

# Assignment 5

Math 345, Prof. Shi

~~Due: Friday, Oct 10 (5pm)~~ Do not turn in, and solution will be available Friday, Oct 10.

1. A relatively simple equation modeling the growth of solid tumors is given by the Gompertz growth law  $\frac{dN}{dt} = re^{-at}N$  where  $N(t)$  is the tumor cell population at time  $t$ .

- (a) Give a reason for considering the growth rate to be  $re^{-at}$ .
- (b) Find the solution in the general situation (assuming  $r, a$  arbitrary parameters and  $N(0) = N_0$ ).

2. The competition between bookstores Nile.com and Barnes & Bobel can be described by a nonlinear model:

$$\begin{aligned}\frac{dx}{dt} &= 3x - 3y, \\ \frac{dy}{dt} &= -x + 2y(1 - y).\end{aligned}$$

(We only consider  $x \geq 0, y \geq 0$ .)

- (a) Find and sketch the nullclines of the system, and mark the direction of the equation vector field on the nullclines.
- (b) Find all equilibrium points of the system.
- (c) At each equilibrium point, linearize the system and identify the type of the linearized system. (stable node, unstable node, stable spiral, unstable spiral, saddle, or cannot be linearly determined etc.)
- (d) Use `pplane` or another phase portrait plotter to plot the phase portrait. What is the outcome of competition? Describe asymptotic behavior for typical initial values.

3. The Lotka-Volterra mutualism model is given by

$$\begin{aligned}\frac{dN_1}{dt} &= r_1 N_1 \frac{K_1 - N_1 + \beta_{12} N_2}{K_1}, \\ \frac{dN_2}{dt} &= r_2 N_2 \frac{K_2 - N_2 + \beta_{21} N_1}{K_2}.\end{aligned}$$

We consider the two populations  $N_1 \geq 0$  and  $N_2 \geq 0$ , and all parameters are positive.

- (a) Use  $s = r_1 t$ ,  $u = N_1/K_1$  and  $v = N_2/K_2$  as new dimensionless variables to simplify the equations into a new one: (express  $a, b, c$  in terms of old parameters)

$$\begin{aligned}u' &= u(1 - u + bv), \\ v' &= av(1 - v + cu).\end{aligned}$$

- (b) Find equilibrium points of the system. Identify two parameter regimes in which the system has 3 or 4 non-negative equilibrium points.

- (c) For each parameter regime, at each equilibrium point, linearize the system and identify the type of the linearized system. (stable node, unstable node, stable spiral, unstable spiral, saddle, or cannot be linearly determined etc.)
- (d) Use `ppplane` or another phase portrait plotter to plot two phase portraits with  $a = b = 1$ , and  $c$  values belonging two regimes. What is the outcome of cooperation? Describe asymptotic behavior for typical initial values.