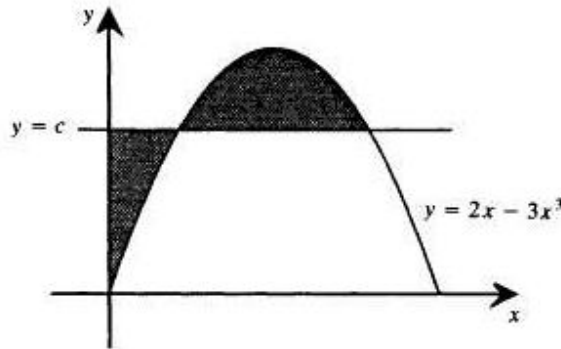


## Problem Set 1

Discussion: Sept. 8.

- (VT, 1992-2) Assume that  $x_1 > y_1 > 0$  and  $y_2 > x_2 > 0$ . Find a formula for the shortest length  $l$  of a planar path that goes from  $(x_1, y_1)$  to  $(x_2, y_2)$  and that touches both the  $x$ -axis and the  $y$ -axis. Justify your answer.
- (VT, 2005-4) A cubical box with sides of length 7 has vertices at  $(0,0,0)$ ,  $(7,0,0)$ ,  $(0,7,0)$ ,  $(7,7,0)$ ,  $(0,0,7)$ ,  $(7,0,7)$ ,  $(0,7,7)$ ,  $(7,7,7)$ . The inside of the box is lined with mirrors and from the point  $(0,1,2)$ , a beam of light is directed to the point  $(1,3,4)$ . The light then reflects repeatedly off the mirrors on the inside of the box. Determine how far the beam of light travels before it first returns to its starting point at  $(0,1,2)$ .
- (VT, 1984-2) Consider any three consecutive positive integers. Prove that the cube of the largest cannot be the sum of the cubes of the other two.
- (VT, 2008-2) How many sequences of 1's and 3's sum to 16? (Examples of such sequences are  $\{1, 3, 3, 3, 3, 3\}$  and  $\{1, 3, 1, 3, 1, 3, 1, 3\}$ .)
- (Putnam, 1993-A1) The horizontal line  $y = c$  intersects the curve  $y = 2x - 3x^3$  in the first quadrant as in the Figure. Find  $c$  so that the areas of the two shaded regions are equal.



- (Putnam, 2001-A1) Consider a set  $S$  and a binary operation  $*$ , *i.e.*, for each  $a, b \in S$ ,  $a * b \in S$ . Assume  $(a * b) * a = b$  for all  $a, b \in S$ . Prove that  $a * (b * a) = b$  for all  $a, b \in S$ .
- (Putnam, 2002-A2) Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.
- Prove that there exists a multiple of 2009 whose decimal expansion contains only digits 1 and 0.
- (MIT homework 9) (1.5 star) Find the missing term:  
10; 11; 12; 13; 14; 15; 16; 17; 20; 22; 24; 31; 100; ?; 10000
- (MIT homework 9) (1.5 star) Explain the rule which generates the following sequence:  
2; 3; 10; 12; 13; 20; 21; 22; 23; 24; 25; 26; 27; 28; 29; 30; 31; 32; 33; 34; 35; 36; 37; 38; 39; 200; 201; 202;  $\dots$   
(Hint: Don't think mathematically!)