Lecture 9 (Math 345)

Mixing Problem (Math 112, Math 302)

A tank contains 1000 L of salt water with 15 kg of dissolved salt. Salt water with 0.3 kg/L enters the tank at a rate of 5 L/min. ~~The sam~~ The salt mixes immediately and the same amount of mixture is drained from the tank (5L/min). How much salt is in the tank after t minutes?

$A(t)$ = amount of salt. $C(t)$ = concentration of salt $= \dfrac{A(t)}{V}$

$$\begin{cases} \dfrac{dA}{dt} = 5 \cdot 0.3 - 5 \cdot \dfrac{A}{1000} = 1.5 - 0.005A \\ A(0) = 15 \end{cases}$$

$0.3\ \mathrm{kg}/L = C_0$ (concentration of feed)

$1000\ L = V$ (volume)

$5L/\mathrm{min} = F$ (inflow rate = outfolw rate)

$$\frac{dA}{dt} = F \cdot C_0 - F \cdot \frac{A}{V}$$

$$\frac{d(C \cdot V)}{dt} = F \cdot C_0 - F \cdot C \quad \Rightarrow \quad \frac{dC}{dt} = \frac{F \cdot C_0}{V} - \frac{F \cdot C}{V}$$

"salt" $\longrightarrow$ "nutrient"

variable	dimension	parameter	dimension
t	T	C_0	ML^{-3}
$C(t)$	ML^{-3}	F	L^3T^{-1}
		V	L^3

Now combining bacteria and nutrient.

$$\frac{dN}{dt} = k(C)N \qquad \text{and} \qquad \frac{dC}{dt} = \frac{F\cdot C_0}{V} - \frac{F\cdot C}{V}$$

$$\frac{dc}{dt} = -\alpha k(C)N$$

Assume $k(C) = aC$

variable	dimension	parameter	dimension
t	T	C_0	ML^{-3}
C	ML^{-3}	~~F~~	~~L^3T^{-1}~~
N	PL^{-3}	~~V~~	~~L^3~~
		a	$L^3M^{-1}T^{-1}$
		α	MP^{-1}
		D	T^{-1}

$$\dim(k(C)) = T^{-1}$$

$$\dim(k(c)) = \dim a \cdot \dim C$$

$$T^{-1} = \dim a \cdot ML^{-3}$$

$$\dim\left(\frac{dc}{dt}\right) = ML^{-3}T^{-1} = -\dim(\alpha)\underbrace{k(C)}_{T^{-1}}\underbrace{N}_{PL^{-3}}$$

dimension check

$$\dim(k(c)) = \dim\left(\frac{F}{V}\right)$$

$$\begin{cases} \dfrac{dN}{dt} = k(c)N - \dfrac{FN}{V} \\[2ex] \dfrac{dc}{dt} = -\alpha k(C)N + \dfrac{FC_0}{V} - \dfrac{FC}{V} \end{cases}$$

<u>chemostate model</u> of bacteria and nutrient.

$$\frac{F}{V} = D \qquad \begin{cases} \dfrac{dN}{dt} = aCN - DN \\[2ex] \dfrac{dC}{dt} = -\alpha aCN + D(C_0 - C) \end{cases}$$

<u>Nondimensionalization</u>

$$N^* = c_1 N, \quad C^* = c_2 C,$$

$$t^* = c_3 t$$

$$t^* = Dt, \quad C^* = \frac{C}{C_0}, \quad N^* = \frac{\alpha N}{C_0}$$

$$\frac{dN}{dN^*} = \frac{C_0}{\alpha}, \quad \frac{dt^*}{dt} = D \qquad C = C_0 C^*, \ N = \frac{C_0}{\alpha} N^*$$

$$\frac{dC}{dC^*} = C_0$$

$$\frac{dN}{dt} = \frac{dN}{dN^*}\frac{dN^*}{dt^*}\frac{dt^*}{dt} = \frac{C_0}{\alpha}\frac{dN^*}{dt^*}\cdot D = aC_0C^*\frac{C_0}{\alpha}N^* - D\cdot\frac{C_0}{\alpha}N^*$$

$$\Rightarrow \frac{dN^*}{dt^*} = \frac{aC_0}{D}C^*N^* - N^*$$

$$\frac{dC}{dt} = \frac{dC}{dC^*}\frac{dC^*}{dt^*}\frac{dt^*}{dt} = C_0\cdot\frac{dC^*}{dt^*}\cdot D = -\alpha aC_0C^*\frac{C_0}{\alpha}N^* + D(C_0 - C_0C^*)$$

$$\Rightarrow \frac{dC^*}{dt^*} = -\frac{aC_0}{D}C^*N^* + 1 - C^*$$

Define $A = \frac{aC_0}{D}$, dropping $*$ for simplicity

$$\begin{cases} \frac{dN}{dt} = ACN - N \\ \frac{dC}{dt} = -ACN + 1 - C \end{cases}$$

parameter before: 4 after 1 (3 change of variables)

Equilibrium: $\begin{cases} ACN - N = 0 & ① \\ -ACN + 1 - C = 0 & ② \end{cases}$

① $\Rightarrow N = 0$ or $AC = 1 \Rightarrow C = \frac{1}{A}$

If $N = 0$, ② $\Rightarrow 1 - C = 0 \Rightarrow C = 1 \quad (N, C) = (0, 1)$

If $C = \frac{1}{A}$, ② $-N + 1 - \frac{1}{A} \Rightarrow N = 1 - \frac{1}{A} \quad (N, C) = (1 - \frac{1}{A}, \frac{1}{A})$

"Extinction equilibrium" ~~(N_*, C_*)~~ $(N, C) = (0, 1)$

no bacteria, nutrient = feed

"Existence equilibrium" $(N, C) = \left(1 - \frac{1}{A}, \frac{1}{A}\right)$

if $A > 1$ so $1 - \frac{1}{A} > 0$,

What is A? $A = \frac{a C_0}{D} > 1$

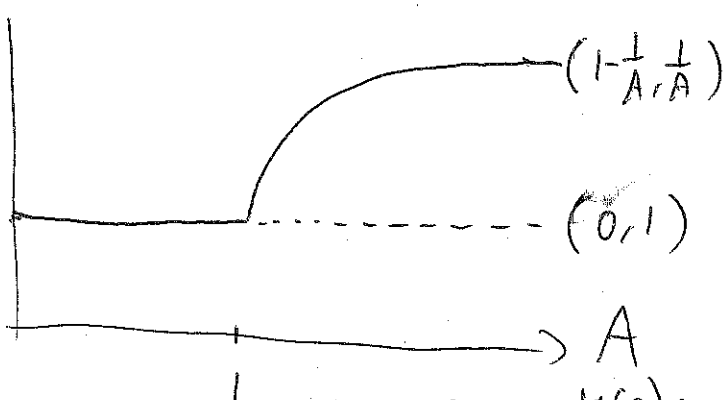
$\Rightarrow$ If a is large (reproduction rate high),

or C_0 is large (nutrient feed is high)

or D is small (flow rate is slow)

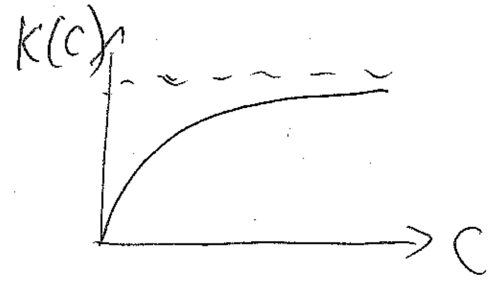
then bacteria will survive! otherwise they will die.

bifarcation



Stability discuss later

Homework $K(C) = \frac{K_{max} C}{K_n + C}$



Saturation reproduction rate