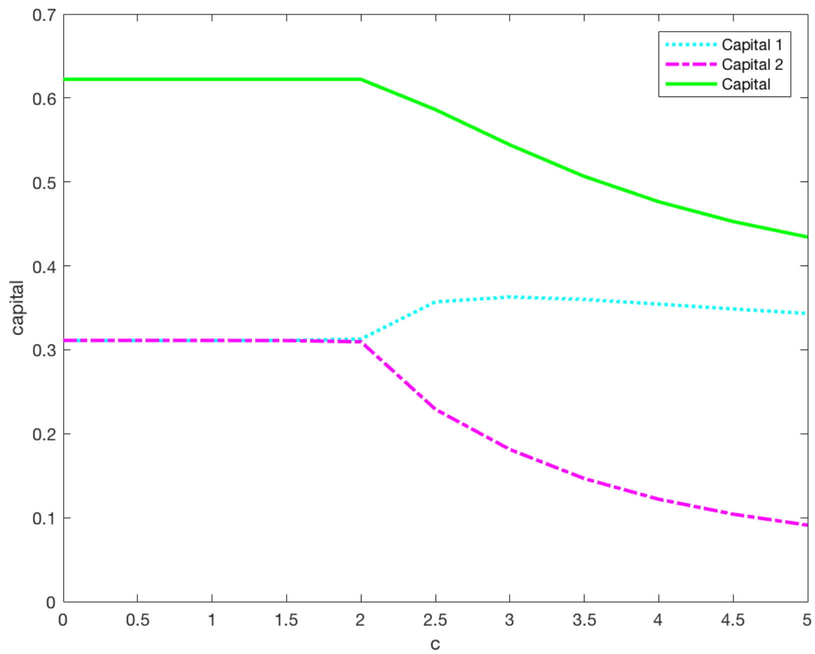

Figure 5.8: Equilibrium points of capital in two regions for different c values. Parameters used: $d_k = 0.1$, $d_l = 0.2$, $A = 2$, $\phi = 0.5$, $\delta = 3$, $a = 0.7$, $b = 1$. Initial value: $(K_{10}, K_{20}, L_{10}, L_{20}) = (0.25, 0.6, 0.9, 0.1)$.

Chapter 6

Conclusion

We aim to study the how capital induced labor movement between two geographic regions such as states or countries would affect their economic systems. By constructing mathematical models and doing quantitative analysis, we hope to provide insight for some economic issues.

Based on classical Solow Economic Growth Model and Logistic Population Model, we propose a four-variable ODE model to describe the economical and population growth in two regions connected through capital and labor movement. We assume that capital flow is proportional to two regions' capital difference and labor flow is proportional to two regions' capital and labor difference.

We analyze the model by studying its equilibrium points and stability, and simulated the model in `Matlab`. We find that when there is no capital induced labor movement, the system always reaches a unique positive point, no matter what initial condition is and the equilibrium point is globally stable. On the other hand, labor or capital diffusion rates can influence the amplitude of equilibrium. When either of the diffusion rate increases, the total labor or the total capital eventually decreases. But overall, the economic growth of two regions is better in an open economy with labor movement than that of a closed economy without labor movement. But this does not hold true for the capital exchange.

The overall economic growth of two regions is worse in an open economy with capital movement that that of a closed economy without capital movement, We also study the case when there is a capital induced labor movement. We find that when the capital induced labor movement rate is small, the two regions will reach the same positive equilibrium point and it is stable. The total capital remains the same and capital-induced labor movement does not affect overall economic growth of two regions. However, when the capital induced labor movement rate is large, even when the two region have similar growth conditions and initial capital amounts, it can cause the capital and labor to concentrate in one region, which leads to imbalance in economic development. The total amount of capital and labor in two regions will also decrease. Therefore, when capital-induced labor movement is large, it decreases overall economic growth of two regions.

In the future, we hope to collect real data of two regions such as capital and labor in America and Mexico to verify our conclusions. From quantitative analysis in `Matlab`, we can see that no matter with or without capital induced labor movement, the total capital and labor in two regions will eventually decrease. It contradicts with our intuition and we want to figure out the reason. Moreover, in this research, we study the symmetric cases in capital induced labor movement. We also want to study the equilibrium conditions in asymmetric cases, which will be far more complicated.

Chapter 7

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