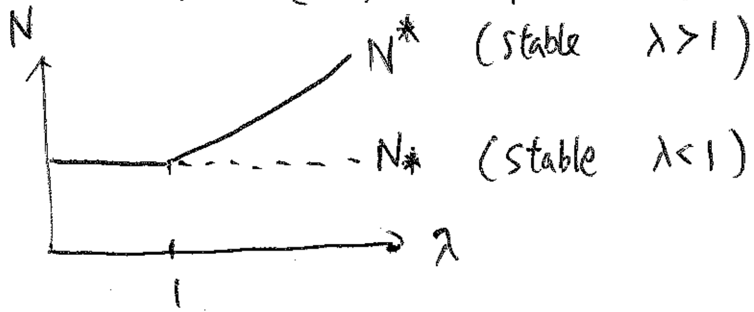
Lecture 6 (Math 345)

Bellow's model $\quad X_{n+1} = \dfrac{\lambda X_n}{1+X_n^b} \qquad \left(N_{n+1} = \dfrac{\lambda N_n}{1+N_n^b} \right)$

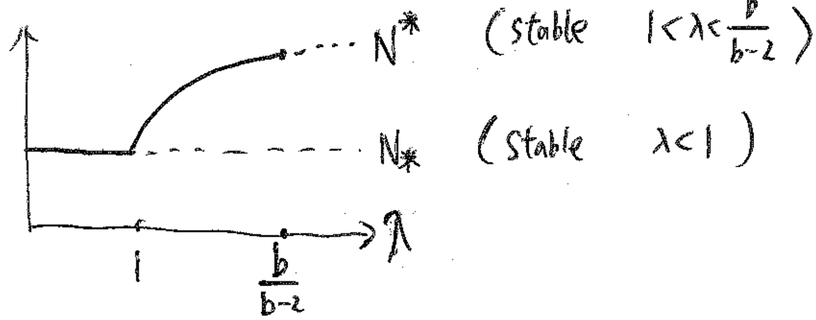
Equilibria: $N_* = 0$ (for all $\lambda > 0$, $b > 0$)

$N^* = (\lambda-1)^{\frac{1}{b}}$ (for $\lambda > 1$)

$0 < b \le 1$



$b > 2$



What happens at $\lambda = \dfrac{b}{b-2}$? $\qquad f'(N_*) = -1$

behavior near N_*: $\left(f'(N_*)\right)^n \Rightarrow (-1)^n$ oscillation!

At $\lambda = \dfrac{b}{b-2}$, a periodic orbit with period-2 emerges, that is called a <u>flip bifurcation</u>

<u>Periodic orbit</u> for $X_{n+1} = f(X_n)$ is a sequence $\{X_n\}$ so that $X_{n+p} = X_n$ for all $n \in \mathbb{N}$, and $p \geq 1$ is the <u>period</u>. (also called a <u>p-cycle</u>)

An <u>equilibrium</u> is a periodic orbit with period 1 (trivial)

<u>Example</u> $X_{n+1} = -X_n \Rightarrow X_1 = 2,\ X_2 = -2,\ X_3 = 2, \cdots$

Solution is a period-2 PO.

So for $\lambda > \frac{b}{b-2}$, Solutions of Bellow's model appears to converge to a 2-cycle.

<u>Another example</u> logistic map

$$X_{n+1} = r X_n (1 - X_n), \quad X_1 \geq 0. \quad r > 0$$

Equilibrium: $X = rX(1-X)$

$X = 0$ or $1 = r(1-X) \Rightarrow 1 - X = \frac{1}{r} \Rightarrow X = 1 - \frac{1}{r}$

(only >0 if $r>1$)

Stability: $f(x) = rx(1-x)$

$f'(x) = r - 2rx$

At $x_* = 0$, $f'(x_*) = r$

So $0 < r < 1$, $0 < f'(x_*) < 1 \Rightarrow$ stable

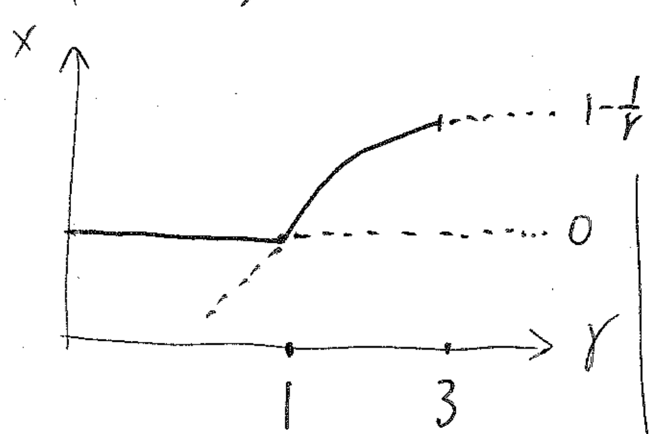
$r > 1$, $\Rightarrow$ unstable

At $x_*^* = 1 - \frac{1}{r}$, $f'(x^*) = r - 2r(1 - \frac{1}{r}) = r - 2r + 2 = 2 - r$

$1 < r < 3$, $-1 < f'(x^*) < 1 \Rightarrow$ stable

$r < 1 \Rightarrow$ unstable

$f'(x^*) > 1$



$r = 1$ transcritical bifurcation

$r = 3$ flip bifurcation

Bifurcation occurs if $f'(x_*) = 1$ or -1

① **Flip bifurcation**: $f'(x_*) = -1$ stable equilibrium $\longrightarrow$ unstable equilibrium, stable 2-cycle

② **transcritical bifurcation**: $f'(x_*) = 1$

① another equilibrium emerges

② two curves of equilibria intersect

③ 2 equilibria for $r < r_1$, 2 equilibria for $r > r_1$

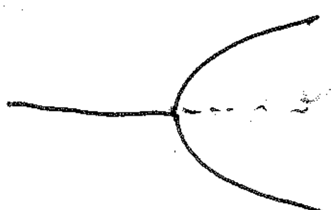
1 equilibrium when $r = r_1$.

③ **pitchfork bifurcation** $f'(x_*) = 1$

1 equilibrium for $r \le r_1$, 3 equilibria for $r > r_1$

Example $x_{n+1} = \dfrac{\lambda x_n}{1 + x_n^2}$ (Bellow's model $b=2$)

$x_* = 0$ $\quad 1 = \dfrac{\lambda}{1+x^2} \Rightarrow 1 + x^2 = \lambda \Rightarrow x^2 = \lambda - 1 = x = \pm\sqrt{\lambda - 1}$



④ **Saddle-node bifurcation** (fold bifurcation)

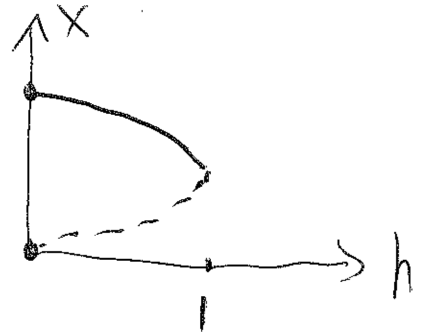
$f'(x_*) = 1$

0 equilibrium for $r > r_1$, 2 equilibria for $r < r_1$

Example $x_{n+1} = 3x_n - x_n^2 - h$ (logistic model with harvesting)

$x = 3x - x^2 - h$

$x^2 - 2x + h = 0$ $\qquad x = \dfrac{2 \pm \sqrt{4 - 4h}}{2} = 1 \pm \sqrt{1-h}$



$h=1$ fold bifurcation point

meaning $h>1$, $\lim_{n\to\infty} X_n \to \infty$

Continuing Logistic map

What is a two-cycle? $X_{n+2} = X_n$

$$\begin{cases} X_{n+1} = rX_n(1-X_n) \\ X_{n+2} = rX_{n+1}(1-X_{n+1}) \end{cases}$$

~~So a 2-cycle x, y, x, y ... satisfies~~ $\cancel{y = rx(1-x)}$

$$X_n = X_{n+2} = r\left(rX_n(1-X_n)\right)\cdot\left(1-rX_n(1-X_n)\right)$$

So in a 2-cycle: $x, y, x, y, \cdots$, x satisfies

$$x = r\left(rx(1-x)\right)\left(1-rx(1-x)\right) \longrightarrow$$

$$1 = r^2(1-x)(1-rx+rx^2)$$

$X_{n+1} = f(X_n)$

$X_{n+2} = f(X_n)$

$\Rightarrow X_{n+2} = f(f(X_n))$

$\Rightarrow x = f(f(x))$

2nd iteration of $f(x)$.

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a 2-cycle is an equilibrium for $Y_{n+1} = f(f(Y_n))$

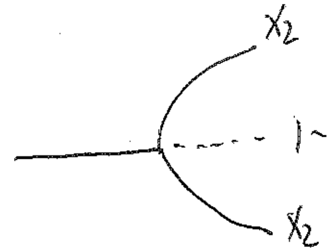
$$r^2(rx^2 - rx + 1)(x-1) + 1 = 0$$

$$r^3\left(x-\left(1-\tfrac{1}{r}\right)\right)\left(x^2 - \left(1+\tfrac{1}{r}\right)x + \left(\tfrac{1}{r}+\tfrac{1}{r^2}\right)\right) = 0$$

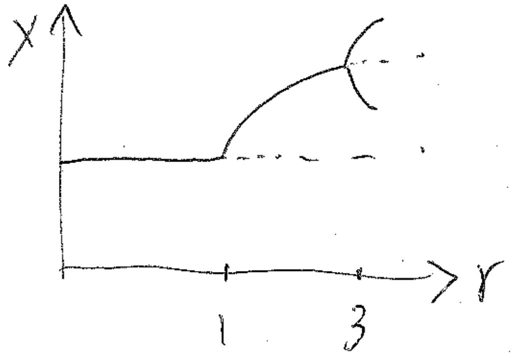
$\downarrow$ $x = 1-\frac{1}{r}$ equilibrium

$\downarrow$ $x = \dfrac{r+1 \pm \sqrt{(r-3)(r+1)}}{2r}$

x_1, x_2



flip bifurcation



Now x_1, x_2 are stable equilibria of

$$y_{n+1} = f(f(y_n))$$

$$f(x_1) = x_2 \quad f(x_2) = f(x_1)$$

$$g(x) = f(f(x))$$

$$g'(x) = f'(f(x)) \cdot f'(x)$$

$$g'(x_2) = f'(x_1) \cdot f'(x_2)$$

If $|g'(x_2)| = |f'(x_1) \cdot f'(x_2)| > 1 \Rightarrow$ then 2-cycle (x_1, x_2) become unstable!

With Calculation, $r = 1+\sqrt{6}$ is the bifurcation point

Another flip bifurcation occurs, $\Rightarrow$ a 4-Cycle appears.

Numerical simulation: $r = 0.9 \ (\to 0)$

$r = 1.5 \ (\to 1 - \frac{1}{r})$

$r = 3.1 \ (\to$ 2-cycle$)$

$1+\sqrt{6} \approx 3.45$

$r = 3.5 \ (\to$ 4-cycle$)$

A cascade of flip bifurcation occurs at $r_1 = 3$, $r_2 = 1+\sqrt{6}$, $r_3 = \cdots$ (2-Cycle, 4-cycle, 8-Cycle $\cdots$)

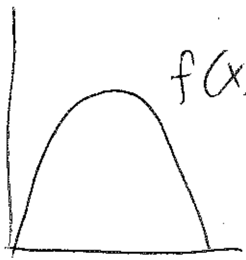
The dynamic becomes chaotic when $r > 3.56995$

Robert May (1976)

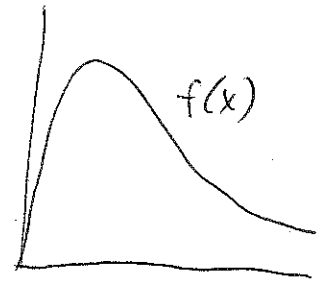
Matlab bifarcation diagram: drawing "eventual orbits"

So $X_{n+1} = f(X_n)$ can have a chaotic dynamics!

logistic



Bellows

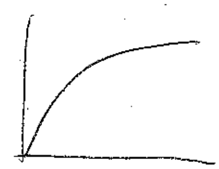


$b > 1$

$f(x)$ is not monotone.

Ecological models $X_{n+1} = f(X_n)$

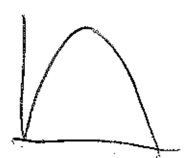
$$f(x) = \frac{\lambda x}{1+x} \quad \text{(Beverton-Holt)}$$



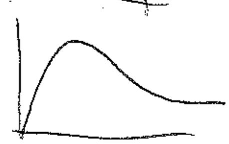
f monotone.

$$f(x) = \frac{\lambda x}{1+x^b} \quad (b>1) \quad \text{(Bellow)}$$

$$f(x) = rx(1-x) \quad \text{(logistic)}$$



$$f(x) = xe^{-rx} \quad \text{(Ricker)}$$



Another model: Population Genetics