Lecture 8 (Math 345)

Differential Equation Models

derivative: rate of change of a quantity $u(t)$: $\frac{du(t)}{dt}$

$u(t)$ = a function of time (population, frequency, etc.)

differential equation: a relation involving $\frac{du}{dt}$ and u and maybe other things.

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① Growth of bacteria

$N(t)$ = number of bacteria at time t.

Assumption: one bacteria divides, $\Rightarrow$ K new bacteria in a unit time

$$N(t+\Delta t) \approx N(t) + KN(t)\Delta t$$

$$\Rightarrow \quad \frac{N(t+\Delta t) - N(t)}{\Delta t} = kN(t)$$

$$\lim_{\Delta t \to 0} \frac{N(t+\Delta t) - N(t)}{\Delta t} = \frac{dN}{dt}$$

$$\underset{\text{growth rate}}{\frac{dN}{dt}} = k \underset{\text{population}}{N}$$

(Malthus equation) Malthus, 1798

K = rate of production per unit time.

= growth rate per capita

Solving the equation (Separation of Varinbles)

$$\frac{dN}{dt} = kN \Rightarrow \frac{dN}{N} = k\,dt \xRightarrow{\text{integrate}} \int \frac{dN}{N} = \int k\,dt$$

$$\Rightarrow \ln N = kt + C \Rightarrow N = e^{kt+C} = e^{kt}\cdot e^{C} = Ce^{kt}$$

Let $t=0$ $\quad N(0) = Ce^{k0} = C \Rightarrow C = N(0)$

$$\Rightarrow N(t) = N(0)\,e^{kt} \quad \text{(exponential growth)}$$

Compare : $\quad N_{n+1} = k N_n \Rightarrow N_n = N_0 \cdot k^n$

Exponential Decay $\quad \frac{dN}{dt} = -kN \Rightarrow N(t) = N(0)e^{-kt} \quad (k>0)$

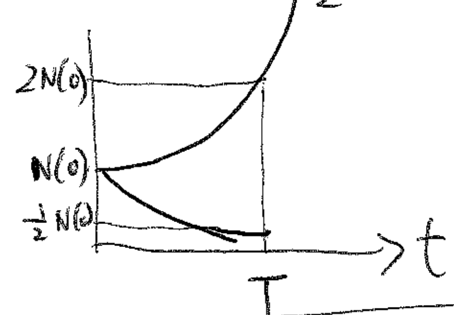
doubling time for exponential growth: $N(T) = 2N(0)$

$$2 = e^{kT} \Rightarrow \ln 2 = kT \Rightarrow T = \frac{\ln 2}{k}$$

Half life for exponential decay ; $N(T) = \frac{1}{2}N(0)$

$$\frac{1}{2} = e^{-kT} \Rightarrow \ln\frac{1}{2} = -kT \Rightarrow T = \frac{\ln 2}{k}$$

same value



② growth depends on the resource, and resource is limited

$N(t)$ = number of bacteria , $C(t)$ = amount of nutrient (resource)

reproduction rate $K = k(C) = aC$

$$\frac{dN}{dt} = K(C)N = aCN$$

$$\frac{dC}{dt} = -\alpha\frac{dN}{dt} = -\alpha aCN$$

(if α units of nutrient are consumed to produce 1 unit population)

This is a system of equations! $N(0) = N_0$, $C(0) = C_0$

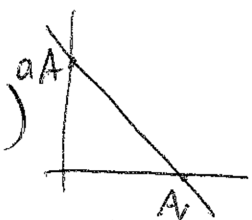
$$\frac{dC}{dt} = -\alpha\frac{dN}{dt} \Rightarrow C(t) = -\alpha N(t) + A$$

$$C(0) = -\alpha N(0) + A \Rightarrow A = C_0 + \alpha N_0$$

$$\Rightarrow C(t) = -\alpha N(t) + C_0 + \alpha N_0 \quad (\text{or } C(t) + \alpha N(t) = C_0 + \alpha N_0)$$

Let $C_0 + \alpha N_0 = A$. $\quad C(t) = A - \alpha N(t)$

$$\Rightarrow \frac{dN}{dt} = \underbrace{a(A-\alpha N)}_{\text{growth rate per capita}}N \qquad K(N) = a(A-\alpha N)$$



growth rate decreases as N increases

Simplifying equation → dimension analysis

Each quantity has a dimension (physical dimension not spatial one)

see Notes

variable	dimension	parameter	dimension
N	number (P)	a	$1/(M\cdot T)$
t	time (T)	A	M
C	mass (M)	α	M/P

Basic Laws for dimensions in an equation

① If $A = B$, then $\dim(A) = \dim(B)$

② If $A + B$, then $\dim(A) = \dim(B)$.

③ $\dim(A \cdot B) = \dim A \cdot \dim B$, $\dim\left(\frac{A}{B}\right) = \frac{\dim A}{\dim B}$

$$C(0) = C_0 \Rightarrow \dim(C) = \dim(C_0) = M$$

$$A = C_0 + \alpha N_0 \Rightarrow \dim(A) = \dim(C_0 + \alpha N_0) = \dim(C_0) = M$$

$$A - \alpha N \Rightarrow \dim(A) = \dim(\alpha \cdot N) = \dim \alpha \cdot \dim N$$

$$\Rightarrow M = \dim \alpha \cdot P \Rightarrow \dim \alpha = \frac{M}{P}$$

$$\dim\left(\frac{dN}{dt}\right) = \frac{P}{T} = \dim\left(a(A - \alpha N) \cdot N\right) = \dim(a) \cdot M \cdot P$$

$$\Rightarrow \dim a = \frac{1}{M \cdot T}$$

Change the variables $N^* = C_1 \cdot N$, $t^* = C_2 \cdot t$

So that N^*, t^* are dimensionless ($\dim N^* = 1$, $\dim t^* = 1$)

C_1, C_2 are combinations of a, A, α

$$t^* = aAt \quad \left(\dim t^* = \dim a \cdot \dim A \cdot \dim t = \frac{1}{M \cdot T} \cdot M \cdot T = 1\right)$$

$$N^* = \frac{\alpha}{A} \cdot N \quad \left(\dim N^* = \frac{\dim \alpha}{\dim A} \cdot \dim N = \frac{M/P}{M} \cdot P = 1\right)$$

$$\frac{dN}{dN^*} = \frac{A}{\alpha} \qquad \frac{dt^*}{dt} = aA$$

Chain rule : $(f(g(t)))' = f'(g(t)) \cdot g'(t)$

$$\frac{df}{dg}\frac{dg}{dt} = \frac{df}{dt}$$

$$\frac{dN}{dt} = a\,(A-\alpha N)\,N \quad \Rightarrow \frac{dN}{dt} = \frac{dN}{dN^*}\,\frac{dN^*}{dt^*}\cdot\frac{dt^*}{dt}$$

$$= \boxed{\frac{A}{\alpha}}\,\frac{dN^*}{dt^*}\cdot aA$$

$$= a\,\left(A-\alpha\frac{A}{\alpha}N^*\right)\cdot\boxed{\frac{A}{\alpha}}\,N^*$$

$$\Rightarrow A\frac{dN^*}{dt^*} = (A-AN^*)\,N^* = \cancel{A}\,(1-N^*)\,N^*$$

We can drop $*$ $\Rightarrow$ $\frac{dN}{dt} = \cancel{A}\,(1-N)\,N$ (dimensionless equation)

Before : 3 parameters (a, A, α) After: 0 parameter (A)

This process is called <u>nondimensionalization</u>

New equation $\frac{dN}{dt} = (1-N)\,N$

Solve: $\frac{dN}{(1-N)N} = dt \quad \Rightarrow \left(\frac{1}{1-N} + \frac{1}{N}\right)dN = dt$

$$\Rightarrow \ln\frac{N}{1-N} = t + C \Rightarrow \frac{N}{1-N} = ce^t$$

$$\Rightarrow N = (1-N)\,ce^t = ce^t - cNe^t \Rightarrow N + ce^tN = ce^t$$

$$\Rightarrow N = \frac{ce^t}{1+ce^t} \Rightarrow N(0) = \frac{c}{1+c} \Rightarrow C = N(0) + N(0)\,c$$

$$\Rightarrow C = \frac{N(0)}{1-N(0)} \Rightarrow N(t) = \frac{\frac{N(0)}{1-N(0)}e^t}{1+\frac{N(0)}{1-N(0)}e^t} = \frac{N(0)e^t}{1-N(0)+N(0)e^t} = \frac{N(0)}{(1-N(0))e^{-t}+N(0)}$$

Solution of $N' = (1-N)N \Rightarrow$

$$N(t) = \frac{N(0)}{(1-N(0))e^{-t} + N(0)} \Rightarrow N^*(t) = \frac{N^*(0)}{(1-N^*(0))e^{-t^*} + N^*(0)}$$

Back to $\frac{dN}{dt} = a(A - \alpha N)N$

$$\frac{\alpha}{A} \cdot N(t) = \frac{\frac{\alpha}{A} N(0)}{\left(1 - \frac{\alpha}{A} N(0)\right) e^{-aAt} + \frac{\alpha}{A} N(0)}$$

$$N(t) = \frac{N(0)}{\left(1 - \frac{\alpha}{A} N(0)\right) e^{-aAt} + \frac{\alpha}{A} N(0)}$$

Qualitative Analysis

$$N' = (1-N)N$$

Steady state $N(t) = N_1$ (constant solution)

$N'(t) = 0 \quad (1-N)N = 0 \Rightarrow N = 0$ or $N = 1$

Stability? linear equation $N' = kN$ ($k > 0$ exponential growth; $k < 0$ —— decay)

$$N' = f(N) \approx f(N_1) + f'(N_1)(N - N_1) \ \circleddash$$

$\tilde{N} = N - N_1 \quad \tilde{N}' = f'(N_1)\tilde{N}$

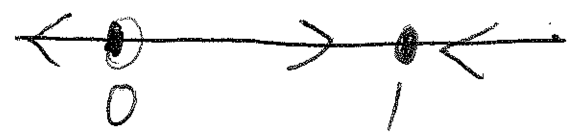
So $f'(N_1) > 0 \Rightarrow$ unstable

$f'(N_1) < 0 \Rightarrow$ stable,

$f(N) = (1-N)N = N - N^2 \qquad f'(N) = 1-2N$

$f'(0) = 1 > 0$ (unstable) $\quad f'(1) = -1 < 0$ (stable)

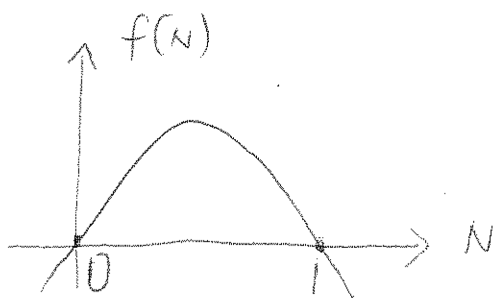
<u>phase line</u>



$\rightarrow\bullet\leftarrow$ stable $\quad \leftarrow\circ\rightarrow$ unstable

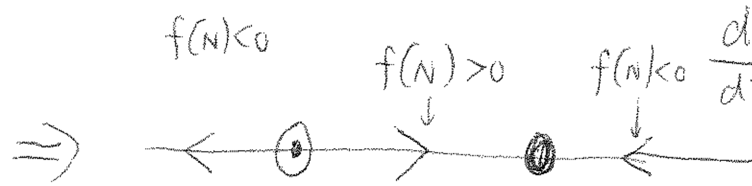
In a phase line, a dot represents an equilibrium (steady state)

$\rightarrow\bullet\leftarrow$ stable $\quad \leftarrow\circ\rightarrow$ unstable, direction indicates the solution increases or decreases $\longrightarrow$ increase $\longleftarrow$ decrease



$f'(1) > 0 \qquad f'(1) < 0$

$\Rightarrow$



$\frac{dN}{dt} = f(N)$

Conclusions for logistic model $N' = (1-N)N$

① Solution: $N(t) = \dfrac{N(0)}{(1-N(0))e^{-t} + N(0)}$

$\lim_{t\to\infty} N(t) = 1 \quad$ if $N(0) > 0$.

② Two equilibria: $N=0$ is unstable

$N = 1$ is stable

③ biological meaning: population tends to a limit (equilibrium) $N = 1$

(called carrying capacity

④ In original model, carrying capacity $N = \dfrac{A}{\alpha}$