Assignment 7
Math 345, Prof. Shi

Due: Friday, Nov 2 (5pm)

1. A study of Yale University freshmen described an influenza epidemic with $S_0 = 0.911$ and $S_\infty = 0.513$. Here we measure the number of susceptibles as a fraction of the total population size. Assume that we use an SIR epidemic model: (with $S + I = N = 1$)

$$\frac{dS}{dt} = -\beta SI, \quad \frac{dI}{dt} = \beta SI - \alpha I.$$ 

(a) Estimate the basic reproduction number $R_0$ and determine if there is an epidemic.
(b) What fraction of students would have to be immunized to prevent an epidemic?
(c) What was the maximum percentage of Yale students suffering from influenza?

2. Consider an SIS epidemic model: $S \rightarrow I \rightarrow S$, which can be described by the system of equations:

$$\frac{dS}{dt} = -\beta SI + \alpha I, \quad \frac{dI}{dt} = \beta SI - \delta I.$$ 

Here $\beta$ and $\alpha$ have the same meaning as in the SIR model. Let the total population be $N$.

(a) Use change of variables: $u = \frac{S}{N}$, $v = \frac{I}{N}$, $s = \alpha t$ to obtain dimensionless system:

$$\frac{du}{ds} = -R_0 uv + v, \quad \frac{dv}{ds} = R_0 uv - v.$$ 

Here $R_0$ is the new parameter. Find the expression of $R_0$ in terms of $\alpha, \beta$ and $N$.
(b) Prove that $u(s) + v(s) = 1$ for solution $(u(s), v(s))$ of the new system.
(c) Find all equilibrium points of the system and determine their stability. Notice that any equilibrium $(u, v)$ satisfies $0 \leq u, v \leq 1$ and $u + v = 1$.
(d) Use pplane or other plotting tools to plot the phase portraits for $R_0 < 1$ and $R_0 > 1$.

3. To determine whether an initial infection with HIV will develop into AIDS we introduce a simple model which includes the CD4$^+$ T cells ($T(t)$), the infected CD4$^+$ T cells ($T^*(t)$), and the HIV virus outside the T cells ($V(t)$). They satisfy the equations:

$$\frac{dT}{dt} = A - \beta TV - \mu T, \quad \frac{dT^*}{dt} = \beta TV - \delta T^*, \quad \frac{dV}{dt} = \gamma \delta T^* - \kappa V.$$ 

Here $A$ is the natural production rate of healthy T cells, $\beta$ is the infection rate of healthy T cells by external virus, $\mu, \delta$ and $\kappa$ are the death rates of $T$, $T^*$ and $V$ respectively, and $\gamma$ is the number of virus particles that emerge upon death of infected one CD4$^+$ T cells.

(a) Find the disease free equilibrium (DFE) in form of $E_0 = (T_0, 0, 0)$, and determine the condition on parameters so that $E_0$ is stable.
(b) Find the endemic equilibrium (EE) $E_1 = (T_1, T^*_1, V_1)$ which is positive, and determine the condition on parameters so that $E_1$ exists.
(c) Use Matlab program hiv.m to simulate the solutions with $A = 500$, $\kappa = 1000$, $\mu = 0.5$, $\delta = 1$, and (i) $\beta = 4$, $\gamma = 4$; (ii) $\beta = 1$, $\gamma = 2$. Initial condition: $T(0) = 400$, $T^*(0) = 200$ and $V(0) = 10$, and $0 \leq t \leq 60$. What do you observe? Explain your result.
(d) (extra 1 pt) Use Routh-Hurwitz criterion to prove that $E_1$ is stable whenever it exists (and is positive).