Lecture 5

Nonlinear Difference Equation Models

linear model $\quad X_{n+1} = aX_n + bX_{n-1}, \quad x_n \in \mathbb{R}$

Matrix model $\quad X_{n+1} = AX_n, \quad X_n \in \mathbb{R}^k$

Solvable but not realistic

Nonlinear models: $\quad X_{n+1} = f(X_n, X_{n-1}, \cdots) \quad X_n \in \mathbb{R}$

$X_{n+1} = g(X_n) \quad X_n \in \mathbb{R}^k$

$f: \mathbb{R} \to \mathbb{R}$. or $g: \mathbb{R}^k \to \mathbb{R}^k$ are nonlinear functions.

Simulations $\quad$ Time series, Cobweb $\quad X_{n+1} = f(X_n)$

An equilibrium is X satisfying $X = f(X)$

If $X_1 = X$, then $X_2 = f(X_1) = f(X) = X$, $\cdots$ $X_n = X$

So X is also called a fixed point (or steady state)

Example $\quad$ Bellow's model

$$N_{n+1} = \frac{\lambda N_n}{1 + N_n^b}$$

Equilibrium: $\quad N = \dfrac{\lambda N}{1+N^b}$ or $\dfrac{\lambda N}{1+N^b} - N = 0 \quad N\left(\dfrac{\lambda}{1+N^b} - 1\right) = 0$

$N = 0$ or $\dfrac{\lambda}{1+N^b} = 1 \Rightarrow \lambda = 1 + N^b \Rightarrow N^b = \lambda - 1 \Rightarrow N = (\lambda-1)^{\frac{1}{b}}$

(only exist if $\lambda > 1$)

$N_* = 0$ and $N^* = (\lambda - 1)^{\frac{1}{b}}$

<u>Stability</u> If x_1 is close to an equilibrium x_*, what happens?

$$f(x) \approx f(x_*) + f'(x_*)(x - x_*)$$

$$x_{n+1} = f(x_n) \approx f(x_*) + f'(x_*)(x_n - x_*)$$

$$x_{n+1} - x_* \approx f'(x_*)(x_n - x_*)$$

Shift $y_n = x_n - x_*$ then $y_{n+1} \approx f'(x_*)\, y_n$ (Malthus Eq!)

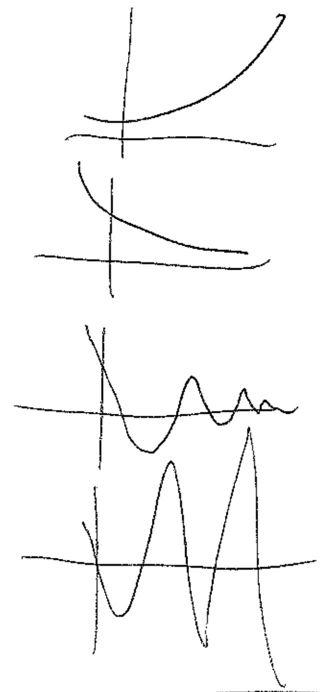
Chap 1 $\Rightarrow$ 4 cases

$f'(x_*) > 1$ exponential growth

$0 < f'(x_*) < 1$ exponential decay

$-1 < f'(x_*) < 0$ oscillatory decay

$f'(x_*) < -1$ oscillatory growth



When $-1 < f'(x_*) < 1$, $\lim_{n\to\infty} (x_n - x_*) = 0$ converging to x_*

x_* is a <u>stable</u> equilibrium.

Otherwise it is <u>unstable</u>

Stable means if x_1 is close to x_*, then $\lim_{n\to\infty} x_n = x_*$

<u>Criterion of stability</u>: If $|f'(x_*)| < 1$, then x_* is stable.

Example Bellow's model

$$f(N) = \frac{\lambda N}{1+N^b} \qquad f'(N) = \frac{\lambda(1+N^b) - \lambda N \cdot bN^{b-1}}{(1+N^b)^2} = \frac{\lambda(1+(1-b)N^b)}{(1+N^b)^2}$$

For $N_* = 0$

$f'(N_*) = \lambda$ So $\lambda > 1 \Rightarrow N_*$ is unstable

$0 < \lambda < 1 \Rightarrow N_*$ is stable.

For $N^* = (\lambda - 1)^{\frac{1}{b}}$ $\qquad f'(N^*) = \dfrac{\lambda(1+(1-b)(\lambda-1))}{(1+\lambda-1)^2} = \dfrac{\lambda(1+\lambda-b\lambda-1+b)}{\lambda^2}$

$(N^*)^b = \lambda - 1$

$$= \frac{\lambda + b - b\lambda}{\lambda} = \frac{(1-b)\lambda + b}{\lambda}$$

$$= 1 - b + \frac{b}{\lambda}$$

Case 1 If $b \le 1$ then $1 - b + \frac{b}{\lambda} > 0$

Indeed $1 - b + \frac{b}{\lambda} = 1 + b\left(\frac{1}{\lambda} - 1\right) < 1$

So $0 < f'(N_*) < 1 \Rightarrow$ stable (exponential monotone decay to N^*)

If $f'(N) > 0$, then $f(N)$ is increasing

Here $f'(N) = \dfrac{\lambda(1+(1-b)N^b)}{(1+N^b)^2} > 0$.

Prove: If $f(N)$ is increasing, then a solution sequence $\{N_n\}$ of $N_{n+1} = f(N_n)$ is either increasing or decreasing

proof Case A $N_2 = f(N_1) > N_1$

then $N_3 = f(N_2) > f(N_1) > N_2$

$\Rightarrow N_{n+1} > N_n$ for $n \ge 2 \Rightarrow$ increasing

Case B $N_2 = f(N_1) < N_1$

then $N_3 = f(N_2) < f(N_1) = N_2$

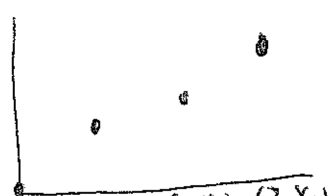
$\Rightarrow N_{n+1} < N_n$ for $n \geq 2$ $\Rightarrow$ decreasing

Theorem in Calculus II: If a sequence is monotone (inc / dec), and it is bounded, then $\lim_{n\to\infty} X_n = X$ exists.

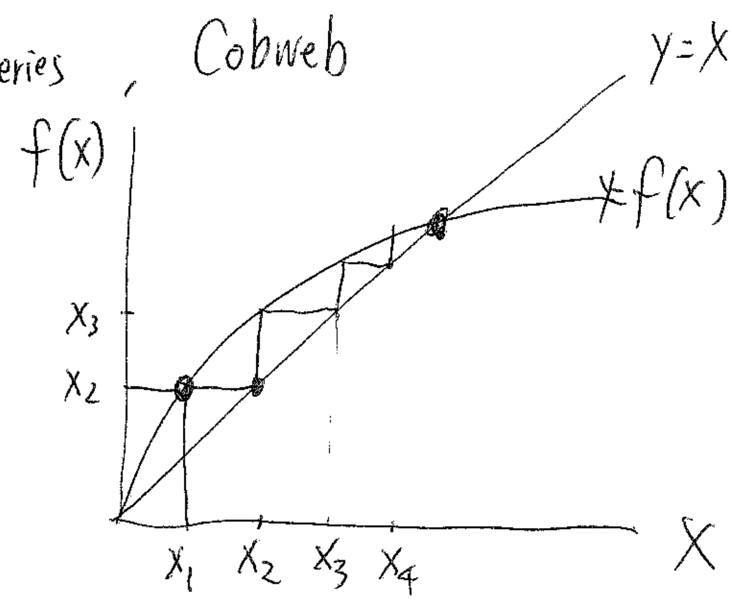
Monotone dynamics: For $N_{n+1} = f(N_n)$, if f is increasing or decreasing, then $\{N_n\}$ is also inc or dec, and it converges to an equilibrium or it goes to ∞.

Case 2 $b > 1$

Numerical approach: time series, Cobweb



time series



$(X_1, X_2) = (X_1, f(X_1))$

$\to (X_2, X_2)$

$\to (X_2, X_3) = (X_2, f(X_2))$

$\to (X_3, X_3)$

$\to (X_n, X_n)$

$\to (X_n, X_{n+1}) = (X_n, f(X_n))$

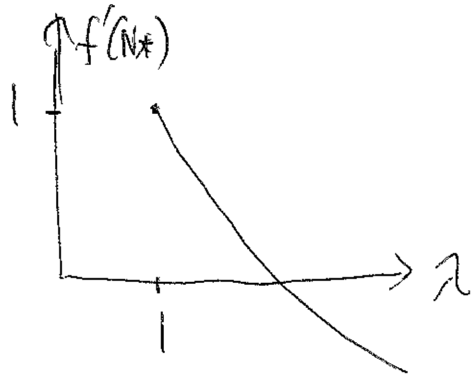
$\to (X_{n+1}, X_{n+1})$

repeat.

For $b > 1$, the sequence may not be monotone.

We use λ as a parameter, fixing $b > 1$.

Then $f'(N^*) = 1 - b + \frac{b}{\lambda}$ becomes ~~larger~~ smaller if λ is large



ⓐ when $f'(N^*) = 0$?

$$1 - b + \frac{b}{\lambda} = 0 \qquad \frac{b}{\lambda} = b - 1$$

$$\Rightarrow \lambda = \frac{b}{b-1}$$

$1 < \lambda < \frac{b}{b-1}$, $0 < f'(N^*) < 1$ $\Rightarrow$ exponential decay.

ⓑ when $f'(N^*) = -1$?

$$1 - b + \frac{b}{\lambda} = -1 \qquad 2 - b + \frac{b}{\lambda} = 0 \qquad \frac{b}{\lambda} = b - 2$$

$$\lambda = \frac{b}{b-2}$$

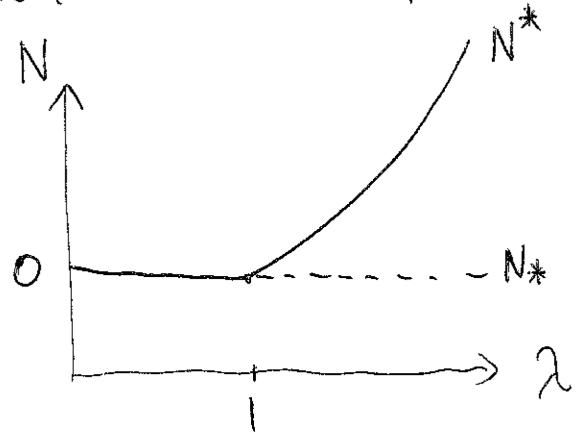
$\frac{b}{b-1} < \lambda < \frac{b}{b-2}$ $\quad -1 < f'(N^*) < 0$ $\Rightarrow$ oscillatory decay (still stable)

but $\lambda > \frac{b}{b-2}$ $\quad f'(N^*) < -1$ $\Rightarrow$ oscillatory growth! (unstable)

Bifurcation Diagram

bifarcation: behavior of solutions changes when parameter changes.

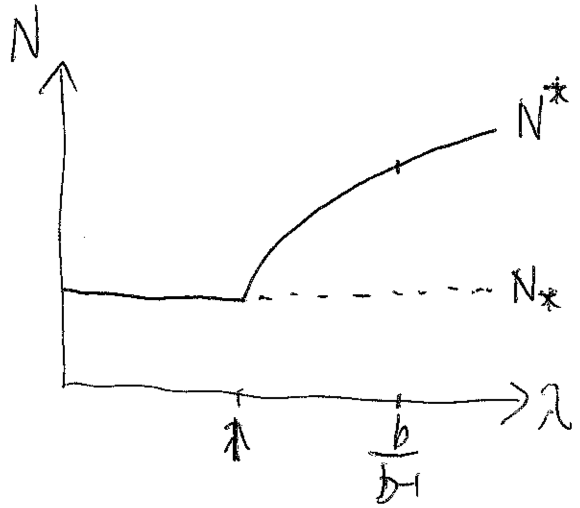
$0 < b \leq 1$



—— stable equilibrium

- - - - - unstable equilibrium

$1 < b < 2$



$\rightarrow$ still stable

but for $\lambda > \frac{b}{b-1}$

oscillatory damping

$b > 2$