Problem Set 3

Discussion Problems Discussion: Sept. 22

1. (a) Four-digit number $S = aabb$ is a square. Find it; (*hint: 11 is a factor of S*)
   (b) If n is a sum of two square, so is $2n$. (*hint: simple algebra*)

2. (a) If n is an even number, then $323|20^n + 16^n - 3^n - 1$; (*hint: factorize 323*)
   (b) If n is an integer, then $9|4^n + 15n - 1$. (*hint: consider cases when n modulus 3*)

3. (a) If $2n + 1$ and $3n + 1$ are squares, then $5n + 3$ is not a prime;
   (b) If $3n + 1$ and $4n + 1$ are squares, then $56|n$. (*hint: follow the idea in presentation problem*)

4. If p is a prime, then $p^2 \equiv 1(mod24)$; (*hint: prove* $24|p^2 - 1$)

5. (a) (VT 1979) Show that for all positive integers n, that 14 divides $3^{4n+2} + 5^{2n+1}$;
   (b) (VT 1981) $2^{48} - 1$ is exactly divisible by what two numbers between 60 and 70?
   (*hint: (a)* $14 = 2 \cdot 7$, *(b) factorizing*)

6. (a) (VT 1982) What is the remainder when $X^{1982} + 1$ is divided by $X - 1$? Verify your answer;
   (b) (MIT training 2 star) Let n be an integer greater than one. Show that $n^4 + 4^n$ is not prime. (*hint: there is a magic identity due to Sophie Germain:* $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$)

7. (VT 1988) Let a be a positive integer. Find all positive integers n such that $b = a^n$ satisfying the condition that $a^2 + b^2$ is divisible by $ab + 1$. (*hint: prove that* $a^m + 1|a^n + 1$, *then* $m|n$.)

8. (VT 2005) Find the largest positive integer n with the property that $n + 6(p^3 + 1)$ is prime whenever p is a prime number such that $2 \leq p < n$. Justify your answer.

More Problems:

1. (Putnam 1986-A2) What is the units (*i.e.*, rightmost) digit of $\left[\frac{10^{20000}}{10^{100}+3}\right]$? Here $[x]$ is the greatest integer $\leq x$.

2. (Putnam 1998-A4) Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

3. (Putnam 1998-B6) Prove that, for any integers a, b, c, there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.

4. (Putnam 1985-A4) Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

5. (Putnam 1955-B4) Do there exist 1,000,000 consecutive integers each of which contains a repeated prime factor?

6. (Putnam 1956-A2) Prove that every positive integer has a multiple whose decimal representation involves all ten digits.

7. (Putnam 1966-B2) Prove that among any ten consecutive integers at least one is relatively prime to each of the others.

8. (MIT training 2.5 star) Let d be any divisor of an integer of the form $n^2 + 1$. Prove that $d - 3$ is not divisible by 4.

9. (MIT training 3 star) What is the last nonzero digit of 10000!?

10. (MIT training 2.5 star) Let n be an integer, and suppose that $n^4 + n^3 + n^2 + n + 1$ is divisible by k. Show that either k or $k - 1$ is divisible by 5.