Periodic solutions

A periodic solution is a solution $(x(t), y(t))$ of

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

such that $x(t+T) = x(t)$ and $y(t+T) = y(t)$ for any t, where T is a fixed number which is a period of the solution.

Periodicity in biology:
life proceeds in a rhythmic and periodic style...
the periodic patterns are not easily disrupted or changed by a noisy and random environment.

First example:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x$$

Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$

$r' = 0$, $\theta' = -1$

Second example:

$$\frac{dx}{dt} = y + x(1 - x^2 - y^2), \quad \frac{dy}{dt} = -x + y(1 - x^2 - y^2)$$

Polar coordinates: $r' = r(1 - r^2)$, $\theta' = -1$

(Logistic equation for periodic solutions)

General polar coordinate equation:

$$\frac{dx}{dt} = \lambda(r)x + w(r)y, \quad \frac{dy}{dt} = -w(r)x + \lambda(r)y$$

Polar coordinates: $r' = r\lambda(r)$, $\theta' = w(r)$

Examples: Predator-prey models (lynx-hare population, shark-fish population,), nerve conduction (Hodgkin-Huxley equations, Fitzhugh-Nagumo model), chemical reactions

Lokta-Volterra predator-prey equation

$$\frac{dx}{dt} = ax - bxy$$
$$\frac{dy}{dt} = -cx + dxy$$

Critiques of the equation: there are infinitely many periodic solutions, and all of them are neutrally stable. Ideally, the system should have a **limit cycle** which all solutions tend to.

Limit cycle: a periodic solution which is stable. (if the initial point is inside and near the cycle, then the solution starting from there spirals out and tends to it; if the point is outside and near the cycle, then the solution starting from there spirals out and tends to it.)

Predator-prey model with Holling type II predation
(saturating interaction)

$$\frac{du}{ds} = u(1-u) - \frac{auv}{u+b}, \quad \frac{dv}{ds} = -cv + \frac{duv}{u+b}. \tag{1}$$

u-nullcline: $u = 0$, $v = \dfrac{(1-u)(u+b)}{a}$

v-nullcline: $v = 0$, $u = \dfrac{bc}{d-c}$

Equilibrium points: $(0,0)$, $(1,0)$, $(\dfrac{bc}{d-c}, \dfrac{bd(d-c-bc)}{a(d-c)^2})$.

Dynamics when $a = d = 1$:

u-nullcline: $u = 0$, $v = (1 - u)(u + b)$
v-nullcline: $v = 0$, $u = \dfrac{bc}{1 - c}$
Equilibrium points: $(0,0)$, $(1,0)$, $(\dfrac{bc}{1-c}, \dfrac{b(1-c-bc)}{(1-c)^2})$.

Case 1: $(c > \dfrac{1}{1+b})$ Two equilibrium points at $(0,0)$ and $(1,0)$, and $(1,0)$ is a sink which attracts any initial values.

Case 2: $(0 < c < \dfrac{1}{1+b})$ Three equilibrium points at $(0,0)$, $(1,0)$, $(u_0, v_0) = (\dfrac{bc}{1-c}, \dfrac{b(1-c-bc)}{(1-c)^2})$. Stability of (u_0, v_0) ?

A brief theory of periodic orbits

Theorem 1 Inside a periodic orbit, there is at least one equilibrium point.

Corollary 2 Let (x_0, y_0) be an equilibrium point. If there is a stable or unstable orbit of this equilibrium going to infinity as $t \to \pm\infty$, then there is no periodic orbit around this equilibrium point.

Corollary 3 Let (x_0, y_0) be an equilibrium point. If the direction of vector field on the nullclines near (x_0, y_0) is not clockwise or counterclockwise, then there is no periodic orbit around this equilibrium point.

Theorem 4 (Poincaré-Bendixon) If there is a "donut" region (a region without equilibrium point) that the vector field is pointing inside at any point on the boundary, then there is a periodic orbit in the "donut" region.

Corollary 5 If in a region, there is only one equilibrium point which is a source or spiral source, and the vector field is pointing inside at any point on the boundary, then there is a periodic orbit around this equilibrium point.

A typical case: the solutions from outside are spiraling in, and the equilibrium point is a spiral source.

Theorem 6 (Poincaré-Andronov-Hopf bifurcation) If a bifurcation occurs such that an equilibrium point changes from a spiral sink to spiral source, and the solutions from outside are always spiraling in, then there is a periodic orbit around the spiral source.

Jules Henri Poincaré (1854-1912):
One of greatest mathematicians in 20th century
http://www-gap.dcs.st-and.ac.uk/ history/Mathematicians/Poincare.ht

$$\frac{du}{ds} = u(1-u) - \frac{uv}{u+b}, \quad \frac{dv}{ds} = -cv + \frac{uv}{u+b}. \tag{2}$$

Stability of (u_0, v_0):

$$J = \begin{pmatrix} \dfrac{c(1-c-bc-b)}{1-c} & -c \\ 1-c-bc & 0 \end{pmatrix}$$

Eigenvalue equation: $\lambda^2 - \dfrac{c(1-c-bc-b)}{1-c}\lambda + c(1-c-bc) = 0$

Spiral sink or sink: $1-c-bc-b < 0$ (or $c > \dfrac{1-b}{1+b}$)

Spiral source or source: $1-c-bc-b > 0$ (or $0 < c < \dfrac{1-b}{1+b}$)

Conclusion: $c = \dfrac{1-b}{1+b}$ is where Hopf bifurcation occurs. When $c > \dfrac{1-b}{1+b}$ but near $\dfrac{1-b}{1+b}$, (u_0, v_0) is a spiral sink, and all solutions tend to (u_0, v_0) in an oscillating fashion; when c passes $\dfrac{1-b}{1+b}$ but near $\dfrac{1-b}{1+b}$, (u_0, v_0) becomes a spiral source, but all solutions away from (u_0, v_0) still tends toward (u_0, v_0) (spiral inward), thus a periodic solution (with small oscillation) emerges around (u_0, v_0), and it is a limit cycle.

Examples:
$b = 0.5$, the Hopf bifurcation point is $c = 1/3 \approx 0.33333$
$b = 0.2$, the Hopf bifurcation point is $c = 2/3 \approx 0.66667$

Theorem: (proved around 1980) For

$$\frac{du}{ds} = u(1-u) - \frac{uv}{u+b}, \quad \frac{dv}{ds} = -cv + \frac{uv}{u+b}, \tag{3}$$

if $c > \frac{1}{1+b}$, then $(1,0)$ is asymptotically stable; if $\frac{1-b}{1+b} < c < \frac{1}{1+b}$, then (u_0, v_0) is asymptotically stable; and if $0 < c < \frac{1-b}{1+b}$, then there exists a unique periodic solution, and it attracts all positive initial values except the equilibrium points $(0,0)$, $(1,0)$ and (u_0, v_0).

Hilbert problem 16: Consider $u' = f(u,v)$, $v' = g(u,v)$. If f and g are polynomials, prove the system has only finite many limit cycles. When f and g are quadratic functions, prove there are at most three limit cycles. (Counterexample is found by S. Shi in 1979, four limit cycles).

Turing instability?

$$\begin{cases} \frac{\partial u}{\partial t} = d_u u_{xx} + u(1-u) - \frac{uv}{u+b}, \quad x \in (0,1), \\ \frac{\partial v}{\partial t} = d_v v_{xx} - cv + \frac{uv}{u+b}, \ \ x \in (0,1), \\ u_x(0,t) = u_x(1,t) = v_x(0,t) = v_x(1,t) = 0, \\ u(x,0) = u_0(x) \geq 0, \ \ v(x,0) = v_0(x) \geq 0, \ \ x \in (0,1). \end{cases} \tag{4}$$

Nope... since $g_v = 0$, we need $f_u \cdot g_v < 0$ for Turing instability.

But many Turing type bifurcations occurs when $0 < c < \dfrac{1-b}{1+b}$ along the equilibrium solution (u_0, v_0). The global dynamics of the reaction-diffusion system is still not known.

Neural modeling: excitable systems

FitzHugh-Nagumo system:

$$\begin{aligned} \epsilon\frac{dv}{dt} &= v(v-a)(1-v)-w, \\ \frac{dw}{dt} &= v-c-bw. \end{aligned} \tag{5}$$

$v(t)$: excitation variable, $w(t)$: potassium conductance.

$a = 0.1$, $b = 0.5$, $c = 0$, $\epsilon = 0.01$: equilibrium $(0,0)$ is asymptotically stable, but it is an excitable system where a large excursion occurs if a perturbation is beyond a threshold.

$a = 0.1$, $b = 0.5$, $c = 0.1$, $\epsilon = 0.01$: equilibrium $(0.1,0)$ is unstable, and a limit cycle exists (relaxation oscillation).

Hopf bifurcation:

A Hopf point is about $c = 0.0525$ where the eigenvalues of equilibrium point is pure imaginary. But this seems not a Hopf bifurcation, since a small periodic solution does not emerge near equilibrium, but a periodic solution with large amplitude appears!

In fact, a Hopf bifurcation does occurs, but it is subcritical, so for c slightly less than $c = 0.0525$, there are two periodic solutions!