Turing Diffusion-induced instability

Junping Shi

College of William and Mary, USA

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Math 442
Many chemical and biological processes can be simulated by mathematical models involving temporal and spatial variables.

**Example:**

![Picture of stem cell](image1.png) ![Reaction-diffusion simulation](image2.png)

(left: a picture of stem cell; right: a reaction-diffusion simulation)

**Reaction-diffusion systems** are mathematical models that describe how the concentration of one or more substances distributed in space changes under the influence of two processes: local chemical reactions in which the substances are converted into each other, and diffusion which causes the substances to spread out in space.
Mathematical Models

\[
\frac{\partial u(x, t)}{\partial t} = D_1 \Delta u(x, t) + f(u(x, t), v(x, t)),
\]
\[
\frac{\partial v(x, t)}{\partial t} = D_2 \Delta v(x, t) + g(u(x, t), v(x, t)).
\]

\( t \): time variable, \( x = (x_1, x_2, \cdots, x_n) \): spatial variable
\( u(x, t), v(x, t) \): density of substances at time \( t \) and location \( x \)
\( \Delta u(x, t) = \text{div}(\nabla u(x, t)) = \sum_{i=1}^{n} \frac{\partial^2 u(x, t)}{\partial x_i^2} \)

\( \Delta u, \Delta v \) are Diffusion: transport of molecules from a region of higher concentration to one of lower concentration by random molecular motion.

\( f(u, v), g(u, v) \) are Reaction: death/birth, chemical reaction/generation
Alan Turing (1912-1954)

One of greatest scientists in 20th century

Designer of Turing machine (a theoretical computer) in 1930s

Designing electromechanical machine which breaks German U-boat Enigma, helping the battle of the Atlantic

Initiate nonlinear theory of biological growth


Phil. Trans. Royal Society London B

http://www.turing.org.uk/
During his relatively brief life, Turing made a unique impact on the history of computing, computer science, artificial intelligence, developmental biology, and the mathematical theory of computability. 2012 will be a celebration of Turings life and scientific impact, with a number of major events taking place throughout the year.

Alan Turing Year

Feb 23, Nature published a special issue for Alan Turing.
Kinetic (K): \( \frac{du}{dt} = f(u, v) \), \( \frac{dv}{dt} = g(u, v) \)

Reaction-diffusion system (R-D): \( u_t = d_1 \Delta u + f(u, v) \), \( v_t = d_2 \Delta v + g(u, v) \)

Here \( u(x, t) \) and \( v(x, t) \) are the density functions of two chemicals (morphogen) or species which interact or react

- A constant solution \( u(t, x) = u_0, v(t, x) = v_0 \) can be a stable solution of (K), but an unstable solution of (R-D). Thus the instability is induced by diffusion. (Diffusion is generally a stabilizing force.)

- On the other hand, there must be stable non-constant equilibrium solutions, or stable non-equilibrium behavior, which have more complicated spatial-temporal structure.

Morphogen
Charles Darwin (1809-1882)

- *Origin of Species*, published in 1859
- Theory of Natural Selection
- *Question Darwin can’t answer:*
  How could complex patterns of life be produced by natural selection?
Discovery of DNA structure

• Francis Crick
  (Cambridge University)
• James Watson
  (Harvard University)
1962 Nobel Prize
(1953 “Nature” paper)

http://www.nature.com/nature/dna50/watsoncrick.pdf
Is DNA the secret of life?

• Inside every living creature on earth, there is a complex molecular DNA code (gene), which prescribes the creature’s form, growth, development, and behavior.

• Genes are not engineering blueprints, they are recipes in a cookbook. Recipes are different from meals.

• The mathematical laws of physical and chemistry control the growing organism's response to its genetic instructions.

   *Ian Stewart “Life’s other secret”*
Example

淮南桔，而淮北枳

(California’s orange cannot grow in Florida)

Although with same genes, the different chemical environment may produce different creatures.
Diffusion

- $u(t,x)$: density function of a chemical
- The chemical will move from high density places to lower density places, this is called **diffusion**
- Diffusion is the mechanism of many molecular or cellular motions
- Diffusion can be described by a heat equation

$$\frac{\partial u}{\partial t} = D \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$
Reaction-diffusion equations

- Let $U(x,t)$ and $V(x,t)$ be the density functions of two chemicals or species which interact or react

\[
U_t = D_U \Delta U + f(U, V), \\
V_t = D_V \Delta V + g(U, V).
\]

Alan Turing (1952 *Phil. Trans. Roy. Soc.* )
“The Chemical Basis of Morphogenesis”

**Morphogenesis** (from the Greek *morphê* shape and *genesis* creation) is one of three fundamental aspects of developmental biology along with the control of cell growth and cellular differentiation. Morphogenesis is concerned with the shapes of tissues, organs and entire organisms and the positions of the various specialized cell types.
Alan Turing (1912-1954)

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http://www.turing.org.uk/
Turing’s idea

\[ U_t = D_U \Delta U + f(U, V), \]
\[ V_t = D_V \Delta V + g(U, V). \]

• A constant solution \( u(t,x)=u_0, v(t,x)=v_0 \) can be a stable solution of (2), but an unstable solution of (2). Thus the instability is induced by diffusion.

• On the other hand, there must be stable non-constant equilibrium solutions which have more complicated spatial structure.
An example of Turing patterns by James Murray (author of books: *Mathematical Biology*)

*Emeritus Professor*
University of Washington, Seattle
Oxford University, Oxford
http://www.amath.washington.edu/people/faculty/murray/
Why do animals’ coats have patterns like spots, or stripes?
Murray’s theory

Murray suggests that a single mechanism could be responsible for generating all of the common patterns observed. This mechanism is based on a reaction-diffusion system of the morphogen prepatterns, and the subsequent differentiation of the cells to produce melanin simply reflects the spatial patterns of morphogen concentration.

**Melanin:** pigment that affects skin, eye, and hair color in humans and other mammals.

**Morphogen:** Any of various chemicals in embryonic tissue that influence the movement and organization of cells during morphogenesis by forming a concentration gradient.
Reaction-diffusion systems

\[ U_t = D_U \Delta U + a - U - \rho R(U, V), \]
\[ V_t = D_V \Delta V + c(b - V) - \rho R(U, V), \]
where \[ R(U, V) = \frac{dUV}{e + fU + gU^2}. \]

Domain: rectangle
Boundary conditions:
  head and tail (no flux),
  body side (periodic)
The full reaction-diffusion system:

\[
\begin{aligned}
&u_t = u_{xx} + u_{yy} + \lambda f(u, v), \quad t > 0, \ (x, y) \in (0, a) \times (0, b), \\
v_t = d(v_{xx} + v_{yy}) + \lambda g(u, v), \quad t > 0, \ (x, y) \in (0, a) \times (0, b), \\
u_x = 0, \ x = 0, a, \ u(x, 0) = u(x, b), \ u_x(x, 0) = u_x(x, b), \\
v_x = 0, \ x = 0, a, \ v(x, 0) = v(x, b), \ v_x(x, 0) = v_x(x, b), \\
u(0, x, y) = u_0(x, y), \ v(0, x, y) = v_0(x, y).
\end{aligned}
\]

Solution of the system:

\[
\left(\begin{array}{c}
u(t, x, y) \\
v(t, x, y)
\end{array}\right) = \sum_{n,m=0}^{\infty} C_{n,m} e^{\mu_{n,m} t} V_{n,m} \cos \left(\frac{n\pi x}{a}\right) \cos \left(\frac{2m\pi y}{b}\right),
\]

where \(\mu_{n,m}\) are the eigenvalues of matrix \(\lambda J - k_{n,m}^2 D\), \(J\) is Jacobian, \(D = \left(\begin{array}{cc}1 & 0 \\0 & d\end{array}\right)\), and \(k_{n,m} = \left(\frac{n^2}{a^2} + \frac{4m^2}{b^2}\right)^{\frac{1}{2}}\pi^2\).
1. From Turing instability theory, the unstable mode and the spatial pattern have same spatial structure.

2. The unstable mode is determined by the parameter pair \((d, \lambda)\), where \(d = D_V/D_U\), \(\lambda = S/D_U\), and \(S\) is proportional to the size of the domain.

3. Spatial patterns can only occur when \(d\) is large; and when \(d\) is large, the patterns appear in the order of eigenvalue sequence \(k_{n,m} = \left(\frac{n^2}{a^2} + \frac{4m^2}{b^2}\right)\pi^2\).

4. When \(b\) is smaller, the striped patterns are more likely, which are constant in \(y\) direction; and when \(b\) is larger, the spot patterns are also possible, which are not constant in \(y\) direction.
Eigenvalues depend on the geometry of the rectangle (Geometry here means the narrowness of the rectangle.)

**Example 1:** \( b/a = 2 \)

\[
\begin{align*}
k_{0,0} &= 0, & k_{1,0} &= k_{0,1} = -\pi^2, & k_{1,1} &= -2\pi^2, & k_{2,0} &= k_{0,2} = -4\pi^2, \\
k_{2,1} &= k_{1,2} = -5\pi^2, & k_{2,2} &= -8\pi^2, & k_{3,0} &= k_{0,3} = -9\pi^2, & \ldots.
\end{align*}
\]

**Example 2:** \( b/a = 20 \)

\[
\begin{align*}
k_{0,0} &= 0, & k_{1,0} &= -\pi^2, & k_{2,0} &= -4\pi^2, & k_{3,0} &= -9\pi^2, \\
k_{4,0} &= -16\pi^2, & \ldots, & k_{10,0} = k_{0,1} = -100\pi^2, & \ldots.
\end{align*}
\]

**Theorem 1**: Snakes always have striped (ring) patterns, but not spotted patterns.

**Turing-Murray Theory**: snake is the example of \( b/a \) is large.
Snake pictures (stripe patterns)
“Theorem 2”: There is no animal with striped body and spotted tail, but there is animal with spotted body and striped tail.

**Turing-Murray theory**: The body is always wider than the tail. The same reaction-diffusion mechanism should be responsible for the patterns on both body and tail. Then if the body is striped, and the parameters are similar for tail and body, then the tail must also be striped since the narrower geometry is easier to produce strips.

Examples: zebra, tiger (striped body and tail), leopard (spotted body and tail), genet, cheetah (spotted body and striped tail)
Spotted body and striped tail or legs

Cheetah (upper), Okapi (lower)

Tiger (upper), Leopard (lower)
Spotted body and striped tail

Genet (left), Giraffe (right)
Natural Patterns of $\cos(kx)$

$\cos(x)$: Valais goat

(single color: $f(x)=1$, a lot of examples)
Cos(2x): Galloway belted Cow
cos(2x): Giant Panda
Other related researches

Patterns of sea shells

Patterns of tropical fishes
Waves in the Belousov-Zhabotinsky reaction

Boris P. Belousov
(Soviet Union, 1951, left)

Anatol M. Zhabotinsky
(Soviet Union, 1961, right)

Chemical reactions can be oscillatory (periodic)!
Real Turing patterns?

• Turing’s patterns are from a theory. Many phenomena may be explained by Turing’s theory, but it does not mean the real biological mechanism is governed by these equations. (we can only say maybe)

• Are there such patterns existing in real chemical reactions? (Biology is more complicated than chemistry)------It was not known for many years, so Turing’s theory was only a theory, after all. But……
Chlorite-Iodide-Malonic Acid (CIMA) reaction

- CIMA reaction spots
- CIMA reaction stripes
- Fish skin
- Leopard body
- Fingerprint
- Zebra stripes