A review of stability and dynamical behaviors of differential equations:

scalar ODE: $u_t = f(u)$, system of ODEs: $\begin{cases} u_t = f(u,v), \\ v_t = g(u,v), \end{cases}$

reaction-diffusion equation:
$u_t = D\Delta u + f(u)$, $x \in \Omega$, with boundary condition

reaction-diffusion system:
$\begin{cases} u_t = D_u \Delta u + f(u,v), \\ v_t = D_v \Delta v + g(u,v), \end{cases}$, $x \in \Omega$, with boundary condition

All equation is in form of $U_t = F(U)$, where u can be a scalar or vector, spatial independent or dependent

Abstract Equation $U_t = F(U)$

Equilibrium solution: U_0 such that $F(U_0) = 0$,
linearized operator: $F'(U_0)$

U_0 is **stable** if the eigenvalues of equation $F'(U_0)w = \lambda w$ are all with negative real parts.

(Linear behavior) Since the equation $U_t = F(U)$ is approximately the linearized equation $U_t = F'(U)(U - U_0)$ near the equilibrium solution $U = U_0$, then $U(t) \approx \sum C_i \exp(\lambda_i t)\phi_i$ near $U = U_0$, where (λ_i, ϕ_i) are the eigenvalue-eigenvector pairs of $F'(U_0)w = \lambda w$.

If $\lambda_1 = \max \lambda_i$, then $U(t) \approx C_1 \exp(\lambda_1 t)\phi_1$ if $U(0) \approx U_0$.

Example 1: Scalar equation

$u_t = f(u)$, $f(u_0) = 0$, linearized operator: $f'(u_0)$

$f'(u_0) < 0$ stable, otherwise unstable

Example 2: Linear system

$\begin{cases} u_t = 2u + 3v, \\ v_t = 4u + 3v, \end{cases}$, linearized system: $J = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix}$

Eigenvalue problem: $\begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix} w = \lambda w$,

eigen-pairs: $\lambda_1 = -1$, $\phi_1 = (1, -1)$; $\lambda_2 = 6$, $\phi_2 = (3, 4)$

Solution: $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \approx c_2 e^{6t} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ if perturbed from equilibrium.

Example 3: Nonlinear ODE system:

$\begin{cases} x_t = 2x - x^2 - xy, \\ y_t = 3y - y^2 - 2xy, \end{cases}$, linearized operator: $J = \left(\begin{array}{cc} 2 - 2x - y & -x \\ -2y & 3 - 2y - 2 \end{array} \right.$

Equilibrium points: $(0,0)$, $(2,0)$, $(3,0)$ and $(1,1)$

For $(1,1)$, eigenvalues $\lambda_1 = -1 + \sqrt{2}$, $\phi_1 = (0.58, -0.82)$; $\lambda_2 = -1 - \sqrt{2}$, $\phi_2 = (0.58, 0.82)$

So near the saddle point $(1,1)$, $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_1 e^{(-1+\sqrt{2})t} \begin{pmatrix} 0.58 \\ -0.82 \end{pmatrix}$ $c_2 e^{(-1-\sqrt{2})t} \begin{pmatrix} 0.58 \\ 0.82 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_1 e^{(-1+\sqrt{2})t} \begin{pmatrix} 0.58 \\ -0.82 \end{pmatrix}$ when perturbed from equilibrium.

Example 4: scalar diffusion equation
$u_t = Du_{xx}$, $x \in (0, \pi)$, $t > 0$, $u(t, 0) = u(t, \pi) = 0$.

Equilibrium solution: $u(x) = 0$.
Linearized operator: $F'(u)[w] = Dw_{xx}$ for w satisfying $w(0) = w(\pi) = 0$,
Eigenvalue problem: $Dw_{xx} = \lambda w$, $x \in (0, \pi)$, $w(0) = w(\pi) = 0$.

Eigenvalue-eigenfunction pairs:
$\lambda_m = -Dm^2$, $\phi_m(x) = \sin(mx)$, $m = 1, 2, \cdots$.

Solution: $u(x,t) = \sum c_m \exp(-Dm^2 t) \sin(mx) \approx c_1 \exp(-Dt) \sin(x)$ near $u = 0$, and $u(x) = 0$ is a stable equilibrium solution

Example 5: scalar diffusive Malthusian equation
$u_t = Du_{xx} + au$, $x \in (0, \pi)$, $t > 0$, $u(t, 0) = u(t, \pi) = 0$.

Equilibrium solution: $u(x) = 0$.
Linearized operator: $F'(u)[w] = Dw_{xx} + aw$ for w satisfying $w(0) = w(\pi) = 0$,
Eigenvalue problem: $Dw_{xx} + aw = \lambda w$, $x \in (0, \pi)$, $w(0) = w(\pi) = 0$.

Eigenvalue-eigenfunction pairs:
$\lambda_m = a - Dm^2$, $\phi_m(x) = \sin(mx)$, $m = 1, 2, \cdots$.

Solution: $u(x, t) = \sum c_m \exp((a - Dm^2)t) \sin(mx) \approx c_1 \exp((a - D)t) \sin(x)$ near $u = 0$. $u(x) = 0$ is a stable equilibrium solution if $a < D$, and it is unstable if $a > D$.
(Thus $a = D$ is a potential bifurcation point.)

Example 6: Fisher equation
$u_t = Du_{xx} + au(1-u)$, $x \in (0,\pi)$, $t > 0$, $u(t,0) = u(t,\pi) = 0$.

Equilibrium solution: $u(x) = 0$.
Linearized operator: $F'(u)[w] = Dw_{xx} + aw$ for w satisfying $w(0) = w(\pi) = 0$,
Eigenvalue problem: $Dw_{xx} + aw = \lambda w$, $x \in (0,\pi)$, $w(0) = w(\pi) = 0$.

Eigenvalue-eigenfunction pairs:
$\lambda_m = a - Dm^2$, $\phi_m(x) = \sin(mx)$, $m = 1, 2, \cdots$.

Solution: $u(x,t) = \sum c_m \exp((a - Dm^2)t)\sin(mx) \approx c_1 \exp((a - D)t)\sin(x)$ near $u = 0$. $u(x) = 0$ is a stable equilibrium solution if $a < D$, and it is unstable if $a > D$.
(Thus $a = D$ is a bifurcation point.)

Fisher equation: (cont.)
For $a > D$, Fisher equation has a positive equilibrium solution $u_a(x)$ (from local bifurcation theory, and global bifurcation of time-mapping).

$u_a(x)$ is a stable equilibrium solution.
(see lecture notes for a proof)

Diffusion is a stabilizing process?
In Fisher equation: $u_t = Du_{xx} + au(1-u)$, $x \in (0, \pi)$, $t > 0$, $u(t, 0) = u(t, \pi) = 0$.

Equilibrium solution $u = 0$ is also an equilibrium solution of $u_t = au(1-u)$, but an unstable one (since $f'(0) > 0$). (eigenvalue $\lambda = af'(0) > 0$)

Equilibrium solution $u(x) = 0$ is stable for Fisher equation when $a < D$, and it is still unstable when $a > D$. (eigenvalues: $\lambda_m = af'(0) - Dm^2$)

Thus diffusion makes the eigenvalue "more negative" and diffusion in scalar equation has a stabilizing effect. (If stable for ODE, also stable for PDE; even unstable for ODE, still can be stable for PDE.)

Next model (most useful one): reaction-diffusion system

$u_t = D_u u_{xx} + f(u,v)$, $v_t = D_v v_{xx} + g(u,v)$,
$u_x(t,0) = u_x(t,1) = 0$, $v_x(t,0) = v_x(t,1) = 0$

If (u_0, v_0) is an equilibrium solution that $f(u_0,v_0) = g(u_0,v_0) = 0$, then (u_0,v_0) is also an equilibrium solution of $u_t = f(u,v)$, $v_t = g(u,v)$.

Linearized operator: ODE: Jacobian $J = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$
PDE: $diag(d_u\Delta, d_v\Delta) + J = \begin{pmatrix} d_u\Delta & 0 \\ 0 & d_v\Delta v \end{pmatrix} + \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$

Eigenvalue problem: $\begin{cases} D_u\Delta\phi + f_u(u_0,v_0)\phi + f_v(u_0,v_0)\psi = \lambda\phi, \\ D_v\Delta\psi + g_u(u_0,v_0)\phi + g_v(u_0,v_0)\psi = \lambda\psi, \\ \phi_x(0) = \phi_x(1) = 0, \ \ \psi_x(0) = \psi(1) = 0. \end{cases}$

If all eigenvalues of J are with negative real parts (so (u_0, v_0) is a stable equilibrium solution for ODE), is (u_0, v_0) also a stable equilibrium solution for PDE?

It seems so since the additional part is consist of diffusion operators only, and diffusion is supposed to stabilizing......

But, as Alan Turing pointed out, (u_0, v_0) could be an unstable equilibrium solution for PDE even if it is stable for ODE! So diffusion has an unstable effect for such system.

How is that possible? Let's calculate now......