Problem Set 2

Discussion: Sept. 15

1. 15 people sit around a table. When they sit down, they did not notice that a name tag is in front of each seat, and they found that no any name tag and the person sitting there match each other. Prove that after some rotation of the sitting order, at least two people will match the name tag where they sit.

2. Six circles with radius 1 is randomly put inside of a circle with radius 6. Prove that at least one more circle with radius 1 can be put inside the big circle without intersecting the other six.

3. A city has 10000 different telephone lines numbered by 4-digit numbers. More than half of the telephone lines are in the downtown. Prove that there are two telephone numbers in the downtown whose sum is again the number of a downtown telephone line.

4. Suppose a musical group has 11 weeks to prepare for opening night, and they intend to have at least one rehearsal each day. However, they decide not to schedule more than 12 rehearsals in any 7-day period, to keep from getting burned out. Prove that there exists a sequence of successive days during which the band has exactly 21 rehearsals.

5. (UIUC 2000) Suppose that $a_1, a_2, \cdots, a_n$ are n given integers. Prove that there exist integers r and s with $0 \leq r < s \leq n$ such that $a_{r+1} + a_{r+2} + \cdots + a_s$ is divisible by n.

6. (a) Show that among any 9 points in a triangle of area 1, there are 3 points that form a triangle of area at most 1/4. (b) Show that given any 9 points in a triangle of area 1, there is a triangle of area at least 1/12 that does not contain any of those 9 points in its interior. (Can you improve 1/12?)

7. The Fibonacci sequence is defined by $a_1 = 1$, $a_2 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Prove that for any integer m, there exists a_k such that a_k ends with m zeros.

8. (Putnam 1994-A4) Let A and B be 2×2 matrices with integer entries such that $A, A+B, A+2B, A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.