Math 345 Lecture 11

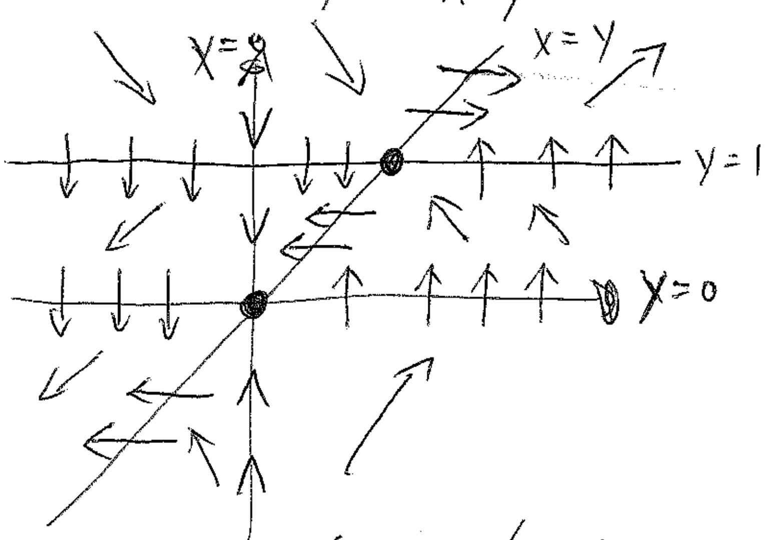
~~Equilibrium~~ $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$ equilibrium $\begin{cases} f(x,y)=0 \\ g(x,y)=0 \end{cases}$ Jacobian $\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$

Nullclines $f(x,y)=0$ a curve in phase plane s.t $x'=0$ (x-nullcline)

$g(x,y)=0$ a curve in phase plane s.t. $y'=0$ (y-nullcline)

Example $x' = x(y^2-y) = x(y-1)y$

$y' = x-y$



① intersection of x-nullcline and y-nullcline is an equilibrium

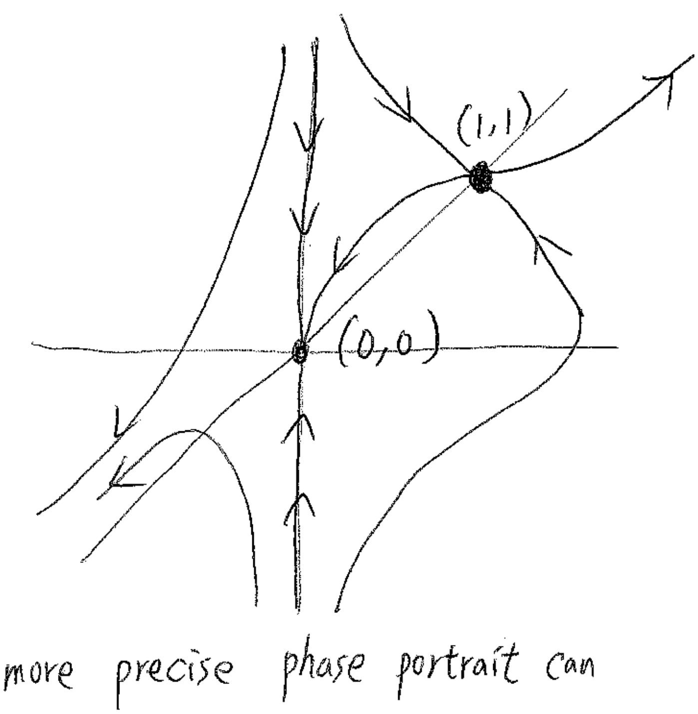
② vector field (f,g) is
vertical on x-nullcline ($x'=0$)
horizontal on y-nullcline ($y'=0$)

$x=0$ $\quad x'=0,\ y'=-y$

$y=0$ $\quad x'=0,\ y'=x$

$y=1,$ $\quad x'=0,\ y'=x-1$

$x=y,$ $\quad x'=y^2(y-1),\ y'=0$



A more precise phase portrait can be drawn using pplane.

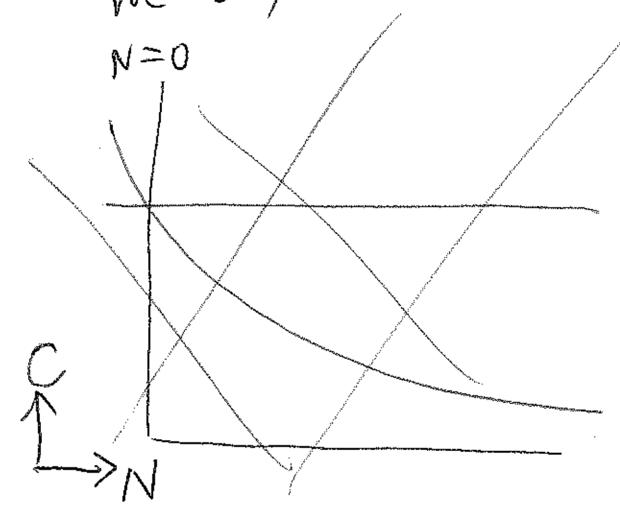
Example chemostat model

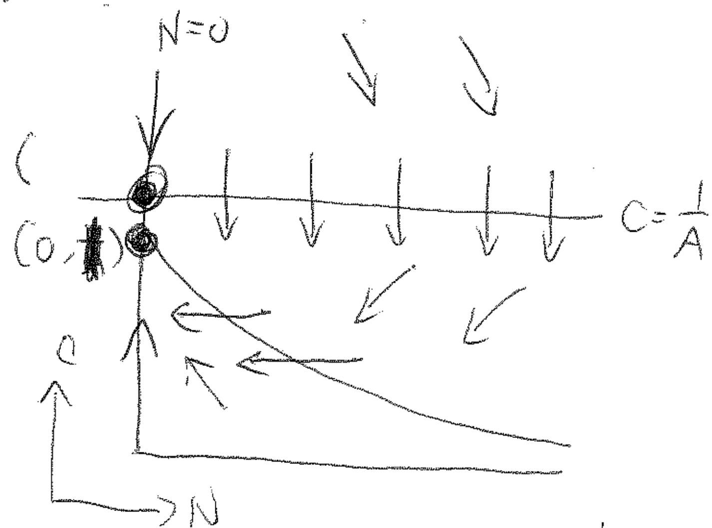
$$N' = ACN - N = N(AC-1)$$
$$C' = -ACN + 1 - C$$

N - nullcline: $N=0$ or $AC=1$ $(C=\frac{1}{A})$

C - nullcline: $-ACN+1-C=0 \Rightarrow C(AN+1)=1 \Rightarrow C=\frac{1}{AN+1}$

we only consider $N \geq 0$, $C \geq 0$





$$\lim_{t\to 0} (N(t), C(t)) = (0,1)$$

extinction

$0 < A < 1$ then $\frac{1}{A} > 1$

$N=0$, $N'=0$, $C' = 1-C$

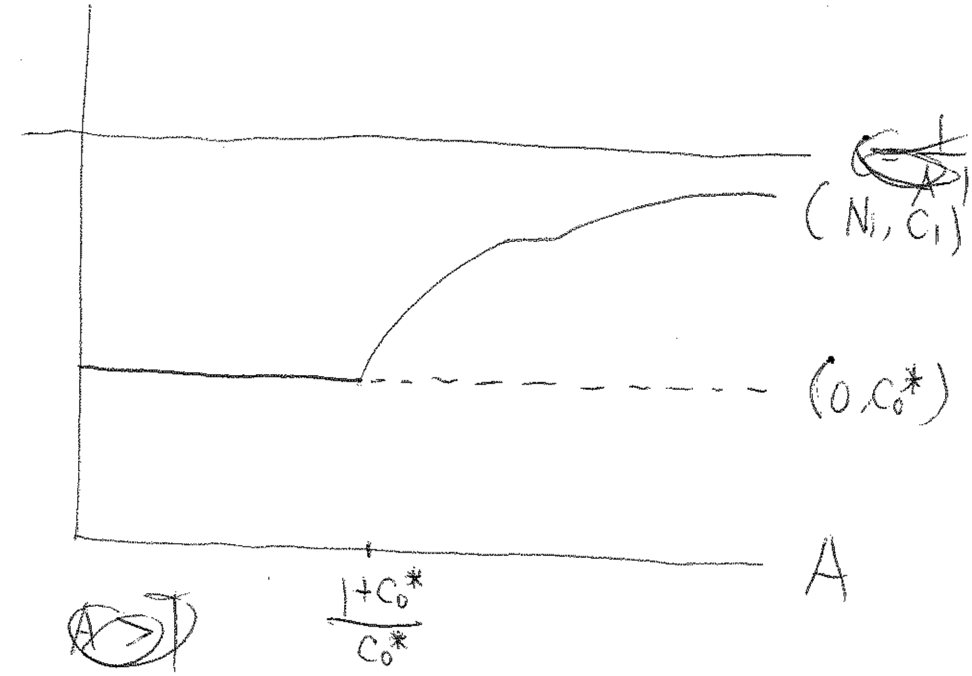
$C=\frac{1}{A}$, $N'=0$, $C' = -N+1-\frac{1}{A}$

≤ 0

$C=\frac{1}{AN+1}$, $C'=0$, $N' = N\cdot\left(A\frac{1}{AN+1}-1\right)$

$= N \cdot \frac{A-AN-1}{AN+1}$

< 0



$$J(0, C_0^*) = \begin{pmatrix} \frac{AC_0^*}{1+C_0^*} - 1 & 0 \\ -\frac{A(C_0^*)^2}{1+C_0^*} & -E \end{pmatrix} \qquad T = \frac{AC_0^*}{1+C_0^*} - 1 - E$$

$$D = 1 - \frac{AC_0^*}{1+C_0^*}$$

indeed $\lambda_1 = -E$, $\lambda_2 = \frac{AC_0^*}{1+C_0^*} - 1$

$$J\left(E - \frac{E}{C_0^*(A-1)}, \frac{1}{A-1}\right) = \begin{pmatrix} 0 & \frac{AN_1}{(1+C_1)^2} \\ -C_0^* & -E - \frac{AN_1C_0^*}{(1+C_1)^2} \end{pmatrix} \qquad T = -E - \frac{AN_1C_0^*}{(1+C_1)^2} < 0$$

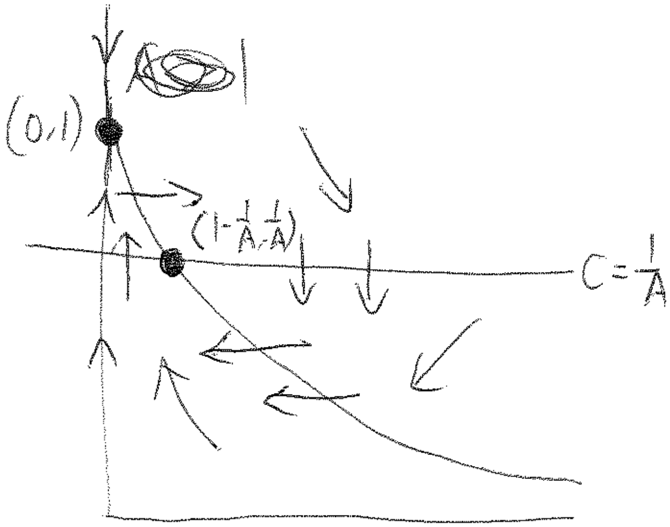
(N_1, C_1)

$$D = \frac{AN_1C_0^*}{(1+C_1)^2} > 0.$$

$(0, C_0^*)$ stable if $\frac{AC_0^*}{1+C_0^*} - 1 < 0$ or $A < \frac{1+C_0^*}{C_0^*}$

unstable if $A > \frac{1+C_0^*}{C_0^*}$

(N_1, C_1) is valid if $A > \frac{1+C_0^*}{C_0^*}$, and it is stable if it is valid.



$A > 1$

$N = 0,\ N' = 0,\ C' = 1 - C$

$C = \frac{1}{A},\ N' = 0,\ C' = -N + 1 - \frac{1}{A}$

$C = \frac{1}{AN+1},\ C' = 0,\ N' = N(AC - 1)$

$N' > 0$ if $C > \frac{1}{A}$

$N' < 0$ if $C < \frac{1}{A}$.

<u>Question</u>: Is there a periodic orbit?

<u>Example</u> chemostat with saturated growth (HW last time).

$$\begin{cases} N' = \dfrac{ANC}{1+C} - N \\ C' = E(C_0^* - C) - \dfrac{AC_0^* NC}{1+C} \end{cases}$$

Equilibrium: $(0, C_0^*)$, $\left(E - \frac{E}{C_0^*(A-1)}, \frac{1}{A-1}\right)$

N-nullcline: $N = 0$ or $\frac{AC}{1+C} - 1 = 0$ $\left(AC = 1 + C \Rightarrow (A-1)C = 1 \Rightarrow C = \frac{1}{A-1}\right)$

C-nullcline: $E(C_0^* - C)(1+C) = AC_0^* NC$

$$\Rightarrow N = \frac{E(C_0^* - C)(1+C)}{AC_0^* C}$$

Jacobian $J = \begin{pmatrix} \frac{AC}{1+C} - 1 & AN \cdot \frac{1}{(1+C)^2} \\ -\frac{AC_0^* C}{1+C} & -E - \frac{ANC_0^*}{(1+C)^2} \end{pmatrix}$

$\left(\frac{C}{1+C}\right)' = \frac{(1+C)\cdot 1 - C}{(1+C)^2} = \frac{1}{(1+C)^2}$